Quantum information in nucleon-nucleon scattering

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Nucleon-nucleon scattering is a fundamental process in low-energy nuclear physics. Although its spin correlations have been studied intensively, few works take the perspective of quantum information science. In this work, I explore the quantum information aspects of spin correlations in the partially polarized neutron-proton scattering in the *S* wave. It is found that the spin mutual information, entanglement and discord of the out-states, when averaged over the possible in-states, all vary in a similar way with respect to the relative momentum. The novel connection between emergent symmetry and spin entanglement found by a previous study [Phys. Rev. Lett. **122**, 102001 (2019)] could then be extended naturally to spin correlations beyond entanglement.

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I. INTRODUCTION

Nucleon-nucleon scattering is one of the most important processes in nuclear physics. It plays an irreplaceable role in understanding the low-energy properties of quantum chromodynamics (QCD) and provides the most direct probe of the nucleon-nucleon interaction. The dynamical properties of the nucleon-nucleon scattering have been studied intensively. Several sophisticated models have been proposed to describe its cross sections, phase shifts, and spin correlations to high precision, such as AV18 [1], CD-Bonn [2], and chiral potentials [3–7].

On the other hand, tremendous progress has been made in quantum information science during the last three decades [8–11]. A number of new figures of merit have been proposed to characterize quantum systems, and those associated with correlations are of particular interest. Generally speaking, correlations refer to the extra information generated by taking separate ingredients as a whole system and could be divided into the classical and quantum ones. Classical correlations may be understood as the correlations with meaningful counterparts in classical mechanics, while quantum correlations may be identified as the complementary set of classical correlations. Entanglement [12] and discord [13–15] are two kinds of quantum correlations, both rooted in the defining features of quantum mechanics. Explicitly, entanglement results from the joint effect of the tensor product structure of multipartite Hilbert space and the fundamental principle of superposition, while discord is closely related to quantum measurements. Typically, a quantum state with nonzero entanglement also has nonzero discord, but not vise versa.

Although numerous efforts have been made to understand spin correlations[1] in the nucleon-nucleon scattering (see, e.g., Refs. [16–28]), few of them take the standpoint of quantum information science.[2] There are a few reasons for exploring this direction, among which the first and foremost is that quantum information science can provide brand-new understanding on a number of conventional facets of nuclear physics and thus boost it from an unconventional perspective. In 2019, Beane *et al.* adopted a quantum information measure called the entanglement power [30] to study the spin entanglement generated by the polarized $n + p$ scattering in the *S* wave [31]. They observed a novel connection between the Wigner and Schrödinger symmetries [32–35] and entanglement minimization, suggesting that low-energy symmetries of QCD could have a quantum information understanding. This unexpected result was soon generalized to the hadron-hadron scattering [31,36,37], the nucleon-nucleus scattering [38], and even the gravitational scattering [39,40]. It was also polished from a more formal viewpoint [41].

In this work, I continue the exploration of the quantum information aspects of spin correlations in the $n + p$ scattering. Technically, in Ref. [31], the in-state is taken for simplicity to be an unentangled polarized state of the proton and neutron. In other words, the in-state under consideration is a *pure* state. In comparison, I study partially polarized $n + p$ scattering, which is more general than the polarized one. The corresponding in-state is parametrized in the spin space by the following

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[1]In this work, the words "spin correlations" are used to refer to the general correlations between two nucleons in the spin space. They are not the same as the so-called spin-correlation parameters (also known as the *C* coefficients) which belong to a specific kind of spin observables introduced to characterize spin correlations.

[2]See Ref. [29] for a pioneering study in the 1970s on the Bell test in the nucleon-nucleon scattering.

density matrix:

$$\rho_{\text{in}} = p_1|n_\uparrow p_\uparrow\rangle\langle n_\uparrow p_\uparrow| + p_2|n_\uparrow p_\downarrow\rangle\langle n_\uparrow p_\downarrow| + p_3|n_\downarrow p_\uparrow\rangle\langle n_\downarrow p_\uparrow| + p_4|n_\downarrow p_\downarrow\rangle\langle n_\downarrow p_\downarrow| \quad (1)$$

$$= \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}, \quad (2)$$

with $\{n_\uparrow, n_\downarrow, p_\uparrow, p_\downarrow\}$ being the spin-up and spin-down nucleons, and $\{p_i\} \equiv \{p_1, p_2, p_3, p_4\}$ being the non-negative classical probabilities for the four bipartite basis states $\{|n_\uparrow p_\uparrow\rangle \equiv (1, 0, 0, 0)^T, \;\; |n_\uparrow p_\downarrow\rangle \equiv (0, 1, 0, 0)^T, \;\; |n_\downarrow p_\uparrow\rangle \equiv (0, 0, 1, 0)^T, |n_\downarrow p_\uparrow\rangle \equiv (0, 0, 0, 1)^T\}$, normalized by $p_1 + p_2 + p_3 + p_4 = 1$. If one of $\{p_i\}$ equals one, the density matrix ρ_{in} becomes a pure state satisfying $\rho_{\text{in}}^2 = \rho_{\text{in}}$ and corresponds to polarized incoming beams. If $p_1 = p_2 = p_3 = p_4 = 1/4$, ρ_{in} represents the completely unpolarized state satisfying $\rho_{\text{in}} = I_4/4$, with I_4 being the 4×4 identity matrix, and corresponds to the completely unpolarized incoming beams. The out-state ρ_{out} after the $n + p$ scattering is then given by

$$\rho_{\text{out}} = S\rho_{\text{in}}S^\dagger, \quad (3)$$

which is in general a mixed state, with S and $S^\dagger$ being the S matrix and its Hermitian conjugate. In the S-wave channel, the S matrix could be parametrized by [31,41]

$$S = \tfrac{1}{2}[\exp(i2\delta_1) + \exp(i2\delta_0)]I_4 + \tfrac{1}{2}[\exp(i2\delta_1) - \exp(i2\delta_0)]\text{SWAP}. \quad (4)$$

