Renormalization of nuclear chiral effective field theory with nonperturbative leading-order interactions

A. M. Gasparyan [*] and E. Epelbaum [†]
Ruhr-Universität Bochum, Fakultät für Physik und Astronomie, Institut für Theoretische Physik II, D-44780 Bochum, Germany

(Received 11 February 2023; accepted 24 March 2023; published 10 April 2023)

We extend the renormalizability study of the formulation of chiral effective field theory with a finite cutoff, applied to nucleon-nucleon scattering, by taking into account nonperturbative effects. We consider the nucleon-nucleon interaction up to next-to-leading order in the chiral expansion. The leading-order interaction is treated nonperturbatively. In contrast to the previously considered case when the leading-order interaction was assumed to be perturbative, new features related to the renormalization of the effective field theory are revealed. In particular, more severe constraints on the leading-order potential are formulated, which can enforce the renormalizability and the correct power counting for the next-to-leading-order amplitude. To illustrate our theoretical findings, several partial waves in the nucleon-nucleon scattering, $^3P_0$, $^3S_1$-$^3D_1$, and $^1S_0$ are analyzed numerically. The cutoff dependence and the convergence of the chiral expansion for those channels are discussed.

DOI: 10.1103/PhysRevC.107.044002

I. INTRODUCTION

Over the last decades, the effective field theory (EFT) approach has become a standard tool in studies of the nucleon-nucleon (NN), few-nucleon, and many-nucleon systems due to the possibility to perform systematically improvable calculations in accordance with the chiral power counting. The chiral power counting implies an expansion of observables in terms of the ratio of the soft and the hard scales $Q = q/\Lambda_b$. The soft scale is given by the pion mass M_π and the external particle 3-momenta $|\vec{p}|$, whereas the hard-scale Λ_b is the breakdown scale of the EFT expansion of the order of the ρ-meson mass.

Starting with the seminal work by Weinberg [1,2], a lot of progress has been achieved in this field, see Refs. [3–8] for reviews.

In realistic calculations, one has to deal with regularization of an infinite number of divergent Feynman diagrams originating from the field theoretic treatment of nonperturbative amplitudes. One of the most practical approaches is related to introducing a finite (of the order of the hard-scale Λ_b) cutoff Λ in momentum space (or a corresponding short distance cutoff in coordinate space). The success of such a scheme is reflected in very accurate calculations at high orders in the chiral expansion, see Refs. [9–11] for recent applications.

A justification of such an approach from the fundamental point of view is complicated by the issue of renormalization and power counting violation due to the appearance of positive powers of the cutoff in the amplitude. Such contributions are generated by loop momenta of the order of the cutoff Λ. There exists a qualitative understanding in the literature [12–15] that such positive powers of Λ in the leading-order (LO) amplitude get compensated by the negative powers of the scale Λ_V stemming from the LO potential, which is also regarded to be of the order of the hard-scale Λ_b: $\Lambda_V \sim \Lambda \sim \Lambda_b$. Further, one believes that at higher chiral orders, the power counting breaking terms can be absorbed by a renormalization (shift) of lower order contact interactions [12]. However, until recently, a rigorous treatment of these problems and a systematic analysis of conditions under which the renormalization program can be carried out has been missing. Such a rigorous treatment is extremely important within the EFT approach, where systematic power counting is utilized to estimate theoretical uncertainties.

We addressed this issues in our study in Ref. [16]. In particular, we considered the LO potential consisting of the long-range one-pion-exchange term and a set of contact interactions that are momentum-independent or quadratic in momenta. The LO potential was regularized by various types of the form factors in momentum space, including local and nonlocal regulators both power-like and Gaussian. This covers most of the schemes considered in the literature.

In Ref. [16], it was assumed that the iterations of the leading-order potential V_0 can be treated perturbatively. More precisely, the series of the LO and the next-to-leading-order (NLO) amplitude in powers of V_0 were assumed to be convergent. However, the convergence rate of the expansion in V_0 might still be slower than the convergence rate of the chiral EFT expansion, which makes it necessary to sum up all (or many) iterations of V_0. However, the NLO potential needs not be iterated in the NLO amplitude. In the physical case of the NN scattering, such a perturbative regime is realized in most of the partial waves. The prominent exceptions are the $^1S_0$, $^3S_1$-$^3D_1$, and $^3P_0$ channels.

Under the above rather general assumptions, we proved the following statements:

*ashot.gasparyan@rub.de
†evgeny.epelbaum@rub.de

(i) The LO amplitude satisfies the dimensional power counting at each order in V_0 and is of chiral order $O(Q^0)$. If necessary, then contact interactions quadratic in momenta can be promoted to leading order.

(ii) The NLO amplitude in P and higher waves satisfies the dimensional power counting at each order in V_0 and is of chiral order $O(Q^2)$.

(iii) The unrenormalized NLO amplitudes in the S-waves (including the $^3S_1$-$^3D_1$ channel) violate the power counting and are of order $O(Q^0)$. To absorb the power-counting breaking terms, we employed the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization procedure and performed the overall subtractions in the diagrams as well as subtractions in all nested subdiagrams. As a result, the renormalized NLO amplitude was shown to satisfy the dimensional power counting and being of chiral order $O(Q^2)$ up to corrections logarithmic in the cutoff at each order in V_0.

In the current work, we extend our analysis to the nonperturbative case, i.e., to the situation when the series in the LO potential V_0 do not converge for the LO and/or NLO amplitude. This will allow us to consider the above-mentioned nonperturbative channels in NN scattering. Our analysis is based on the application of the Fredholm method of solving integral equations, which enables us to match the perturbative and nonperturbative regimes.

Our paper is organized as follows. In Sec. II, we briefly describe our formalism based on the effective Lagrangian, the corresponding effective potential and the way the amplitude is constructed in the nonperturbative case. In Sec. III, we explain the application of the Fredholm method for the LO Lippmann-Schwinger equation. In Sec. IV, we demonstrate the renormalization of the nucleon-nucleon interaction in P waves and higher. The renormalization in the S waves is addressed in Sec. V. Numerical results that illustrate our formal considerations are presented in Sec. VI. The paper ends with a summary. Bounds on the effective potential and various integrals are collected in Appendix.

II. FORMALISM

A. Effective Lagrangian and potential

In this section we briefly describe the formalism of chiral EFT used in our analysis. Some details are omitted and can be found in Ref. [16].

The starting point is the effective chiral Lagrangian represented as a series of all possible terms consistent with the symmetries of the underlying theory [17]. The expansion of the Lagrangian is performed in terms of the quark masses and field derivatives. The effective Lagrangian contains purely pionic terms, single nucleon terms, two-nucleon interactions, etc.:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{NN}^{(0)} + \mathcal{L}_{NN}^{(2)} + \dots, \quad (1)$$

where the superscripts denote chiral orders.

The chiral expansion of the NN amplitude in terms of the small parameter Q is performed according to the Weinberg power counting [2] (with possible modifications based on phenomenological arguments, e.g., promotion of certain higher order contributions to lower orders). The power of Q for a potential (i.e., two-nucleon-irreducible) contribution is determined by a sum over all vertices i in the diagram:

$$D = 2L + \sum_i \left(d_i + \frac{n_i}{2} - 2\right), \quad (2)$$

where L is the number of loops, n_i is the number of nucleon lines at vertex i and d_i is the number of derivatives and pion-mass insertions at vertex i. The chiral order of a $2N$-reducible diagram is equal to the sum of the orders of its irreducible components.

Since the LO contributions appear at order $O(Q^0)$, the corresponding potential terms have to be iterated an infinite number of times. To implement this procedure on a formal level and to regularize multiple-loop integrals, it is convenient to reformulate the effective Lagrangian of two-nucleon interactions in Eq. (1) in terms of the nonlocal regularized potential contributions of the form (see Ref. [16] for details)

$$\mathcal{L}_V(x) = -\int d\vec{y}\, d\vec{y}\,'\frac{1}{2}N^\dagger_{j_1}(x_0,\vec{x}-\vec{y}\,'/2)N^\dagger_{j_2}(x_0,\vec{x}+\vec{y}\,'/2) \times V(\vec{y}\,',\vec{y})_{j_1,j_2;i_1,i_2}N_{i_2}(x_0,\vec{x}+\vec{y}/2)N_{i_1}(x_0,\vec{x}-\vec{y}/2), \quad (3)$$

where i_1, i_2, j_1, j_2 are the combined spin and isospin indices of the corresponding nucleons. This formulation is customary for the few-body and nuclear physics.

The full potential is organized as a series according to the chiral expansion:

$$V = V^{(0)} + V^{(2)} + V^{(3)} + V^{(4)} + \dots. \quad (4)$$

Bare potentials $V^{(i)}$ are split into the renormalized parts V_i and the counter terms δV_i:

$$V^{(i)} = V_i + \delta V_i, \quad \delta V_i = \delta V_i^{(2)} + \delta V_i^{(3)} + \delta V_i^{(4)} + \dots. \quad (5)$$

The counter terms $\delta V_i^{(j)}$ $(j > i)$ absorb the divergent and the power counting violating terms appearing at order $O(Q^j)$.

The LO potential V_0 is regulated (the details are given in Sec. II B and in Appendix A) using a cutoff Λ to make the iterations of V_0 finite. We regard the cutoff value Λ (the largest cutoff among all cutoffs used in the LO potential) to be of the order of the hard-scale $\Lambda \sim \Lambda_b$. Higher order potentials can be considered either regulated or unregulated depending on a particular scheme, which will be discussed in the subsequent sections.

Note that to make some intermediate expressions mathematically well defined, one might need to introduce additional cutoffs that drop out from the final results after performing certain subtractions. Such cutoffs can be chosen to be much larger than Λ (or even infinity large).

To make the formulation of the theory in terms of nonlocal (on the Lagrangian level) regularized potential contributions completely equivalent to the original formulation in terms of local interactions, the regulator corrections $\delta_\Lambda V$ have to be taken into account:

$$\delta_\Lambda V = \sum_i \delta_\Lambda V^{(i)}, \quad \delta_\Lambda V^{(i)} := V^{(i)}_{\Lambda=\infty} - V^{(i)}_\Lambda, \quad (6)$$

where $V^{(i)}_{\Lambda=\infty}$ is the unregulated potential at the chiral order i. One possibility, often implicitly used in practical calculations, is to expand $\delta_\Lambda V$ in powers of $1/\Lambda$ and absorb the resulting terms by higher order contact interactions. This is possible if the potential does not contain nonlocally regularized long-range contributions. Another approach suggested in Ref. [16] is to keep the terms with $\delta_\Lambda V$ explicitly and consider those as perturbation. This allows us to reduce the cutoff dependence and extend the range of possible values of Λ, especially to smaller ones.

B. LO and NLO potentials and regulators

Our treatment of the LO and NLO potentials is identical to Ref. [16].

Weinberg's power counting in Eq. (2) implies that the leading-order $O(Q^0)$ potential $V_0(\vec{p}\,', \vec{p}\,)$ is represented by the sum of the regulated static one-pion-exchange potential and the short-range part:

$$V_0(\vec{p}\,', \vec{p}\,) = V^{(0)}_{1\pi,\Lambda}(\vec{p}\,', \vec{p}\,) + V^{(0)}_{\text{short},\Lambda}(\vec{p}\,', \vec{p}\,), \tag{7}$$

where the short-range part $V^{(0)}_{\text{short},\Lambda}$ may contain momentum-independent contact terms as well as the contact terms quadratic in momentum. The latter are formally of order $O(Q^2)$, as follows from Eq. (2). Nevertheless, it is known that in some channels, e.g., $^1\text{S}_0$ and $^3\text{P}_0$, their promotion to leading order can be motivated by phenomenological arguments, see, e.g., Refs. [18–21].

For the sake of generality, we allow for different forms of regulators: power-like local, power-like nonlocal, Gaussian local, and Gaussian nonlocal regulators as well as all possible combinations of those. In Ref. [16], we argued that for a local part of the LO potential $V_{0,\text{local}}(\vec{q}\,)$, the regulator (if it is also local) can be rather "mild." If the regulated LO potential behaves as

$$V_{0,\text{local}}(\vec{q}\,) \sim \frac{1}{|\vec{q}\,|^2}, \quad \text{for } |\vec{q}| \to \infty, \tag{8}$$

then both LO and NLO amplitudes turn finite after renormalization even if the NLO potential is not regulated. The reason for that is a milder ultraviolet behavior of local structures after performing subtractions. Such a mild regulator cannot be chosen for the nonlocal parts of the LO potential.

Equation (8) implies that in the spin-triplet channels the one-pion-exchange potential can be regulated by a dipole form factor,

$$F_{q,1\pi,\Lambda,1} = \frac{\Lambda^2 - M_\pi^2}{q^2 + \Lambda^2}, \tag{9}$$

whereas for the spin-singlet channels it can even be left unregulated.

Although in practical calculations one typically implements Gaussian or even sharper regulators to guarantee the finiteness of all integrals, we consider separately the above-mentioned situation with a local part of the LO potential having the ultraviolet asymptotics as in Eq. (8) and say that such a potential has a "mild" regulator in contrast to "standard" regulators, i.e. all other cases. This is done to keep the analysis general and to clarify the difference between perturbative and nonperturbative regimes. Moreover, such an analysis is useful to understand the cutoff dependence of the NN amplitude: the milder regulator can be chosen, the weaker cutoff dependence should be expected.

For completeness, we provide the explicit expressions for the LO potential and the corresponding regulators in Appendix A.

The next-to-leading-order potential $V_2(\vec{p}\,', \vec{p}\,)$ contains the short-range part, the two-pion-exchange potential and the regulator corrections to the leading-order potential:

$$V_2(\vec{p}\,', \vec{p}\,) = V^{(2)}_{2\pi}(\vec{p}\,', \vec{p}) + V^{(2)}_{\text{short}}(\vec{p}\,', \vec{p}\,) + \delta_\Lambda V^{(0)}(\vec{p}\,', \vec{p}\,). \tag{10}$$

In Ref. [16], we found that one does not need to regularize the NLO potential to perform the renormalization of the NLO amplitude. Or, equivalently, one can introduce a cutoff $\Lambda_{\text{NLO}} \gg \Lambda$. However, in practical calculations, one can choose $\Lambda_{\text{NLO}} \sim \Lambda_b$ if it improves efficiency of a computational scheme. Both approaches are formally equivalent because the regulator corrections $\delta_\Lambda V^{(2)}$ appear at order $O(Q^4)$ in accordance with the dimensional power counting.

It turns out, that the situation is slightly different in the general nonperturbative case, where for the choice of the "mild" LO regulator we need to keep Λ_{NLO} finite. It can still be larger than Λ, but not arbitrarily large, see discussion in Sec. V.

The explicit expressions for the NLO potential can be found in Appendix B.

C. *NN* amplitudes and contour rotation

In the present study we work predominantly in the partial wave lsj basis, which makes the analysis of the nonperturbative effects more efficient. In the lsj basis, the potential and the amplitude are $n_{\text{PW}} \times n_{\text{PW}}$ matrices, where $n_{\text{PW}} = 1$ ($n_{\text{PW}} = 2$) for the uncoupled (coupled) partial waves. The series for the partial wave LO amplitude and for the unrenormalized NLO amplitude are given by

$$T_0 = \sum_{n=0}^{\infty} T_0^{[n]}, \quad T_0^{[n]} = V_0 K^n = \bar{K}^n V_0, \tag{11}$$

$$T_2 = \sum_{m,n=0}^{\infty} T_2^{[m,n]}, \quad T_2^{[m,n]} = \bar{K}^m V_2 K^n, \tag{12}$$

where G is the free two-nucleon propagator and

$$K = GV_0, \quad \bar{K} = V_0 G. \tag{13}$$

In the nonperturbative case these equations generalize to

$$T_0 = V_0 R = \bar{R} V_0, \tag{14}$$

$$T_2 = \bar{R} V_2 R, \tag{15}$$

where R ($\bar{R}$) is the resolvent of the Lippmann-Schwinger equation (LSE)

$$R = \frac{1}{\mathbb{1} - K}, \quad \bar{R} = \frac{1}{\mathbb{1} - \bar{K}}. \tag{16}$$

The renormalized expression for the NLO amplitude $\mathbb{R}(T_2)$ is obtained by adding the relevant counter term, see Sec. V for details:

$$\mathbb{R}(T_2) = \bar{R}\big(V_2 + \delta V_0^{(2)}\big)R. \tag{17}$$

The explicit form of the LSE, $T_0 = V_0 + V_0 G T_0$, reads

$$(T_0)_{l'l}(p', p; p_{\rm on}) = \sum_{l''} \int \frac{p''^2 dp''}{(2\pi)^3} (V_0)_{l'l''}(p', p'') \times G(p''; p_{\rm on})(T_0)_{l''l}(p'', p; p_{\rm on}),$$
$$G(p''; p_{\rm on}) = \frac{m_N}{p_{\rm on}^2 - p''^2 + i\epsilon}. \tag{18}$$

The indices l, l', l'' denote the orbital angular momentum of the NN system, $p_{\rm on}$ is the on-shell center-of-mass (c.m.) nucleon momentum and p (p') are the initial (final) off-shell c.m. momenta.

It turns out useful to modify the integration path over the off-shell momentum p'' and rotate the contour into the complex plane [22–24]. The new integration contour $\mathcal{C}$ is defined by $p'' = |p''|e^{-i\alpha_\mathcal{C}}$. Our choice for the rotation angle $\alpha_\mathcal{C}$ is determined by the location of singularities of the LO potential in the complex plane [16]:

$$\alpha_\mathcal{C} = \frac{1}{2} \arctan \frac{M_\pi}{(p_{\rm on})_{\rm max}}, \tag{19}$$

where $(p_{\rm on})_{\rm max}$ is the maximal considered on-shell momentum.

The contour rotation enables us to perform direct estimations of the bounds on the partial wave amplitudes avoiding principal value integrals.

D. Bounds on the potentials and the *NN* propagator

By analogy with Ref. [16], we use certain upper bounds for the potentials and the NN propagator that are valid for off-shell momenta lying on the complex contour $\mathcal{C}$ and for the allowed real on-shell momenta. These bounds allow us to estimate the nucleon-nucleon LO and NLO amplitudes and to verify the corresponding power counting.

Following Ref. [16], in the bounds considered below, we introduce dimensionless constants named $\mathcal{M}_i$: $\mathcal{M}_{V_0}$, $\mathcal{M}_G$, etc., which are supposed to be of order one. Analogous constants appear in our final estimates for the amplitudes.

Some of the inequalities should be modified compared to Ref. [16] to be better suited for the nonperturbative analysis. In particular, for the LO potential $V_0(p', p)$, we need bounds that are separable in momenta p and p'.

The inequalities listed below are meant to hold for all matrix elements of the partial wave potentials $V_0(p', p)$ and $V_2(p', p)$ in l, l' space. Their derivation can be found in Appendices C and D.

The LO partial-wave potential obeys the following bounds:

$$|V_0(p', p)| \leqslant \mathcal{M}_{V_0} V_{0,\rm max}\, g(p')h(p), \tag{20}$$
$$|V_0(p', p)| \leqslant \mathcal{M}_{V_0} V_{0,\rm max}\, h(p')g(p),$$

with

$$V_{0,\rm max} = \frac{8\pi^2}{m_N \Lambda_V}, \tag{21}$$

where the exact form of the functions g and h (and the value of $\mathcal{M}_{V_0}$) depends on the partial wave and on the form of a regulator. For $l = 0$ (for the coupled partial waves, we mean by l the lowest possible orbital angular momentum), g and h are given by

$$g(p) = \lambda_{\rm log}(p/\Lambda), \;\; h(p) = 1, \tag{22}$$

for the "mild" regulator, and by

$$g(p) = [\lambda(p/\Lambda)]^2, \quad h(p) = [\lambda(p/\Lambda)]^{-1}, \tag{23}$$

for the "standard" regulators with the functions λ and $\lambda_{\rm log}$ defined as

$$\lambda(\xi) = \theta(1 - |\xi|) + \theta(|\xi| - 1)\frac{1}{|\xi|^2},$$
$$\lambda_{\rm log}(\xi) = \theta(1 - |\xi|) + \theta(|\xi| - 1)\frac{1 + \ln|\xi|}{|\xi|^2}. \tag{24}$$

For higher partial waves, $l \geqslant 1$, we adopt the bounds

$$g(p) = \lambda_{\rm log}(p/\Lambda)/|p|, \quad h(p) = |p|. \tag{25}$$

Notice that while in the latter case one could use a stronger bound and replace $\lambda_{\rm log}$ with λ for the "standard" regulator, this would not affect our conclusions. Therefore, we prefer to employ this unified bound.

For spin-singlet partial waves without a short-range LO contribution, one can improve the above bounds and replace in Eq. (25) $\lambda_{\rm log}(p/\Lambda)$ with $\lambda_{\rm log}(p/M_\pi)$. However, in all such channels the perturbative regime for the LO potential is realized, which has already been analyzed in Ref. [16] and will not be considered here.

Note that for $|p| \leqslant \Lambda$, and, in particular, for the on-shell momentum $|p| = p_{\rm on}$, we have in all cases $g(p) = h(p) = 1$.

It is convenient also to introduce the functions

$$v_0(p', p) = V_0(p', p)[\mathcal{M}_{V_0} V_{0,\rm max}\, h(p')g(p)]^{-1},$$
$$\bar{v}_0(p', p) = V_0(p', p)[\mathcal{M}_{V_0} V_{0,\rm max}\, g(p')h(p)]^{-1}, \tag{26}$$

for which the following bounds hold:

$$|v_0(p', p)| \leqslant 1, \quad |\bar{v}_0(p', p)| \leqslant 1. \tag{27}$$

For the unregulated NLO potential, we adopt the bounds from Ref. [16]. In particular, for $l = 0$:

$$|V_2(p', p)| \leqslant \mathcal{M}_{V_2,0}(|p|^2 + |p'|^2)\tilde{f}_{\rm log}(p', p), \tag{28}$$

with

$$\tilde{f}_{\rm log}(p', p) = \frac{8\pi^2}{m_N \Lambda_V \Lambda_b^2} f_{\rm log}(p', p),$$
$$f_{\rm log}(p', p) = \theta(|p| - M_\pi)\ln\frac{|p|}{M_\pi} + \theta(|p'| - M_\pi)\ln\frac{|p'|}{M_\pi} + 1, \tag{29}$$

where we have dropped the $\log \Lambda/M_\pi$ term in the definition of $f_{\rm log}$, which is unnecessary and was introduced in Ref. [16] for convenience.

Note that in Ref. [16], the NLO potential V_2 was split into two parts

$$V_2(p', p) = \hat{V}_2(p', p) + \tilde{V}_2(p', p), \tag{30}$$

with

$$\hat{V}_2(p', p) = V_2(0, 0), \quad \tilde{V}_2(p', p) = V_2(p', p) - V_2(0, 0), \tag{31}$$

and the inequality in Eq. (28) is, strictly speaking, valid for $\tilde{V}_2$. However, in the present work, we use most of the time the scheme with $V_2(0, 0) = 0$. Therefore, in what follows, we will always assume that $\tilde{V}_2 = V_2$ unless specified otherwise. For alternative schemes, we also provide the bound for $\hat{V}_2$:

$$|\hat{V}_2(p', p)| \leqslant \hat{\mathcal{M}}_{V_2,0} \frac{8\pi^2}{m_N \Lambda_V} \frac{M_\pi^2}{\Lambda_b^2}. \tag{32}$$

For higher partial waves $l > 0$, it is sufficient to implement the p-wave bound:

$$|V_2(p', p)| \leqslant \mathcal{M}_{V_2,1} |p'||p| \tilde{f}_{\log}(p', p). \tag{33}$$

For the regularized NLO potential with the cutoff $\Lambda_{\rm NLO}$, the bounds in Eq. (28) are modified as follows (see Appendix D 3 a):

$$|V_2(p', p)| \leqslant \mathcal{M}_{V_2,0}(|p|^2 + |p'|^2) \tilde{f}_{\log}(p', p) \lambda_{\log}(p'/\Lambda_{\rm NLO}), \text{ or}$$
$$|V_2(p', p)| \leqslant \mathcal{M}_{V_2,0}(|p|^2 + |p'|^2) \tilde{f}_{\log}(p', p) \lambda_{\log}(p/\Lambda_{\rm NLO}). \tag{34}$$

For the two-nucleon propagator $G(p; p_{\rm on}) = m_N/(p_{\rm on}^2 - p^2)$, we use the same bound as in Ref. [16]:

$$|G(p; p_{\rm on})| \leqslant \mathcal{M}_G \frac{m_N}{|p^2|}, \tag{35}$$

with $\mathcal{M}_G = 1/\sin(2\alpha_C)$.

