Landau sum rules with noncentral quasiparticle interactions

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We derive explicit expressions for the Landau sum rules for the case of the most general spin-dependent quasiparticle interaction including all possible tensor interactions. For pure neutron matter, we investigate the convergence of the sum rules at different orders of approximation. Employing modern nuclear Hamiltonians based on chiral effective field theory, we find that the inclusion of noncentral interactions improves the convergence of the sum rules only for low densities ($n \lesssim 0.1$ fm^{-3}). Around nuclear matter saturation density, we find that even ostensibly perturbative nuclear interactions violate the sum rules considerably. By artificially weakening the strength of the nuclear Hamiltonian, the convergence can be improved.

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I. INTRODUCTION

In Landau's theory of normal Fermi liquids [1–6], the properties of a system made of strongly interacting fermions are described in terms of weakly interacting quasiparticles. For low-energy and long-wavelength excitations, the quasiparticles interact near the Fermi surface via a particle-hole (ph) interaction that is fully characterized by the so-called Landau parameters. Some of these parameters are related to macroscopic properties of the system such as the specific heat, the incompressibility, or the spin susceptibility, which in systems such as liquid helium-3 or cold atoms are obtained directly from experimental measurements [7,8]. In isospin-symmetric nuclear matter, these quantities can be deduced only indirectly from experiments, and therefore the Landau parameters are often calculated from phenomenological or microscopic interactions. In principle, the Landau interaction is the long-wavelength limit of the true interaction, so that it can provide a useful benchmark to compare different interactions and methods. The Landau parameters are constrained under general grounds. First, they must satisfy certain sum rules which follow from the Pauli exclusion principle applied to the forward scattering amplitude [9–11]. Second, the stability of the spherical Fermi surface against small deformations can be expressed in terms of inequalities involving combinations of the Landau parameters [3,5].

Since the first application of the theory to atomic nuclei [5], it has been recognized that the most important contributions to the quasiparticle interaction come in the form of central interactions that depend on quasiparticle spin and isospin. The inclusion of a tensor component to the quasiparticle interaction was made by Dabrowski and Haensel [12–14], and it was shown that tensor terms modify the stability criteria [15] and forward scattering sum rules [16]. In these references, the noncentral component was assumed to have the same form as the one-pion exchange interaction

$$S_{12}(\hat{\mathbf{k}}_{12}) = 3(\hat{\mathbf{k}}_{12}\cdot\boldsymbol{\sigma})(\hat{\mathbf{k}}_{12}\cdot\boldsymbol{\sigma}') - (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'), \tag{1}$$

where $\mathbf{k}_{12} = \mathbf{k} - \mathbf{k}'$, and $\mathbf{k}$, σ refer to the momentum and spin of the quasiparticles. Several authors have highlighted the importance of tensor terms to properly describe the properties of various Fermi liquids such as nuclear matter [17,18], atomic nuclei [19], and cold atom [20,21] systems. Within nuclear physics, the Landau theory has also been used successfully to study the properties of excited states by describing the nuclear response function in both the static [22–24] and dynamic cases [25,26].

Schwenk and Friman [17] have pointed out that in a many-body medium the presence of the Fermi sea defines a preferred frame, and two additional tensor contributions to the quasiparticle interaction can be constructed from the center-of-mass momentum vector $\mathbf{P}_{12} = \mathbf{k} + \mathbf{k}'$. These have the form

$$K_{12}(\hat{\mathbf{P}}_{12}) = 3(\hat{\mathbf{P}}_{12}\cdot\boldsymbol{\sigma})(\hat{\mathbf{P}}_{12}\cdot\boldsymbol{\sigma}') - (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'), \tag{2}$$

$$A_{12}(\hat{\mathbf{k}}_{12}, \hat{\mathbf{P}}) = (\boldsymbol{\sigma}\cdot\hat{\mathbf{P}}_{12})(\boldsymbol{\sigma}'\cdot\hat{\mathbf{k}}_{12}) - (\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}_{12})(\boldsymbol{\sigma}'\cdot\hat{\mathbf{P}}_{12}) \tag{3}$$

and are designated as the center-of-mass tensor and cross-vector interactions, respectively. The latter arises at second

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order in perturbation theory from the coupling of spin-orbit terms in the free-space interaction with any other non-spin-orbit term [27]. As discussed in Ref. [18], these three tensor terms contribute with a similar magnitude and thus they should be treated on the same footing. By performing specific calculations, it is then possible to quantify their impact on specific observables, as done in Ref. [28] for the case of the response function for pure neutron matter.