In quantum information science, the identity matrix I_4 is also known as the identity gate for two qubits, and $\text{SWAP} = (I_4 + \sum_{k=1}^{3}\sigma_k \otimes \sigma_k)/2$ is the standard SWAP gate, with $\{\sigma_k\}$ being the three Pauli matrices ($k = 1, 2, 3$). The phases δ_0 and δ_1 in Eq. (4) are the ${}^1S_0$ and ${}^3S_1$ phase shifts, whose values can be either predicted by theoretical models or extracted from experimental data. As ρ_{out} is a mixed state, the entanglement measures for pure states may not be directly applicable. To overcome this problem, I adopt an entanglement measure called negativity [42], which is applicable for both pure and mixed states and has an analytic expression for ρ_{out}. Moreover, the spin mutual information [8,9] and spin discord [13–15] are also calculated for ρ_{out}, motivated by the attempt to generalize the results of Ref. [31] beyond entanglement. This may pave the way to a more complete picture for emergent symmetries and spin correlations in low-energy QCD.

This paper is organized as follows: In Sec. II, the mutual information, bipartite separability, negativity, and geometric discord are reviewed briefly, which are useful figures of merit in quantum information science. In Sec. III, the quantum information aspects of spin correlations in the partially polarized $n + p$ scattering in the S wave are studied, with the analytic and numerical results presented and discussed in detail. Section IV summarizes and concludes.

II. QUANTUM INFORMATION

A. Mutual information

In this work, it is the quantum mutual information that is under consideration instead of its classical counterpart, and this should not cause any confusion as quantum mechanics is the most natural framework to analyze the nucleon-nucleon scattering. Given the density matrix ρ_{AB} for a general bipartite state, the mutual information $\mathcal{I}(A{:}B)$ is defined by [8,9]

$$\mathcal{I}(A{:}B) \equiv \mathcal{S}(\rho_A) + \mathcal{S}(\rho_B) - \mathcal{S}(\rho_{AB}), \quad (5)$$

with $\mathcal{S}(\rho) \equiv -\text{tr}[\rho\log_2(\rho)]$ being the von Neumann entropy for the density matrix ρ, $\rho_A \equiv \text{tr}_B(\rho_{AB})$ being the reduced density matrix for Subsystem A, and $\rho_B \equiv \text{tr}_A(\rho_{AB})$ being the reduced density matrix for Subsystem B. The usefulness of mutual information relies on the property that it could be regarded as a quantitative measure for the total amount of correlations (classical + quantum) in ρ_{AB} [43].

B. Bipartite separability and negativity

A bipartite state (pure or mixed) is called separable if its density matrix could be decomposed into the following form:

$$\rho_{AB} = \sum_i p_i\rho_A^{(i)} \otimes \rho_B^{(i)}, \quad i = 1, 2, 3 \ldots. \quad (6)$$

Here, $\{\rho_A^{(i)}\}$ and $\{\rho_B^{(i)}\}$ are a number of density matrices for Subsystems A and B, and $\{p_i\}$ are the non-negative classical probabilities normalized by $\sum_i p_i = 1$. The essential feature of the separable ρ_{AB} is that it can be prepared via local operations and classical communication (LOCC). If ρ_{AB} cannot be decomposed into the form of Eq. (6), it is called entangled and thus cannot be prepared by utilizing LOCC only.

Given a general bipartite density matrix ρ_{AB}, it is crucial to determine whether it is separable or not. Often, ρ_{AB} depends on some additional parameters. It is interesting to work out under which conditions these parameters give rise to a separable or entangled ρ_{AB}. A convenient criterion for this purpose is the Peres-Horodecki criterion [44,45]. With the *orthonormal* basis vectors $\{|k_A l_B\rangle \equiv |k_A\rangle \otimes |l_B\rangle\}$ for the bipartite state, a general density matrix ρ_{AB} could be parametrized by

$$\rho_{AB} = \sum_{klmn}(\rho_{AB})_{k_A l_B;m_A n_B}|k_A l_B\rangle\langle m_A n_B|, \quad (7)$$

$$(\rho_{AB})_{k_A l_B;m_A n_B} \equiv \langle k_A l_B|\rho_{AB}|m_A n_B\rangle, \quad (8)$$

and the partial transpose T_A of ρ_{AB} is defined by

$$\rho_{AB}^{T_A} \equiv \sum_{klmn}(\rho_{AB})_{m_A l_B;k_A n_B}|k_A l_B\rangle\langle m_A n_B|. \quad (9)$$

Then, the Peres-Horodecki criterion states that the necessary condition for a separable ρ_{AB} is that all the eigenvalues of $\rho_{AB}^{T_A}$ are non-negative. In other words, if $\rho_{AB}^{T_A}$ has one or more negative eigenvalues, ρ_{AB} must be entangled. In the two-qubit system, it is proven that the Peres-Horodecki criterion can be promoted to be both the sufficient and necessary condition for separability [45].