III. LEADING-ORDER LIPPMANN-SCHWINGER EQUATION

In this section we outline the Fredholm method for solving integral equations and derive the bounds on the resolvents of the LSE and on the LO amplitude in the nonperturbative case. The resolvents R and $\bar{R}$ of the partial-wave LSE, see Eq. (16), can be represented by means of the Fredholm formula [25,26] as

$$R = (\mathbb{1} - K)^{-1} = \mathbb{1} + \frac{Y}{D},$$
$$\bar{R} = (\mathbb{1} - \bar{K})^{-1} = \mathbb{1} + \frac{\bar{Y}}{D}, \tag{36}$$

where the Fredholm determinant D is a number and depends only on the on-shell momentum $D = D(p_{\rm on})$, whereas the minor Y ($\bar{Y}$) is a matrix in the l, l' space and an operator in the space of the off-shell momenta: $Y = Y_{ji}(p', p; p_{\rm on})$. The quantities Y, $\bar{Y}$ and D can be expanded into convergent series in powers of the LO potential V_0:

$$Y = \sum_{n=1}^{\infty} Y^{[n]}, \quad \bar{Y} = \sum_{n=1}^{\infty} \bar{Y}^{[n]}, \quad D = \sum_{n=0}^{\infty} D^{[n]}. \tag{37}$$

In what follows, we will consider the resolvent R and the minor Y. The results are trivially generalized for $\bar{R}$ and $\bar{Y}$.

The terms $D^{[n]}$ and $Y^{[n]}$ can be written as [25,26]

$$D^{[n]}(p_{\rm on}) = \frac{(-1)^n}{n!} \sum_{i_1,\dots,i_n} \int \prod_{k=1}^{n} \frac{p_k^2 dp_k}{(2\pi)^3} \times [\det_{D,n}(K)]_{i_1,\dots i_n}(p_1, \dots, p_n; p_{\rm on}), \tag{38}$$

and

$$Y_{i'i}^{[n+1]}(p', p; p_{\rm on}) = \frac{(-1)^n}{n!} \sum_{i_1,\dots,i_n} \int \prod_{k=1}^{n} \frac{p_k^2 dp_k}{(2\pi)^3} [\det_{Y,n+1}(K)]_{i,i_1,\dots i_n,i'} \times (p, p_1, \dots, p_n, p'; p_{\rm on}), \tag{39}$$

where the matrix indices $i, i_1, ..i_n$ and i' correspond to the orbital angular momentum $l = j \pm 1$ for coupled partial waves and $l = j$ for uncoupled partial waves. In the above equations, the determinants for an operator X with matrix elements $X(p', p; p_{\rm on})$ (or $X(p', p)$ if it is independent of $p_{\rm on}$) are defined as

$$[\det_{D,n}(X)]_{i_1,\dots,i_n}(p_1, \dots, p_n; p_{\rm on}) = \begin{vmatrix} X_{i_1,i_1}(p_1, p_1; p_{\rm on}) & \cdots & X_{i_n,i_1}(p_1, p_n; p_{\rm on}) \\ \cdots & \cdots & \cdots \\ X_{i_1,i_n}(p_n, p_1; p_{\rm on}) & \cdots & X_{i_n,i_n}(p_n, p_n; p_{\rm on}) \end{vmatrix}, \tag{40}$$

and

$$[\det_{Y,n+1}(X)]_{i,i_1,\dots,i_n,i'}(p, p_1, \dots, p_n, p'; p_{\rm on}) = \begin{vmatrix} X_{i'i}(p', p; p_{\rm on}) & X_{i_1 i}(p_1, p; p_{\rm on}) & \cdots & X_{i_n i}(p_n, p; p_{\rm on}) \\ X_{i' i_1}(p', p_1; p_{\rm on}) & X_{i_1 i_1}(p_1, p_1; p_{\rm on}) & \cdots & X_{i_n i_1}(p_n, p_1; p_{\rm on}) \\ \cdots & \cdots & \cdots & \cdots \\ X_{i' i_n}(p', p_n; p_{\rm on}) & X_{i_1 i_n}(p_1, p_n; p_{\rm on}) & \cdots & X_{i_n i_n}(p_n, p_n; p_{\rm on}) \end{vmatrix}. \tag{41}$$

Rescaling V_0 as in Eq. (26), we obtain

$$K_{i'i}(p', p; p_{\rm on}) = (v_0)_{i'i}(p', p) \mathcal{M}_{V_0} V_{0,\max}\, g(p') h(p) G(p'; p_{\rm on}), \tag{42}$$

so that

$$D^{[n]}(p_{\rm on}) = \frac{(-1)^n}{n!} (\mathcal{M}_{V_0} V_{0,\max})^n \sum_{i_1,\dots,i_n} \int \left[\prod_{k=1}^{n} \frac{p_k^2 dp_k}{(2\pi)^3} g(p_k) h(p_k) G(p_k; p_{\rm on}) \right] [\det_{D,n}(v_0)]_{i_1,\dots,i_n}(p_1, \dots, p_n), \tag{43}$$

and

$$Y_{i'i}^{[n+1]}(p', p; p_{\rm on}) = \frac{(-1)^n}{n!}(\mathcal{M}_{V_0}V_{0,\max})^{n+1}g(p')h(p)G(p'; p_{\rm on})\sum_{i_1,\ldots,i_n}\int\left[\prod_{k=1}^{n}\frac{p_k^2 dp_k}{(2\pi)^3}g(p_k)h(p_k)G(p_k; p_{\rm on})\right] \times [\det_{Y,n+1}(v_0)]_{i,i_1,\ldots,i_n,i'}(p, p_1, \ldots, p_n, p'). \tag{44}$$

A. Upper bounds for the Fredholm determinant

First, we analyze the series for the Fredholm determinant D. Since the matrix elements $v_{0;ji}(p', p)$ are bounded by [see Eq. (27)]

$$|v_{0;ji}(p', p)| \leqslant 1, \tag{45}$$

the Hadamard's inequality for determinants gives [25,26]

$$|\det_{D,n}(v_0)| \leqslant n^{n/2}. \tag{46}$$

Therefore, using Stirling's formula, we can estimate $D^{[n]}$ as follows:

$$|D^{[n]}| \leqslant \frac{1}{n!}\Sigma^n n^{n/2} \leqslant \frac{1}{\sqrt{2\pi n}}\left(\frac{e\Sigma}{\sqrt{n}}\right)^n = \frac{1}{\sqrt{2\pi}e\Sigma}\left(\frac{e\Sigma}{\sqrt{n}}\right)^{n+1} = \frac{1}{\sqrt{2\pi}e\Sigma}\exp\left[-(n+1)\ln\frac{\sqrt{n}}{e\Sigma}\right] =: \mathcal{M}_{D,n}, \tag{47}$$

where Σ is defined as

$$\mathcal{M}_{V_0}V_{0,\max}n_{\rm PW}\left|\int\frac{p^2dp}{(2\pi)^3}g(p)h(p)G(p; p_{\rm on})\right| \leqslant \frac{\mathcal{M}_{V_0}\mathcal{M}_G}{\Lambda_V}n_{\rm PW}\int\frac{d|p|}{\pi}g(p)h(p) =: \Sigma. \tag{48}$$

Since $g(p)$ and $h(p)$ depend only on the ratio p/Λ, we can write

$$\Sigma = \mathcal{M}_\Sigma\frac{\Lambda}{\Lambda_V}, \tag{49}$$

where the numerical value of the constant $\mathcal{M}_\Sigma$ depends on a particular form of $g(p)$ and $h(p)$.

If we assume $\Lambda \sim \Lambda_V$, then $\Sigma \sim 1$ up to a numerical factor. The situation when $\Sigma < 1$ corresponds to a convergent series for the LO amplitude in terms of V_0. In contrast, for the nonperturbative regime that we consider, we have $\Sigma \geqslant 1$.

The maximal value of $D^{[n]}$ is achieved at some $n = n_{D_{\max}}$ and can be estimated by differentiating Eq. (47) with respect to n:

$$n_{D_{\max}} \approx e\Sigma^2, \quad |D^{[n]}| \leqslant \mathcal{M}_{D^{[n]},\max} \approx \frac{e^{e\Sigma^2/2}}{\sqrt{2\pi}e\Sigma}, \tag{50}$$

which is formally a number of order one, but it grows very rapidly with Σ.

The whole series for D is also bounded by a constant of order one:

$$|D| \leqslant \mathcal{M}_D, \tag{51}$$

which can be estimated by replacing the sum with an integral and using Laplace's method:

$$\mathcal{M}_D = \sum_{n=0}^{\infty}\mathcal{M}_{D,n} \approx \int_0^\infty dt\mathcal{M}_{D,t} \approx \sqrt{2\pi}\left(-\frac{\partial^2\ln\mathcal{M}_{D,t}}{\partial t^2}\right)^{-1/2}\mathcal{M}_{D,t}|_{t=n_{D_{\max}}} \approx \sqrt{2}e^{e\Sigma^2/2}, \tag{52}$$

which agrees rather well with the series summed numerically [see Eq. (47)]. For example, for $\Sigma = 1$, both results give $\mathcal{M}_D \approx 5$.

The bounds (47) and (52) are rather weak and very conservative. If Σ is not close to one, then the numerical values for $\mathcal{M}_D$ become very large. However, in realistic calculations, we can see that D does actually not exceed the values of order one. Clearly, one can always perform a numerical check to verify whether our approach to the renormalizability of the *NN* amplitude based on the Fredholm method is reliable. Note also that for the $^1S_0$ and $^3S_1$-$^3D_1$ *NN* channels, one can expect Σ to be close to one (ignoring the fine-tuning between attractive and repulsive forces) because the first (quasi) bound states in these channels are very shallow. This is roughly confirmed by an analysis of the Weinberg eigenvalues in Ref. [9].

There are particular cases when the estimate in Eq. (47) can be readily improved. For example, for purely local LO potentials, the quantities D and Y correspond to the Jost function and the regular solution of the Schrödinger equation in configuration space and the terms in their expansion, $D^{[n]}$ and $Y^{[n]}$, decrease as $1/n!$. However, if the LO consists of only a short-range separable potential (or is dominated by such a contribution), then the series for D and Y contain a finite number of terms. In our general discussion, we will simply assume that Eq. (51) holds.

We will also need an estimate for the series remainder:

$$\delta_n D = \sum_{k=n+1}^{\infty} D^{[n]}. \tag{53}$$

From Eq. (47), we can conclude that for sufficiently large n,

$$n > n_0 \equiv \tilde{\mathcal{M}}_{\delta D}, \tag{54}$$

the terms $D^{[n]}$ and, therefore, also the remainder $\delta_n\Delta$ decrease faster than exponential

$$\delta_n D \leqslant e^{-\mathcal{M}_{\delta D}n}, \tag{55}$$

with any $\mathcal{M}_{\delta D}$, which we will use in our further estimates. The value $\tilde{\mathcal{M}}_{\delta D}$ depends on $\mathcal{M}_{\delta D}$ and on Σ. Based on Eq. (47), we can conclude that the exponential decrease starts only for

$$\tilde{\mathcal{M}}_{\delta D} > (e\Sigma)^2, \tag{56}$$

which, being formally a number of order one, becomes extremely large unless $\Sigma \approx 1$. However, as follows from the discussion above, in realistic calculations, such an exponentially suppressed regime can be reached much earlier. In fact, in the numerical calculation presented in Sec. VI, the relative error $\delta_n D/D$ becomes less than one percent in most cases for $n = 3$ or 4.

B. Bounds for the minor Y

By analogy with the Fredholm determinant D, we can perform the same analysis for the minor Y starting from the definition in Eq. (44). Using again the Hadamard's inequality,

$$|\det_{Y,n}(v_0)| \leqslant n^{n/2}, \tag{57}$$

we get the bound for $Y^{[n]}$:

$$\left|Y_{ji}^{[n]}(p', p; p_{\rm on})\right| \leqslant \mathcal{M}_{Y,n}|G(p', p_{\rm on})|\frac{8\pi^2\mathcal{M}_{V_0}}{m_N\Lambda_V}g(p')h(p) \tag{58}$$

with

$$\mathcal{M}_{Y,n} = \frac{1}{(n-1)!}\Sigma^{n-1}n^{n/2} \leqslant \frac{e}{\sqrt{2\pi}}\left(\frac{e\Sigma}{\sqrt{n}}\right)^{n-1}. \tag{59}$$

Further, taking into account the bound for the propagator in Eq. (35), we obtain

$$\begin{aligned}\left|Y_{ji}^{[n]}(p', p; p_{\rm on})\right| &\leqslant \mathcal{M}_{Y,n}\frac{8\pi^2\mathcal{M}_{V_0}\mathcal{M}_G}{\Lambda_V|p'|^2}g(p')h(p)\\ &=: \frac{8\pi^2\mathcal{M}_Y}{\Lambda_V|p'|^2}\mathcal{M}_{Y,n}\, g(p')h(p).\end{aligned} \tag{60}$$

Analogous to Eq. (52), the whole series for Y can be estimated to be

$$\begin{aligned}|Y_{ji}(p', p; p_{\rm on})| &\leqslant \frac{8\pi^2\mathcal{M}_Y}{\Lambda_V|p'|^2}Y_{\max}\, g(p')h(p)\\ &=: \frac{8\pi^2\mathcal{M}_{Y_{\max}}}{\Lambda_V|p'|^2}\, g(p')h(p),\end{aligned} \tag{61}$$

where

$$Y_{\max} = \sum_{k=0}^{\infty}\mathcal{M}_{Y,n} \leqslant \sqrt{2}e\,\Sigma\, e^{e\Sigma^2/2}. \tag{62}$$

The remainder $\delta_n Y_{\max}$, defined as

$$\delta_n Y_{\max} = \sum_{k=n+1}^{\infty}\mathcal{M}_{Y,n}, \tag{63}$$

can be bounded similarly to $\delta_n D$ by an exponent with an arbitrary base:

$$\delta_n Y_{\max} \leqslant e^{-\mathcal{M}_{\delta Y}\, n}, \quad \text{for } n > \tilde{\mathcal{M}}_{\delta Y}, \tag{64}$$

with some $\tilde{\mathcal{M}}_{\delta Y}$. As in the case of $\delta_n D$, the estimated value of $\tilde{\mathcal{M}}_{\delta Y} \sim (e\Sigma)^2$ becomes very large for Σ significantly larger than one. However, in the actual calculations, its numerical value is typically much more natural, see the discussion in the previous subsection. The same comment applies also to the bound in Eq. (62) for $Y_{\max}$.

The remainder $\delta_n Y(p', p; p_{\rm on})$ follows from Eq. (64):

$$\begin{aligned}|\delta_n Y_{ji}(p', p; p_{\rm on})| &= \left|\sum_{k=n}^{\infty}Y_{ji}^{[n]}(p', p)\right|\\ &\leqslant \frac{8\pi^2\mathcal{M}_Y}{\Lambda_V|p'|^2}\delta_n Y_{\max}\, g(p')h(p)\\ &=: \frac{8\pi^2\mathcal{N}_{\delta_n Y}}{\Lambda_V|p'|^2}\, g(p')h(p).\end{aligned} \tag{65}$$

The bounds for $\bar{Y}(p', p; p_{\rm on})$ are obtained from Eqs. (61) and (65) by interchanging $p \leftrightarrow p'$.

C. Bounds for the LO amplitude

After these preparations, we are finally in the position to deduce the bounds for the on-shell LO amplitude, which can be represented as

$$T_0 = V_0 R = \frac{N_0}{D}, \quad N_0 = V_0 D + V_0 Y. \tag{66}$$

First, consider the quantity N_0 defined explicitly as follows:

$$\begin{aligned}(N_0)_{ji}(p_{\rm on}) &= (V_0)_{ji}(p_{\rm on}, p_{\rm on})D(p_{\rm on})\\ &\quad + \sum_{i'}\int\frac{p'^2 dp'}{(2\pi)^3}(V_0)_{ji'}(p_{\rm on}, p')Y_{i'i}(p', p_{\rm on}; p_{\rm on}).\end{aligned} \tag{67}$$

Applying the bounds from Eqs. (20), (51), and (61), we obtain

$$\begin{aligned}&|(N_0)_{ji}(p_{\rm on})|\\ &\quad\leqslant \mathcal{M}_{V_0}V_{0,\max}\left[\mathcal{M}_D + \frac{n_{\rm PW}\mathcal{M}_{Y_{\max}}}{\Lambda_V}\int\frac{d|p|}{\pi}g(p)h(p)\right]\\ &\quad\leqslant \mathcal{M}_{V_0}V_{0,\max}\left(\mathcal{M}_D + \frac{\mathcal{M}_{Y_{\max}}\Sigma}{\mathcal{M}_{V_0}\mathcal{M}_G}\right) =: \mathcal{M}_{N_0}V_{0,\max}.\end{aligned} \tag{68}$$

Now, we can analyze the bounds for the LO amplitude T_0. Since T_0 is the ratio of N_0 and D, it is important how the Fredholm determinant D is bounded from below. From the definition in Eq. (38), it follows that all terms $D^{[n]}$ should be in general of order $O(Q^0)$. However, in a realistic situation, there might be certain cancellations among terms in the series, and the actual numerical value of $D(p_{\rm on})$ might turn out to be very small. This can happen when there is a shallow bound or quasibound state, which leads to an enhancement of the amplitude at threshold. Such a situation only takes place in the ${}^1S_0$ of NN scattering. Therefore, in our analysis for higher partial waves with $l \geqslant 1$, we regard the Fredholm determinant as being "natural":

$$|D(p_{\rm on})| \geqslant \mathcal{M}_{D,\min}, \tag{69}$$

where $\mathcal{M}_{D,\min}$ is a constant of order one. From Eqs. (68) and (69), we conclude that for $l \geqslant 1$, the LO amplitude is bounded by

$$|(T_0)_{ji}| \leqslant \mathcal{M}_{T_0}V_{0,\max} \tag{70}$$

and satisfies the same power counting as V_0, i.e., is of order $O(Q^0)$.

For the S-wave channels, we allow for the real part of D to be small, while still bounded from below at least at threshold. Moreover, we assume that the imaginary part of D, which is proportional to $p_{\rm on}$, is not a subject to additional cancellations. In particular, we exclude the situation when both N and D are equal to zero, i.e., the presence of a Castillejo-Dalitz-Dyson (CDD) pole [27,28]. We combine these conditions into the following constraint:

$$|D(p_{\rm on})| \geqslant \mathcal{M}_{D,\min}\left(\kappa + \frac{p_{\rm on}}{\Lambda_V}\right), \tag{71}$$

where $\kappa > 0$ is not necessarily of order one, but can be numerically small. The factor $1/\Lambda_V$ in front of $p_{\rm on}$ follows from the upper bound for the imaginary part of D.

The LO amplitude T_0 is enhanced compared to V_0, which can be written as

$$|(T_0)_{ji}| \leqslant \mathcal{M}_{T_0}\kappa^{-1}V_{0,\max} \tag{72}$$

or

$$|(T_0)_{ji}| \leqslant \mathcal{M}_{T_0}\frac{\Lambda_V}{p_{\rm on}}V_{0,\max}, \tag{73}$$

depending on the value of the on-shell momentum $p_{\rm on}$. The latter bound is in fact a unitary limit for the LO amplitude up to a numerical factor of order one, which justifies the coefficient $1/\Lambda_V$ in Eq. (71), see the definition of $V_{0,\max}$ in Eq. (D21). Equation (73) means that the LO amplitude becomes effectively of order $O(Q^{-1})$ in agreement with findings of Refs. [29,30].

To summarize, we have applied the Fredholm method to decompose the resolvent of the LS equation and derived the bounds for the Fredholm determinant D, the minor Y and the on-shell LO amplitude. The bounds involve undetermined dimensionless constants of order one, which can be calculated for each particular situation.

IV. NEXT-TO-LEADING-ORDER AMPLITUDE IN THE NONPERTURBATIVE CASE: P AND HIGHER PARTIAL WAVES

In this section we consider the on-shell ($p = p' = p_{\rm on}$) NLO amplitude T_2 for orbital angular momenta $l \geqslant 1$ and derive the corresponding bounds in the nonperturbative regime. We represent the amplitude T_2 using the Fredholm decomposition of the resolvent in Eq. (36) as follows:

$$T_2 = \bar{R}V_2R = V_2 + T_{2,Y}/D + T_{2,\bar{Y}}/D + T_{2,\bar{Y}Y}/D^2 =: \frac{N_2}{D^2}, \tag{74}$$

with

$$T_{2,Y} = V_2Y, \quad T_{2,\bar{Y}} = \bar{Y}V_2, \quad T_{2,\bar{Y}Y} = \bar{Y}V_2Y, \tag{75}$$

or more explicitly:

$$\begin{aligned}
T_{2,Y}(p',p;p_{\rm on}) &= \int \frac{p_1^2dp_1}{(2\pi)^3}V_2(p',p_1)Y(p_1,p;p_{\rm on}),\\
T_{2,\bar{Y}}(p',p;p_{\rm on}) &= \int \frac{p_1'^2dp_1'}{(2\pi)^3}\bar{Y}(p',p_1';p_{\rm on})V_2(p_1',p),\\
T_{2,\bar{Y}Y}(p',p;p_{\rm on}) &= \int \frac{p_1^2dp_1}{(2\pi)^3}\frac{p_1'^2dp_1'}{(2\pi)^3}\bar{Y}(p',p_1';p_{\rm on})V_2(p_1',p_1)Y(p_1,p;p_{\rm on}).
\end{aligned} \tag{76}$$

First, consider $T_{2,Y}$. The bounds for V_2 and Y in Eqs. (33) and (61) give

$$|T_{2,Y}(p',p;p_{\rm on})| \leqslant \int \frac{|p_1|^2d|p_1|}{(2\pi)^3}|V_2(p',p_1)||Y(p_1,p;p_{\rm on})| \leqslant \mathcal{M}_{V_2,1}n_{\rm PW}\frac{8\pi^2\mathcal{M}_{Y_{\max}}}{\Lambda_V}|p'|h(p)\int \frac{|p_1|d|p_1|}{(2\pi)^3}\tilde{f}_{\log}(p',p_1)g(p_1). \tag{77}$$

The functions g and h for P and higher partial waves are given in Eq. (25), which results in the following inequality:

$$\begin{aligned}
|T_{2,Y}(p',p;p_{\rm on})| &\leqslant \mathcal{M}_{V_2,1}n_{\rm PW}\frac{8\pi^2\mathcal{M}_{Y_{\max}}}{\Lambda_V}|p'||p|\int \frac{d|p_1|}{(2\pi)^3}\tilde{f}_{\log}(p',p_1)\lambda_{\log}(p_1/\Lambda)\\
&= \mathcal{M}_{V_2,1}n_{\rm PW}\frac{8\pi^2\mathcal{M}_{Y_{\max}}}{\Lambda_V}\frac{8\pi^2}{m_N\Lambda_V\Lambda_b^2}|p'||p|\int \frac{d|p_1|}{(2\pi)^3}f_{\log}(p',p_1)\lambda_{\log}(p_1/\Lambda)\\
&= \frac{\mathcal{M}_{V_2,1}n_{\rm PW}\mathcal{M}_{Y_{\max}}}{\Lambda_V}\frac{8\pi^2}{m_N\Lambda_V\Lambda_b^2}|p'||p|\left\{\left[1+\theta(|p'|-M_\pi)\ln\frac{|p'|}{M_\pi}\right]I_{\lambda_{\log},1} + I_{\lambda_{\log},2}\right\},
\end{aligned} \tag{78}$$

where the typical integrals $I_{\lambda_{\log},1}$ and $I_{\lambda_{\log},2}$ are defined and estimated in Appendix F and we have used Eq. (29). Using those estimates, we obtain

$$|T_{2,Y}(p',p;p_{\rm on})| \leqslant \mathcal{M}_{2,Y}\frac{8\pi^2}{m_N\Lambda_V\Lambda_b^2}|p'||p|\frac{\Lambda}{\Lambda_V}\left[1+\theta(|p'|-M_\pi)\ln\frac{|p'|}{M_\pi} + \ln\frac{\Lambda}{M_\pi}\right], \tag{79}$$

which reduces to

$$|T_{2,Y}(p_{\rm on})| \leqslant \mathcal{M}_{2,Y;{\rm on}} \frac{8\pi^2}{m_N \Lambda_V \Lambda_b^2} \frac{\Lambda}{\Lambda_V} p_{\rm on}^2 \ln \frac{\Lambda}{M_\pi}, \tag{80}$$

for the on-shell momenta $p = p' = p_{\rm on}$. The bounds for $T_{2,\bar{Y}}$ are the same as for $T_{2,Y}$.