In the present article, we address a problem presented in Ref. [27] concerning the violation of specific sum rules [16] for Landau parameters extracted from modern nuclear Hamiltonians based on chiral effective field theory. In that article, the authors observed a clear violation of the forward scattering sum rule in pure neutron matter when the quasiparticle interaction was computed up to second order in perturbation theory. On the one hand, this might not seem surprising since the quasiparticle scattering amplitude and sum rules depend nonlinearly on the Fermi liquid parameters. In particular, Babu and Brown [29] emphasized that an infinite number of terms would be needed in the quasiparticle interaction to match the exchange diagrams generated in the integral equation for the quasiparticle scattering amplitude.[1] On the other hand, starting from the same perturbative nucleon-nucleon interactions, it was found (1) that the equation of state of pure neutron matter is well converged at second order in perturbation theory [30] and (2) that Weinberg eigenvalue analyses [31,32] of the free-space and in-medium particle-particle scattering amplitudes imply a rapid convergence of the Born series. Therefore, one might expect a second-order perturbation theory calculation of the Fermi liquid parameters to be sufficient for a well converged quasiparticle scattering amplitude. In Ref. [27] it was assumed that the violation of the sum rule might be due to an incorrect approximation done in deriving it, i.e., the lack of an explicit tensor term in the derivation of the equations. To address such an issue, in the present work we perform a complete derivation of the sum rules following Ref. [16], but including explicitly the effects of the most general tensor terms as detailed in Ref. [17].

II. SUM RULES

In order to simplify the notation, we start by considering a uniform system made of spin-saturated neutrons. Such a system, hereafter named pure neutron matter (PNM), is very convenient in order to perform sophisticated many-body calculations in order to test various models without the additional difficulties related to finite size effects as in atomic nuclei; moreover PNM can be considered as a first approximation to describe macroscopic systems such as neutron stars. The current formalism can also be easily extended to an infinite system made of an equal number of neutrons and protons: symmetric nuclear matter (SNM). The details on how to perform such an extension are given at the end of this section.

[1]In contrast, a calculation of the quasiparticle scattering amplitude directly at some given order in perturbation theory should automatically satisfy the sum rules.

In the long-wavelength limit, the quasiparticles are restricted to be at the surface of the Fermi sphere ($|\mathbf{k}| = |\mathbf{k}'| = k_F$) so that the interaction depends only on k_F and the relative angle θ between vectors $\mathbf{k}$ and $\mathbf{k}'$. It is convenient to deal with dimensionless quantities. We thus divide the interaction by the density of states $N_0 = k_F m^*/\pi^2$, where k_F and m^* are the Fermi momentum and quasiparticle effective mass, respectively. The PNM ph interaction is written as

$$\mathcal{F}(\mathbf{k},\sigma;\mathbf{k}',\sigma') = F(\theta) + G(\theta)(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}') + H(\theta)\frac{\mathbf{k}_{12}^2}{k_F^2}S_{12}(\hat{\mathbf{k}}_{12}) + K(\theta)\frac{\mathbf{P}_{12}^2}{k_F^2}S_{12}(\hat{\mathbf{P}}_{12}) + L(\theta)\frac{(\mathbf{P}_{12}\cdot\mathbf{k}_{12})}{k_F^2}A_{12}(\hat{\mathbf{k}}_{12},\hat{\mathbf{P}}_{12}). \tag{4}$$

The dimensionless functions F, G, H, K, L encode the angular dependence of the quasiparticle interaction in the different spin channels and can be expanded in the form

$$F(\theta) = \sum_\ell F_\ell P_\ell(\cos\theta), \tag{5}$$

and similarly for the other functions. The coefficients $F_\ell, G_\ell, H_\ell, K_\ell, L_\ell$ are the dimensionless Landau parameters.

Tensor terms have been written in Eq. (4) following the conventional definition given in Ref. [14]. However, some authors [17,23,27,33] have defined them without the prefactors $\mathbf{k}_{12}^2/k_F^2$, $\mathbf{P}_{12}^2/k_F^2$, and $(\mathbf{P}_{12}\cdot\mathbf{k}_{12})/k_F^2$ because this leads to a faster convergence [23] in the sense that the absolute values of the parameters H_ℓ, K_ℓ, L_ℓ decrease as ℓ increases. Although the physical information contained in the ph interaction is the same in both cases, the Landau parameters are different. Both sets of parameters are actually connected through a recurrence relation [23,28]. However, the form (4) is more suited to the deduction of the sum rules.

Analogously to Eq. (4), we write the expression for the forward scattering amplitude as

$$\mathcal{A}(\mathbf{k},\sigma;\mathbf{k}',\sigma') = B(\theta) + C(\theta)(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}') + D^{(H)}(\theta)\frac{\mathbf{k}_{12}^2}{k_F^2}S_{12}(\hat{\mathbf{k}}) + D^{(K)}(\theta)\frac{\mathbf{P}_{12}^2}{k_F^2}S_{12}(\hat{\mathbf{P}}_{12}) + D^{(L)}(\theta)\frac{(\mathbf{P}_{12}\cdot\mathbf{k}_{12})}{k_F^2}A_{12}(\hat{\mathbf{k}}_{12},\hat{\mathbf{P}}_{12}). \tag{6}$$