When the mixed state ρ_{AB} is entangled, the amount of entanglement could be quantified by different measures, among which the negativity has the advantage of good calculability and is adopted in this work. Given ρ_{AB}, the negativity is defined as [42]

$$\mathcal{N}(\rho_{AB}) \equiv -\sum_i \lambda_i, \quad (10)$$

with λ_i being the *negative* eigenvalues of the partially transposed density matrix $\rho_{AB}^{T_A}$. According to the Peres-Horodecki criterion, the negativity $\mathcal{N}(\rho_{AB})$ is exactly equal to zero for the separable ρ_{AB}, as $\rho_{AB}^{T_A}$ has no negative eigenvalues.

C. Geometric discord

A pure state is quantum correlated if and only if it is entangled. However, this is not true for a mixed state, and there are separable mixed states with nonzero quantum correlations, indicating the existence of quantum correlations beyond entanglement. Discord is one example of these quantum correlations beyond entanglement and could be measured by the so-called geometric discord [46],

$$\mathcal{D}_G(B{:}A) \equiv \min_{\sigma\in\mathrm{CQ}} ||\rho_{AB} - \sigma_{AB}||^2. \tag{11}$$

Here, $||\rho||^2 \equiv \mathrm{tr}(\rho\rho^\dagger)$ is the Hilbert-Schmidt distance, and σ_{AB} is a typical classical-quantum state parametrized by

$$\sigma_{AB} = \sum_i p_i |i_A\rangle\langle i_A| \otimes \rho_B^{(i)}, \tag{12}$$

with $\{|i_A\rangle\}$ being the orthonormal basis vectors for A, $\{\rho_B^{(i)}\}$ being the density matrices for B, and $\{p_i\}$ being the classical probabilities normalized by $\sum_i p_i = 1$. The "classical-quantum" nature of σ_{AB} originates from the fact that σ_{AB} remains undisturbed under the local von Neumann measurements $\{|i_A\rangle\langle i_A|\}$ in Subsystem A but not in Subsystem B. Similarly, one could also define the quantum-classical and classical-classical states. The minimization in Eq. (11) is taken over all the possible σ_{AB}s. Besides the geometric discord, several other measures are available for discord, such as the original definition of quantum discord given by Refs. [47,48]. The main advantage of the geometric discord is that it allows an analytic evaluation for two-qubit systems. Suppose the two-qubit density matrix is parametrized by the Bloch representation

$$\rho_{AB} = \frac{1}{4}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \mathcal{T}_{\mu\nu}\sigma_\mu \otimes \sigma_\nu, \tag{13}$$

with $\sigma_\mu = (I_2, \sigma_1, \sigma_2, \sigma_3)$ and $\mathcal{T}_{\mu\nu} = \mathrm{tr}(\rho_{AB}\sigma_\mu \otimes \sigma_\nu)$. It is found that the corresponding geometric discord is given in the closed form [46]

$$\mathcal{D}_G(B{:}A) = \frac{1}{4}\left(\sum_{j=1}^{3}\sum_{\nu=0}^{3} \mathcal{T}_{j\nu}^2 - \lambda_{\max}\right), \tag{14}$$

where $\lambda_{\max}$ is the maximal eigenvalue of the 3×3 matrix $\boldsymbol{L}_A \equiv \boldsymbol{a}\cdot\boldsymbol{a}^T + \boldsymbol{E}\cdot\boldsymbol{E}^T$, with $\boldsymbol{a} = (\mathcal{T}_{10}, \mathcal{T}_{20}, \mathcal{T}_{30})^T$ and $\boldsymbol{E}_{ij} = \mathcal{T}_{ij}$ $(i, j = 1, 2, 3)$. On the contrary, the original definition of the quantum discord involves a minimization that can only be done numerically for ρ_{out}, making it less convenient for my study.

III. SPIN CORRELATIONS

A. Spin mutual information

After some symbolic simplification, ρ_{out} in Eq. (3) is given explicitly by

$$\rho_{\text{out}} = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}[p_2 + p_3 + (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)] & \frac{i}{2}(p_2 - p_3)\sin(2\delta_0 - 2\delta_1) & 0 \\ 0 & -\frac{i}{2}(p_2 - p_3)\sin(2\delta_0 - 2\delta_1) & \frac{1}{2}[p_2 + p_3 - (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)] & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}. \tag{15}$$

Here, $p_1, \ldots, p_4$ are the classical probabilities for different combinations of spin polarizations in the in-state ρ_{in}, and δ_0 and δ_1 are the phase shifts for the $n + p$ scattering in the ${}^1S_0$ and ${}^3S_1$ waves. Implicitly, δ_0 and δ_1 depend on the relative momentum p. The reduced density matrices for the neutron and proton turn out to be diagonal and are given by

$$\rho_{\text{out}}^n = \mathrm{diag}\left\{p_1 + \tfrac{1}{2}[p_2 + p_3 + (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)],\, p_4 + \tfrac{1}{2}[p_2 + p_3 + (-p_2 + p_3)\cos(2\delta_0 - 2\delta_1)]\right\}, \tag{16}$$

$$\rho_{\text{out}}^p = \mathrm{diag}\left\{p_1 + \tfrac{1}{2}[p_2 + p_3 + (-p_2 + p_3)\cos(2\delta_0 - 2\delta_1)],\, p_4 + \tfrac{1}{2}[p_2 + p_3 + (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)]\right\}. \tag{17}$$