Next, we analyze $T_{2,\bar{Y}Y}$:

$$\begin{aligned}|T_{2,\bar{Y}Y}(p', p; p_{\rm on})| &\leqslant \int \frac{|p_1|^2 d|p_1|}{(2\pi)^3} \left| \frac{|p_1'|^2 d|p_1'|}{(2\pi)^3} \right| \bar{Y}(p', p_1'; p_{\rm on})||V_2(p_1', p_1)||Y(p_1, p; p_{\rm on})| \\ &\leqslant \mathcal{M}_{V_2,1} n_{\rm PW}^2 \left(\frac{8\pi^2 \mathcal{M}_{Y_{\rm max}}}{\Lambda_V} \right)^2 h(p')h(p) \int \frac{|p_1| d|p_1|}{(2\pi)^3} \frac{|p_1'| d|p_1'|}{(2\pi)^3} \tilde{f}_{\rm log}(p', p_1) g(p_1') g(p_1).\end{aligned} \tag{81}$$

The integrals over p_1 and p_1' factorize, giving rise to the same set of integrals as in $T_{2,Y}$. The analog of Eq. (80) for $T_{2,\bar{Y}Y}$ in the on-shell kinematics is given by

$$|T_{2,\bar{Y}Y}(p_{\rm on})| \leqslant \mathcal{M}_{2,\bar{Y}Y;{\rm on}} \frac{8\pi^2}{m_N \Lambda_V \Lambda_b^2} \frac{\Lambda^2}{\Lambda_V^2} p_{\rm on}^2 \ln \frac{\Lambda}{M_\pi}. \tag{82}$$

Combining the bounds for V_2, $T_{2,Y}$, $T_{2,\bar{Y}}$ and $T_{2,\bar{Y}Y}$ and setting $\Lambda \sim \Lambda_V$, we obtain

$$\begin{aligned}|T_2(p_{\rm on})| \leqslant \tilde{\mathcal{M}}_2 &\frac{8\pi^2}{m_N \Lambda_V \Lambda_b^2} p_{\rm on}^2 \\ &\times \ln \frac{\Lambda}{M_\pi} [1 + D(p_{\rm on})^{-1} + D(p_{\rm on})^{-2}].\end{aligned} \tag{83}$$

Since we assume that for the P and higher partial waves the Fredholm determinant is bounded from below by a constant of order one, see Eq. (69), Eq. (83) takes the form

$$|T_2(p_{\rm on})| \leqslant \mathcal{M}_2 \frac{8\pi^2}{m_N \Lambda_V \Lambda_b^2} p_{\rm on}^2 \ln \frac{\Lambda}{M_\pi}. \tag{84}$$

Thus, the NLO amplitude is of order $O(Q^2)$ up to a factor $\ln \Lambda/M_\pi$, which agrees with the dimensional power counting. This result reproduces the one obtained in Ref. [16] for the case of a perturbative LO interaction.

A. Promoting a contact term to leading order

In this subsection we consider separately the scenario with promoting leading P-wave contact terms to the LO potential. As already discussed in Sec. II B, phenomenological arguments may require a promotion of contact interactions quadratic in momenta to the LO potential, even though they are formally of order $O(Q^2)$. A typical example is the $^3P_0$ partial wave, where the promotion of the contact interaction to leading order is often considered as necessary.

Below, we discuss the subtlety related to the freedom of choosing the renormalization condition, i.e., deciding what part of the considered contact interaction should be included into the LO potential and what part of it should be left in the NLO potential.

The LO partial wave contact interaction in the P-wave channel i is given by

$$V^{(0)}_{{\rm short},\Lambda,i}(p', p) = C_i V_{C_i,\Lambda}(p', p), \tag{85}$$

where $V_{C_i,\Lambda}(p', p)$ is the partial wave projection of the regulated contact term (see Appendix A) relevant for the considered channel. The corresponding NLO contact interaction has the same structure:

$$V^{(2)}_{{\rm short},\Lambda,i}(p', p) = C_{2,i} V_{C_i,\Lambda}(p', p). \tag{86}$$

In our estimates, we always assume that the LO low-energy constants (LECs) are of natural size,

$$C_i = \frac{\mathcal{M}_{C_i}}{\Lambda_b^2} \frac{8\pi^2}{m_N \Lambda_V}, \tag{87}$$

see Appendix of Ref. [16] (the factor of 4π corresponds to the partial-wave basis), so that the contact interactions quadratic in momenta are of order $\sim p^2/\Lambda^2 \sim O(Q^2)$ and are suppressed for small momenta. As a consequence, the regulator corrections to the contact interactions quadratic in momenta are effects of order $O(Q^4)$ and can be neglected in the present study. This is why we adopt the same regulator for $V^{(0)}_{{\rm short},\Lambda,i}$ and $V^{(2)}_{{\rm short},\Lambda,i}$ even though, in principle, one could employ a larger cutoff for the NLO terms or even use the unregulated potential. Nevertheless, if the contact interactions quadratic in momenta are promoted to leading order, then their contribution in the iterations of the LO potential at momenta $p \sim \Lambda$ is of the same order as those of the momentum-independent contact interactions and of the one-pion-exchange potential as long as we treat $\Lambda \sim \Lambda_b$.

The freedom to choose the renormalization scheme manifests itself schematically as follows: if we perform the transformation

$$C_i \to C_i + \delta C_i, \quad C_{2,i} \to C_{2,i} - \delta C_i, \quad \delta C_i \ll C_i, \tag{88}$$

and expand the LO and NLO amplitudes in Eqs. (14) and (15) in δC, then the linear in δC terms cancel:

$$\delta T_0 \approx -\delta T_2 \approx \delta C \bar{R} V_{C_i,\Lambda} R, \tag{89}$$

where we have neglected higher order effects, such as the terms proportional simultaneously to δC and the NLO potential.

As was shown in this section, there are no power counting breaking contributions in P waves at NLO stemming from the iterations of the LO potential. This means that $C_{2,i}$ is the renormalized quantity, where we assume that the divergent contributions to the two-pion-exchange diagrams are subtracted within some scheme, e.g., as is done for our choice of the nonpolynomial two-pion-exchange contribution, see Eq. (B2). Then, one obvious choice for the renormalization condition is

$$C_{2,i} = 0. \tag{90}$$

However, at higher orders, power counting breaking terms will appear also in P waves, and one will have to absorb them by performing renormalization of the same contact interaction. Therefore, to be consistent with our subtraction scheme for the S waves, we impose the renormalization condition on C_i and $C_{2,i}$ by requiring that the NLO amplitude in channels with $l = 1$ vanishes at threshold faster than $p_{\rm on}^2$:

$$(T_2)_{11}(p_{\rm on})/p_{\rm on}^2\big|_{p_{\rm on}=0} = 0. \tag{91}$$

Instead of the threshold point $p_{\rm on} = 0$, one can also take another renormalization point below or above threshold within the applicability of our approach.

A potential problem related to the above renormalization condition was discussed in great detail in Ref. [31] when studying schemes with large or infinite cutoffs. It arises near "exceptional" cutoff values for which the contribution of the contact interaction to the NLO amplitude is unnaturally small:

$$\big(\bar{R}V_{C_i,{\rm short},\Lambda}R\big)(p_{\rm on})/p_{\rm on}^2\big|_{p_{\rm on}=0} \approx 0, \tag{92}$$

which, in turn, leads to an unnaturally large value of $C_{2,i}$. In such a case, the power counting is violated unless the zero of the function on the left-hand side of Eq. (92) is factorizable (i.e., it appears at all energies). The condition in Eq. (92) can take place, e.g., in the spin-triplet channels with attractive one-pion-exchange potential such as ${}^3{\rm P}_0$ if the adopted cutoff value is too large. Then one starts to feel the singular nature of the one-pion-exchange potential, which is reflected in oscillations of the scattering wave function at short distances. Note that this effect does not directly correspond to the appearance of spurious bound states, although the two issues are related to each other.

In Ref. [31], several particular cases were discussed when the condition in Eq. (92) can be avoided or the corresponding zero is factorizable. However, we are interested in the general case, in which the practical solution of the problem would be to explicitly verify that the LO potential is chosen in such a way that the condition in Eq. (92) is not fulfilled. In such a case, the NLO amplitude will satisfy the expected power counting. In fact, for the regulators mentioned in the discussion in Sec. VI and many other choices tested by us, if the cutoff value is of the order of the hard scale, then Eq. (92) is never fulfilled. A simple indication that the cutoff of the LO potential is not "exceptional" is the naturalness of the renormalized NLO low-energy constants.

To summarize, we have shown that the P-wave NLO amplitudes formally satisfy the dimensional power counting in the nonperturbative regime. This holds also for the case when a contact interaction quadratic in momenta is promoted to LO if one makes sure that a certain condition on the LO potential is satisfied.

V. NONPERTURBATIVE RENORMALIZATION OF THE AMPLITUDE AT NLO: S WAVES

In this section we consider the renormalization of the NLO amplitude in the nonperturbative regime for S waves. As in the perturbative case considered in Ref. [16], subtractions have to be made to absorb contributions that violate power counting. We will start with generalizing the perturbative result of Ref. [16] and then analyze under which conditions a particular power counting can be established.

A. General formula

Analogous to the situation discussed in Sec. IV A, there is freedom to choose the momentum-independent part of the NLO potential

$$\hat{V}_2(p', p) = V_2(0, 0), \tag{93}$$

because it can be partly or completely absorbed by the LO potential. In the perturbative case, the NLO amplitude corresponding to $\hat{V}_2$ does not contain any power counting breaking contributions in contrast to the remaining part $\tilde{T}_2$ that is generated by

$$\tilde{V}_2(p', p) = V_2(p', p) - \hat{V}_2(p', p). \tag{94}$$

In what follows, we will mostly consider the scheme with $\hat{V}_2(p', p) = 0$, which is well suited for compensating possible threshold enhancement of the LO amplitude due to nonperturbative effects. Alternative schemes will be briefly discussed separately. Therefore, when using the results of Ref. [16], we will assume

$$\tilde{V}_2 = V_2, \quad \tilde{T}_2 = T_2. \tag{95}$$

First, we recall some notation from Ref. [16]. For an operator $X = X_{l'l}(p', p; p_{\rm on})$, where $l(l')$ is the initial (final) orbital angular momentum, we define the subtraction operation $\mathbb{T}$:

$$\mathbb{T}(X) = X_{00}(0, 0, 0)V_{\rm ct}, \tag{96}$$

where the contact term is given by

$$V_{\rm ct} = |\chi\rangle\langle\chi|, \tag{97}$$

$$\langle p, lsj|\chi\rangle = \delta_{l,0}. \tag{98}$$

We assume that the counter term is unregulated or regulated with some $\Lambda_{\rm ct} \gg \Lambda$. Analogously, we introduce the subtraction operation $\mathbb{T}^{m_i,n_i}$ for subdiagrams (m_i, n_i) of the diagram (m, n) corresponding to $T_2^{[m,n]}$. We follow the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) subtraction scheme [32–34] and represent the renormalized amplitude via the forest formula:

$$\mathbb{R}\big(T_2^{[m,n]}\big) = T_2^{[m,n]} + \sum_{U_k\in\mathcal{F}^{m,n}} \left(\prod_{(m_i,n_i)\in U_k} -\mathbb{T}^{m_i,n_i}\right) T_2^{[m,n]}, \tag{99}$$

where $\mathcal{F}^{m,n}$ represents the set of all forests, i.e., the set of all possible distinct sequences of nested subdiagrams

(m_i, n_i):

$$\begin{aligned} &U_k = ((m_{k;1}, n_{k;1}), (m_{k;2}, n_{k;2}), \dots), \\ &m \geqslant m_{k;i+1} \geqslant m_{k;i} \geqslant 0, \quad n \geqslant n_{k;i+1} \geqslant n_{k;i} \geqslant 0, \\ &n + m > 0. \end{aligned} \tag{100}$$

In Ref. [16], it was proved that each term in the expansion in V_0 of the renormalized NLO amplitude satisfies the dimensional power counting and is bounded by

$$\left|\mathbb{R}\left(T_2^{[m,n]}\right)(p_{\rm on})\right| \leqslant \frac{8\pi^2 \mathcal{M}_{T_2}}{m_N \Lambda_V} \Sigma_{2,0}^{m+n} \frac{p_{\rm on}^2}{\Lambda_b^2} \ln \frac{\Lambda}{M_\pi}, \tag{101}$$

where

$$\Sigma_{2,0} = 2\mathcal{M}_{\max} \frac{\Lambda}{\Lambda_V} \tag{102}$$

is a quantity of order one ($\Sigma_{2,0} \geqslant 1$ in the nonperturbative case).

To resume the series

$$\mathbb{R}(T_2)(p_{\rm on}) = \sum_{m,n=0}^{\infty} \mathbb{R}(T_2^{[m,n]})(p_{\rm on}), \tag{103}$$

we perform some rearrangement of Eq. (99), as explained below.

It is convenient to introduce the following notation:

$$\begin{aligned} |\bar{\psi}\rangle = \bar{R}|\chi\rangle, \quad \langle\psi| = \langle\chi|R, \\ \psi_l(p; p_{\rm on}) = \langle\psi|p, lsj\rangle = \langle p, lsj|\bar{\psi}\rangle. \end{aligned} \tag{104}$$

For on-shell momenta $p = p_{\rm on}$, the explicit form of ψ_l reads

$$\begin{aligned} \psi_l(p_{\rm on}) &:= \psi_l(p_{\rm on}; p_{\rm on}) \\ &= \delta_{l,0} + \int \frac{p^2 dp}{(2\pi)^3} G(p; p_{\rm on}) (T_0)_{0,l}(p, p_{\rm on}; p_{\rm on}), \end{aligned} \tag{105}$$

and it coincides with the scattering wave function at the origin $(r = 0)$.

Now, consider the sum of all unrenormalized diagrams:

$$T_2 = \bar{R} V_2 R, \tag{106}$$

and perform first all single overall subtractions:

$$\delta T_2^{(1),\rm overall} = -\mathbb{T}(T_2) = -(T_2)_{00}(0, 0; 0)|\chi\rangle\langle\chi|, \tag{107}$$

where the superscript (1) denotes the number of subtractions.

If we add all possible rescatterings with the LO potential, then we will obtain all terms with single subtractions in subdiagrams:

$$\begin{aligned} \delta T_2^{(1)} &= \bar{R}\delta T_2^{(1),\rm overall} R = -(T_2)_{00}(0, 0; 0)\bar{R}|\chi\rangle\langle\chi|R \\ &= -(T_2)_{00}(0, 0; 0))|\bar{\psi}\rangle\langle\psi|. \end{aligned} \tag{108}$$

Analogously, the sum of all double nested subtractions (one of which is an overall subtraction) is given by

$$\begin{aligned} \delta T_2^{(2),\rm overall} &= -\mathbb{T}\left(\delta T_2^{(1)} - \delta T_2^{(1),\rm overall}\right) \\ &= (T_2)_{00}(0, 0; 0)[\psi_0(0)^2 - 1]|\chi\rangle\langle\chi| \\ &= -[\psi_0(0)^2 - 1]\delta T_2^{(1),\rm overall}, \end{aligned} \tag{109}$$

where the constant term $\delta T_2^{(1)}$ was already subtracted in the previous step and should be excluded. All terms with double nested subtractions in subdiagrams are obtained by adding the rescattering contributions:

$$\delta T_2^{(2)} = \bar{R}\delta T_2^{(2),\rm overall} R = -[\psi_0(0)^2 - 1]\delta T_2^{(1)}. \tag{110}$$

Continuing with further multiple nested subtractions, we obtain recursion relations:

$$\begin{aligned} \delta T_2^{(n+1),\rm overall} &= -\mathbb{T}\left(\delta T_2^{(n)} - \delta T_2^{(n),\rm overall}\right) \\ &= -[\psi_0(0)^2 - 1]\delta T_2^{(n),\rm overall}, \end{aligned} \tag{111}$$

and

$$\delta T_2^{(n+1)} = -[\psi_0(0)^2 - 1]\delta T_2^{(n)}, \tag{112}$$

where the superscripts (n) and $(n+1)$ denote the number of nested subtractions. The terms $T_2^{(n)}$ can be summed up to

$$\delta T_2 = \sum_{n=1}^{\infty} \delta T_2^{(n)} = \delta T_2^{(1)} \sum_{n=0}^{\infty} [1 - \psi_0(0)^2]^n = \delta T_2^{(1)} \frac{1}{\psi_0(0)^2}. \tag{113}$$

Finally,

$$\mathbb{R}(T_2) = T_2 + \delta T_2 = T_2 - \frac{(T_2)_{00}(0, 0; 0)}{\psi_0(0)^2}|\bar{\psi}\rangle\langle\psi|. \tag{114}$$

Taking the on-shell matrix elements of $\mathbb{R}(T_2)$, we obtain

$$\mathbb{R}(T_2)_{l'l}(p_{\rm on}) = (T_2)_{l'l}(p_{\rm on}) + \delta C \psi_{l'}(p_{\rm on})\psi_l(p_{\rm on}), \tag{115}$$

with the counter term constant

$$\delta C = -\frac{(T_2)_{00}(0)}{\psi_0(0)^2}. \tag{116}$$

Equation (115) can also be obtained directly without referring to the perturbative result from the renormalization condition:

$$\mathbb{R}(T_2)_{l'l}(0) = 0. \tag{117}$$

Therefore, the perturbative and nonperturbative results match in the regime where both are applicable.

Similarly to the analysis of higher partial waves in Sec. IV, we use the Fredholm decomposition of the resolvent of the LS equation and introduce the quantities N_2 and ν,

$$(T_2)_{l'l}(p_{\rm on}) = \frac{(N_2)_{l'l}(p_{\rm on})}{D(p_{\rm on})^2}, \quad \psi_l(p_{\rm on}) = \frac{\nu_l(p_{\rm on})}{D(p_{\rm on})}. \tag{118}$$

The counter term constant can be expressed as

$$\delta C = -\frac{(N_2)_{00}(0)}{\nu_0(0)^2}. \tag{119}$$

Then, the renormalized amplitude $\mathbb{R}(T_2)$ reads

$$\mathbb{R}(T_2)_{l'l}(p_{\rm on}) = \frac{1}{D(p_{\rm on})^2}[(N_2)_{l'l}(p_{\rm on}) + \delta C\, \nu_{l'}(p_{\rm on})\nu_l(p_{\rm on})]$$
$$= \frac{\mathbb{R}(N_2)_{l'l}(p_{\rm on})}{D(p_{\rm on})^2} = \frac{\mathbb{R}(\tilde{N}_2)_{l'l}(p_{\rm on})}{D(p_{\rm on})^2\, \nu_0(0)^2}, \tag{120}$$

where, for convenience, the following quantities have been introduced:

$$\mathbb{R}(N_2)_{l'l}(p_{\rm on}) = (N_2)_{l'l}(p_{\rm on}) + \delta C\, \nu_{l'}(p_{\rm on})\nu_l(p_{\rm on}), \tag{121}$$

$$\mathbb{R}(\tilde{N}_2)_{l'l}(p_{\rm on}) = (N_2)_{l'l}(p_{\rm on})\nu_0(0)^2 - (N_2)_{00}(0)\nu_{l'}(p_{\rm on})\nu_l(p_{\rm on}). \tag{122}$$

B. Power counting with the naturalness condition for $\nu_0(0)$

In this subsection we analyze the expression for the renormalized NLO amplitude $\mathbb{R}(T_2)$ in Eq. (120) and determine what power counting it satisfies under which conditions. Considering different constraints on various quantities entering $\mathbb{R}(T_2)$, we can understand to what extent the renormalizability of the amplitude depends on details of the short-range dynamics.

We assume that the Fredholm determinant $D(p_{\rm on})$ satisfies the bound in Eq. (71), which includes also the case of a shallow (quasi-) bound state. For the function $D(p_{\rm on})^2$, we can write

$$|D(p_{\rm on})^2| \geqslant \mathcal{M}_{D,\min}^2 \kappa^2, \tag{123}$$

or, if κ is very small, then

$$|D(p_{\rm on})^2| \geqslant \mathcal{M}_{D,\min}^2 \frac{p_{\rm on}^2}{\Lambda_V^2}. \tag{124}$$

We will also need the upper bound for the quantity $\nu_l(p_{\rm on})$, see Eq. (E21):

$$\nu_l(p_{\rm on}) \leqslant \mathcal{M}_\nu. \tag{125}$$

First, we consider the "natural" case when the quantity $\nu_0(0)$ is bounded not only from above as in Eq. (125), but also from below by some constant of order one:

$$\nu_0(0) \geqslant \mathcal{M}_{\nu,\min}, \tag{126}$$

which also implies the natural value of the counter term constant δC, see Eq. (119), similarly to the condition of the absence of "exceptional" cutoffs in Sec. IV A. Then as follows from Eq. (120), to analyze the power counting that the renormalized amplitude $\mathbb{R}(T_2)$ satisfies, it is sufficient to find bounds for $\mathbb{R}(\tilde{N}_2)$.

As we show in Appendix E, the quantity $\mathbb{R}(\tilde{N}_2)$ can be expanded into a convergent series in terms of V_0:

$$\mathbb{R}(\tilde{N}_2)(p_{\rm on}) = \sum_{m,n=0}^{\infty} [\mathbb{R}(\tilde{N}_2)(p_{\rm on})]^{[m,n]}$$
$$= \sum_{m,n=0}^{n_{\max}} [\mathbb{R}(\tilde{N}_2)(p_{\rm on})]^{[m,n]} + \delta_{n_{\max}}[\mathbb{R}(\tilde{N}_2)(p_{\rm on})]$$
$$=: S_{\tilde{N}_2,n_{\max}}(p_{\rm on}) + \delta_{n_{\max}}[\mathbb{R}(\tilde{N}_2)(p_{\rm on})], \tag{127}$$

and the remainder $\delta_n[\mathbb{R}(\tilde{N}_2)(p_{\rm on})]$ decreases faster than exponential with any base $\mathcal{M}_{\delta\tilde{N}_2}$ starting with some $n = \tilde{\mathcal{M}}_{\delta\tilde{N}_2}$ [see Eq. (E22)]:

$$|\delta_n[\mathbb{R}(\tilde{N}_2)]| \leqslant \frac{8\pi^2}{m_N\Lambda_V}\mathcal{N}_{\tilde{N}_2} e^{-\mathcal{M}_{\delta\tilde{N}_2} n}, \quad \text{for } n > \tilde{\mathcal{M}}_{\delta\tilde{N}_2}. \tag{128}$$

The prefactor $\mathcal{N}_{\tilde{N}_2}$ is given by

$$\mathcal{N}_{\tilde{N}_2} = \frac{\Lambda^2}{\Lambda_b^2}\ln\frac{\Lambda}{M_\pi} \tag{129}$$

in the case of the "standard" regulators of the LO potential. For the "mild" regulator, it depends also on the regulator of the NLO potential $\Lambda_{\rm NLO}$:

$$\mathcal{N}_{\tilde{N}_2} = \frac{\Lambda\Lambda_{\rm NLO}}{\Lambda_b^2}\ln\frac{\Lambda_{\rm NLO}}{\Lambda}\ln\frac{\Lambda_{\rm NLO}}{M_\pi}, \tag{130}$$

and, in contrast to the perturbative regime, the regulator $\Lambda_{\rm NLO}$ cannot be set to infinity (in general) but can be chosen $\Lambda_{\rm NLO} \gg \Lambda$. Note that we do not consider the choice $\Lambda_{\rm NLO} \sim \Lambda$ for the "mild" LO regulator because in such a case, we would simply reproduce the variant with the "standard" regulators. The appearance of $\Lambda_{\rm NLO}$ in the expression for $\mathcal{N}_{\tilde{N}_2}$ is an indication of a potentially stronger cutoff dependence of the NLO amplitude in the nonperturbative regime.