The scattering amplitude and the ph interaction are related via an integral equation [3,16],

$$\mathcal{A}(\mathbf{k},\sigma;\mathbf{k}',\sigma') = \mathcal{F}(\mathbf{k},\sigma;\mathbf{k}',\sigma') - \frac{1}{2}\mathrm{Tr}_{\sigma''}\int\frac{d\hat{\mathbf{k}}''}{4\pi}\mathcal{F}(\mathbf{k},\sigma;\mathbf{k}''\sigma'')\mathcal{A}(\mathbf{k}''\sigma'';\mathbf{k}'\sigma'), \tag{7}$$

where the factor 1/2 is related to the spin degrees of freedom. This expression follows by carefully taking the low-energy, long-wavelength limit of the Bethe-Salpeter equation resumming particle-hole ladder diagrams. Its solution is easily obtained in the spin-independent channel, with the well-known result

$$B_\ell = \frac{F_\ell}{1+\frac{1}{2\ell+1}F_\ell}. \tag{8}$$

In the absence of tensor terms, the parameters C_ℓ are similarly obtained as

$$C_\ell = \frac{G_\ell}{1 + \frac{1}{2\ell+1}G_\ell}. \tag{9}$$

Such a result was already given in Ref. [16], where we refer the reader for additional details. However, in the presence of tensor terms, the sum rule involving the spin-dependent Landau parameters becomes more complicated. In order to obtain the modified expression, we have to recouple the orbital angular momentum and the spin of the quasiparticles to a good total angular momentum J. The ph interaction written in the angular momentum coupled form reads

$$\mathcal{F}^J_{\ell\ell'} = \frac{1}{2}\frac{1}{4\pi}[(2\ell+1)(2\ell'+1)]^{1/2} \times \sum_{mm'm_sm'_s}(\ell,1,J;m,m_s,M)(\ell',1,J;m',m'_s,M) \times \int d\hat{k}\, d\hat{k}' Y^*_{\ell m}(\hat{k})Y_{\ell' m'}(\hat{k}')\langle 1,m_s|\mathcal{F}(\mathbf{k},\sigma;\mathbf{k}',\sigma')|1,m'_s\rangle, \tag{10}$$

where $|1,m_s\rangle$ is the spin state of the ph pair, and $(L,S,J;M_L,M_S,M)$ is a Clebsh-Gordan coefficient. The explicit expression for $\mathcal{F}^J_{\ell\ell'}$ in terms of Landau parameters is given in Appendix A. Proceeding analogously for the scattering amplitude, Eq. (7) takes the form

$$\mathcal{A}^J_{\ell\ell'} = \mathcal{F}^J_{\ell\ell'} - \sum_{\ell''}\frac{1}{2\ell''+1}\mathcal{F}^J_{\ell\ell''}\mathcal{A}^J_{\ell''\ell'}. \tag{11}$$

Solving this equation, one fully determines the forward scattering amplitude in terms of Landau parameters.

A. Scattering amplitude and Landau parameters

The solutions of the system (11) were given in Ref. [16]. For completeness, we rewrite them here. One must distinguish two cases: first, for $(\ell,\ell') = (J\pm 1, J\mp 1)$ one deals with a system of coupled equations, whose solutions are written as

$$\mathcal{A}^J_{\ell\ell} = \frac{1}{\Delta^J_{\ell\ell'}}\left[\mathcal{F}^J_{\ell\ell}\left(1+\frac{\mathcal{F}^J_{\ell'\ell'}}{2\ell'+1}\right) - \frac{1}{2\ell'+1}\left(\mathcal{F}^J_{\ell\ell'}\right)^2\right], \tag{12}$$

$$\mathcal{A}^J_{\ell\ell'} = \frac{1}{\Delta^J_{\ell\ell'}}\mathcal{F}^J_{\ell\ell'}, \tag{13}$$

$$\Delta^J_{\ell\ell'} = \left(1+\frac{\mathcal{F}^J_{\ell\ell}}{2\ell+1}\right)\left(1+\frac{\mathcal{F}^J_{\ell'\ell'}}{2\ell'+1}\right) - \frac{1}{(2\ell+1)(2\ell'+1)}\left(\mathcal{F}^J_{\ell\ell'}\right)^2. \tag{14}$$

In the second case we have $J=\ell=\ell'$ and $J=0$, $\ell=\ell'=1$. Only a single equation remains, whose solution is

$$\mathcal{A}^J_{\ell\ell} = \frac{\mathcal{F}^J_{\ell\ell}}{1+\mathcal{F}^J_{\ell\ell}/(2\ell+1)}. \tag{15}$$