Therefore, the spin mutual information of the out-state ρ_{out} is given by

$$\begin{aligned} \mathcal{I}_{\text{out}}(n{:}p) = {} & p_1\log_2(p_1) + p_2\log_2(p_2) + p_3\log_2(p_3) + p_4\log_2(p_4) \\ & - \left\{p_1 + \tfrac{1}{2}[p_2 + p_3 + (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)]\right\}\log_2\left(p_1 + \tfrac{1}{2}[p_2 + p_3 + (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)]\right) \\ & - \left\{p_4 + \tfrac{1}{2}[p_2 + p_3 + (-p_2 + p_3)\cos(2\delta_0 - 2\delta_1)]\right\}\log_2\left(p_4 + \tfrac{1}{2}[p_2 + p_3 + (-p_2 + p_3)\cos(2\delta_0 - 2\delta_1)]\right) \\ & - \left\{p_1 + \tfrac{1}{2}[p_2 + p_3 + (-p_2 + p_3)\cos(2\delta_0 - 2\delta_1)]\right\}\log_2\left(p_1 + \tfrac{1}{2}[p_2 + p_3 + (-p_2 + p_3)\cos(2\delta_0 - 2\delta_1)]\right) \\ & - \left\{p_4 + \tfrac{1}{2}[p_2 + p_3 + (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)]\right\}\log_2\left(p_4 + \tfrac{1}{2}[p_2 + p_3 + (p_2 - p_3)\cos(2\delta_0 - 2\delta_1)]\right). \end{aligned} \tag{18}$$

Similarly, the spin mutual information of the in-state ρ_{in} is given by

$$\begin{aligned} \mathcal{I}_{\text{in}}(n{:}p) = {} & p_1\log_2(p_1) + p_2\log_2(p_2) + p_3\log_2(p_3) + p_4\log_2(p_4) - (p_1 + p_2)\log_2(p_1 + p_2) - (p_3 + p_4)\log_2(p_3 + p_4) \\ & - (p_1 + p_3)\log_2(p_1 + p_3) - (p_2 + p_4)\log_2(p_2 + p_4). \end{aligned} \tag{19}$$

To quantify the spin mutual information generated by the $n+p$ scattering, it may be meaningful to work out the difference between $\mathcal{I}_{\rm out}(n{:}p)$ and $\mathcal{I}_{\rm in}(n{:}p)$,

$$\Delta\mathcal{I}(n{:}p) \equiv \mathcal{I}_{\rm out}(n{:}p) - \mathcal{I}_{\rm in}(n{:}p), \tag{20}$$

and its average over all the possible in-states in the form of Eq. (2) (constrained by $p_1+p_2+p_3+p_4=1$),

$$\Delta\mathcal{I}_{\rm av}(n{:}p) \equiv 6\int_0^1 dp_1\int_0^1 dp_2\int_0^1 dp_3\int_0^1 dp_4 \times \Delta\mathcal{I}(n{:}p)\,\delta(p_1+p_2+p_3+p_4-1). \tag{21}$$

The prefactor 6 is given by the reciprocal of the multiple integral $\int_0^1 dp_1\int_0^1 dp_2\int_0^1 dp_3\int_0^1 dp_4\,\delta(p_1+p_2+p_3+p_4-1)$, which is the volume of the probability space allowed by classical probability theory. In the following parts, it is $\Delta\mathcal{I}_{\rm av}(n{:}p)$ and its variants defined in Eqs. (34)–(35) that will be used to characterize the $n+p$ scattering rather than the original $\mathcal{I}_{\rm out}(n{:}p)$ and $\mathcal{I}_{\rm in}(n{:}p)$, and it is named as the average spin mutual information for simplicity, without stressing that it is the spin mutual information difference between the out- and in-states. This should not cause any confusion.

B. Spin entanglement

Given $\rho_{\rm out}$ in Eq. (15), the four eigenvalues of the partially transformed density matrix $\rho_{\rm out}^{T_A}$ are found to be

$$\lambda_{1,2} = \tfrac{1}{2}[p_2+p_3 \pm (p_2-p_3)\cos(2\delta_0-2\delta_1)], \tag{22}$$

$$\lambda_{3,4} = \tfrac{1}{2}\left[p_1+p_4 \pm \sqrt{(p_1-p_4)^2+(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1)}\right]. \tag{23}$$

Here, λ_1 and λ_3 take the positive sign, while λ_2 and λ_4 take the minus sign. Obviously, λ_1, λ_2, λ_3 are non-negative definitely, and only the non-negativity of λ_4 is questionable. The requirement of $\lambda_4 \geqslant 0$ can be mathematically reduced to the condition

$$(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1) \leqslant 4p_1p_4. \tag{24}$$