The general conservative estimate for $\tilde{\mathcal{M}}_{\delta\tilde{N}_2}$ yields $\tilde{\mathcal{M}}_{\delta\tilde{N}_2} \gtrsim (e\Sigma)^2$, which is rather large. In realistic calculations, it turns out to be much smaller, see the discussion in Sec. III A and the numerical results in Sec. VI.

However, expanding Eq. (120) in V_0 gives

$$[\mathbb{R}(\tilde{N}_2)(p_{\rm on})]^{[m,n]} = \sum_{m_1=0}^{m}\sum_{m_2=0}^{m-m_1}\sum_{n_1=0}^{n}\sum_{n_2=0}^{n-n_1} D^{[m-m_1-m_2]}(p_{\rm on})$$
$$\times D^{[n-n_1-n_2]}(p_{\rm on})\nu_0(0)^{[m_2]}\nu_0(0)^{[n_2]}$$
$$\times \mathbb{R}\big(T_2^{[m_1,n_1]}\big)(p_{\rm on}). \tag{131}$$

Using the perturbative bounds on $\mathbb{R}(T_2^{[m,n]})$ in Eq. (101) and Eqs. (51) and (125), we obtain

$$[\mathbb{R}(\tilde{N}_2)(p_{\rm on})]^{[m,n]} \leqslant \mathcal{M}_D^2\mathcal{M}_\nu^2 \sum_{m_1=0}^{m}\sum_{n_1=0}^{n}\big|\mathbb{R}(T_2^{[m_1,n_1]})(p_{\rm on})\big|$$
$$\leqslant \frac{8\pi^2\mathcal{M}_{T_2}\mathcal{M}_D^2\mathcal{M}_\nu^2}{m_N\Lambda_V}\frac{p_{\rm on}^2}{\Lambda_b^2}$$
$$\times \ln\frac{\Lambda}{M_\pi}\sum_{m_1=0}^{m}\sum_{n_1=0}^{n}\Sigma_{2,0}^{m_1+n_1}. \tag{132}$$

Performing the summation up to $n = n_{\max}$, we obtain

$$\begin{aligned}\left|S_{\tilde{N}_2,n_{\max}}(p_{\rm on})\right| &\leqslant \frac{8\pi^2\mathcal{M}_{T_2}\mathcal{M}^2_{D^{[n]},\max}}{m_N\Lambda_V}\frac{p_{\rm on}^2}{\Lambda_b^2}\\ &\quad\times \ln\frac{\Lambda}{M_\pi}\sum_{m,n=0}^{n_{\max}}\sum_{m_1=0}^{m}\sum_{n_1=0}^{n}\Sigma_{2,0}^{m_1+n_1}\\ &\leqslant \frac{8\pi^2\mathcal{M}_{N_2;2}}{m_N\Lambda_V}\frac{p_{\rm on}^2}{\Lambda_b^2}\ln\frac{\Lambda}{M_\pi}n_{\max}^4\Sigma_{2,0}^{2n_{\max}}\\ &=: \frac{8\pi^2\mathcal{M}_S}{m_N\Lambda_V}\frac{p_{\rm on}^2}{\Lambda_b^2}\Phi_{\log}. \end{aligned} \tag{133}$$

Given that the remainder $\delta_n[\mathbb{R}(\tilde{N}_2)]$ can be made arbitrarily small by choosing a sufficiently large $n_{\max}$, e.g.,

$$|\delta_n[\mathbb{R}(\tilde{N}_2)]| \leqslant \frac{8\pi^2}{m_N\Lambda_V}\frac{M_\pi^2\kappa^2}{\Lambda_b^2}, \tag{134}$$

whereas the sum in Eq. (133) has the bound similar to the one for the perturbative amplitude up to numerical constants of order one and possible factors logarithmic in Λ, $\Phi_{\log}$, we can conclude that $\mathbb{R}(\tilde{N}_2)$ is bounded as

$$|\mathbb{R}(\tilde{N}_2)(p_{\rm on})| \leqslant \frac{8\pi^2\mathcal{M}_{\tilde{N}_2}}{m_N\Lambda_V}\left[\frac{p_{\rm on}^2}{\Lambda_b^2}\Phi_{\log} + \frac{M_\pi^2}{\Lambda_b^2}\kappa^2\right]. \tag{135}$$

Whether this picture is indeed realized for the realistic NN interaction, i.e., whether $\mathcal{M}_{\tilde{N}_2}$ is really (and not only formally) of the order of one, is straightforward to verify by explicit numerical checks of the series for $\mathbb{R}(\tilde{N}_2)$ as we do partly in Sec. VI.

For completeness, we show below that Eq. (135) holds formally in the chiral limit, i.e., for the expansion parameter $Q \ll 1$. What we have to prove is that there exists such a value of $n_{\max}$ that the remainder $\delta_{n_{\max}}[\mathbb{R}(\tilde{N}_2)]$ satisfies Eq. (134), and, at the same time, the prefactor

$$\chi = n_{\max}^4\Sigma_{2,0}^{2n_{\max}} \tag{136}$$

in Eq. (133) does not contain inverse powers of Q and, therefore, does not destroy the power counting.

The choice

$$n_{\max} \geqslant \max(k_0, \bar{k}_0), \tag{137}$$

with

$$k_0 = \tilde{\mathcal{M}}_{\delta\tilde{N}_2}, \quad \bar{k}_0 = -\frac{1}{\mathcal{M}_{\delta\tilde{N}_2}}\ln\frac{M_\pi^2\kappa^2}{\Lambda_b^2\mathcal{N}_{\tilde{N}_2}}, \tag{138}$$

guarantees that Eq. (134) holds, as follows from Eq. (128). Note that the inequality $\bar{k}_0 > k_0$ holds only for extremely small $Q = M_\pi/\Lambda_b$. However, in the actual calculations, this can happen also for physical values of Q.

The factor χ is then given by

$$\chi = \tilde{\mathcal{M}}^4_{\delta\tilde{N}_2}\Sigma_{2,0}^{2\tilde{\mathcal{M}}_{\delta\tilde{N}_2}} \tag{139}$$

if $n_{\max} = k_0$, and by

$$\chi = \frac{1}{\mathcal{M}^4_{\delta\tilde{N}_2}}\left(\ln\frac{M_\pi^2\kappa^2}{\Lambda_b^2\mathcal{N}_{\tilde{N}_2}}\right)^4\left(\frac{M_\pi^2\kappa^2}{\Lambda_b^2\mathcal{N}_{\tilde{N}_2}}\right)^{-2\frac{\ln\Sigma_{2,0}}{\mathcal{M}_{\delta\tilde{N}_2}}} \tag{140}$$

if $n_{\max} = \bar{k}_0$. In the latter case, if $\mathcal{M}_{\delta\tilde{N}_2}$ is chosen to be $\mathcal{M}_{\delta\tilde{N}_2} \gg \ln\Sigma_{2,0}$, then the factor $(\frac{M_\pi^2\kappa^2}{\Lambda_b^2\mathcal{N}_{\tilde{N}_2}})^{-2\frac{\ln\Sigma_{2,0}}{\mathcal{M}_{\delta\tilde{N}_2}}}$ can be neglected.

Thus, we conclude that Eq. (135) holds with

$$\Phi_{\log} = \begin{cases}\mathcal{M}_{\log}\ln\frac{\Lambda}{M_\pi}, & n_{\max} = k_0,\\ \mathcal{M}_{\log}\ln\frac{\Lambda}{M_\pi}\left(\ln\frac{M_\pi^2\kappa^2}{\Lambda_b^2\mathcal{N}_{\tilde{N}_2}}\right)^4, & n_{\max} = \bar{k}_0.\end{cases} \tag{141}$$

Now we come back to the expression for the renormalized NLO amplitude in Eq. (120). For small on-shell momenta $p_{\rm on}$, i.e., when

$$\left|S_{\tilde{N}_2,n_{\max}}(p_{\rm on})\right| \leqslant |\delta_n[\mathbb{R}(\tilde{N}_2)]|, \tag{142}$$

Eqs. (123), (126), and (134) give

$$\left|\mathbb{R}(T_2)_{l'l}(p_{\rm on})\right| \leqslant \frac{8\pi^2\mathcal{M}_{T_2,\rm low}}{m_N\Lambda_V}\frac{M_\pi^2}{\Lambda_b^2}, \tag{143}$$

which means that in this energy region, $\mathbb{R}(T_2)$ is of order $O(Q^2)$.

As the on-shell momentum increases, i.e.,

$$\left|S_{\tilde{N}_2,n_{\max}}(p_{\rm on})\right| \geqslant |\delta_n[\mathbb{R}(\tilde{N}_2)]|, \tag{144}$$

we should use Eq. (133) instead of Eq. (134) to obtain

$$|\mathbb{R}(T_2)_{l'l}(p_{\rm on})| \leqslant \frac{8\pi^2\mathcal{M}_{T_2,\rm high}}{m_N\Lambda_V}\frac{p_{\rm on}^2}{\Lambda_b^2}\frac{\Phi_{\log}}{\kappa^2}, \tag{145}$$

which is enhanced compared to $O(Q^2)$ by a factor $1/\kappa^2$. In the worst case of the unitary limit, we obtain from Eq. (124):

$$|\mathbb{R}(T_2)_{l'l}(p_{\rm on})| \leqslant \frac{8\pi^2\mathcal{M}_{T_2,\rm high}}{m_N\Lambda_V}\frac{\Lambda_V^2}{\Lambda_b^2}\Phi_{\log}, \tag{146}$$

which corresponds effectively to $\mathbb{R}(T_2) \sim O(Q^0)$. This is still one order higher than the LO amplitude $O(Q^{-1})$, see Eq. (73), but the convergence rate is rather low in this case. A natural way to reduce the effect of the numerical enhancement of the LO amplitude and to improve convergence is to promote some part of the NLO potential to leading order, which will make the numerical constant $\mathcal{M}_{T_2,\rm high}$ smaller. The simplest recipe would be to promote the contact interactions quadratic in momentum. As already mentioned, this approach is suggested, e.g., for the $^1S_0$ partial wave. We will discuss this possibility in Sec. V D.

C. Local LO potential in a spin-singlet channel and analogous cases

Above, we considered the general case of the LO potential under an additional assumption on its short-range part formulated in Eq. (126) in terms of the naturalness of $\nu_0(0)$. It is instructive to consider one particular case, when the LO potential in a spin-singlet channel is fully local. Then, this condition is satisfied automatically. Moreover, for a local LO potential, the following identity holds:

$$\nu_0(p_{\rm on}) \equiv 1, \tag{147}$$

which follows from the fact that the scattering wave function at the origin $\psi_{p_{\rm on}}$ coincides with the inverse of the Jost function $f(p_{\rm on})$ and the inverse of the Fredholm determinant [25]:

$$\psi(p_{\rm on}) = f(p_{\rm on})^{-1} = D(p_{\rm on})^{-1}, \tag{148}$$

and the definition (118). Therefore, we have [see the definitions in Eqs. (122) and (121)]

$$\mathbb{R}(\tilde{N}_2)(p_{\rm on}) = \mathbb{R}(N_2)(p_{\rm on}) = \Delta N_2(p_{\rm on}) = N_2(p_{\rm on}) - N_2(0). \tag{149}$$

The whole discussion in the previous subsection applies for the case of a local LO potential, except the absence of the additional condition (126). In the general case, when the constraint in Eq. (126) is not satisfied, we still can have a situation similar to the local single-channel potential if we assume that the series for $\mathbb{R}(N_2)$ [not for $\mathbb{R}(\tilde{N}_2)$] converges and the bound analogous to Eq. (135) holds:

$$|\mathbb{R}(N_2)(p_{\rm on})| \leqslant \frac{8\pi^2 \mathcal{M}_{\tilde{N}_2}}{m_N \Lambda_V}\left[\frac{p_{\rm on}^2}{\Lambda_b^2}\Phi_{\rm log} + \frac{M_\pi^2}{\Lambda_b^2}\kappa^2\right]. \tag{150}$$

This is possible if the smallness of $\nu_0(0)$ in the denominator of $\mathbb{R}(N_2)$ is compensated by a corresponding small factor in the numerator, see Eq. (121). Whether this indeed takes place can be verified numerically in any particular case. From Eq. (150), we can deduce the same bounds for the renormalized NLO amplitude as in Eqs. (143), (145), and (146).

We made this comment to emphasize that the naturalness constraint on $\nu_0(0)$ is not necessary to guarantee renormalizability of the NLO amplitude, but is the most simple one from the practical point of view.

D. Promoting a momentum dependent contact term to leading order

In this subsection we analyze the situation when it is necessary to promote the momentum dependent S-wave contact term to leading order. For definiteness, we consider the $^1S_0$ partial wave, where such a promotion has been shown to significantly improve the convergence of the chiral EFT expansion, see Refs. [18,35]. Since this is a spin-singlet channel, we omit the l, l' indices in this subsection. We also omit all channel indices.

The whole analysis in the preceding subsections remains valid in this case, except that similarly to the promotion of the subleading term in the P waves considered in Sec. IV A, there is freedom choosing what part of such a contact term should be included in LO potential V_0 and what part remains in the NLO potential V_2.

We rewrite Eq. (121) by explicitly separating the part with the contact term quadratic in momenta:

$$\begin{aligned}\mathbb{R}(N_2)(p_{\rm on}) &= N_2(p_{\rm on}) + \delta C \nu(p_{\rm on})^2 \\ &=: \Delta N_2(p_{\rm on}) + \delta C \nu(p_{\rm on})^2 + C_2 N_{C_2}(p_{\rm on}),\end{aligned} \tag{151}$$

with

$$N_{C_2}(p_{\rm on}) = [\bar{R} V_C R](p_{\rm on}) D(p_{\rm on})^2. \tag{152}$$

The potential V_C is the contact interaction quadratic in momenta that projects onto the $^1S_0$ partial wave. This potential can remain regulated because the regulator corrections to it are of higher order.

Following our subtraction scheme at $p_{\rm on} = 0$, we introduce two renormalization conditions to fix δC and C_2:

$$\mathbb{R}(N_2)(0) = 0, \qquad \left.\frac{d^2\mathbb{R}(N_2)(p_{\rm on})}{dp_{\rm on}^2}\right|_{p_{\rm on}=0} = 0. \tag{153}$$

Note that N_2 is an analytic function of $p_{\rm on}^2$ at $p_{\rm on} = 0$.

Of course, Eq. (153) can be also formulated in terms of the amplitudes:

$$\mathbb{R}(T_2)(0) = 0, \qquad \left.\frac{d^2\mathbb{R}(T_2)(p_{\rm on})}{dp_{\rm on}^2}\right|_{p_{\rm on}=0} = 0. \tag{154}$$

Analogous to the situation in P waves, the above renormalization conditions can lead to a problem for "exceptional" cutoffs when Eqs. (153) become inconsistent, which happens not only when $\nu(0) = 0$ but also when the following equation is satisfied [31]:

$$\left.\left[\frac{d^2 N_{C_2}(p_{\rm on})}{dp_{\rm on}^2} - 2N_{C_2}(p_{\rm on})\nu(p_{\rm on})\frac{d^2\nu(p_{\rm on})}{dp_{\rm on}^2}\right]\right|_{p_{\rm on}=0} = 0. \tag{155}$$

As in the case of the P waves, an indirect indication that the cutoff is not close to an "exceptional" value is the naturalness of the NLO LECs. In our numerical calculation in Sec. VI, we found no "exceptional" cutoffs for the cutoff values of the order or below the hard scale.

E. Other subtraction schemes

In all analyses of the nonperturbative regime, we have always adopted the prescription to perform subtractions at threshold, see Eq. (117). In this subsection we briefly discuss other possibilities. Choosing different subtraction points, e.g., the deuteron pole position for the $^3S_1$-$^3D_1$ channel, is equivalent to setting, in contrast to Eq. (93), $\hat{V}_2 \neq 0$:

$$\hat{V}_2(p', p) = \hat{\kappa}^2 \frac{8\pi^2}{m_N \Lambda_V}\frac{M_\pi^2}{\Lambda_b^2}, \tag{156}$$

where $\hat{\kappa}$ is a constant of order one, see Eq. (D30). Since this potential is just an S-wave contact term, the corresponding NLO amplitude is given by

$$\begin{aligned}(\hat{T}_2)_{l'l}(p_{\rm on}) &= \hat{\kappa}^2 \frac{8\pi^2}{m_N \Lambda_V}\frac{M_\pi^2}{\Lambda_b^2}\psi_{l'}(p_{\rm on})\psi_l(p_{\rm on}) \\ &= \hat{\kappa}^2 \frac{8\pi^2}{m_N \Lambda_V}\frac{M_\pi^2}{\Lambda_b^2}\frac{\nu_{l'}(p_{\rm on})\nu_l(p_{\rm on})}{D(p_{\rm on})^2}.\end{aligned} \tag{157}$$

From Eqs. (123) and (125), we obtain the following bound:

$$|(\hat{T}_2)_{l'l}(p_{\rm on})| \leqslant \frac{8\pi^2}{m_N \Lambda_V}\frac{\mathcal{M}_\nu^2}{\mathcal{M}_{D,\rm min}^2}\frac{M_\pi^2}{\Lambda_b^2}\frac{\hat{\kappa}^2}{\kappa^2}. \tag{158}$$

For the perturbative case considered in Ref. [16] and for the case without an enhancement of the LO amplitude, the amplitude $\hat{T}_2$ satisfies the dimensional power counting: $T_2 \sim O(Q^2)$. However, in the situation when the LO amplitude is enhanced, the additional factor $\hat{\kappa}^2/\kappa$ in Eq. (158) relative to Eq. (72) spoils convergence even at threshold. We will have the worst situation in the unitary limit with $\kappa \ll 1$.

Thus, we conclude that for a reasonable convergence in the case of an enhanced LO amplitude, one should choose a subtraction scheme not much different from ours, i.e., such that $\hat{\kappa}/\kappa \sim 1$.

To summarize, we have shown that renormalization of the NLO amplitude for the S waves can be done explicitly also in the nonperturbative regime by analyzing the Fredholm decomposition of the amplitudes. In contrast to the perturbative case discussed in Ref. [16], additional constraints on the LO potential have to be fulfilled to ensure renormalizability and convergence of the chiral expansion. Then, the power counting works also in the situation when the LO amplitude is enhanced at threshold, although to make the scheme more efficient, it might be necessary to promote certain contributions to leading order.

VI. NUMERICAL RESULTS

In this section we illustrate our theoretical findings by explicit numerical calculation of the NLO NN amplitude in the three channels where the LO interaction should be treated nonperturbatively: $^3\mathrm{P}_0$, $^3\mathrm{S}_1$ -$^3\mathrm{D}_1$, and $^1\mathrm{S}_0$. The results for other channels were presented in Ref. [16].

We adopt the same values for the numerical constants as in Ref. [16]: the pion decay constant $F_\pi = 92.1$ MeV, the isospin average nucleon and pion masses $m_N = 938.9$ MeV, $M_\pi = 138.04$ MeV and the effective nucleon axial coupling constant $g_A = 1.29$. The calculations have been performed using *Mathematica* [36].

For the regularization of the LO and NLO potentials, we adopt the scheme similar to the one used in realistic calculation in Ref. [9] at fifth order in the chiral expansion, which allows us to have a direct interpretation of the numerical values of the cutoffs. In particular, we use the local Gaussian regulator for the one-pion-exchange potential and the nonlocal Gaussian regulator for all contact interactions with the same cutoff Λ, see Appendix A. For the sake of simplicity, we also employ the local Gaussian regulator in the form of the overall factor $F_{\Lambda_{\rm NLO},\exp}(q)$ for the two-pion-exchange potential. As in Ref. [16], the cutoff value $\Lambda_{\rm NLO}$ is set to the hard-scale $\Lambda_{\rm NLO} = 600$ MeV. This choice for the chiral expansion breakdown scale is consistent with the recent studies in the few-nucleon sector [37–40].

The momentum-independent contact interactions at NLO are included without a regulator in accordance with our power counting. The contact interactions quadratic in momenta are regulated with the same cutoff $\Lambda_{\rm NLO}$ at LO and at NLO in contrast to our choice in Ref. [16], where, for simplicity, we left the corresponding NLO contact terms unregulated. Both options are legitimate since the regulator corrections to the contact interactions quadratic in momenta is an effect of a higher order, $O(Q^4)$. By the same reason, the regulator corrections to the LO contact interactions quadratic in momenta are not taken into account.

The cutoff values for the one-pion-exchange potential and for the momentum-independent LO contact interactions are varied in the regions below and above $\Lambda = 450$ MeV, which was found to be the optimal cutoff value in Ref. [9]. The lower region corresponds to extremely soft cutoffs, where explicit regulator corrections to the LO potential are likely to be important. The upper region contains relatively hard (of the order of the hard scale) cutoffs as well as cutoffs above Λ_b, for which we expect slower convergence in terms of the Fredholm expansion and, therefore, potential problems with interpretation within our renormalization scheme.

The free parameters are determined by a fit to the empirical phase shifts from the Nijmegen partial wave analysis [41] up to $E_{\rm lab} = 150$ MeV. The phase shifts and the mixing parameters are calculated through the following unitarization procedure. First, the nonunitary NLO T matrix is transformed to the S matrix via

$$S_{l'l}(p_{\rm on}) = 1 - i\frac{m_N p_{\rm on}}{8\pi^2} T_{l'l}(p_{\rm on}). \tag{159}$$

The diagonal phase shifts in the Stapp parametrization of the S matrix [42] are determined as (modulo π)

$$\delta_{ll} = \tfrac{1}{2}\arg(S_{ll}), \tag{160}$$

whereas the mixing parameter ϵ_{l+1} is obtained from the off-diagonal element of the S matrix:

$$S_{l+2,l} = i\sin(2\epsilon_{l+1})\exp(i\delta_l + i\delta_{l+2}). \tag{161}$$

The dependence of the results on a particular unitarization scheme is a higher-order effect, provided the chiral expansion for the amplitude is convergent.

The numerical analysis we perform does not aim at achieving a perfect description of the data as we work only at next-to-leading order in the chiral expansion. Rather, we are interested in the convergence and renormalization issues. In particular, we make sure that for the cutoff values we employ, no spurious bound states appear and no "exceptional" cutoffs discussed in Secs. IV and V lie within this range. The latter fact manifests itself in the natural values of the fitted next-to-leading-order LECs. The natural values of the NLO LECs are also an indication of the "naturalness" of the quantity $\nu_0(0)$, which is the simplest condition for the renormalizability of the S-wave NLO amplitudes, see Sec. V B. The natural size is roughly given by

$$\frac{8\pi^2}{m_N\Lambda_b}, \tag{162}$$

for the LECs accompanying momentum-independent contact terms and by

$$\frac{8\pi^2}{m_N\Lambda_b^3}, \tag{163}$$

for the LECs of contact terms quadratic in momenta. Obviously, naturalness is not a mathematically strict criterion. However, a sign of potential problems would be a rapid growth with cutoff of one or several LECs.

