Energy-independent complex single-*P*-waves *NN* potential from the Marchenko equation

N. A. Khokhlov*
Peter the Great St. Petersburg Polytechnic University, RU-195251 St. Petersburg, Russia

(Received 5 April 2022; revised 29 January 2023; accepted 24 March 2023; published 3 April 2023)

I extend the previous results on solving the inverse problem of quantum scattering theory (Marchenko theory, fixed-l inversion). In particular, a set of isosceles triangular-pulse functions is applied to expand the Marchenko equation input kernel in a separable form. The separable form enables a reduction of the Marchenko equation to a system of linear equations for the output kernel expansion coefficients. In the general case of a single partial wave, a linear expression of the input kernel is obtained in terms of the Fourier series coefficients of $q^{1-m}(1-S(q))$ functions in the finite range of the momentum $0 \leqslant q \leqslant \pi/h$ [$S(q)$ is the scattering matrix; $m = 0, 1, \ldots, 2l$; l is the angular orbital momentum]. Thus, I show that the partial S matrix on the finite interval determines the potential function of the corresponding radial Schrödinger equation with h-step accuracy. A numerical algorithm was developed to implement the method. The developed procedure was applied to the partial-wave analysis data of NN elastic scattering up to 3 GeV. Calculated energy-independent complex partial potentials describe these data for single P waves.

DOI: 10.1103/PhysRevC.107.044001

I. INTRODUCTION

Extracting the interparticle interaction potential from scattering data is a fundamental problem in nuclear physics. Various methods are used for such extraction. Fitting parameters of the phenomenological potential can solve this problem. Such an adjustment is easily implemented in the case of a small number and low accuracy of the known experimental data. An accurate description of a large number of experimental data requires exact methods of solving the inverse problem (IP) of quantum scattering. The development of such precise and unambiguous methods remains a fundamental challenge [1–5]. The ill-posedness of the problem significantly complicates its numerical solution.

The fixed-l IP considered here is usually solved within Marchenko, Krein, and Gelfand-Levitan theories [6–13]. In these approaches, the IP is reduced to solving Fredhölm integral equations of the second kind. von Gevamb and Kohlhoff successfully applied Marchenko and Gelfand-Levitan theories to extract NN partial potentials from partial-wave analysis (PWA) data [14,15]. They used the PWA data up to the inelastic threshold ($E_{\text{lab}} \approx 280$ MeV) and approximated the corresponding partial S matrices (spectral densities) by rational fraction expansions (Padé approximants). In this case, the input kernels of the integral equations are represented as finite separable series of the Riccati-Hankel functions products (separable kernels). A Fredhölm integral equation of the second kind with a separable kernel is solved analytically. The partial potentials, in this case, are also expressed through Riccati-Hankel functions (Bargman-type potentials). A similar approach in frames of the Marchenko theory was used later to extract optical model NN partial potentials from elastic NN scattering PWA data up to 3 GeV [16,17]. This method for solving the inverse problem includes four numerical procedures. The first procedure is the PWA, the second procedure is the approximation of the S matrix, the third procedure is the solution of the integral equation, and the fourth procedure is the differentiation of the output kernel to calculate the potential. Errors of each step are accumulated, estimating the error in the calculation of the potential is not an easy task. For $l \geqslant 1$, Marchenko inversion is unstable for $r < 1$ fm, and Gelfand-Levitan inversion is unstable for $r > 8$ fm [14,15]. Description of PWA data within errors for energies up to 3 GeV requires the use of high-order Padé approximants [16,17]. An increase in the accuracy of the approximation and, accordingly, an increase in the order of the Padé approximant can lead to significant changes in the potential. The Padé approximant is not the best choice since approximants of different orders that give close S-matrix values at the PWA points can differ significantly between these points. Thus, the convergence of methods that use rational fraction expansions of the S matrix (spectral density) is not apparent.

This paper generalizes the algebraic method [18,19] for solving the IP. In this method, the Marchenko equation input kernel is expanded into a separable series in an isosceles triangular-pulse function set. Then, one would obtain a linear expression of the expansion coefficients in terms of the Fourier series coefficients of $q^{1-m}(1-S(q))$ ($m = 0, 1, \ldots, 2l$) functions on a finite range of the momentum $0 \leqslant q \leqslant \pi/h$. Thus, the Marchenko equation with a separable kernel is solved, which can be performed analytically like in Refs. [14–17]. Theory of the Fourier's series substantiates convergence of the procedure with decreasing step h.