According to the Peres-Horodecki criterion, this is also the sufficient and necessary condition for $\rho_{\rm out}$ to be separable. Complementarily, the sufficient and necessary condition for an entangled $\rho_{\rm out}$ is given by

$$(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1) > 4p_1p_4. \tag{25}$$

To quantify the incidence of entangled states produced by the $n+p$ scattering, the volume of spin entanglement is defined under the inspiration of Ref. [49],

$$\mathcal{E}_V \equiv 6\int_0^1 dp_1\int_0^1 dp_2\int_0^1 dp_3\int_0^1 dp_4\,\Theta[(p_2-p_3)^2\sin^2 \times(2\delta_0-2\delta_1)-4p_1p_4]\,\delta(p_1+p_2+p_3+p_4-1), \tag{26}$$

which is the relative volume of the in-states that lead to the entangled states after the $n+p$ scattering over all the possible in-states in the probability space spanned by p_1, p_2, p_3, and p_4. Here, $\Theta(x)$ is the standard Heaviside step function, satisfying $\Theta(x)=1$ if $x>0$ and $\Theta(x)=0$ if $x<0$.

The spin negativity, which measures the amount of spin entanglement in $\rho_{\rm out}$, is then given by

$$\mathcal{N}(\rho_{\rm out}) = \tfrac{1}{2}\left[\sqrt{(p_1-p_4)^2+(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1)} - (p_1+p_4)\right] \tag{27}$$

$$\text{if } (p_2-p_3)^2\sin^2(2\delta_0-2\delta_1) > 4p_1p_4,$$
$$= 0 \text{ if } (p_2-p_3)^2\sin^2(2\delta_0-2\delta_1) \leqslant 4p_1p_4. \tag{28}$$

Similar to the average spin mutual information $\Delta\mathcal{I}_{\rm av}(n{:}p)$, the average spin negativity could be defined as follows:

$$\mathcal{N}_{\rm av} \equiv 6\int_0^1 dp_1\int_0^1 dp_2\int_0^1 dp_3\int_0^1 dp_4\,\mathcal{N}(\rho_{\rm out}) \times\delta(p_1+p_2+p_3+p_4-1), \tag{29}$$

which gives a global measure for the entanglement generation capability of the neutron-proton interaction independent of the mixed in-states $\rho_{\rm in}$.

C. Spin discord

With $\rho_{\rm out}$ given in Eq. (15), the matrix elements of $\{\mathcal{T}_{\mu\nu}\}$ in the Bloch representation are given by

$$\begin{aligned}
\mathcal{T}_{00} &= p_1+p_2+p_3+p_4 = 1,\\
\mathcal{T}_{03} &= p_1-p_4-(p_2-p_3)\cos(2\delta_0-2\delta_1),\\
\mathcal{T}_{12} &= (p_2-p_3)\sin(2\delta_0-2\delta_1),\\
\mathcal{T}_{21} &= -(p_2-p_3)\sin(2\delta_0-2\delta_1),\\
\mathcal{T}_{30} &= p_1-p_4+(p_2-p_3)\cos(2\delta_0-2\delta_1),\\
\mathcal{T}_{33} &= p_1-p_2-p_3+p_4,
\end{aligned} \tag{30}$$

while all the other $\mathcal{T}_{\mu\nu}$s are equal to zero. The 3×3 matrix $\boldsymbol{L}_A$ is found to be diagonal, with

$$\begin{aligned}
\boldsymbol{L}_A = \mathrm{diag}\{&(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1),\\
&\times(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1),\\
&\times(p_1-p_2-p_3+p_4)^2+[p_1-p_4\\
&+(p_2-p_3)\cos(2\delta_0-2\delta_1)]^2\}.
\end{aligned}$$

As a result, $\lambda_{\rm max}$ in Eq. (14) is given by

$$\begin{aligned}
\lambda_{\rm max} &= \max\{(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1), (p_1-p_2-p_3+p_4)^2\\
&+[p_1-p_4+(p_2-p_3)\cos(2\delta_0-2\delta_1)]^2\}.
\end{aligned} \tag{31}$$

The corresponding geometric discord is given by

$$\begin{aligned}
\mathcal{D}_G(p{:}n) = \tfrac{1}{8}\{&4p_1^2+5p_2^2+5p_3^2+4p_4^2-4p_1p_2\\
&-4p_1p_3-2p_2p_3-4p_2p_4-4p_3p_4\}\\
&+\tfrac{1}{8}(p_2-p_3)[4(p_1-p_4)\cos(2\delta_0-2\delta_1)\\
&-(p_2-p_3)\cos(4\delta_0-4\delta_1)]\\
&-\tfrac{1}{4}\max\{(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1),\\
&\times(p_1-p_2-p_3+p_4)^2+[p_1-p_4\\
&+(p_2-p_3)\cos(2\delta_0-2\delta_1)]^2\}.
\end{aligned} \tag{32}$$

Similar to $\Delta\mathcal{I}_{\rm av}(n{:}p)$ and $\mathcal{N}_{\rm av}$, the average geometric discord could be defined by

$$\mathcal{D}_{\rm av}(p{:}n) \equiv 6\int_0^1 dp_1\int_0^1 dp_2\int_0^1 dp_3\int_0^1 dp_4\,\mathcal{D}_G(p{:}n)\times\delta(p_1+p_2+p_3+p_4-1), \tag{33}$$

which averages $\mathcal{D}_G(p{:}n)$ over all the possible in-states given by Eq. (2).