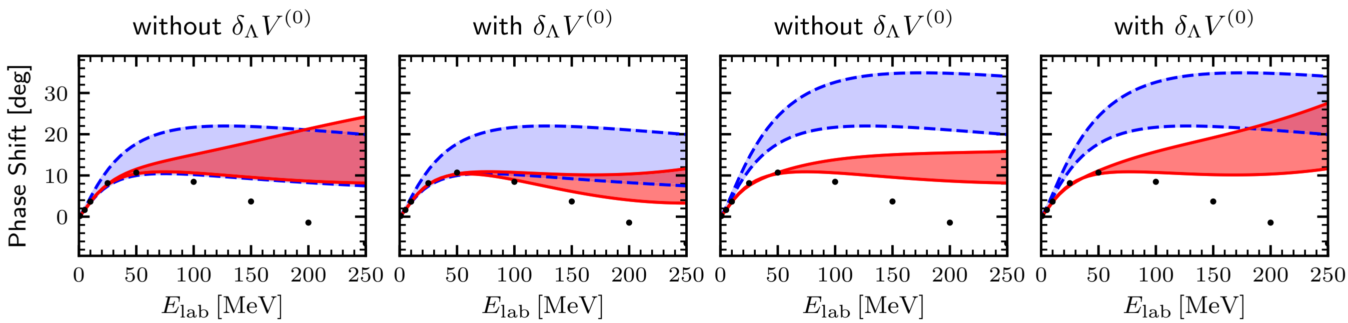

FIG. 1. The results of the leading-order (blue dashed lines) and next-to-leading-order (red solid lines) calculations for the $^3P_0$ partial wave without promoting the contact interaction. The bands indicate the variation of the one-pion-exchange cutoff within the range $\Lambda_{1\pi} \in (300, 450)$ MeV for two left plots and within the range $\Lambda_{1\pi} \in (450, 600)$ MeV for two right plots. The second and fourth plots correspond to the NLO potential with the regulator correction $\delta_\Lambda V^{(0)}$, while the results in the first and third plots are obtained without this term. The empirical phase shifts shown by black solid dots are from Ref. [41]. The plots were created using Matplotlib [43].

Understanding the power counting for the renormalized amplitudes in terms of the convergence of the Fredholm expansion is demonstrated by looking at the convergence of the Fredholm determinant expanded in terms of the LO potential. Convergence of other elements of the Fredholm formulas for the LO and the NLO amplitudes can be analyzed in a similar manner. Their convergence rates are typically comparable with the one for the Fredholm determinant. An absolute value of Fredholm determinant much larger than 1 is also a problem for our interpretation of the power counting, especially for the channels with the enhanced LO amplitude. In such a case, the numerators in the Fredholm formulas N_0, N_2 will also be very large, contradicting the power counting that we suggest. On the contrary, we expect the absolute value of the Fredholm determinant for those channels to be smaller than 1.

A. $^3P_0$ channel

We begin our discussion with the $^3P_0$ partial wave and first follow the dimensional power counting. That means that at leading order, we include only the one-pion-exchange potential and no further terms are promoted. At next-to-leading order, there is one free parameter $C_{2,^3P_0}$ that determines the strength of the NLO contact interaction. The results for the LO and NLO calculations are presented in Fig. 1. In contrast to other plots in this section, we restrict ourselves to the values of the cutoff $\Lambda \leqslant 600$ MeV because for larger cutoffs, the calculated phase shifts deviate too strongly from the data points.

For soft cutoffs values below $\Lambda = 450$ MeV, the convergence of the chiral EFT expansion and the description of the data are reasonable. Moreover for such cutoffs, the LO amplitude can be regarded as perturbative, in the sense that the series in V_0 converges very rapidly, and already a single iteration of the LO potential provides an accuracy of one percent. Therefore, the analysis of Ref. [16] can be applied. One can also see that the band for next-to-leading order corresponding to the variation of the cutoff gets considerably narrower if the regulator correction to the one-pion-exchange potential is taken into account explicitly. Further discussion of the fully perturbative approach in the $^3P_0$ channel can be found in Refs. [44,45].

As one increases the cutoff value, the convergence of expansion of the amplitude in powers of V_0 becomes much slower. This is not problem for our formalism as we formulated the power counting in the nonperturbative case in Sec. IV. However, as one can see in Fig. 1, the disagreement with the data gets more severe and the convergence of the chiral EFT expansion deteriorates. In fact, such a strong deviation of the LO phase shifts from the data leads to a strong violation of unitarity. Another indication of the inefficiency of the resulting EFT expansion is a rather small value of the Fredholm determinant. At threshold, it equals $D \approx 0.4$ for $\Lambda = 600$ MeV compared to $D \approx 1$ for $\Lambda = 300$–450 MeV.

Large contributions from higher orders makes it more efficient to promote the NLO contact interaction to leading order, see also Refs. [20,21]. In fact, the case of very soft cutoffs considered above, which shows a reasonable convergence of the chiral expansion, can also be viewed as a modification of the short-range part of the LO potential analogous to promotion of a contact interaction. Note that our motivation for promoting the NLO contact term is not the requirement of the existence of an infinite cutoff limit as advocated, e.g., in Ref. [20], but rather a large strength of the LO one-pion-exchange potential in this channel. Specifically, we demand that the difference between the LO results and empirical values of the phase shifts can be corrected by a perturbative inclusion of higher-order interactions.

In the scheme with a contact term at LO, there is also one free parameter to be determined from the fit, namely $C_{^3P_0}$, whereas the NLO constant $C_{2,^3P_0}$ is fixed by the renormalization condition in Eq. (91). The corresponding results are shown in Fig. 2. As one can see, the convergence pattern when going from LO to NLO becomes much better. Taking into account the regulator correction to the one-pion-exchange potential $\delta_\Lambda V^{(0)}$ explicitly leads to narrower cutoff-variation bands at NLO, especially for soft cutoffs.

The expansion of the Fredholm determinant in powers of V_0 converges rather rapidly for the cutoffs $\Lambda \leqslant 600$ MeV: at order $(V_0)^3$, a one-percent accuracy is achieved. For $\Lambda \approx 800$ MeV, the same accuracy requires expansion up to order $(V_4)^4$. The absolute value of the Fredholm determinant varies within the range 0.7–2.3 increasing for higher values of the cutoff. The numerical values of the constant $C_{2,^3P_0}$ in the units

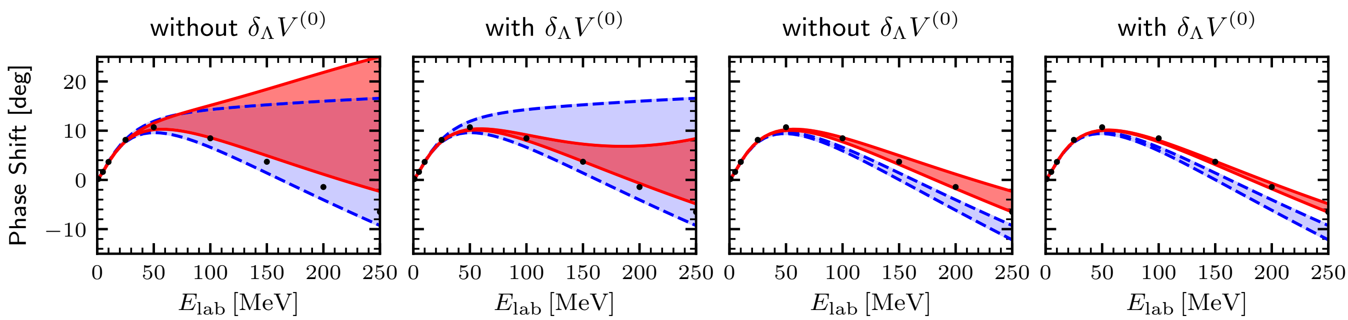

FIG. 2. The results of the leading-order (blue dashed lines) and next-to-leading-order (red solid lines) calculations for the $^3P_0$ partial wave with the contact term promoted to leading order. The bands indicate the variation of the one-pion-exchange cutoff within the range $\Lambda_{1\pi} \in (300, 450)$ MeV for two left plots and within the range $\Lambda_{1\pi} \in (450, 800)$ MeV for two right plots. The second and fourth plots correspond to the NLO potential with the regulator correction $\delta_\Lambda V^{(0)}$, while the results in the first and third plots are obtained without this term. The empirical phase shifts shown by black solid dots are from Ref. [41].

of Eq. (163) is reasonably natural for the choice of the hard-scale $\Lambda_b = 600$ MeV at least for lower Λ values. Specifically, $C_{2,^3P_0} \approx 2$ for $\Lambda \approx 450$ MeV but increases to $C_{2,^3P_0} \approx 30$ for $\Lambda \approx 800$ MeV.

Combining the above results, we conclude that for the cutoffs below or of the order of the hard scale, the renormalization of the NLO amplitude can be understood within the approach developed in this paper. For higher values of the cutoff, the renormalizability of the theory becomes questionable.

B. $^3S_1$-$^3D_1$ channel

Next, we consider the system of the coupled $^3S_1$-$^3D_1$ partial waves. The LO potential is obviously nonperturbative due to the presence of the shallow deuteron bound state. The enhancement of the LO amplitude at threshold is not as strong as, e.g., in the $^1S_0$ channel. Therefore, we assume that within the renormalization scheme specified in Eq. (117), the dimensional power counting should work. That means that the LO potential contains only the one-pion-exchange and the momentum-independent contact term contributions.

There are three parameters to be determined from the fit: the LO constant $C_{^3S_1}$, the NLO constant at the diagonal contact term quadratic in momenta, $C_{2,^3S_1,p^2}$, and the NLO constant accompanying the off-diagonal contact term C_{2,ϵ_1}. The NLO momentum-independent contact term with the constant $C_{2,^3S_1}$ is fixed from the renormalization condition in Eq. (117). The above-mentioned three parameters are determined by fitting the phase shifts in the diagonal $^3S_1$ channel and the mixing parameter ϵ_1, i.e., the channels with contact terms in the potential. The $^3D_1$ phase shift comes out as a prediction.

The results of the fit for various cutoffs are shown in Fig. 3. In general, we observe a reasonable convergence of the chiral expansion except for the ϵ_1 channel where the LO as well as the full contributions are rather small.

As expected for soft cutoffs $\Lambda \leqslant 450$ MeV, taking into account the explicit regulator corrections $\delta_\Lambda V^{(0)}$ for the one-pion-exchange potential and the leading contact term significantly reduces the cutoff dependence at next-to-leading order.

Given the relatively large number of free parameters and possible fine-tuning, it is necessary to explicitly verify the renormalizability criteria specified above.

First, we check the naturalness of the NLO LECs in the units specified in Eqs. (162) and (163). The absolute values of the constants $C_{2,^3S_1,p^2}$ and C_{2,ϵ_1} do not exceed 12 for all considered values of the cutoffs. The maximal absolute value of the constant $C_{^3S_1,p^2}$ is about 6 for $\Lambda \leqslant 600$ MeV, but it starts rising very fast and reaches the value of $C_{^3S_1,p^2} \approx 20$ for $\Lambda = 800$ MeV (and continues rising rapidly).

The Fredholm determinant converges with a one-percent accuracy at orders $(V_0)^3$–$(V_0)^5$ for $\Lambda \leqslant 600$ MeV and at order $(V_0)^6$ for $\Lambda = 800$ MeV. The absolute value of the Fredholm determinant at threshold (at $E_{\rm lab} = 250$ MeV) varies in the range 0.6–0.8 (1.8–3.6) for $\Lambda \leqslant 600$ MeV and is as large as 1.6 (7.5) for $\Lambda = 800$ MeV.

Summarizing the above observations, our numerical results confirm the renormalizability of the NLO amplitude in the $^3S_1$-$^3D_1$ channels for the cutoffs below or of the order of the hard scale. For higher values of the cutoffs, the renormalizability in the sense discussed in the present paper is not guaranteed.

C. $^1S_0$ channel

Finally, we discuss the $^1S_0$ partial wave. The enhancement of the LO amplitude due to the extremely shallow quasibound state is very strong. Nevertheless, we start with trying to adopt the dimensional power counting and do not promote any additional contact interaction to leading order. Therefore, the LO potential consists of the one-pion-exchange contribution and the leading contact term. Two parameters are determined from the fit: the LO constant $C_{^1S_0}$ and the NLO constant $C_{2,^1S_0,p^2}$ corresponding to the contact term quadratic in momenta. The NLO constant $C_{2,^1S_0}$ is fixed from the renormalization condition in Eq. (117). The results are shown in Fig. 4. As in the case of the $^3P_0$ partial wave, the convergence of the chiral expansion is acceptable only for small values of the cutoff $\Lambda \leqslant 450$ MeV. For larger values of the cutoffs, the LO contribution is too large compared to the data, which leads to a strong violation of unitarity. The regulator corrections to the one-pion-exchange potential and the leading contact term

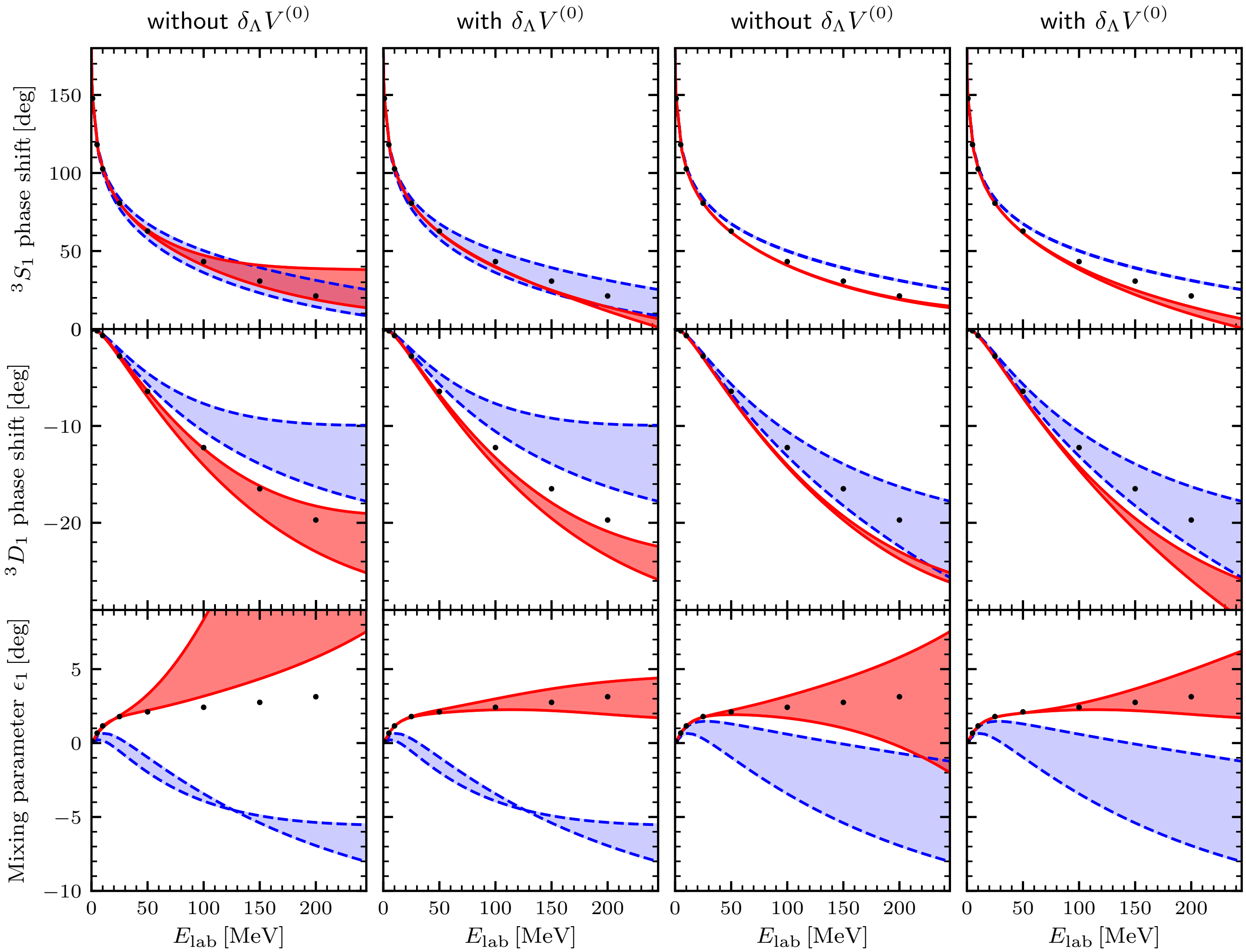

FIG. 3. The results of the leading-order (blue dashed lines) and next-to-leading-order (red solid lines) calculations for the $^3S_1$ -$^3D_1$ channels with the contact term promoted to leading order. The bands indicate the variation of the cutoff of the LO potential within the range $\Lambda \in (300, 450)$ MeV for two left columns and within the range $\Lambda \in (450, 800)$ MeV for two right columns. The second and fourth columns correspond to the NLO potential with the regulator correction $\delta_\Lambda V^{(0)}$, while the results in the first and third columns are obtained without this term. The data are as in Figs. 1 and 2.

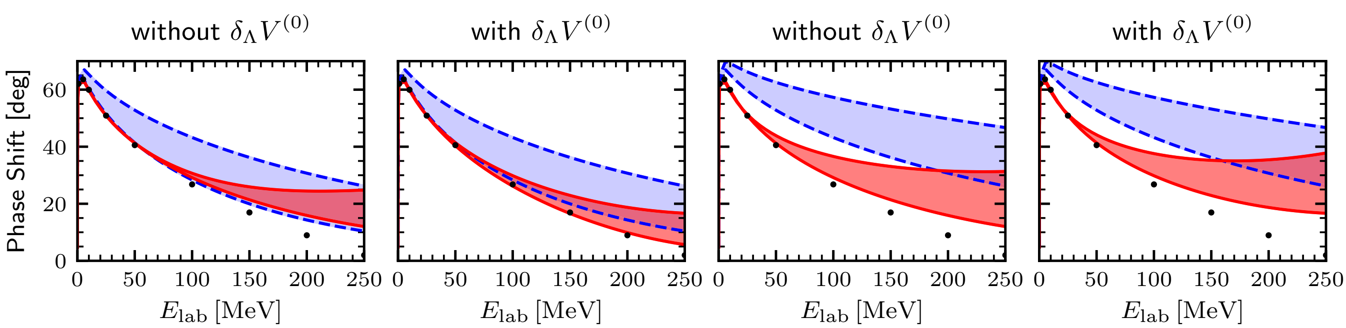

FIG. 4. The results of the leading-order (blue dashed lines) and next-to-leading-order (red solid lines) calculations for the $^1S_0$ partial wave without promoting the contact interaction quadratic in momentum. The bands indicate the variation of the cutoff of the LO potential within the range $\Lambda \in (300, 450)$ MeV for two left plots and within the range $\Lambda \in (450, 800)$ MeV for two right plots. The second and fourth plots correspond to the NLO potential with the regulator correction $\delta_\Lambda V^{(0)}$, while the results in the first and third plots are obtained without this term. The data are as in Figs. 1 and 2.

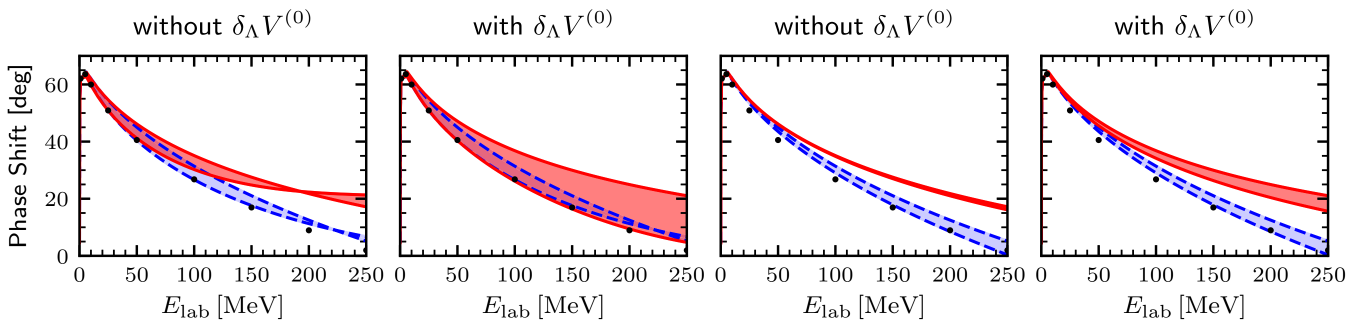

FIG. 5. The results of the leading-order (blue dashed lines) and next-to-leading-order (red solid lines) calculations for the $^1S_0$ partial wave with the contact interaction quadratic in momentum promoted to leading order. The bands indicate the variation of the cutoff of the LO potential within the range $\Lambda \in (300, 450)$ MeV for two left plots and within the range $\Lambda \in (450, 800)$ MeV for two right plots. The second and fourth plots correspond to the NLO potential with the regulator correction $\delta_\Lambda V^{(0)}$, while the results in the first and third plots are obtained without this term. The data are as in Figs. 1 and 2.

practically do not affect the size of the bands corresponding to the variations of the cutoff, which is also a sign of a slow convergence. As the cutoff increases, the Fredholm determinant at threshold changes from 0.7 to 0.3. Therefore, the slow convergence of the chiral expansion for the NLO amplitude is expected from our analysis in Sec. V. Nevertheless, the series for the Fredholm determinant converges rapidly: the one-percent accuracy is obtained at order $(V_0)^3$. The naturalness of the NLO LECs in the units of Eqs. (162) and (163) is also reasonably fulfilled: the absolute value of the constant $C_{2,^1S_0}$ does not exceed 2, and the absolute value of the constant $C_{2,^1S_0,p^2}$ is below 25.

A large deviation of the LO results from the data is a motivation for promoting the subleading contact interaction to leading order (as the simplest solution), see Refs. [18,19]. As we argued in the discussion of the $^3P_0$ partial wave, adopting soft values of the cutoff $\Lambda \leqslant 450$ MeV in the scheme with one contact term at leading order is a sizable modification of the short-range part of the LO potential and is, to some extent, equivalent to the promotion of an additional contact term.

Now, we consider the scheme with the contact interaction quadratic in momenta being promoted to the LO potential. There are still two parameters to fit: $C_{^1S_0}$ and $C_{^1S_0,p^2}$. The constants $C_{2,^1S_0}$ and $C_{2,^1S_0,p^2}$ are fixed from the renormalization conditions in Eq. (154). The results for the scheme with two contact terms in the LO potential are presented in Fig. 5. For higher Λ values, the convergence pattern for the EFT expansion in this scheme is significantly better than in the scheme without promotion of the momentum-dependent contact term. The cutoff dependence is weak for the cutoff values $\Lambda \geqslant 450$ MeV. For soft cutoffs, it may seem that explicit regulator corrections makes the cutoff dependence stronger. However, this is probably accidental because, as one can see, the cutoff dependence for the case without regulator corrections is nonlinear and varies nontrivially with momentum. This is caused by various cancellations due to the fine-tuning of two contact terms.

The absolute value of the Fredholm determinant at threshold is $D \approx 0.1$ for all considered cutoffs, which is in agreement with our expectations for the strongly enhanced LO amplitude. The expansion of the Fredholm determinant in powers of the LO potential approaches an accuracy of one percent at order $(V_0)^3$ for the cutoffs below or equal to the hard scale and at order $(V_0)^4$ for $\Lambda = 800$ MeV.

For all analyzed cutoffs, the naturalness constraint for the NLO constants is reasonably well satisfied without an obvious tendency to its violation, which can be explained by a regular behavior of the spin-singlet one-pion-exchange potential at short distances.

To summarize, the numerical calculations for the channels $^3P_0$, $^3S_1$-$^3D_1$, and $^1S_0$ are in agreement with our theoretical analysis of the renormalization of the NLO amplitude with a finite cutoff. We observed a reasonable convergence of the chiral EFT expansion. However, for the $^3P_0$ and $^1S_0$ partial waves a more efficient scheme within the considered EFT formulation is obtained when the subleading contact interactions are promoted to leading order, as has already been discussed in the literature. The naturalness constraints on the NLO LECs and on the value of the Fredholm determinants are fulfilled for the cutoff values below or of the order of the hard scale. The convergence rate of the Fredholm determinants in powers of the LO potential also appears to be sufficiently rapid for such values of the cutoffs. This allows us to interpret the renormalizability of the NLO amplitude within the method developed in the current paper. When the cutoff approaches the value $\Lambda \approx 800$ MeV or higher, the renormalizability constraints are not clearly fulfilled anymore, even though the convergence of the amplitude might still be reasonable.

Thus, we conclude that the preferable choice of the cutoff values is roughly $\Lambda \lesssim 600$ MeV. For very soft cutoffs $\Lambda =$ 300–450 MeV, the regulator corrections to the LO potential should be explicitly taken into account to remove the regulator artifacts.

VII. SUMMARY

We have extended our previous study in Ref. [16] and analyzed the renormalization of the nucleon-nucleon amplitude at NLO in chiral EFT in the case when the LO interaction is nonperturbative. Our scheme is based on the formulation of chiral EFT with a finite cutoff derived from the effective Lagrangian.