*nikolakhokhlov@yandex.ru; nakhokhlov@gmail.com

NN potentials used as an input for (semi)microscopic construction of nuclear optical potentials (OPs) are usually real and only describe the *NN* PWA data below the inelastic threshold [20–24]. However, in the absence of a microscopic theory, complex partial *NN* potentials are required to describe nuclear reactions with energies of the *NN* relative motion above the threshold [17]. In current models, real partial potentials are modified by energy-dependent imaginary terms [16,25]. The imaginary terms are absent below the threshold. Therefore, the partial wave function is real below the threshold up to an r-independent factor. For a realistic optical *NN* potential, this constraint seems too restrictive. The generally recognized fundamental interaction potential used in quantum mechanics is the electromagnetic potential. The classical limits of the electromagnetic potential is real-valued. A real-valued potential usually gives a Hermitian Hamiltonian, but exceptions exist [26]. A Hermitian Hamiltonian ensures the conservation of probability and the real-valued energy spectrum of the system. Thus, the standard axiomatics of quantum mechanics requires Hermiticity of the Hamiltonian. The strong interaction and the corresponding *NN* interaction are absent in the classical limit. It is reasonable to consider the strong *NN* interaction to be symmetric with respect to space-time (PT) reflection. The Hamiltonian of a PT-symmetric system is not always Hermitian [26–29]. There are no logical or physical reasons to exclude the possibility of a PT-symmetric non-Hermitian Hamiltonian for a nucleon system.

The optical model potential (pseudopotential) definition does not guarantee that it will be Hermitian [30,31]. One can only assert that Hamiltonian eigenfunctions corresponding to eigenenergies below the threshold satisfy the Hermiticity condition. However, the Hermiticity condition

$$\int \psi^*(V^+ - V)\psi dx = 0 \tag{1}$$

implies $V^+ - V = 0$ only if Eq. (1) holds for an arbitrary function ψ, which is not true for OPs. Using the phase-equivalent Krein transformations, one can obtain an energy-independent complex potential giving a unitary S matrix [7]. The use of optical potentials limited by the condition $V^+ - V = 0$ below the inelastic threshold is a physically unreasonable limitation.

Previously, it was shown that the Marchenko theory applies to the description of elastic *nD* scattering from zero energy up to energies well above the threshold. In this case, Marchenko theory produces energy-independent complex partial *nD* potentials [32,33]. They used a rational parametrization, the same as in Refs. [14–17] for the unitary S matrix. The small value of the threshold (the deuteron's binding energy $E_{\text{c.m.}} \approx 2.226\,\text{MeV}$) does not allow us to judge the applicability of the Marchenko theory in the general case. I analyzed the Marchenko theory [7,8] ($l = 0$), [9] ($l > 0$) and found that the theory applies not only to unitary $S(q)$ matrices but also to nonunitary S matrices. The algebraic form of the Marchenko equation [18] allows for the calculation of an energy-independent complex local partial potential corresponding to a partly unitary and partly nonunitary S matrix. This approach was previously utilized to analyze ${}^1S_0$ data of elastic *NN* scattering PWA [19]. In this study, I extend the investigation of Ref. [19] to analyze single P-wave *NN* PWA data (up to $E_{\text{lab}} \approx 3\,\text{GeV}$) and demonstrate that these data are described by energy-independent complex partial potentials. The reconstructed partial S and Pwaves potentials constitute an energy-independent soft core OP that describes elastic *NN* scattering PWA data up to 3 GeV.

II. MARCHENKO EQUATION IN AN ALGEBRAIC FORM

The radial Schrödinger equation is

$$\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r) + q^2\right)\psi(r,q) = 0. \tag{2}$$

The Marchenko equation [7,8] is a Fredhölm integral equation of the second kind:

$$F(x,y) + L(x,y) + \int_x^{+\infty} L(x,t)F(t,y)dt = 0. \tag{3}$$

The kernel function is defined by the following expression:

$$\begin{aligned} F(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} h_l^+(qx)[1 - S(q)]h_l^+(qy)dq \\ &\quad + \sum_{j=1}^{n_b} h_l^+(\tilde{q}_j x)M_j^2 h_l^+(\tilde{q}_j y) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} h_l^+(qx)Y(q)h_l^+(qy)dq, \end{aligned} \tag{4}$$

where $h_l^+(z)$ is the Riccati-Hankel function, and

$$Y(q) = \left[1 - S(q) - i\sum_{j=1}^{n_b} M_j^2(q - \tilde{q}_j)^{-1}\right]. \tag{5}$$

Experimental data entering the kernel are

$$\{S(q), (0 < q < \infty), \tilde{q}_j, M_j, j = 1, \ldots, n\}, \tag{6}$$

where $S(q) = e^{2\imath\delta(q)}$ is a scattering matrix dependent on the momentum q. The S matrix defines asymptotic behavior at $r \to +\infty$ of regular at $r = 0$ solutions of Eq. (2) for $q \geqslant 0$; $\tilde{q}_j^2 = E_j \leqslant 0, E_j$ is jth bound state energy ($-\imath\tilde{q}_j \geqslant 0$); M_j is jth bound state asymptotic constant.

The potential function of Eq. (2) is obtained from the solution of Eq. (3)

$$V(r) = -2\frac{dL(r,r)}{dr}. \tag{7}$$

Many methods for solving Fredhölm integral equations use a series expansion of the equation kernel [34–40]. I also use this approach.