As stated in Ref. [16], the denominators of $\mathcal{A}^J_{\ell\ell}$ provide the simple generalization of the stability criteria associated with the Landau parameters [24]:

$$1+\frac{1}{2l+1}\mathcal{F}^J_{ll} > 0. \tag{16}$$

Notice that when $\ell = 1$ this inequality is not generally valid. As Kiselev *et al.* have shown [34], in case one deals with operators that do not correspond to conserved quantities in the long-wavelength limit, there are nonquasiparticle contributions to the response functions. With these results, the parameters C_ℓ are then obtained using the relation

$$C_\ell = \frac{1}{3}\sum_J \frac{2J+1}{2\ell+1}\mathcal{A}^J_{\ell\ell}. \tag{17}$$

It is interesting to observe that in this sum the tensor parameters $D^{(H)}_\ell, D^{(K)}_\ell, D^{(L)}_\ell$ are exactly canceled out. However, C_ℓ does depend on the Landau tensor parameters H_ℓ, K_ℓ, L_ℓ via the matrix elements $\mathcal{F}^J_{\ell\ell'}$.

B. Sum rules

The Pauli exclusion principle imposes a restriction on the scattering amplitude and hence on the Landau parameters. Indeed, antisymmetry of the wave function requires that the forward scattering amplitude satisfies

$$P(\mathbf{k}\leftrightarrow\mathbf{k}')P_\sigma\mathcal{A}(\mathbf{k},\sigma;\mathbf{k}',\sigma') = -\mathcal{A}(\mathbf{k}',\sigma';\mathbf{k},\sigma), \tag{18}$$

where the $P(\mathbf{k}\leftrightarrow\mathbf{k}')$ and P_σ are exchange operators of momentum and spin, respectively. In other words, in PNM the *forward* scattering amplitude should vanish in the triplet state when $\mathbf{k}=\mathbf{k}'$. A very interesting consequence is that in this configuration there is no contribution coming from the coupling $S=0$ and $S=1$ as discussed in Ref. [27]. Looking to Eq. (7) one can see that the tensor term involving $D^{(K)}$ does not vanish in that case, contrary to terms involving $D^{(H)}$ and $D^{(L)}$. Working with states of good angular momentum, as in Eq. (10), one obtains the following triplet sum rule:

$$\sum_\ell\left(B_\ell + C_\ell - 2\left[\frac{D^{(K)}_\ell}{2\ell+1} + 2\frac{D^{(K)}_{\ell+1}}{2\ell+3} + \frac{D^{(K)}_{\ell+2}}{2\ell+5}\right] \times \frac{3\sqrt{(\ell+1)(\ell+2)(2\ell+1)(2\ell+5)}}{2\ell+3}\right) = 0. \tag{19}$$

We notice that this equation depends on all Landau parameters $F_\ell, G_\ell, H_\ell, K_\ell, L_\ell$, whose dependence is included through C_ℓ by solving Eq. (7).

From these results, it is possible to generalize to the SNM case. First of all, isospin should be included in both the ph interaction and the forward scattering amplitude. In Eqs. (4)–(6) one has to duplicate all terms, multiplied with $(\boldsymbol{\tau}\cdot\boldsymbol{\tau}')$, and labeling the new functions as $B', C', D^{(H')}, D^{(K')}, D^{(L')}$. The ph interaction and the scattering amplitude must include an isospin index, and in Eq. (10) one uses the notation $\mathcal{F}^{JT}_{\ell\ell'}$ for the ph interaction (and similarly for the scattering amplitude). The Pauli principle constrains the spin-singlet/isospin-triplet and spin-triplet/isospin-singlet states, leading to two sum

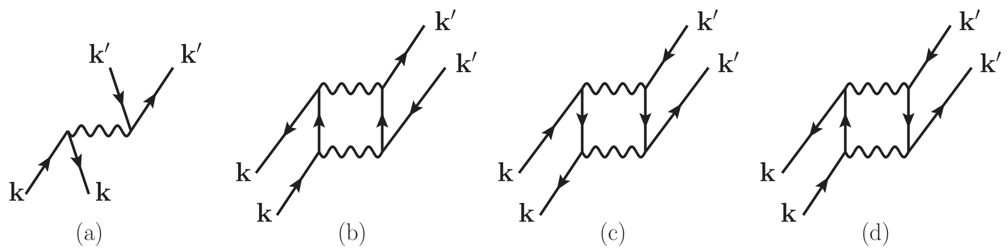

FIG. 1. First- and second-order diagrammatic contributions to the quasiparticle interaction consisting of (a) the first-order term and the second-order (b) particle-particle, (c) hole-hole, and (d) particle-hole terms.

rules:

$$\sum_\ell (B_\ell - 3C_\ell - 3B'_\ell + 9C'_\ell) = 0, \tag{20}$$

$$\begin{aligned}\sum_\ell \Bigg(& B_\ell + C_\ell - 2\left[\frac{D_\ell^{(K)}}{2\ell+1} + 2\frac{D_{\ell+1}^{(K)}}{2\ell+3} + \frac{D_{\ell+2}^{(K)}}{2\ell+5}\right] \\ & \times \frac{3\sqrt{(\ell+1)(\ell+2)(2\ell+1)(2\ell+5)}}{2\ell+3} \\ & + B'_\ell + C'_\ell - 2\left[\frac{D_\ell^{(K')}}{2\ell+1} + 2\frac{D_{\ell+1}^{(K')}}{2\ell+3} + \frac{D_{\ell+2}^{(K')}}{2\ell+5}\right] \\ & \times \frac{3\sqrt{(\ell+1)(\ell+2)(2\ell+1)(2\ell+5)}}{2\ell+3}\Bigg) = 0. \end{aligned} \tag{21}$$

Notice that the singlet sum rule, Eq. (20), implicitly depends on the tensor Landau parameters through Eq. (11).