In the previous paper, the power counting for the renormalized NLO amplitude was justified for the case when the series for the iterations of the LO potential are (rapidly enough) convergent, i.e., for the perturbative case. The corresponding subtractions in the form of the LO S-wave contact terms that absorb the power counting breaking contributions were identified. Starting from P waves, the NLO amplitudes were found not to require any subtractions in agreement with dimensional power counting.

The method of analysis of the power counting in the nonperturbative regime relies on the Fredholm formula for the solution of the integral equations, which represents the numerators and denominators of the amplitudes as individually convergent series in powers of the LO potential. To implement the Fredholm decomposition, we first had to derive stronger bounds on the LO potential compared to the ones used in the perturbative case. In contrast to the perturbative regime, it turned out that the minimal "mild" regulator can, in general, not be employed if the NLO potential remains unregulated. This implies a potentially stronger cutoff dependence in the nonperturbative case.

The results for the P and higher partial waves in the NN system reproduce to a large extend our previous findings. The dimensional power counting for the LO and NLO amplitudes is formally satisfied without subtractions unless there is an enhancement of the LO amplitude due to the presence of a shallow (quasi-)bound state, which is not the case for the physical channels. Nevertheless, in some cases, the promotion of NLO contact terms to leading order can be motivated by phenomenological arguments as, e.g., in the $^3\mathrm{P}_0$ channel. In the latter situation, however, one has to choose the LO potential in such a way as to avoid the appearance of "exceptional" cutoffs, for which the renormalization breaks down. The simplest way to verify that the adopted value of the cutoff is not close to "exceptional" is to make sure that the NLO LECs are of a natural size.

For the S waves, we have shown that the series for the subtractions at next-to-leading order, obtained in the perturbative case, can be resummed in a closed form. Such a resummation is equivalent to the condition for the renormalized NLO amplitude to vanish at threshold. Using the Fredholm formula allowed us to analyze also the case when the LO amplitude is enhanced at threshold compared to the dimensional power counting estimate. This happens in the $^3\mathrm{S}_1$-$^3\mathrm{D}_1$ and $^1\mathrm{S}_0$ channels where shallow bound and quasibound states are present. The dimensional power counting for the NLO amplitudes is still valid in those cases if certain additional constraints on the LO potential are fulfilled. Again, these constraints eventually reveal themselves in the naturalness of the NLO LECs. However, the convergence of the chiral expansion in the channels with enhanced LO amplitude may become significantly slower, especially in the $^1\mathrm{S}_0$ channel, where the enhancement is most pronounced. To improve the convergence, one can, analogous to the $^3\mathrm{P}_0$ partial wave, promote a subleading contact term to the LO potential with the same warning regarding "exceptional" cutoffs.

Finally, we have illustrated our theoretical findings by numerical calculations of the NN phase shifts at next-to-leading order by fitting the unknown free parameters to the empirical data. We considered three channels with nonperturbative dynamics, namely $^3\mathrm{P}_0$, $^3\mathrm{S}_1$-$^3\mathrm{D}_1$ and $^1\mathrm{S}_0$, and varied the LO cutoff in the range of $\Lambda = 300$–800 MeV. We observed reasonable convergence of the chiral expansion, especially when the subleading contact terms are promoted in the $^3\mathrm{P}_0$ and $^1\mathrm{S}_0$ channels.

As criteria for the interpretation of the renormalizability of the NLO amplitude in terms of the Fredholm expansion, we used the naturalness of the NLO LECs and of the Fredholm determinant as well as the convergence rate of the expansion of the latter in powers of the LO potential. It turns out that all these constraints are fulfilled as long as the cutoff values are chosen below or of the order of the hard-scale $\Lambda_b \approx 600$ MeV. For particularly soft cutoffs $\Lambda = 300$–450 MeV, taking into account explicit regulator corrections to the LO potential compensates for the regulator artifacts and reduces the cutoff dependence.

When the cutoff increases beyond the hard scale, the renormalizability constraints start being violated. Therefore, we conclude that the cutoff values $\Lambda \lesssim \Lambda_b$ are preferable from the point of view of the renormalization of EFT.

Further development of our approach goes in the direction of extending it beyond next-to-leading order in the chiral expansion. It is also important to generalize the scheme to few-nucleon systems and the processes involving electroweak interactions.

ACKNOWLEDGMENTS

We thank Jambul Gegelia for helpful discussions and for useful comments on the manuscript. This work was supported by DFG (Grant No. 426661267), by BMBF (Contract No. 05P21PCFP1), by ERC AdG NuclearTheory (Grant No. 885150), by the MKW NRW under the funding code NW21-024-A, and by the EU Horizon 2020 research and innovation programme (STRONG-2020, Grant Agreement No. 824093).

APPENDIX A: LEADING-ORDER POTENTIAL

The short-range part of the leading-order potential in its general form can be chosen to include the momentum-independent contact interactions and contact terms quadratic in momenta (altogether nine terms), multiplied by the power-like nonlocal form factor of an appropriate power n:

$$V^{(0)}_{\mathrm{short},\Lambda}(\vec{p}\,', \vec{p}\,) = \sum_i C_i\, V_{C_i}\, F_{\Lambda_i,n_i}(p', p), \tag{A1}$$

where V_{C_i} is any basis for the contact terms, e.g., the partial wave basis, and the regulators are given by

$$F_{\Lambda,n}(p', p) = F_{\Lambda,n}(p')F_{\Lambda,n}(p), \quad F_{\Lambda,n}(p) = [F_\Lambda(p)]^n,$$
$$F_\Lambda(p) = \frac{\Lambda^2}{p^2+\Lambda^2}. \tag{A2}$$

One can also introduce a regulator of a Gaussian form by replacing $F_{\Lambda,n}(p)$ with

$$F_{\Lambda,\mathrm{exp}}(p) = \exp\left(-p^2/\Lambda^2\right). \tag{A3}$$

Alternatively, one could introduce local short-range interactions (for the terms that depend only on $\vec{q}$, except for the spin-orbit term) using the appropriate basis [46] and the local regulator

$$F_{q,\Lambda,n}(\vec{p}\,',\vec{p}\,) = [F_\Lambda(q)]^n = \left(\frac{\Lambda^2}{q^2+\Lambda^2}\right)^n, \tag{A4}$$

or with the regulator in the Gaussian form $F_{\Lambda,\exp}(q)$.

The long-range part of the LO potential is represent by the one-pion-exchange contribution, which is split into the triplet, singlet, and contact parts

$$V_{1\pi}^{(0)} = -\left(\frac{g_A}{2F_\pi}\right)^2 \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,\frac{\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}}{q^2+M_\pi^2} =: V_{1\pi,\mathrm{t}}^{(0)} + V_{1\pi,\mathrm{s}}^{(0)} + V_{1\pi,\mathrm{ct}}^{(0)}, \tag{A5}$$

with

$$V_{1\pi,\mathrm{s}}^{(0)} = \left(\frac{g_A}{2F_\pi}\right)^2 \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,\frac{(\vec{\sigma}_1\cdot\vec{\sigma}_2-1)}{4}\,\frac{M_\pi^2}{q^2+M_\pi^2},$$
$$V_{1\pi,\mathrm{ct}}^{(0)} = -\left(\frac{g_A}{2F_\pi}\right)^2 \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,\frac{(\vec{\sigma}_1\cdot\vec{\sigma}_2-1)}{4}. \tag{A6}$$

All three parts, if necessary, are regularized individually. The contact part $V_{1\pi,\mathrm{ct}}$ can be absorbed by the leading-order $^1\mathrm{S}_0$ contact term and thus needs not be considered separately. The triplet and singlet potentials can be regularized by means of the nonlocal form factor [see Eq. (A2)]:

$$V_{1\pi,\Lambda}^{(0)}(\vec{p}\,',\vec{p}\,) = V_{1\pi,\mathrm{s}}^{(0)}(\vec{p}\,',\vec{p}\,)F_{\Lambda_\mathrm{s},n_\mathrm{s}}(p',p) + V_{1\pi,\mathrm{t}}^{(0)}(\vec{p}\,',\vec{p}\,)F_{\Lambda_\mathrm{t},n_\mathrm{t}}(p',p), \tag{A7}$$

or by means of the local regulator:

$$V_{1\pi,\Lambda}^{(0)}(\vec{p}\,',\vec{p}\,) = V_{1\pi,\mathrm{s}}^{(0)}(\vec{p}\,',\vec{p}\,)F_{q,1\pi,\Lambda_\mathrm{s}}(\vec{p}\,',\vec{p}\,) + V_{1\pi,\mathrm{t}}^{(0)}(\vec{p}\,',\vec{p}\,)F_{q,1\pi,\Lambda_\mathrm{t}}(\vec{p}\,',\vec{p}\,), \tag{A8}$$

with

$$F_{q,1\pi,\Lambda_\mathrm{s}}(\vec{p}\,',\vec{p}\,) = \left(\frac{\Lambda_\mathrm{s}^2-M_\pi^2}{q^2+\Lambda_\mathrm{s}^2}\right)^{n_\mathrm{s}},$$
$$F_{q,1\pi,\Lambda_\mathrm{t}}(\vec{p}\,',\vec{p}\,) = \left(\frac{\Lambda_\mathrm{t}^2-M_\pi^2}{q^2+\Lambda_\mathrm{t}^2}\right)^{n_\mathrm{t}}. \tag{A9}$$

Note that in Ref. [16], a more general form of the local regulator was considered.

The spin-singlet part of the one-pion-exchange potential can, in principle, be left unregulated. This is, however, only relevant for the spin-singlet channels without short-range interactions. All such channels can be regarded as having perturbative LO potential and were already analyzed in Ref. [16]. For the spin-singlet channel considered in this work, $^1\mathrm{S}_0$, the effects of a regulator will be driven by the contact interaction in any case.

To regularize the spin-triplet part of the one-pion-exchange potential in the LO Lippmann-Schwinger equation, it is sufficient to introduce a dipole ($n_\mathrm{t}=1$) regulator, which we refer to as the "mild" regulator. All other options, i.e., $n_\mathrm{t}\geqslant 2$ are referred to as the "standard" regulators.

One can also adopt the local Gaussian regulator for the one-pion-exchange potential:

$$F_{q,1\pi,\exp,\Lambda}(\vec{p}\,',\vec{p}\,) = \exp\left[-\left(q^2+M_\pi^2\right)/\Lambda^2\right]. \tag{A10}$$

APPENDIX B: NEXT-TO-LEADING-ORDER POTENTIAL

The short-range part of the next-to-leading-order potential is given by the sum of contact terms analogous to Eq. (A1):

$$V_{\mathrm{short}}^{(2)}(\vec{p}\,',\vec{p}\,) = \sum_i C_{2,i}\,V_{C_i}. \tag{B1}$$

The nonpolynomial part of the two-pion-exchange potential is given by (it is equivalent to the one provided in Ref. [47] up to polynomial terms)

$$\begin{aligned} V_{2\pi}^{(2)}(\vec{p}\,',\vec{p}\,) = &-\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{384\pi^2F_\pi^4}\tilde{L}(q)\Bigg[4M_\pi^2\left(5g_A^4-4g_A^2-1\right) \\ &+q^2\left(23g_A^4-10g_A^2-1\right)+\frac{48g_A^4M_\pi^4}{4M_\pi^2+q^2}\Bigg] \\ &+\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{8\pi^2F_\pi^4}\frac{g_A^4M_\pi^2q^2}{4M_\pi^2+q^2} \\ &-\frac{3g_A^4}{64\pi^2F_\pi^4}\tilde{L}(q)[\vec{\sigma}_1\cdot\vec{q}\vec{\sigma}_2\cdot\vec{q}-q^2\vec{\sigma}_1\cdot\vec{\sigma}_2], \end{aligned} \tag{B2}$$

where

$$\tilde{L}(q) := L(q)-L(0) = L(q)-1,$$
$$L(q) = \frac{1}{q}\sqrt{4M_\pi^2+q^2}\log\frac{\sqrt{4M_\pi^2+q^2}+q}{2M_\pi}. \tag{B3}$$

The regulator of the NLO potential, not shown explicitly in the above expressions, can be a combination of any local or nonlocal forms. For the two-pion-exchange potential, one can also employ a spectral function regularization by introducing a finite upper limit in the dispersion representation of $\tilde{L}(q)$:

$$\tilde{L}(q) = q^2\int_{2M_\pi}^{\Lambda_\rho}\frac{d\mu}{\mu^2}\frac{\sqrt{\mu^2-4M_\pi^2}}{q^2+\mu^2}. \tag{B4}$$

APPENDIX C: BOUNDS ON THE PLANE-WAVE POTENTIAL

1. Bounds on the substructures

Below, we list the inequalities for the building blocks of the LO and NLO potentials obtained in Ref. [16].

The components of the initial and final nucleon c.m. momenta p and p' are defined as

$$\vec{p} = p\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \vec{p}\,' = p'\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}, \tag{C1}$$

where p is either $p=p_\mathrm{on}$ or lies on the complex contour $p\in\mathcal{C}$: $p=|p|\exp(-i\alpha_C)$, and p' is either $p'=p_\mathrm{on}$ or $p'=|p'|\exp(-i\alpha_C)$.

For the function

$$f_\mu(p',p,x) = \frac{1}{q^2+\mu^2} = \frac{1}{p'^2+p^2-2pp'x+\mu^2}, \tag{C2}$$

with $\mu \geqslant M_\pi$, the following bounds hold

$$|f_\mu(p', p, x)| \leqslant \frac{\mathcal{M}_f}{|p|^2 + |p'|^2 - 2|p||p'|x + \mu^2}, \tag{C3}$$

$$|q_i q_j f_\mu(p', p, x)| \leqslant \mathcal{M}_f, \tag{C4}$$

$$|(\vec{k} \times \vec{q})_i f_\mu(p', p, x)| \leqslant \mathcal{M}_f (1 - x^2)^{-1/2}, \quad i, j = 1, 2, 3. \tag{C5}$$

The subtraction remainders defined as

$$\Delta_p^{(n)} f(p', p) = f(p', p) - \sum_{i=0}^{n} \frac{\partial^i f(p', p)}{i!(\partial p)^i}\bigg|_{p=0} p^i,$$
$$\Delta_{p'}^{(n)} f(p', p) = f(p', p) - \sum_{i=0}^{n} \frac{\partial^i f(p', p)}{i!(\partial p')^i}\bigg|_{p'=0} (p')^i, \tag{C6}$$

satisfy the following inequalities:

$$\left|\Delta_p^{(n)} f_\mu(p', p)\right| \leqslant \mathcal{M}_{f,n} \left|\frac{p}{p'}\right|^{n+1} |f_\mu(p', p)|, \quad \text{if } |p'| > |p|,$$
$$\left|\Delta_{p'}^{(n)} f_\mu(p', p)\right| \leqslant \mathcal{M}_{f,n} \left|\frac{p'}{p}\right|^{n+1} |f_\mu(p', p)| \quad \text{if } |p| > |p'|. \tag{C7}$$

For a more general structure

$$\Psi_{k,m,\{\mu_i\}}(p', p, x) = Q_k(p', p, x) F_{\Lambda,m}(p', p) f_{\{\mu_i\}}(p', p, x), \tag{C8}$$

where the form factor $F_{\Lambda,m}$ is defined in Eq. (A2), $f_{\{\mu_i\}}$ is a product of several f_μ

$$f_{\{\mu_i\}}(p', p, x) = \prod_{i=1,r} f_{\mu_i}(p', p, x), \tag{C9}$$

and Q_k is a homogeneous polynomial of degree k, one can deduce the bounds for derivatives:

$$\left|p^n \frac{\partial^n \Psi_{k,m,\{\mu_i\}}(p', p, x)}{\partial p^n}\bigg|_{p=0}\right| \leqslant \mathcal{M}_{\partial\Psi}^{k,n} \left|p'^k F_{\Lambda,m-\frac{n+1}{2}}(p') f_{\{\mu_i\}}(p', 0, x)\right| \left|\frac{p}{p'}\right|^n,$$
$$\left|(p')^n \frac{\partial^n \Psi_{k,m,\{\mu_i\}}(p', p, x)}{\partial (p')^n}\bigg|_{p'=0}\right| \leqslant \mathcal{M}_{\partial\Psi}^{k,n} \left|p^k F_{\Lambda,m-\frac{n+1}{2}}(p) f_{\{\mu_i\}}(0, p, x)\right| \left|\frac{p'}{p}\right|^n, \quad n \geqslant 0, \tag{C10}$$

and for the subtraction remainders:

$$\left|\Delta_p^{(n)} \Psi_{k,m,\{\mu_i\}}(p', p, x)\right| \leqslant \mathcal{M}_\Psi^{k,n} \left|\frac{p}{p'}\right|^{n+1} (|\Psi_{k,m,\{\mu_i\}}(p', p, x)| + |p'^k F_{\Lambda,m-\frac{n+1}{2}}(p') f_{\{\mu_i\}}(p', 0, x)|), \quad \text{if } |p'| > |p|,$$
$$\left|\Delta_{p'}^{(n)} \Psi_{k,m,\{\mu_i\}}(p', p, x)\right| \leqslant \mathcal{M}_\Psi^{k,n} \left|\frac{p'}{p}\right|^{n+1} (|\Psi_{k,m,\{\mu_i\}}(p', p, x)| + |p^k F_{\Lambda,m-\frac{n+1}{2}}(p) f_{\{\mu_i\}}(0, p, x)|), \quad \text{if } |p| > |p'|. \tag{C11}$$

2. Bounds on the plane-wave leading-order potential

In this section we provide bounds for the leading-order potential. We will need slightly stronger bounds than those obtained in Ref. [16]. In particular, we will need bounds that factorize in initial and finale momenta in the partial wave basis. To obtain them, we will partly keep the angular dependence in binding functions.

The derivation is only slightly different from that of Ref. [16], which we demonstrate for the case of the spin-triplet one-pion-exchange potential.

The locally regularized one-pion exchange potential in the spin-triplet channels can be bounded using equations of Appendix (C1) by the following inequality:

$$\left|V_{1\pi,\mathrm{t}}^{(0)}(\vec{p}\,', \vec{p}\,)\right| \leqslant \left|\frac{g_A^2}{4F_\pi^2} \sum_{i,j} \mathcal{M}_{\mathrm{t},ij} \frac{q_i q_j}{q^2 + M_\pi^2} \left(\frac{\Lambda_\mathrm{t}^2 - M_\pi^2}{q^2 + \Lambda_{\mathrm{t},1}^2}\right)^{n_\mathrm{t}}\right|$$
$$\leqslant \frac{2\pi M_\mathrm{t}}{m_N \Lambda_V} F_{\Lambda,n_\mathrm{t}}(|p'|, |p|, x), \tag{C12}$$

where we have introduced the form factors

$$F_{\Lambda,n}(|p'|, |p|, x) = (F_\Lambda(|p'|, |p|, x))^n,$$
$$F_\Lambda(|p'|, |p|, x) = \frac{\Lambda^2}{|p|^2 + |p'|^2 - 2|p||p'|x + \Lambda^2}. \tag{C13}$$

In Eq. (C12), we replaced Λ_t with the largest cutoff Λ among all regulators in the LO potential, which is possible due to the inequality:

$$F_{\Lambda_1}(|p'|, |p|, x) < F_{\Lambda_2}(|p'|, |p|, x) \quad \text{for } \Lambda_1 < \Lambda_2. \tag{C14}$$

If the triplet one-pion exchange potential is regularized by the nonlocal form factor, then we obtain

$$\left|V_{1\pi,\mathrm{t}}^{(0)}(\vec{p}\,', \vec{p}\,)\right|$$
$$\leqslant \left|\frac{g_A^2}{4F_\pi^2} \sum_{i,j} \mathcal{M}_{\mathrm{t},ij} \frac{q_i q_j}{q^2 + M_\pi^2} \left(\frac{\Lambda^2}{p'^2 + \Lambda^2} \frac{\Lambda^2}{p^2 + \Lambda^2}\right)^{n_\mathrm{t}}\right|$$
$$\leqslant \frac{2\pi M_\mathrm{t}}{m_N \Lambda_V} F_{\Lambda,n_\mathrm{t}}(|p'|) F_{\Lambda,n_\mathrm{t}}(|p|). \tag{C15}$$

Analogously, we obtain bounds for other LO contributions as in Ref. [16] retaining the angular dependence of local form factors and the powers of the form factors.

Finally, the full leading-order potential satisfies

$$|V_0(\vec{p}\,', \vec{p}\,)| \leqslant \frac{\mathcal{M}_{V_0}}{4\pi} V_{0,\max}(p', p, x),$$
$$|V_0(\vec{p}\,', \vec{p}\,)| \leqslant \frac{\mathcal{M}_{V_0}}{4\pi} V_{0,\max}(p, p', x), \tag{C16}$$

where we have introduced

$$V_{0,\max}(p', p, x) = \frac{8\pi^2}{m_N \Lambda_V}[F_{\Lambda,n}(|p'|, |p|, x) + F_{\Lambda,n}(|p'|)], \tag{C17}$$

with n being the smallest power among all regulators in the LO potential. The cases of the "mild" and the "standard" regulators correspond to $n = 1$ and $n \geqslant 2$, respectively. The difference of Eq. (C17) from an analogous bound in Ref. [16] is that the powers of both local and nonlocal form factors are retained and the x-dependence of the local form factor is kept. The bounds for the Gaussian regulators can be reduced to the ones for the power-like regulators as was shown in Ref. [16].

For the spin-singlet channels without a short-range interaction, the bounds in Eq. (C17) can be improved by replacing Λ with M_π. However, as mentioned above, those channels were already covered in our previous study.

The remainders $\Delta_p^{(n)} V_0(\vec p\,', \vec p\,)$ for $|p'| > |p|$ can be estimated using Eq. (C11):

$$\begin{aligned}
\left|\Delta_p^{(n)} V_0(\vec p\,', \vec p\,)\right| &\leqslant \frac{\mathcal{M}_{\Delta V_0,n}}{4\pi}\left|\frac{p}{p'}\right|^{n+1} V_{0,\max}(p', p, x) \quad \text{if } |p'| > |p|,\\
\left|\Delta_{p'}^{(n)} V_0(\vec p\,', \vec p\,)\right| &\leqslant \frac{\mathcal{M}_{\Delta V_0,n}}{4\pi}\left|\frac{p'}{p}\right|^{n+1} V_{0,\max}(p, p', x) \quad \text{if } |p| > |p'|.
\end{aligned} \tag{C18}$$

From Eq. (C10) one obtains the estimates for the derivatives of the leading-order potential:

$$\left| p^m \frac{\partial^m V_0(\vec p\,', \vec p\,)}{(\partial p)^m}\right|_{p=0}\Bigg| \leqslant \frac{2\pi \mathcal{M}_{\partial V_0,n}}{m_N \Lambda_V} F_{\tilde\Lambda,n}(|p'|)\left|\frac{p}{p'}\right|^m \leqslant \frac{\mathcal{M}_{\partial V_0,n}}{4\pi}\left|\frac{p}{p'}\right|^m V_{0,\max}(p', p, x), \tag{C19}$$

$$\left| p'^m \frac{\partial^m V_0(\vec p\,', \vec p\,)}{(\partial p')^m}\right|_{p'=0}\Bigg| \leqslant \frac{2\pi \mathcal{M}_{\partial V_0,n}}{m_N \Lambda_V} F_{\tilde\Lambda,n}(|p|)\left|\frac{p'}{p}\right|^m \leqslant \frac{\mathcal{M}_{\partial V_0,n}}{4\pi}\left|\frac{p'}{p}\right|^m V_{0,\max}(p, p', x), \tag{C20}$$

including the case $m = 0$, where we have used that the local form factor satisfies

$$F_\Lambda(p', 0, x) = F_\Lambda(p'). \tag{C21}$$

Applying Eq. (C19) [Eq. (C20)] to the definition of $\Delta_p^{(n)} V_0(\vec p\,', \vec p\,)$ $[\Delta_{p'}^{(n)} V_0(\vec p\,', \vec p\,)]$ in Eq. (C6) for $|p| > |p'|$ ($|p'| > |p|$), and combining it with Eq. (C18), we obtain the following bounds for the remainders:

$$\begin{aligned}
\left|\Delta_p^{(n)} V_0(\vec p\,', \vec p\,)\right| &\leqslant \frac{\mathcal{M}_{\Delta V_0,n}}{4\pi}\left|\frac{p}{p'}\right|^{n+1} V_{0,\max}(p', p, x),\\
\left|\Delta_{p'}^{(n)} V_0(\vec p\,', \vec p\,)\right| &\leqslant \frac{\mathcal{M}_{\Delta V_0,n}}{4\pi}\left|\frac{p'}{p}\right|^{n+1} V_{0,\max}(p, p', x),
\end{aligned} \tag{C22}$$

which are valid for all considered p and p'. All above general formulas do not include the case when the LO potential contains a locally regulated spin-orbit short-range interaction such as

$$V_{C_5}^{(0)}(\vec p\,', \vec p\,) = C_5 \frac{i}{2}(\vec\sigma_1 + \vec\sigma_2)\cdot(\vec k \times \vec q\,)\left(\frac{\Lambda_5^2}{q^2 + \Lambda_5^2}\right)^{n_5}, \tag{C23}$$

with $n_5 > 1$ (or with the Gaussian form factor). Following the arguments provided in Ref. [16], one can formulate the same bounds as in Eqs. (C16) and (C22) for the quantity $\tilde V_{C_5}^{(0)}$ defined as

$$\begin{aligned}
V_{C_5}^{(0)}(\vec p\,', \vec p\,) &= \tilde V_{C_5}^{(0)}(\vec p\,', \vec p\,)\frac{i}{2}(\vec\sigma_1 + \vec\sigma_2)\cdot \vec n_\phi / \sin\theta,\\
\vec n_\phi &= (-\sin\phi, \cos\phi, 0),
\end{aligned} \tag{C24}$$

which makes it possible, after the partial-wave projection, to treat this interaction on the same footing as all other LO terms.