D. Numerics and discussion

In Secs. III A–III C, some symbolic results are given for the spin mutual information, entanglement, and discord. This section is devoted to developing the corresponding numerical understanding. In the literature, different realistic models of the nucleon-nucleon interactions generally give different theoretical results of the $^1S_0$ and $^3S_1$ phase shifts, especially at the high relative momenta close to 400 MeV. For higher relative momenta, the partons inside nucleons become active, and the low-energy picture of the nucleon-nucleon scattering could be less reliable. To test the model dependence of the final numerical results, six different theoretical data sets are adopted, including PWA93 [50], ESC96 [51,52], NijmI [53], NijmII [53], Reid93 [53], and Nijm93 [53], all available from the NN-OnLine website maintained by the Nijmegen group [54].

In Fig. 1, the variations of the average spin mutual information $\Delta\mathcal{I}_{\rm av}(n{:}p)$, the volume of spin entanglement $\mathcal{E}_V$, the average spin negativity $\mathcal{N}_{\rm av}$, and the average spin discord $\mathcal{D}_{\rm av}(p{:}n)$ are plotted with respect to the relative momentum p for the six models of the phase shifts. It is important to emphasize that the maximal values of these figures of merit in Fig. 1 are all normalized to $\frac{1}{3}$, the maximal value of $\mathcal{E}_V$. This allows an intuitive comparison of the horizontal positions of their minimal and maximal points without the distraction from their vertical scales. It is straightforward to see that, in spite of their different physical nature, the four figures of merit vary in a similar way in all of the six models, going up and down twice at relative momenta less than 100 MeV and generally being suppressed at relative momenta ranging from 100 to 400 MeV. Moreover, their minimal and maximal points actually take place at the same relative momenta. Take Nijm93 as an example. All the four figures of merit get their minimal values of zero at $p = 0$, 19.45 MeV and their maximal values at $p = 6.10$, 63.98 MeV. The similar results hold also for the other models. In each panel, an additional quantity of $\frac{1}{3}\sin^2(2\delta_0 - 2\delta_1)$ is also plotted for comparison (the green long-dashed line, labeled $\mathcal{E}_P$), which is actually proportional to the spin entanglement power derived in Ref. [31] for the pure-state scattering. One can see that $\mathcal{N}_{\rm av}$ almost coincides with $\frac{1}{3}\sin^2(2\delta_0 - 2\delta_1)$, while $\Delta\mathcal{I}_{\rm av}(n{:}p)$ and $\mathcal{D}_{\rm av}(p{:}n)$ lie close to it. Also, it is straightforward to notice that the minimal and maximal points of $\frac{1}{3}\sin^2(2\delta_0 - 2\delta_1)$ coincide perfectly with those of all the four figures of merit. This makes it possible to work out the exact conditions for the minimal and maximal points of these figures of merit. Explicitly, the minimal points take place when $\delta_0 - \delta_1 = \frac{N\pi}{2}$, while the maximal points take place when $\delta_0 - \delta_1 = \frac{\pi}{4} + \frac{N\pi}{2}$, with $N = 0$, $\pm 1, \pm 2, \ldots$.

In Fig. 1, the maximal value of $\mathcal{E}_V$ (the orange short-dashed line) is found to be $\frac{1}{3}$ in all the six models. This value could be easily reproduced by bringing the maximum condition $\delta_0 - \delta_1 = \frac{\pi}{4} + \frac{N\pi}{2}$ into its definition in Eq. (26) and calculating the corresponding multiple integral explicitly. As mentioned before, $\mathcal{E}_V$ measures the proportion of the specific in-states which lead to the entangled out-states after the $n + p$ scattering. Therefore, this result shows that a majority ($\geqslant$66.7%) of the in-states remain unentangled after the $n + p$ scattering, while the in-states that give rise to the entangled out-states belong to the minority ($\leqslant$33.3%). On the other hand, when the minimum condition $\delta_0 - \delta_1 = \frac{N\pi}{2}$ holds, it is easy to check that $\mathcal{E}_V$ equals zero exactly. It means that, under this condition, none of the in-states become entangled after the scattering. Indeed, the S matrix is then reduced to the identity gate if N is even and the SWAP gate if N is odd. In either case, the out-state $\rho_{\rm out}$ is unentangled.

In Fig. 2, a componential analysis is given for the average spin mutual information $\Delta\mathcal{I}_{\rm av}(n{:}p)$. Two different contributions are distinguished, including $\Delta\mathcal{I}^{\rm en}_{\rm av}(n{:}p)$ contributed by the entangled out-states and $\Delta\mathcal{I}^{\overline{\rm en}}_{\rm av}(n{:}p)$ contributed by the entangled out-states,

$$\Delta\mathcal{I}^{\rm en}_{\rm av}(n{:}p) \equiv 6\int_0^1 dp_1\int_0^1 dp_2\int_0^1 dp_3\int_0^1 dp_4\,\Delta\mathcal{I}(n{:}p)\times\Theta[(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1)-4p_1p_4]\times\delta(p_1+p_2+p_3+p_4-1), \tag{34}$$

$$\Delta\mathcal{I}^{\overline{\rm en}}_{\rm av}(n{:}p) \equiv 6\int_0^1 dp_1\int_0^1 dp_2\int_0^1 dp_3\int_0^1 dp_4\,\Delta\mathcal{I}(n{:}p)\times\Theta[4p_1p_4-(p_2-p_3)^2\sin^2(2\delta_0-2\delta_1)]\times\delta(p_1+p_2+p_3+p_4-1). \tag{35}$$