III. PURE NEUTRON MATTER RESULTS FROM CHIRAL EFFECTIVE FIELD THEORY

Having extended the formalism of the sum rules to include all possible noncentral interactions, we are now in a position to test the sum rules when using Landau parameters extracted from a microscopic nuclear Hamiltonian. Previous works [18,27,35] have included chiral effective field theory three-body forces in the calculation of the Landau parameters. In the present study, it is sufficient to consider only two-body forces, since the sum rules should be valid for any model of the nuclear force. The inclusion of three-body forces would result in more realistic values of the Fermi liquid parameters, but their presence would not remedy sum rule convergence problems arising already at the level of two-body forces alone. We include contributions to the quasiparticle interaction up to second order in perturbation theory, consisting of a single first-order antisymmetrized term together with three second-order antisymmetrized terms with particle-particle, hole-hole, and particle-hole intermediate states, as shown in Fig. 1. For simplicity we consider only a free-particle spectrum for the intermediate-state energies appearing in the expressions for the second-order perturbation theory diagrams.

As discussed in Ref. [35], the Landau parameters extracted with this approach do not satisfy the sum rules without the tensor terms; meaning that the truncation at second order may not be sufficient to have a full convergence. In addition, Eq. (7) is an integral equation that iterates the quasiparticle interaction to all orders in a series of reducible particle-hole bubble diagrams. The exchange terms needed to enforce antisymmetry, however, should be included as particle-hole irreducible diagrams in the direct interaction to all orders. Hence, keeping only the first-order contribution and the second-order particle-hole contribution to the quasiparticle interaction should provide a better approximation for fulfilling the sum rules, which we will test below.

Since the Landau parameters depend on the density of the system, we have performed our calculations at four densities $\rho_n = 0.05, 0.1, 0.15, 0.2$ fm^{-3}. As an example, we report in Table I the Landau parameters obtained from the N3LO-450 chiral nucleon-nucleon interaction of Ref. [36], considering all relevant contributions the quasiparticle interaction at first and second order. As we see from the table, the convergence of the tensor Landau parameters defined in Eq. (4) is extremely

TABLE I. Dimensionless Landau parameters at $\rho_n = 0.1$ fm^{-3} including all contributions at first and second order.

ℓ	F_ℓ	G_ℓ	H_ℓ	K_ℓ	L_ℓ
0	−0.7921	1.0593	0.1009	−0.0433	0.0253
1	0.1568	0.5663	0.0047	0.0236	0.1136
2	0.0342	0.1855	−0.0828	0.0501	0.0945
3	0.1239	0.1388	−0.0590	0.0181	0.0151
4	0.0235	0.0706	−0.0460	0.0018	−0.0137
5	0.0904	0.0554	−0.0084	0.0044	−0.0459
6	0.0107	0.0110	0.0003	0.0032	−0.0467
7	0.0669	0.0112	0.0195	0.0020	−0.0311
8	0.0070	−0.0098	0.0139	0.0013	−0.0183
9	0.0453	−0.0021	0.0185	0.0005	−0.0122
10	−0.0018	−0.0113	0.0091	0.0003	−0.0081
11	0.0308	−0.0009	0.0114	0.0001	−0.0066
12	−0.0067	−0.0077	0.0039	0.0001	−0.0051
13	0.0217	0.0009	0.0067	0.0000	−0.0046
14	−0.0089	−0.0052	0.0011	0.0000	−0.0037
15	0.0163	0.0018	0.0041	−0.0000	−0.0034
16	−0.0095	−0.0038	−0.0002	0.0000	−0.0029
17	0.0129	0.0021	0.0028	0.0000	−0.0026
18	−0.0094	−0.0030	−0.0008	−0.0000	−0.0023
19	0.0107	0.0021	0.0020	0.0000	−0.0021
20	−0.0087	−0.0025	−0.0011	0.0000	−0.0019