3. Bounds on the plane-wave next-to-leading-order potential

For the NLO potential, we use the bounds obtained in Ref. [16]. The NLO potential is split into two parts:

$$V_2(\vec p\,', \vec p\,) = \hat V_2(\vec p\,', \vec p\,) + \tilde V_2(\vec p\,', \vec p\,), \tag{C25}$$

with

$$\hat V_2(\vec p\,', \vec p\,) = V_2(0, 0), \quad \tilde V_2(\vec p\,', \vec p\,) = V_2(\vec p\,', \vec p\,) - V_2(0, 0), \tag{C26}$$

which are bound as

$$|\hat V_2(\vec p\,', \vec p\,)| \leqslant \hat{\mathcal{M}}_{V_2} \frac{2\pi}{m_N \Lambda_V}\frac{M_\pi^2}{\Lambda_b^2}, \tag{C27}$$

and

$$\begin{aligned}
|\tilde V_2(\vec p\,', \vec p\,)| &\leqslant \frac{2\pi \mathcal{M}_{V_2}}{m_N \Lambda_V}\frac{|p|^2 + |p'|^2}{\Lambda_b^2} f_{\log}(p', p)\\
&= \frac{\mathcal{M}_{V_2}}{4\pi}(|p|^2 + |p'|^2)\tilde f_{\log}(p', p),
\end{aligned} \tag{C28}$$

with

$$\begin{aligned}
\tilde f_{\log}(p', p) &= \frac{8\pi^2}{m_N \Lambda_V \Lambda_b^2} f_{\log}(p', p),\\
f_{\log}(p', p) &= \theta(|p| - M_\pi)\ln\frac{|p|}{M_\pi} + \theta(|p'| - M_\pi)\ln\frac{|p'|}{M_\pi}\\
&\quad + \ln\frac{\tilde\Lambda}{M_\pi} + 1.
\end{aligned} \tag{C29}$$

In the function $f_{\log}(p', p)$, the term $\ln\frac{\tilde\Lambda}{M_\pi}$ was introduced for convenience so we can omit it (or set $\tilde\Lambda = M_\pi$).

We also allow for a regulator (local or nonlocal) for the NLO potential. We can introduce it simply as a factor, so that the bounds in Eq. (C28) are modified as

$$|\tilde{V}_2(\vec{p}\,', \vec{p}\,)| \leqslant \frac{\mathcal{M}_{V_2}}{4\pi}(|p|^2 + |p'|^2)\tilde{f}_{\log}(p', p) \times [F_{\Lambda_{\rm NLO}}(|p'|, |p|, x) + F_{\Lambda_{\rm NLO}}(|p'|)], \quad (C30)$$

where we combined local and nonlocal regulators into one factor. The first power ($n = 1$) of the form factors is sufficient for our estimates. The cases of higher powers (or Gaussian cutoffs) are included automatically, because

$$F_{\Lambda_{\rm NLO},n}(|p|) \leqslant F_{\Lambda_{\rm NLO}}(|p|),$$
$$F_{\Lambda_{\rm NLO},n}(|p'|, |p|, x) \leqslant F_{\Lambda_{\rm NLO}}(|p'|, |p|, x), \quad (C31)$$

for $n > 1$. If different values of the cutoff are used for different NLO contributions, then $\Lambda_{\rm NLO}$ can be chosen to be the largest value.

APPENDIX D: BOUNDS ON THE PARTIAL-WAVE POTENTIAL

Below, we repeat the arguments of Ref. [16] for deriving the bounds on the partial-wave potential, but take into account an angular dependence of the binding functions.

The partial-wave potential is obtained from the plain-wave potential via

$$V^{sj}_{l',l}(p', p) = \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \int d\Omega\, \langle jl's|\lambda_1'\lambda_2'\rangle\langle\lambda_1'\lambda_2'|V(\vec{p}\,', \vec{p}\,)|\lambda_1\lambda_2\rangle\langle\lambda_1\lambda_2|jls\rangle d^j_{\lambda_1-\lambda_2,\lambda_1'-\lambda_2'}(\theta),$$
$$\langle\lambda_1\lambda_2|jls\rangle = \left(\frac{2l+1}{2j+1}\right)^{\frac{1}{2}} C(l, s, j; 0, \lambda_1 - \lambda_2)C(1/2, 1/2, s; \lambda_1, -\lambda_2), \quad (D1)$$

where λ_i, λ_i' are the helicities of the corresponding nucleons.

Due to unitarity of the transformation, the following constraints hold:

$$|\langle\lambda_1\lambda_2|jls\rangle| \leqslant 1, \quad |\langle 1/2, s_z|\lambda\rangle| \leqslant 1, \quad |d^j_{\lambda,\lambda'}(\theta)| \leqslant 1. \quad (D2)$$

Therefore, if the plain-wave potential is bounded by some angle-dependent function $\phi(p', p, x)$,

$$|V(\vec{p}\,', \vec{p}\,)| \leqslant M_k\phi(p', p, x), \quad (D3)$$

then, for the partial-wave potential, we obtain

$$\left|V^{sj}_{l',l}(p', p)\right| \leqslant 2\pi\tilde{M}_k \int_{-1}^{1} dx\, \phi(p', p, x). \quad (D4)$$

For the special case of the locally regulated spin-orbit contact interaction, a bound of the same type can be obtained if one replaces $|V(\vec{p}\,', \vec{p}\,)|$ by $|\tilde{V}(\vec{p}\,', \vec{p}\,)| = |V(\vec{p}\,', \vec{p}\,)|\sqrt{1-x^2}$, see Appendix C 2 and Ref. [16].

1. Bounds on the form factor $F_{\mu,n}(q)$ integrated over x

In this subsection we derive the bounds on the local form factors

$$F_\mu(q) = \frac{\mu^2}{q^2 + \mu^2}, \quad F_{\mu,2}(q) = F_\mu(q)^2, \quad (D5)$$

integrated over the angle variable x, which are relevant when considering bounds for the partial-wave potentials. The form factors $F_{\mu,n}(q)$ with $n > 2$ satisfy (at least) the same bounds as $F_{\mu,2}(q)$, which is sufficient for our estimates. The same is true for the form factors of the Gaussian form, which was analyzed in detail in Ref. [16].

From Eq. (C3), it follows

$$|q^2 + \mu^2| \geqslant \mathcal{M}_f^{-1}(|p|^2 + |p'|^2 - 2|p||p'|x + \mu^2) = \mathcal{M}_f^{-1}[(|p'|x - |p|)^2 + |p'|^2(1 - x^2) + \mu^2]$$
$$\geqslant \mathcal{M}_f^{-1}[|p'|^2(1 - x)/2 + \mu^2]. \quad (D6)$$

For $|p'| \geqslant \mu$, we obtain

$$\left|\int_{-1}^{1} F_\mu(q)dx\right| = \left|\int_{-1}^{1} \frac{\mu^2 dx}{q^2 + \mu^2}\right| \leqslant 2\mathcal{M}_f\mu^2 \int_{-1}^{1} \frac{dx}{|p'|^2(1-x) + 2\mu^2} = \frac{2\mathcal{M}_f\mu^2}{|p'|^2}\ln(1 + |p'|^2/\mu^2)$$
$$\leqslant \frac{2\mathcal{M}_f\mu^2}{|p'|^2}\ln\frac{2|p'|^2}{\mu^2} < \frac{2\mathcal{M}_f\mu^2}{|p'|^2}\left(1 + \ln\frac{|p'|^2}{\mu^2}\right) \quad (D7)$$

and

$$\left|\int_{-1}^{1} F_{\mu,2}(q)dx\right| = \left|\int_{-1}^{1} \frac{\mu^4 dx}{(q^2+\mu^2)^2}\right| \leqslant 4\mathcal{M}_f \mu^4 \int_{-1}^{1} \frac{dx}{[|p'|^2(1-x)+2\mu^2]^2} = \frac{2\mathcal{M}_f \mu^2}{|p'|^2+\mu^2} < \frac{2\mathcal{M}_f \mu^2}{|p'|^2}, \tag{D8}$$

whereas for $|p'| < \mu$, we can simply use

$$\left|\int_{-1}^{1} dx F_{\mu,n}(q)\right| \leqslant \int_{-1}^{1} dx (\mathcal{M}_f)^n = 2(\mathcal{M}_f)^n, \quad n = 1, 2. \tag{D9}$$

Combining Eq. (D9) with Eq. (D7) or Eq. (D8) and introducing the functions

$$\begin{aligned} \lambda(\xi) &= \theta(1-|\xi|) + \theta(|\xi|-1)\frac{1}{|\xi|^2}, \\ \lambda_{\log}(\xi) &= \theta(1-|\xi|) + \theta(|\xi|-1)\frac{1+\ln|\xi|}{|\xi|^2}, \end{aligned} \tag{D10}$$

we arrive at the following bounds (obviously symmetric under the interchange $p \leftrightarrow p'$):

$$\left|\int_{-1}^{1} F_{\mu}(q)dx\right| \leqslant \mathcal{M}_{F,1}\lambda_{\log}(p'/\mu), \text{ and the same for } p \leftrightarrow p' \tag{D11}$$

and

$$\left|\int_{-1}^{1} F_{\mu,2}(q)dx\right| \leqslant \mathcal{M}_{F,2}\lambda(p'/\mu), \text{ and the same for } p \leftrightarrow p'. \tag{D12}$$

For the function $F_{\mu,2}(q)$, we can also obtain another bound:

$$\left|\int_{-1}^{1} F_{\mu,2}(q)dx\right| \leqslant \mathcal{M}_{F,2}\lambda(p'/\mu)^2/\lambda(p/\mu), \text{ and the same for } p \leftrightarrow p'. \tag{D13}$$

To prove Eq. (D13), we consider three cases.

(1) $|p'| \leqslant \mu$. In this case, $\lambda(p'/\mu) = 1$. Since $\lambda(p/\mu) \leqslant 1$, Eq. (D13) follows from Eq. (D12).
(2) $|p| \geqslant |p'| > \mu$. In this case, $\lambda(p'/\mu) \geqslant \lambda(p/\mu)$ and Eq. (D12) yields Eq. (D13).
(3) $|p| < |p'|$ and $|p'| > \mu$. Consider the definition of the subtraction remainder $\Delta_p^{(1)}$ in Eq. (C7):

$$F_{\mu,2}(q) = F_{\mu,2}(p') + p\frac{\partial F_{\mu,2}(q)}{\partial p}\bigg|_{p=0} + \Delta_p^{(1)}F_{\mu,2}(q). \tag{D14}$$

Now, we estimate the three terms in the last equation individually:

$$\left|\int_{-1}^{1} F_{\mu,2}(p')dx\right| \leqslant \int_{-1}^{1} |F_{\mu,2}(p')|dx \leqslant \frac{2\mu^4}{|p'|^4} = 2\lambda(p'/\mu)^2 \leqslant 2\lambda(p'/\mu)^2/\lambda(p). \tag{D15}$$

From the fact that $\frac{\partial F_{\mu,2}(q)}{\partial p}|_{p=0} \propto x$, it follows

$$\int_{-1}^{1} p\frac{\partial F_{\mu,2}(q)}{\partial p}\bigg|_{p=0} dx = 0. \tag{D16}$$

The bound from Eq. (C7) gives

$$\left|\int_{-1}^{1} \Delta_p^{(1)}F_{\mu,2}(q)\right|dx \leqslant \mathcal{M}_{f,1}\frac{|p|^2}{|p'|^2}\int_{-1}^{1} |F_{\mu,2}(q)|dx, \tag{D17}$$

which [see Eq. (D8)] leads to

$$\left|\int_{-1}^{1} \Delta_p^{(1)}F_{\mu,2}(q)\right|dx \leqslant 2\mathcal{M}_{f,1}\frac{|p|^2\mu^2}{|p'|^4} = 2\mathcal{M}_{f,1}\frac{|p|^2}{\mu^2}\lambda(p'/\mu)^2 \leqslant 2\mathcal{M}_{f,1}\lambda(p'/\mu)^2/\lambda(p/\mu). \tag{D18}$$

Finally,

$$\left|\int_{-1}^{1} F_{\mu,2}(q)dx\right| \leqslant \left|\int_{-1}^{1} F_{\mu,2}(p')dx\right| + \left|\int_{-1}^{1} \Delta_p^{(1)}F_{\mu,2}(q)\right|dx \leqslant 2(\mathcal{M}_{f,1}+1)\lambda(p'/\mu)^2/\lambda(p/\mu). \tag{D19}$$

Combining all three cases, we obtain Eq. (D13).

2. Bounds on the partial-wave leading-order potential

We represent the bounds for the partial-wave LO potential in the separable form:

$$|V_0(p', p)| \leqslant \mathcal{M}_{V_0} V_{0,\max}\, g(p')h(p), \quad |V_0(p', p)| \leqslant \mathcal{M}_{V_0} V_{0,\max}\, h(p')g(p), \tag{D20}$$

with

$$V_{0,\max} = \frac{8\pi^2}{m_N \Lambda_V}, \tag{D21}$$

where the exact form of functions g and h (and the value of $\mathcal{M}_{V_0}$) depends on the partial wave and on the form of a regulator.

Introducing the functions

$$v_0(p', p) = V_0(p', p)[\mathcal{M}_{V_0} V_{0,\max}\, h(p')g(p)]^{-1}, \quad \bar{v}_0(p', p) = V_0(p', p)[\mathcal{M}_{V_0} V_{0,\max}\, g(p')h(p)]^{-1}, \tag{D22}$$

we obtain the bounds

$$|v_0(p', p)| \leqslant 1, \quad |\bar{v}_0(p', p)| \leqslant 1. \tag{D23}$$

The above inequalities are meant to hold for all matrix elements of $V_0(p', p)$ in the l, l' space.

a. S wave

Using the bounds for the plane-wave leading-order potential in Eq. (C16) and performing the partial-wave projection according to Eqs. (D4), (D11), and (D13), we obtain for $l = 0$ (for the coupled partial waves, we mean by l the lowest orbital angular momentum):

$$g(p) = \lambda_{\log}(p/\Lambda), \quad h(p) = 1, \tag{D24}$$

for the "mild" regulator, and

$$g(p) = [\lambda(p/\Lambda)]^2, \quad h(p) = [\lambda(p/\Lambda)]^{-1}, \tag{D25}$$

for the "standard" regulators.

Note that for $|p| \leqslant \Lambda$, in particular, for the on-shell momentum $|p| = p_{\rm on}$, we have $g(p) = h(p) = 1$.

b. Higher partial waves

For $l > 0$, we can use the fact that for $m < l$,

$$\left.\frac{\partial^m V_0(p', p)}{(\partial p)^m}\right|_{p=0} = \left.\frac{\partial^m V_0(p', p)}{(\partial p')^m}\right|_{p'=0} = 0, \tag{D26}$$

and thus

$$\Delta_p^{(m)} V_0(p', p) = \Delta_{p'}^{(m)} V_0(p', p) = V_0(p', p). \tag{D27}$$

For the case of the "mild" regulator utilizing Eq. (C22) and performing the partial-wave projection according to Eqs. (D4) and (D11), we derive

$$g(p) = \lambda_{\log}(p/\Lambda)/|p|^{\tilde{l}}, \quad h(p) = |p|^{\tilde{l}}, \tag{D28}$$

with $\tilde{l} \leqslant l$. Since

$$\lambda(p/\Lambda) \leqslant \lambda_{\log}(p/\Lambda), \tag{D29}$$

the same bounds can be used for the "standard" regulators, see Eq. (D12).

For the purposes of the present paper, it is sufficient to choose $\tilde{l} = 1$.

3. Bounds on the partial-wave next-to-leading-order potential

a. S wave

For $l = 0$, the bounds on the NLO partial-wave potential are the same as in Ref. [16]:

$$|\hat{V}_2(p', p)| \leqslant \hat{\mathcal{M}}_{V_2,0} \frac{8\pi^2}{m_N \Lambda_V} \frac{M_\pi^2}{\Lambda_b^2}, \tag{D30}$$

and

$$|\tilde{V}_2(p', p)| \leqslant \mathcal{M}_{V_2,0}(|p|^2 + |p'|^2)\tilde{f}_{\log}(p', p), \tag{D31}$$

when one employs the "standard" regulators for the LO potentials.

In the case of the "mild" regulator of the LO potential, we use the partial-wave projected regularized expression, applying Eq. (D11) to Eq. (C30):

$$|\tilde{V}_2(p', p)| \leqslant \mathcal{M}_{V_2,0}(|p|^2 + |p'|^2)\tilde{f}_{\log}(p', p)\lambda_{\log}(p'/\Lambda_{\rm NLO}), \quad \text{or}$$
$$|\tilde{V}_2(p', p)| \leqslant \mathcal{M}_{V_2,0}(|p|^2 + |p'|^2)\tilde{f}_{\log}(p', p)\lambda_{\log}(p/\Lambda_{\rm NLO}). \tag{D32}$$

b. Higher partial waves

For $l \geqslant 1$, we simply adopt the bounds from Ref. [16]

$$|\tilde{V}_2(p', p)| \leqslant \mathcal{M}_{V_2,\tilde{l}}\left|\frac{p}{p'}\right|^{\tilde{l}}|p'|^2\tilde{f}_{\log}(p', p), \tag{D33}$$

$$|\tilde{V}_2(p', p)| \leqslant \mathcal{M}_{V_2,\tilde{l}}\left|\frac{p'}{p}\right|^{\tilde{l}}|p|^2\tilde{f}_{\log}(p', p), \tag{D34}$$

where $0 \leqslant \tilde{l} \leqslant l$.

For $\tilde{l} = 1$, both above equations coincide:

$$|\tilde{V}_2(p', p)| \leqslant \mathcal{M}_{V_2,1}|p'||p|\tilde{f}_{\log}(p', p). \tag{D35}$$

For the purposes of the present paper, it is sufficient to take the choice $\tilde{l} = 1$.

APPENDIX E: BOUNDS ON VARIOUS PARTS OF THE *S*-WAVE NLO AMPLITUDE

In this Appendix we provide bounds for various parts of the unrenormalized and renormalized S-wave NLO amplitude and their series remainders. The unrenormalized NLO amplitude is decomposed by factoring out the Fredholm determinant as in Eq. (74):

$$T_2(p', p; p_{\rm on}) = N_2(p', p; p_{\rm on})/D(p_{\rm on})^2, \quad N_2 = V_2D^2 + T_{2,Y}D + T_{2,\bar{Y}}D + T_{2,\bar{Y}Y}, \tag{E1}$$

with

$$T_{2,Y}(p', p; p_{\rm on}) = \int \frac{p_1^2 dp_1}{(2\pi)^3}V_2(p', p_1)Y(p_1, p; p_{\rm on}),$$
$$T_{2,\bar{Y}}(p', p; p_{\rm on}) = \int \frac{p_1'^2 dp_1'}{(2\pi)^3}\bar{Y}(p', p_1'; p_{\rm on})V_2(p_1', p),$$
$$T_{2,\bar{Y}Y}(p', p; p_{\rm on}) = \int \frac{p_1^2 dp_1}{(2\pi)^3}\frac{p_1'^2 dp_1'}{(2\pi)^3}\bar{Y}(p', p_1'; p_{\rm on})V_2(p_1', p_1)Y(p_1, p; p_{\rm on}). \tag{E2}$$

Below, we derive the bounds for the quantities $T_{2,Y}$, $T_{2,\bar{Y}}$, and $T_{2,\bar{Y}Y}$ for the cases of the "standard" and the "mild" regulators of the LO potential.

1. "Standard" regulator

For the "standard" regulators of the LO potential, in particular, for the local regulators of the spin-triplet part of the one-pion-exchange potential of power $n \geqslant 2$, the binding functions g and h have the form [see Eq. (D25)]

$$g(p_1) = \lambda(p_1/\Lambda)^2, \quad h(p) = 1, \text{ if } p < \Lambda. \tag{E3}$$

From the bounds on V_2 [Eq. (D31)] and V_0 [Eq. (D20)], we obtain

$$|T_{2,Y}(p', p; p_{\rm on})| \leqslant \mathcal{M}_{V_2,0}n_{\rm PW}\frac{8\pi^2\mathcal{M}_{Y_{\max}}}{\Lambda_V}\int\frac{(|p_1|^2 + |p'|^2)d|p_1|}{(2\pi)^3}\tilde{f}_{\log}(p', p_1)\lambda(p_1/\Lambda)^2$$
$$= \mathcal{M}_{V_2,0}n_{\rm PW}\frac{8\pi^2\mathcal{M}_{Y_{\max}}}{m_N\Lambda_V^2\Lambda_b^2}\int\frac{d|p_1|}{\pi}(|p_1|^2 + |p'|^2)f_{\log}(p', p_1)\lambda(p_1/\Lambda)^2$$
$$= \mathcal{M}_{V_2,0}n_{\rm PW}\frac{8\pi^2\mathcal{M}_{Y_{\max}}}{m_N\Lambda_V^2\Lambda_b^2}\left\{[|p'|^2I_{\lambda,1a} + I_{\lambda,1b}]\left[1 + \theta(|p'| - M_\pi)\ln\frac{|p'|}{M_\pi}\right] + |p'|^2I_{\lambda,2a} + I_{\lambda,2b}\right\}, \tag{E4}$$

where the typical integrals I_i are defined and estimated in Appendix F. Setting all external momenta on shell, $p = p' = p_{\rm on}$, and using $p_{\rm on} \ll \Lambda$, gives

$$|T_{2,Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \mathcal{M}_{T_2,Y}\mathcal{M}_{Y_{\rm max}}}{m_N \Lambda_V^2 \Lambda_b^2}\Lambda^3 \ln\frac{\Lambda}{M_\pi}, \tag{E5}$$

or, assuming $\Lambda \sim \Lambda_V$,

$$|T_{2,Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \tilde{\mathcal{M}}_{T_2,Y}\mathcal{M}_{Y_{\rm max}}}{m_N \Lambda_V}\frac{\Lambda^2}{\Lambda_b^2} \ln\frac{\Lambda}{M_\pi}. \tag{E6}$$

Symmetrically, the same bound holds for $T_{2,\bar{Y}}(p', p; p_{\rm on})$.