In the second lines of Eqs. (34) and (35) are the Peres-Horodecki conditions for the entangled and unentangled out-states given by Eqs. (25) and (24). While both components vary similarly to $\Delta\mathcal{I}_{\rm av}(n{:}p)$, one can see that $\Delta\mathcal{I}^{\rm en}_{\rm av}(n{:}p)$ is generally greater than or equal to $\Delta\mathcal{I}^{\overline{\rm en}}_{\rm av}(n{:}p)$ for the whole range of relative momenta under consideration. To put it another way, in spite of the minor role of the entangled out-states in absolute numbers (i.e., the small values of $\mathcal{E}_V$), they make a major contribution to the total spin correlations measured by $\Delta\mathcal{I}_{\rm av}(n{:}p)$. Also, it is interesting to note that the excess of $\Delta\mathcal{I}^{\rm en}_{\rm av}(n{:}p)$ over $\Delta\mathcal{I}^{\overline{\rm en}}_{\rm av}(n{:}p)$ gets suppressed at relative momenta ranging from 100 to 400 MeV. A similar analysis has also been applied to the spin discord $\mathcal{D}_{\rm av}(p{:}n)$, the measure for quantum correlations beyond entanglement. The results turn out to be qualitatively similar and are left out to save space.

Last but not least, I would like to discuss the possible implications of the above results on the novel connection between emergent symmetry and spin entanglement. In Ref. [31], it is found that the minimization of spin entanglement (measured by the entanglement power) takes place at $\delta_0 - \delta_1 = \frac{N\pi}{2}$. When $\delta_0 = \delta_1$, the $n + p$ system is equipped with the Wigner