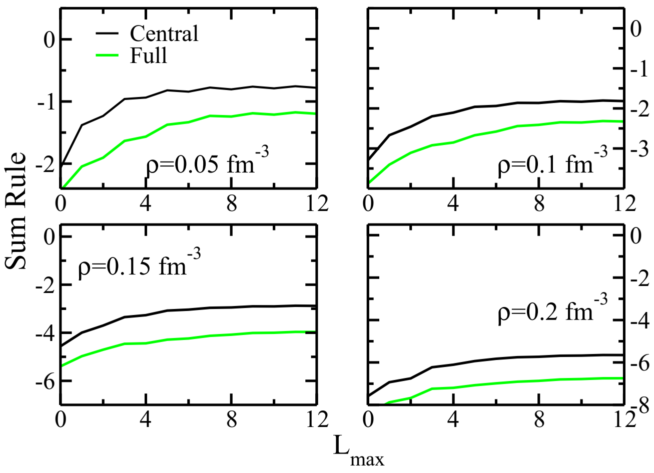

FIG. 2. Convergence of the sum rule given in Eq. (19) at several densities as a function of the maximum value of the partial wave included in the calculations. The Landau parameters have been obtained taking into account the full interaction [case "1+2(full)"].

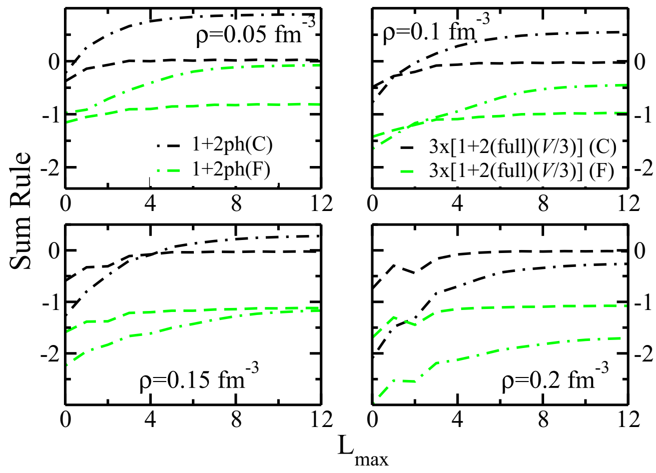

FIG. 3. Same as Fig. 3, but for cases "1+2(ph)" (dash-dotted) and "3×[1+2(full) $(V/3)$]" (dashed lines). See text for details.

slow and one has to go to very high values of the multipolar expansion in order to observe convergence.

In order to test the relevance of the tensor interactions on the forward scattering sum rules, we perform three types of calculations:

(1) Central (C). We test the sum rule (SR) ignoring both the tensor coupling and the tensor terms of the interaction, as done in Ref. [16].
(2) Full (F). We take into account all the terms in Eq. (19) and all the terms in the quasiparticle interaction.

We also need to distinguish some cases in the way we manipulate the Landau parameters obtained from the χEFT, in particular we consider the cases

(1) Case "1+2(full)." We take into account the full interaction at first and second order, by also including all the particle-particle, particle-hole and hole-hole terms.
(2) Case "1+2(ph)." We keep only the first-order and second-order particle-hole diagrams, leaving out the second-order particle-particle and hole-hole diagrams.
(3) Case "3×[1+2(full) $(V/3)$]." We take into account the full interaction at first and second order, by also including all the particle-particle, particle-hole, and hole-hole terms, but we re-scale each order by a factor $1/\alpha$, here $\alpha = 3$. Notice that, to avoid a trivial reduction of the SR, we rescale the final result by a factor 3.

The idea of rescaling the Landau parameters by a factor $1/\alpha$ at each order is useful in order to make the residual interaction more perturbative so that the diagrammatic truncation does not introduce further errors on the convergence. Case "1+2(ph)," instead, is useful to illustrate the role of the pp and hh interaction at second order.

In Fig. 2, we illustrate the evolution of the sum rule given in Eq. (19) as a function of the maximum partial wave included in the calculations and for case "1+2(full)," as discussed above. The SR converges quite fast to a value that is quite large and nonzero around $L_{\max} = 8$. Higher order Landau parameters give essentially no contribution to the SR. Contrary to what was hypothesized in Ref. [18], neither the tensor terms or the tensor couplings can improve the situation in general.

In order to assess the role of the various perturbation theory contributions to the total SR, in Fig. 3, we show the evolution of the SR for cases "1+2(ph)" and "3×[1+2(full) $(V/3)$]." By leaving out the particle-particle and hole-hole diagrams, we observe a clear improvement of the SR at all densities, where the values get much closer to zero. Finally, by reducing the strength of the interaction (case "3×[1+2(full) $(V/3)$]"), one sees that the values of the SR get very close to zero, even after rescaling by a factor of 3 for the total sum. But even in this case the effect of the tensor forces is very mild as already argued in Ref. [37].

For completeness we report the numerical values of the SR in Table II at convergence in L. As already observed from the figure, one can appreciate here the case "1+2(full)" shows the largest violations of the SR, which mainly originates from

TABLE II. Numerical values of the sum rule Eq (19) for different densities and cases as discussed in the text.