Next, we consider the contribution $T_{2,\bar{Y}Y}$:

$$\begin{aligned}
|T_{2,\bar{Y}Y}(p', p; p_{\rm on})| &\leqslant \mathcal{M}_{V_2,0}\left(\frac{8\pi^2 n_{\rm PW}\mathcal{M}_{Y_{\rm max}}}{\Lambda_V}\right)^2 \int \frac{d|p_1|}{(2\pi)^3}\frac{d|p_1'|}{(2\pi)^3}\left(|p_1|^2 + |p_1'^2|\right)\tilde{f}_{\rm log}(p_1', p_1)\lambda(p_1/\Lambda)^2\lambda(p_1'/\Lambda)^2 \\
&= \mathcal{M}_{V_2,0} n_{\rm PW}^2 \frac{8\pi^2 \mathcal{M}_{Y_{\rm max}}^2}{m_N \Lambda_V^3 \Lambda_b^2}\int \frac{d|p_1|d|p_1'|}{\pi^2}(|p_1|^2 + |p_1'|^2) f_{\rm log}(p', p_1)\lambda(p_1/\Lambda)^2\lambda(p_1'/\Lambda)^2 \\
&= \mathcal{M}_{V_2,0} n_{\rm PW}^2 \frac{8\pi^2 \mathcal{M}_{Y_{\rm max}}^2}{m_N \Lambda_V^3 \Lambda_b^2} 2(I_{\lambda,1a}I_{\lambda,1b} + I_{\lambda,2a}I_{\lambda,1b} + I_{\lambda,2b}I_{\lambda,1a}).
\end{aligned} \tag{E7}$$

Setting all external momenta on shell, $p = p' = p_{\rm on}$, and using $p_{\rm on} \ll \Lambda$, we obtain

$$|T_{2,\bar{Y}Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \mathcal{M}_{T_2,\bar{Y}Y}\mathcal{M}_{Y_{\rm max}}^2}{m_N \Lambda_V^3 \Lambda_b^2}\Lambda^4 \ln\frac{\Lambda}{M_\pi}, \tag{E8}$$

or, assuming $\Lambda \sim \Lambda_V$:

$$|T_{2,\bar{Y}Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \tilde{\mathcal{M}}_{T_2,\bar{Y}Y}\mathcal{M}_{Y_{\rm max}}^2}{m_N \Lambda_V}\frac{\Lambda^2}{\Lambda_b^2} \ln\frac{\Lambda}{M_\pi}. \tag{E9}$$

2. "Mild" regulator

For the "mild" regulator of the LO potential, including the case when the spin-triplet one-pion-exchange contribution is regularized by the local dipole regulator, the binding functions g and h have the form [see Eq. (D24)]

$$g(p_1) = \lambda_{\rm log}(p_1/\Lambda), \quad h(p) = 1, \text{ if } p < \Lambda. \tag{E10}$$

By analogy with Eq. (E4) from the bounds on the regularized V_2 [Eq. (D32)] and V_0 [Eq. (D20)], we obtain

$$\begin{aligned}
|T_{2,Y}(p', p; p_{\rm on})| &\leqslant \mathcal{M}_{V_2,0} n_{\rm PW}\frac{8\pi^2 \mathcal{M}_{Y_{\rm max}}}{m_N \Lambda_V^2 \Lambda_b^2}\int \frac{d|p_1|}{\pi}(|p_1|^2 + |p'|^2) f_{\rm log}(p', p_1)\lambda_{\rm log}(p_1/\Lambda)\lambda_{\rm log}(p_1/\Lambda_{\rm NLO}) \\
&= \mathcal{M}_{V_2,0} n_{\rm PW}\frac{8\pi^2 \mathcal{M}_{Y_{\rm max}}}{m_N \Lambda_V^2 \Lambda_b^2}\left\{[|p'|^2 I_{\lambda_{\rm log},1a} + I_{\lambda_{\rm log},1b}]\left[1 + \theta(|p'| - M_\pi)\ln\frac{|p'|}{M_\pi}\right] + |p'|^2 I_{\lambda_{\rm log},2a} + I_{\lambda_{\rm log},2b}\right\},
\end{aligned} \tag{E11}$$

where the typical integrals I_i are defined and estimated in Appendix F. Setting all external momenta on shell, $p = p' = p_{\rm on}$, and using $p_{\rm on} \ll \Lambda \ll \Lambda_{\rm NLO}$, yields

$$|T_{2,Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \mathcal{M}_{T_2,Y}\mathcal{M}_{Y_{\rm max}}}{m_N \Lambda_V^2 \Lambda_b^2}\Lambda^2 \Lambda_{\rm NLO} \ln\frac{\Lambda_{\rm NLO}}{\Lambda}\ln\frac{\Lambda_{\rm NLO}}{M_\pi}, \tag{E12}$$

or, assuming $\Lambda \sim \Lambda_V$:

$$|T_{2,Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \tilde{\mathcal{M}}_{T_2,Y}\mathcal{M}_{Y_{\rm max}}}{m_N \Lambda_V}\frac{\Lambda\Lambda_{\rm NLO}}{\Lambda_b^2} \ln\frac{\Lambda_{\rm NLO}}{\Lambda}\ln\frac{\Lambda_{\rm NLO}}{M_\pi}. \tag{E13}$$

Symmetrically, the same bound holds for $T_{2,\bar{Y}}(p', p; p_{\rm on})$.

Analogous to Eq. (E7), the following bound holds for $T_{2,\bar{Y}Y}$:

$$\begin{aligned}
|T_{2,\bar{Y}Y}(p', p; p_{\rm on})| &\leqslant \mathcal{M}_{V_2,0} n_{\rm PW}^2 \frac{8\pi^2 \mathcal{M}_{Y_{\rm max}}^2}{m_N \Lambda_V^3 \Lambda_b^2}\int \frac{d|p_1|d|p_1'|}{\pi^2} f_{\rm log}(p_1', p_1)\lambda_{\rm log}(p_1/\Lambda)\lambda_{\rm log}(p_1'/\Lambda) \\
&\quad \times [|p_1|^2 \lambda_{\rm log}(p_1/\Lambda_{\rm NLO}) + |p_1'|^2 \lambda_{\rm log}(p_1'/\Lambda_{\rm NLO})] \\
&= \mathcal{M}_{V_2,0} n_{\rm PW}^2 \frac{8\pi^2 \mathcal{M}_{Y_{\rm max}}^2}{m_N \Lambda_V^3 \Lambda_b^2} 2\left(I_{\lambda_{\rm log},1}I_{\lambda_{\rm log},1b} + I_{\lambda_{\rm log},2}I_{\lambda_{\rm log},1b} + I_{\lambda_{\rm log},2b}I_{\lambda_{\rm log},1}\right).
\end{aligned} \tag{E14}$$

Setting all external momenta on shell, $p = p' = p_{\rm on}$, and using $p_{\rm on} \ll \Lambda \ll \Lambda_{\rm NLO}$, we obtain

$$|T_{2,\bar{Y}Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \mathcal{M}_{T_2,\bar{Y}Y}\mathcal{M}^2_{Y_{\rm max}}}{m_N \Lambda_V^3 \Lambda_b^2} \Lambda^3 \Lambda_{\rm NLO} \ln\frac{\Lambda_{\rm NLO}}{\Lambda} \ln\frac{\Lambda_{\rm NLO}}{M_\pi}, \tag{E15}$$

or, assuming $\Lambda \sim \Lambda_V$:

$$|T_{2,\bar{Y}Y}(p_{\rm on})| \leqslant \frac{8\pi^2 \tilde{\mathcal{M}}_{T_2,\bar{Y}Y}\mathcal{M}^2_{Y_{\rm max}}}{m_N \Lambda_V} \frac{\Lambda \Lambda_{\rm NLO}}{\Lambda_b^2} \ln\frac{\Lambda_{\rm NLO}}{\Lambda} \ln\frac{\Lambda_{\rm NLO}}{M_\pi}. \tag{E16}$$

3. Bounds on the function $\nu(p_{\rm on})$

In this subsection we provide bounds on the function $\nu_l(p_{\rm on})$, defined in Eq. (118). We introduce another function $\nu_{Y,l}$ as follows:

$$\nu_l(p_{\rm on}) = D(p_{\rm on})[\delta_{l,0} + \nu_{Y,l}(p_{\rm on})], \tag{E17}$$

which equals [see Eq. (105)]

$$\nu_{Y,l}(p_{\rm on}) = \int \frac{p_1^2 dp_1}{(2\pi)^3} Y_{0,l}(p_1, p_{\rm on}; p_{\rm on}). \tag{E18}$$

Using Eq. (61), we derive the following bound for the function $n_{Y,l}$ in the case of the "standard" regulator of the LO potential (see Appendix D 2 a):

$$|\nu_{Y,l}(p_{\rm on})| \leqslant \frac{\mathcal{M}_{Y_{\rm max}}}{\Lambda_V} \int \frac{d|p_1|}{\pi} g(p_1/\Lambda)h(p_{\rm on}) = \frac{\mathcal{M}_{Y_{\rm max}}}{\Lambda_V} \int \frac{d|p_1|}{\pi} \lambda(p_1/\Lambda)^2 = \frac{\mathcal{M}_{Y_{\rm max}}}{\Lambda_V} I_{\lambda,1a} = \mathcal{M}_{Y_{\rm max}} \mathcal{M}_\lambda \frac{\Lambda}{\Lambda_V}, \tag{E19}$$

where we have utilized the bounds for typical integrals provided in Appendix F.

Assuming $\Lambda \sim \Lambda_V$ yields

$$|\nu_{Y,l}(p_{\rm on})| \leqslant \mathcal{M}_{Y_{\rm max}} \tilde{\mathcal{M}}_\lambda. \tag{E20}$$

For the "mild" regulator of the LO potential, one should replace $\lambda(p_1/\Lambda)^2$ with $\lambda_{\rm log}(p_1/\Lambda)$ and $I_{\lambda,1a}$ with $I_{\lambda_{\rm log},1}$ in Eq. (E19). Since our bounds for $I_{\lambda_{\rm log},1}$ and $I_{\lambda,1a}$ are the same, see Eqs. (F2) and (F4), Eq. (E20) holds also for the "mild" regulator.

Since the Fredholm determinant D is bounded by a constant of order one [Eq. (51)], the same is true for the function $\nu_l(p_{\rm on})$:

$$\nu_l(p_{\rm on}) \leqslant \mathcal{M}_\nu, \tag{E21}$$

as follows from Eqs. (E20) and (E17).

4. Series remainders

From the bounds on the matrix elements of the operator Y ($\bar{Y}$) and its series remainders [Eqs. (61) and (65)] as well as the bounds on the Fredholm determinant D and its series remainders [Eqs. (51) and (53)], it is straightforward to deduce also the bounds for the series remainders of the quantities $T_{2,Y}$, $T_{2,\bar{Y}}$, $T_{2,\bar{Y}Y}$, and ν_Y by just replacing $\mathcal{M}_{Y_{\rm max}}$ with $\mathcal{N}_{\delta_n Y} = \mathcal{M}_Y \delta_n Y_{\rm max}$ and $\mathcal{M}^2_{Y_{\rm max}}$ with $2\mathcal{M}_{Y_{\rm max}}\mathcal{N}_{\delta_n Y} + \mathcal{N}^2_{\delta_n Y}$. Being proportional to $\delta_n Y_{\rm max}$ or $(\delta_n Y_{\rm max})^2$, $T_{2,Y}$, $T_{2,\bar{Y}}$, $T_{2,\bar{Y}Y}$, and ν_Y decrease faster than exponential with any base, see Eq. (64). The series remainder of the Fredholm determinant possesses the same property, see Eq. (55). Therefore, from Eq. (E1) we conclude that N_2 also decreases faster than exponential as well as the renormalized quantity $\mathbb{R}(\tilde{N}_2)$ [Eq. (122)], because those are polynomials in $T_{2,Y}$, $T_{2,\bar{Y}}$, $T_{2,\bar{Y}Y}$, ν_Y, and D.

To be specific, the following bound holds:

$$|\delta_n[\mathbb{R}(\tilde{N}_2)]| = \left| \sum_{k_1,k_2=0}^{\infty} \mathbb{R}(\tilde{N}_2)^{[k_1,k_2]} - \sum_{k_1,k_2=0}^{n} \mathbb{R}(\tilde{N}_2)^{[k_1,k_2]} \right| \leqslant \frac{8\pi^2}{m_N \Lambda_V} \mathcal{N}_{\tilde{N}_2} e^{-\mathcal{M}_{\delta\tilde{N}_2} n}, \quad \text{for } n > \tilde{\mathcal{M}}_{\delta\tilde{N}_2}, \tag{E22}$$

where $\tilde{\mathcal{M}}_{\delta\tilde{N}_2}$ is of order $\tilde{\mathcal{M}}_{\delta\tilde{N}_2} \gtrsim (e\Sigma)^2$ in the general case but is typically much smaller in realistic calculations. The prefactors $\mathcal{N}_{\tilde{N}_2}$ follow from Eqs. (E6), (E9), (E13), (E16), (E21) and (51):

$$\mathcal{N}_{\tilde{N}_2} = \frac{\Lambda^2}{\Lambda_b^2} \ln\frac{\Lambda}{M_\pi} \tag{E23}$$

in the case of the "standard" regulators of the LO potential and

$$\mathcal{N}_{\tilde{N}_2} = \frac{\Lambda \Lambda_{\rm NLO}}{\Lambda_b^2} \ln\frac{\Lambda_{\rm NLO}}{\Lambda} \ln\frac{\Lambda_{\rm NLO}}{M_\pi} \tag{E24}$$

in the case of the "mild" regulator.

APPENDIX F: BOUNDS ON TYPICAL INTEGRALS

In this Appendix we provide the bounds for typical integrals that appear in the course of evaluation of various amplitudes. The integrals

$$I_{\lambda_{\log},1} = \int \frac{d|p|}{\pi}\lambda_{\log}(p/\Lambda), \quad I_{\lambda_{\log},1a} = \int \frac{d|p|}{\pi}\lambda_{\log}(p/\Lambda_{\rm NLO})\lambda_{\log}(p/\Lambda),$$
$$I_{\lambda_{\log},2} = \int \frac{d|p|}{\pi}\lambda_{\log}(p/\Lambda)\theta(|p|-M_\pi)\ln\frac{|p|}{M_\pi}, \quad I_{\lambda_{\log},2a} = \int \frac{d|p|}{\pi}\lambda_{\log}(p/\Lambda_{\rm NLO})\lambda_{\log}(p/\Lambda)\theta(|p|-M_\pi)\ln\frac{|p|}{M_\pi}, \tag{F1}$$

with functions λ and $\lambda_{\log}$ defined in Eq. (D10) can be bounded as follows:

$$I_{\lambda_{\log},1} = \Lambda\int\frac{d\xi}{\pi}\lambda_{\log}(\xi) =: \mathcal{M}_\lambda\Lambda, \quad I_{\lambda_{\log},1a} < I_{\lambda_{\log},1} = \mathcal{M}_\lambda\Lambda,$$
$$I_{\lambda_{\log},2} = \frac{1}{\pi}\left(2+\Lambda+2\Lambda\ln\frac{\Lambda}{M_\pi}\right) \leqslant \mathcal{M}_{\lambda,2}\Lambda\ln\frac{\Lambda}{M_\pi}, \quad I_{\lambda_{\log},2a} < I_{\lambda_{\log},2} \leqslant \mathcal{M}_{\lambda,2}\Lambda\ln\frac{\Lambda}{M_\pi}. \tag{F2}$$

Analogously, for the integrals

$$I_{\lambda,1} = \int\frac{d|p|}{\pi}\lambda(p/\Lambda), \quad I_{\lambda,1a} = \int\frac{d|p|}{\pi}\lambda(p/\Lambda)^2, \quad I_{\lambda,1b} = \int\frac{|p|^2d|p|}{\pi}\lambda(p/\Lambda)^2,$$
$$I_{\lambda,2} = \int\frac{d|p|}{\pi}\lambda(p/\Lambda)\theta(|p|-M_\pi)\ln\frac{|p|}{M_\pi}, \quad I_{\lambda,2a} = \int\frac{d|p|}{\pi}\lambda(p/\Lambda)^2\theta(|p|-M_\pi)\ln\frac{|p|}{M_\pi}$$
$$I_{\lambda,2b} = \int\frac{|p|^2d|p|}{\pi}\lambda(p/\Lambda)^2, \theta(|p|-M_\pi)\ln\frac{|p|}{M_\pi}, \tag{F3}$$

we obtain the following bounds:

$$I_{\lambda,1} = \Lambda\int\frac{d\xi}{\pi}\lambda(\xi) < \Lambda\int\frac{d\xi}{\pi}\lambda_{\log}(\xi) = \mathcal{M}_\lambda\Lambda, \quad I_{\lambda,1a} < I_{\lambda,1} \leqslant \mathcal{M}_\lambda\Lambda,$$
$$I_{\lambda,1b} = \Lambda^3\int\frac{\xi^2d\xi}{\pi}\lambda(\xi)^2 < \Lambda^3\int\frac{d\xi}{\pi}\lambda(\xi) < \Lambda^3\int\frac{d\xi}{\pi}\lambda_{\log}(\xi) = \mathcal{M}_\lambda\Lambda^3, \quad I_{\lambda,2} < I_{\lambda_{\log},2} \leqslant \mathcal{M}_{\lambda,2}\Lambda\ln\frac{\Lambda}{M_\pi},$$
$$I_{\lambda,2a} < I_{\lambda,2} \leqslant \mathcal{M}_{\lambda,2}\Lambda\ln\frac{\Lambda}{M_\pi}, \quad I_{\lambda,2b} < \Lambda^2 I_{\lambda,2} \leqslant \mathcal{M}_{\lambda,2}\Lambda^3\ln\frac{\Lambda}{M_\pi}. \tag{F4}$$

Next, we estimate the integral

$$I_{\lambda_{\log},1b} = \int\frac{|p|^2d|p|}{\pi}\lambda_{\log}(p/\Lambda_{\rm NLO})\lambda_{\log}(p/\Lambda). \tag{F5}$$

Direct estimation under the assumption $\Lambda_{\rm NLO} \gg \Lambda$ gives

$$I_{\lambda_{\log},1b} = \frac{2}{\pi}\Lambda^2\Lambda_{\rm NLO}\ln\frac{\Lambda_{\rm NLO}}{\Lambda} + O(\Lambda_{\rm NLO}) \leqslant \mathcal{M}_{\lambda,1a}\Lambda^2\Lambda_{\rm NLO}\ln\frac{\Lambda_{\rm NLO}}{\Lambda}. \tag{F6}$$

Finally, we derive a bound for the integral

$$I_{\lambda_{\log},2b} = \int\frac{|p|^2d|p|}{\pi}\lambda_{\log}(p/\Lambda_{\rm NLO})\lambda_{\log}(p/\Lambda)\theta(|p|-M_\pi)\ln\frac{|p|}{M_\pi}. \tag{F7}$$

Direct calculation yields

$$I_{\lambda_{\log},2b} = \frac{1}{\pi}\Lambda^2\Lambda_{\rm NLO}\ln\frac{\Lambda_{\rm NLO}}{\Lambda}\ln\frac{\Lambda_{\rm NLO}}{M_\pi} + O(\Lambda_{\rm NLO}\ln\Lambda_{\rm NLO}/M_\pi) \leqslant \mathcal{M}_{\lambda,1a}\Lambda^2\Lambda_{\rm NLO}\ln\frac{\Lambda_{\rm NLO}}{\Lambda}\ln\frac{\Lambda_{\rm NLO}}{M_\pi}. \tag{F8}$$

[1] S. Weinberg, Phys. Lett. B **251**, 288 (1990).
[2] S. Weinberg, Nucl. Phys. B **363**, 3 (1991).
[3] P. F. Bedaque and U. van Kolck, Annu. Rev. Nucl. Part. Sci. **52**, 339 (2002).
[4] E. Epelbaum, H.-W. Hammer, and Ulf-G. Meißner, Rev. Mod. Phys. **81**, 1773 (2009).
[5] R. Machleidt and D. Entem, Phys. Rep. **503**, 1 (2011).
[6] E. Epelbaum and U.-G. Meißner, Annu. Rev. Nucl. Part. Sci. **62**, 159 (2012).
[7] E. Epelbaum, H. Krebs, and P. Reinert, Front. Phys. **8**, 98 (2020).

[8] H. W. Hammer, S. König, and U. van Kolck, Rev. Mod. Phys. **92**, 025004 (2020).

[9] P. Reinert, H. Krebs, and E. Epelbaum, Eur. Phys. J. A **54**, 86 (2018).

[10] D. R. Entem, R. Machleidt, and Y. Nosyk, Phys. Rev. C **96**, 024004 (2017).

[11] P. Reinert, H. Krebs, and E. Epelbaum, Phys. Rev. Lett. **126**, 092501 (2021).

[12] G. P. Lepage, in *Proceedings of the 8th Jorge Andre Swieca Summer School on Nuclear Physics, Sao Jose dos Campos, Campos do Jordao, Brazil, January 26–February 7* (World Scientific Pub Co Inc., Singapore, 1997), pp. 135–180.

[13] J. Gegelia, J. Phys. G: Nucl. Part. Phys. **25**, 1681 (1999).

[14] J. Gegelia and S. Scherer, Int. J. Mod. Phys. A **21**, 1079 (2006).

[15] E. Epelbaum and U.-G. Meißner, Few-Body Syst. **54**, 2175 (2013).

[16] A. M. Gasparyan and E. Epelbaum, Phys. Rev. C **105**, 024001 (2022).

[17] S. Weinberg, Physica A **96**, 327 (1979).

[18] E. Epelbaum, A. M. Gasparyan, J. Gegelia, and H. Krebs, Eur. Phys. J. A **51**, 71 (2015).

[19] B. Long, Phys. Rev. C **88**, 014002 (2013).

[20] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C **72**, 054006 (2005).

[21] M. C. Birse, Phys. Rev. C **76**, 034002 (2007).

[22] J. H. Hetherington and L. H. Schick, Phys. Rev. **137**, B935 (1965).

[23] R. Aaron and R. D. Amado, Phys. Rev. **150**, 857 (1966).

[24] R. T. Cahill and I. H. Sloan, Nucl. Phys. A **165**, 161 (1971) [Erratum: Nucl. Phys. A **196**, 632 (1972)].

[25] R. G. Newton, *Scattering Theory of Waves and Particles* (Springer-Verlag, Berlin, 1982).

[26] F. Smithies, *Integral Equations*, Cambridge Tracts in Mathematics and Mathematical Physics (Cambridge University Press, Cambridge, UK, 1958).

[27] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[28] P. W. Johnson and R. L. Warnock, J. Math. Phys. **22**, 385 (1981).

[29] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B **424**, 390 (1998).

[30] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. B **534**, 329 (1998).

[31] A. M. Gasparyan and E. Epelbaum, Phys. Rev. C **107**, 034001 (2023).

[32] N. N. Bogoliubov and O. S. Parasiuk, Acta Math. **97**, 227 (1957).

[33] K. Hepp, Commun. Math. Phys. **2**, 301 (1966).

[34] W. Zimmermann, Commun. Math. Phys. **15**, 208 (1969).

[35] B. Long and C.-J. Yang, Phys. Rev. C **86**, 024001 (2012).

[36] Wolfram Research, Inc., *Mathematica*, Version 12.0 (Champaign, IL, 2019), https://www.wolfram.com/mathematica.

[37] E. Epelbaum, H. Krebs, and U.-G. Meißner, Eur. Phys. J. A **51**, 53 (2015).

[38] R. J. Furnstahl, N. Klco, D. R. Phillips, and S. Wesolowski, Phys. Rev. C **92**, 024005 (2015).

[39] E. Epelbaum *et al.*, Eur. Phys. J. A **56**, 92 (2020).

[40] E. Epelbaum, PoS **CD2018**, 006 (2019).

[41] V. G. J. Stoks, R. A. M. Klomp, M. C. M. Rentmeester, and J. J. de Swart, Phys. Rev. C **48**, 792 (1993).

[42] H. Stapp, T. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[43] J. D. Hunter, Comput. Sci. Eng. **9**, 90 (2007).

[44] M. C. Birse, Phys. Rev. C **74**, 014003 (2006).

[45] S. Wu and B. Long, Phys. Rev. C **99**, 024003 (2019).

[46] A. Gezerlis, I. Tews, E. Epelbaum, M. Freunek, S. Gandolfi, K. Hebeler, A. Nogga, and A. Schwenk, Phys. Rev. C **90**, 054323 (2014).

[47] E. Epelbaum, W. Glöckle, and U. -G. Meißner, Nucl. Phys. A **747**, 362 (2005).