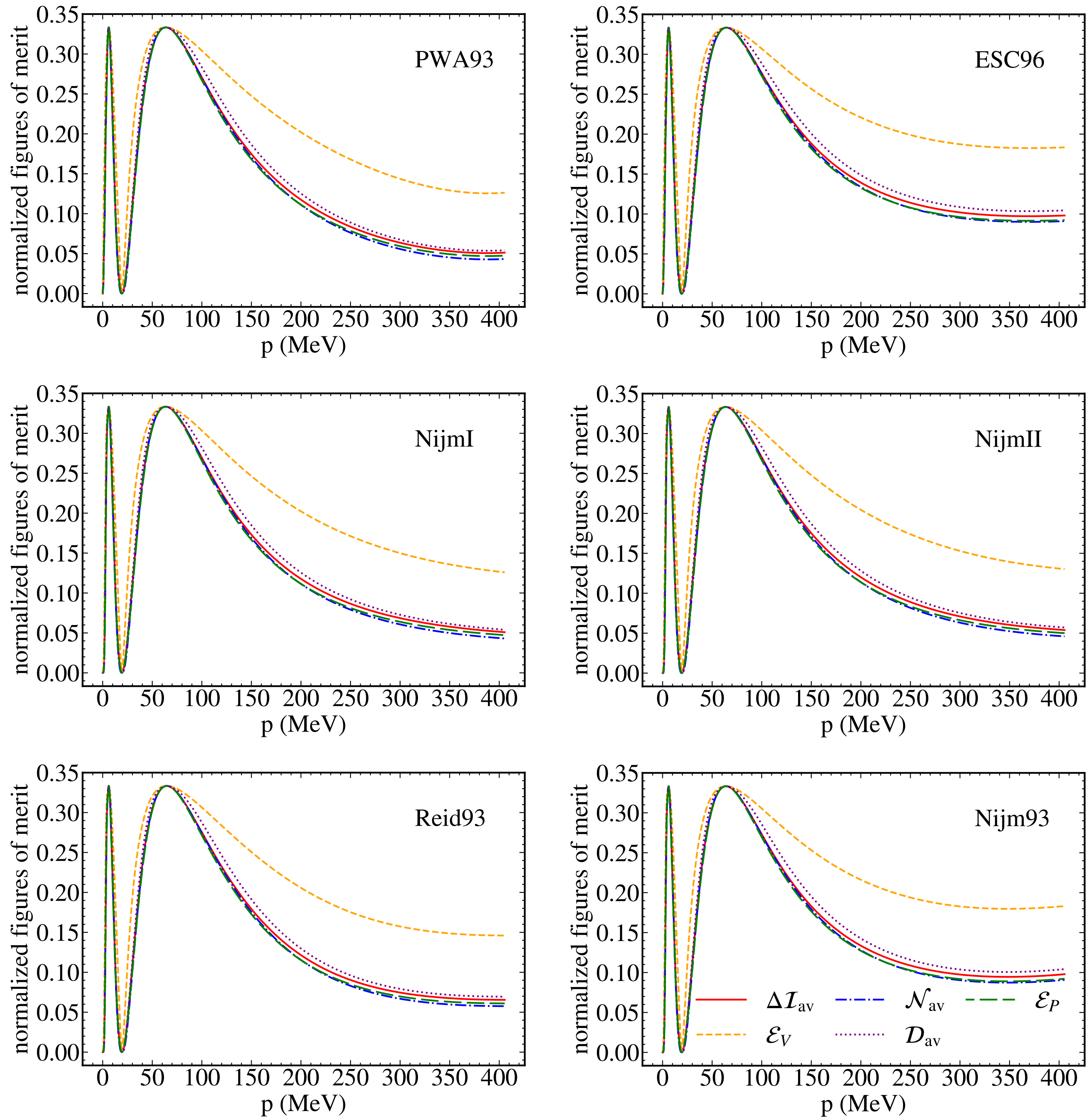

FIG. 1. The variations of the average spin mutual information $\Delta\mathcal{I}_{\rm av}(n{:}p)$ (red solid line), the volume of spin entanglement $\mathcal{E}_V$ (orange short-dashed line), the average spin negativity $\mathcal{N}_{\rm av}$ (blue dash-dotted line), and the average spin discord $\mathcal{D}_{\rm av}(p{:}n)$ (purple dotted line) with respect to the relative momenta for six different models, namely, PWA93 [50], ESC96 [51,52], NijmI [53], NijmII [53], Reid93 [53], and Nijm93 [53]. The maximal values of all the curves are normalized to $\frac{1}{3}$, the maximal value of $\mathcal{E}_V$. For comparison, $\frac{1}{3}\sin^2(2\delta_0 - 2\delta_1)$ is also plotted in each panel (green long-dashed line, labeled $\mathcal{E}_P$, almost coincident with $\mathcal{N}_{\rm av}$).

SU(4) symmetry. When $(\delta_0, \delta_1) = (0, 0)$, $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, 0)$, and $(\frac{\pi}{2}, \frac{\pi}{2})$, the $n + p$ system is equipped with the Schrödinger symmetry. Such a coincidence of (δ_0, δ_1) with the minimal spin entanglement and emergent symmetries thus may suggest a quantum-information understanding of emergent symmetries in low-energy QCD. This analysis could be naturally extend to $\Delta\mathcal{I}_{\rm av}(n{:}p)$ and $\mathcal{D}_{\rm av}(p{:}n)$, as they obey exactly the same minimization condition as the spin entanglement. However, the physical interpretations of $\Delta\mathcal{I}_{\rm av}(n{:}p)$ and $\mathcal{D}_{\rm av}(p{:}n)$ are not the same as entanglement. As mentioned before, $\Delta\mathcal{I}_{\rm av}(n{:}p)$ measures the total amount of spin correlations generated by the $n + p$ scattering, including both classical and quantum contributions, while $\mathcal{D}_{\rm av}(p{:}n)$ represents a general kind of quantum correlations beyond entanglement.

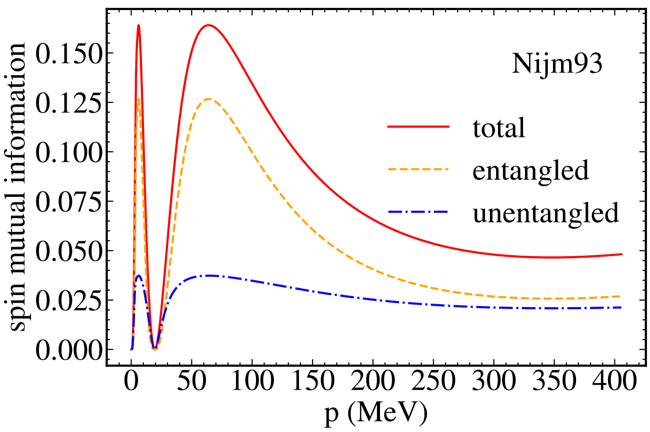

FIG. 2. Different components of the average spin mutual information. The red solid line represents $\Delta\mathcal{I}_{\rm av}(n{:}p)$, the average spin mutual information. The orange dashed line represents $\Delta\mathcal{I}_{\rm av}^{\rm en}(n{:}p)$, the average spin mutual information contributed by the entangled out-states. The blue dash-dotted line represents $\Delta\mathcal{I}_{\rm av}^{\overline{\rm en}}(n{:}p)$, the average spin mutual information contributed by the unentangled out-states. The phase-shift data behind this figure are given by Nijm93.

Therefore, my study shows that the connection between emergent symmetry and spin entanglement holds for spin correlations beyond entanglement as well.

IV. CONCLUSIONS

In this work, I explore the spin correlations of the partially polarized $n + p$ scattering in the S wave from the perspective of quantum information science. Several figures of merit are adopted to characterize the spin correlations, including the mutual information, entanglement, and discord. Among these three concepts, mutual information measures the total amount of classical+quantum correlations, entanglement is the most well-known example of quantum correlations, and discord is a general kind of quantum correlations beyond entanglement. Both analytic and numerical results are derived. It is found that, when averaged over the possible in-states, they all vary in a similar way in spite of their different physical meanings, with the minimal and maximal points taking place at the same relative momenta. As a result, the novel connection between emergent symmetry and spin entanglement found by Ref. [31] could be extended naturally to spin correlations beyond entanglement. My study complements the previous theoretical studies on the nucleon-nucleon scattering and may help to reveal new aspects of spin correlations and low-energy QCD symmetries from the quantum information viewpoint.

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