Density (fm^{-3})	Case	Central	Full
0.05	1+2(full)	−0.780	−1.196
	1+2(ph)	0.884	−0.075
	3×[1+2(full) $(V/3)$]	0.020	−0.816
0.10	1+2(full)	−1.818	−2.327
	1+2(ph)	0.550	−0.449
	3×[1+2(full) $(V/3)$]	−0.027	−0.980
0.15	1+2(full)	−2.8882	−3.965
	1+2(ph)	0.278	−1.170
	3×[1+2(full) $(V/3)$]	−0.026	−1.122
0.20	1+2(full)	−5.651	−6.745
	1+2(ph)	−0.262	−1.704
	3×[1+2(full) $(V/3)$]	−0.018	−1.081

the central terms; the tensor terms are not important enough to compensate for such a deviation.

IV. CONCLUSIONS

We have derived the Landau parameter sum rules including explicitly all tensor terms. The result is general and applies to any system of fermions where Landau theory is applicable, such as liquid helium-3 and cold atom systems. We have also considered in more detail the application of our results to an interesting system for nuclear physics calculations, i.e., the case of pure neutron matter. At second order in perturbation theory, we find that the sum rules are still strongly violated, even employing chiral effective field theory Hamiltonians that have been shown to be perturbative in calculations of the equation of state. We have shown that the effects coming from the additional tensor terms that depend on the center-of-mass momentum are typically one order of magnitude smaller than the contributions arising from the central terms and thus not sufficient to explain the violation of Pauli principle reported in Ref. [27] at moderate densities. The origin of such a violation of the Pauli principle appears to be missing higher-order diagrams in the direct channel that are needed to cancel the exchange diagrams that are iterated to all orders in the quasiparticle scattering amplitude. By keeping only the particle-hole bubble diagram at second order or by reducing the overall strength of the nuclear Hamiltonian, we find an improvement in satisfying the sum rule.

APPENDIX A: THE PH INTERACTION WITH GOOD J

The matrix elements of the Landau interaction are written as

$$\begin{aligned}\mathcal{F}^{J}_{\ell,\ell'} &= \delta_{\ell\ell'}G_\ell + \left[\frac{H_\ell+K_\ell}{2\ell+1}+\frac{H_{\ell'}+K_{\ell'}}{2\ell'+1}\right]\frac{3(2\ell+1)(2\ell'+1)}{2J+1}(1\ell J;000)(1\ell' J;000)\\ &\quad -\left(H_J-K_J+\frac{2}{3}L_J\right)\frac{3(2\ell+1)(2\ell'+1)}{(2J+1)^2}(1\ell J;000)(1\ell' J;000) - 3[(2\ell+1)(2\ell'+1)]^{1/2}(-)^{1+\ell-J}\\ &\quad \times\sum_\lambda\left(H_\lambda-K_\lambda-\frac{2}{3}L_\lambda\right)(1\lambda\ell;000)(1\lambda\ell';000)W(J\ell'\ell\lambda;11) - 2\delta_{\ell\ell'}(H_\ell+K_\ell) + 2\delta_{\ell\ell'}\sum_\lambda(H_\lambda-K_\lambda)(1\ell\lambda;000)^2,\end{aligned} \tag{A1}$$

where $W(abcd;ef)$ is a $6j$ coefficient. An analogous expression is valid for the scattering amplitude, by replacing the Landau parameters with the corresponding C_ℓ, $D^{(H)}_\ell$, $D^{(K)}_\ell$, $D^{(L)}_\ell$. For simplicity, we provide here some useful results:

$$\mathcal{F}^{\ell+1,0}_{\ell,\ell} = G_\ell - \frac{\ell}{2\ell-1}H_{\ell-1} + 2\frac{\ell}{2\ell+3}H_\ell - \frac{\ell(2\ell-1)}{(2\ell+3)^2}H_{\ell+1}, \tag{A2}$$

$$\mathcal{F}^{\ell+1,0}_{\ell+2,\ell+2} = G_{\ell+2} - \frac{(\ell+3)(2\ell+7)}{(2\ell+3)^2}H_{\ell+1} + \frac{2(\ell+3)}{2\ell+3}H_{\ell+2} - \frac{\ell+3}{2\ell+7}H_{\ell+3}, \tag{A3}$$

$$\mathcal{F}^{\ell+1,0}_{\ell,\ell+2} = \left[-\frac{H_\ell}{2\ell+1}+2\frac{H_{\ell+1}}{2\ell+3}-\frac{H_{\ell+2}}{2\ell+5}\right]\frac{3[(2\ell+1)(2\ell+5)(\ell+1)(\ell+2)]^{1/2}}{2\ell+3}, \tag{A4}$$

$$\mathcal{F}^{\ell,0}_{\ell,\ell} = G_\ell + \frac{2\ell+3}{2\ell-1}H_{\ell-1} - 2H_\ell + \frac{2\ell-1}{2\ell+3}H_{\ell+1}. \tag{A5}$$

Notice that in the spin-independent channel the tensor does not act, and these expressions simplify to $\mathcal{F}^{J}_{\ell,\ell'} = \delta_{\ell,\ell'}\delta_{J,\ell}G_\ell$ and $\mathcal{A}^{J=\ell}_{\ell,\ell'} = \delta_{\ell,\ell'}\delta_{J,\ell}C_\ell$.

APPENDIX B: SOLVING THE SYSTEM TO GET C, $D^{(K)}$ IN TERMS OF LANDAU PARAMETERS

To simplify the notation, let us define new parameters

$$X_\ell = D^{(H)}_\ell - D^{(K)}_\ell, \quad Y_\ell = D^{(H)}_\ell + D^{(K)}_\ell, \quad Z_\ell = \frac{2}{3}D^{(L)}_\ell. \tag{B1}$$

Then, the system of equations can be written as sets of four equations,

$$-\frac{(2\ell+3)(\ell+1)}{(2\ell-1)^2}X_{\ell-1} - \frac{3(\ell+1)}{2\ell-1}Z_{\ell-1} + C_\ell + \frac{2(\ell+1)}{2\ell-1}Y_\ell = \mathcal{A}^{\ell-1}_{\ell,\ell} + \frac{\ell+1}{2\ell+3}X_{\ell+1} - \frac{3(\ell+1)}{2\ell+3}Z_{\ell+1}, \tag{B2}$$

$$-\frac{\ell}{2\ell-1}X_{\ell-1} + \frac{3\ell}{2\ell-1}Z_{\ell-1} + C_\ell + \frac{2\ell}{2\ell+3}Y_\ell = \mathcal{A}^{\ell+1}_{\ell,\ell} + \frac{\ell(2\ell-1)}{(2\ell+3)^2}X_{\ell+1} + \frac{3\ell}{2\ell+3}Z_{\ell+1}, \tag{B3}$$

$$\frac{2\ell+3}{2\ell-1}X_{\ell-1} - \frac{3}{2\ell-1}Z_{\ell-1} + C_\ell - 2Y_\ell = \mathcal{A}^{\ell}_{\ell,\ell} - \frac{2\ell-1}{2\ell+3}X_{\ell+1} - \frac{3}{2\ell+3}Z_{\ell+1}, \tag{B4}$$

$$-\frac{1}{2\ell+1}Y_\ell = \mathcal{A}^{\ell+1}_{\ell,\ell+2}\frac{2\ell+3}{3\sqrt{(\ell+1)(\ell+2)(2\ell+1)(2\ell+5)}} - \frac{2}{2\ell+3}X_{\ell+1} + \frac{1}{2\ell+5}Y_{\ell+2}. \tag{B5}$$

One starts by solving the system for $\ell = L_m$, assuming $X_{L_m+1} = 0,\ Y_{L_m+2} = 0,\ Z_{L_m+1} = 0$. This fixes $X_{L_m-1}, Z_{L_m-1}, C_{L_m}, Y_{L_m}$. Next, one solves the system for $\ell = L_m - 1$ assuming $X_{L_m} = 0,\ Y_{L_m+1} = 0,\ Z_{L_m} = 0$. This fixes $X_{L_m-2}, Z_{L_m-2}, C_{L_m-1}, Y_{L_m-1}$. For a lower value of ℓ, one uses the previously obtained values of $X_{\ell+1}, Y_{\ell+2}, Z_{\ell+1}$, and solving the system gives $X_{\ell-1}, Z_{\ell-1}, C_\ell, Y_\ell$.

For $\ell = 1$, the system is

$$-10X_0 - 6Z_0 + C_1 + 4Y_1 = \mathcal{A}^0_{1,1} + \frac{2}{5}X_2 - \frac{6}{5}Z_2, \tag{B6}$$

$$-X_0 + 3Z_0 + C_1 + \frac{2}{5}Y_1 = \mathcal{A}^2_{1,1} + \frac{1}{25}X_2 + \frac{3}{5}Z_2, \tag{B7}$$

$$5X_0 - 3Z_0 + C_1 - 2Y_1 = \mathcal{A}^1_{1,1} - \frac{1}{5}X_2 - \frac{3}{5}Z_2 \tag{B8}$$

$$-\frac{1}{3}Y_1 = \frac{5}{9\sqrt{14}}\mathcal{A}^2_{1,3} - \frac{2}{5}X_2 + \frac{1}{7}Y_3. \tag{B9}$$

For $\ell = 0$, there are two equations:

$$C_0 = \mathcal{A}^1_{0,0}, \tag{B10}$$

$$-Y_0 = \frac{1}{\sqrt{10}}\mathcal{A}^1_{0,2} - \frac{2}{3}X_1 + \frac{1}{5}Y_2. \tag{B11}$$

Finally

$$D^{(K)}_\ell = -(Y_\ell - X_\ell)/2. \tag{B12}$$

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