I introduce auxiliary functions

$$F_m(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{\imath qz}Y(q)dq}{q^m}, \tag{8}$$

then

$$\frac{d^k F_m(z)}{dz^k} = \imath^k F_{m-k}(z), \ (k = 1, 2, \ldots, m). \tag{9}$$

Using the transformations

$$\hat{K}_{z,l} f(z) = z^{l+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^l [z^{-1} f(z)]$$

$$\equiv (-1)^l \sum_{n=0}^{l} \frac{(2l-n)!}{n!(l-n)!}(-2z)^{n-l}\frac{d^n f(z)}{dz^n}, \tag{10}$$

one would get (see Eqs. (10.1.23)–(10.1.26) [41])

$$\hat{K}_{z,l} e^{\pm \imath qz} = q^l h_l^{\pm}(qz) \tag{11}$$

and

$$\hat{K}_{y,l}\hat{K}_{x,l} F_{2l}(x+y)$$

$$= \sum_{n_1,n_2=0}^{l} \frac{(2l-n_1)!}{n_1!(l-n_1)!}\frac{(2l-n_2)!}{n_2!(l-n_2)!}$$

$$\times (-2x)^{n_1-l}(-2y)^{n_2-l}\imath^{n_1+n_2} F_{2l-n_1-n_2}(x+y)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} h_l^+(qx) Y(q) h_l^+(qy) dq \equiv F(x,y). \tag{12}$$

Assuming the finite range R of the potential function $V(r)$, I approximate $F_m(x+y)$ as follows:

$$F_m(x+y) \approx \sum_{k=-2N}^{2N} f_{m,k} H_k(x+y) \tag{13}$$

$$\approx \sum_{k,j=0}^{N} \Delta_k(x) f_{m,k+j} \Delta_j(y), \tag{14}$$

where $f_{m,k} \equiv F_m(kh)$, h is some step, and $R = Nh$. The used basis sets are

$$H_0(x) = \left.\begin{cases} 1 & \text{if } 0 \leqslant x \leqslant h, \\ 0 & \text{otherwise,} \end{cases}\right\}, \qquad H_n(x) = H_0(x - hn) \tag{15}$$

$$\Delta_0(x) = \left.\begin{cases} 1 - |x - 0.25|/h & \text{if } |x - 0.25| \leqslant h, \\ 0 & \text{otherwise} \end{cases}\right\}, \qquad \Delta_n(x) = \Delta_0(x - hn). \tag{16}$$

The basis set $\Delta_i(x)\Delta_j(y)$ is shifted by the vector $(0.25h, 0.25h)$ compared to the set used previously [18,19]. The basis sets are illustrated in Fig. 1. Decreasing the step h, one can approach $F_m(x+y)$ arbitrarily close at all points with both sets. Coefficients $f_{m,k}$ are the same for both approximations Eqs. (13), (14).

The Fourier transform of the basis set Eq. (13)

$$\tilde{\mathrm{H}}_k(q) = \int_{-\infty}^{\infty} \mathrm{H}_k(x) e^{-\imath qx} dx = \frac{\imath(e^{-\imath qh} - 1)}{q e^{\imath qhk}}. \tag{17}$$

The Fourier transform of Eq. (8) yields

$$\frac{Y(q)}{q^m} \approx \sum_{k=-2N}^{2N} f_{m,k}\tilde{\mathrm{H}}_k(q) = \sum_{k=-2N}^{2N} f_{m,k}\frac{\imath(e^{-\imath qh} - 1)}{q e^{\imath qhk}}. \tag{18}$$

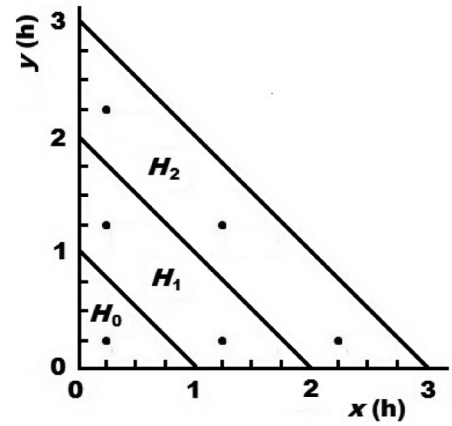

FIG. 1. The basis set $H_n \equiv H_n(x+y)$ [Eq. (15)] is shown as trapezoid (triangle for $n = 0$) regions, where $H_n(x+y) = 1$, and elsewhere $H_n(x+y) = 0$. The regions are bounded by lines $x = 0$, $y = 0$, and $x + y = h(n-1)$. The basis set $\Delta_i(x)\Delta_j(y)$ [Eq. (16)] is shown as projections (points) of the corresponding regular square pyramids apexes on the xy plane. $\Delta_i(x)\Delta_j(y) = 1$ at $x = (0.25 + i)h$, $y = (0.25 + j)h$ (apex of the ij pyramid). The pyramids bases are $(2h \times 2h)$ squares on the xy plain with sides parallel to the x and y axes. $\Delta_i(x)\Delta_j(y) = 0$ on sides of the corresponding squares (and outside them).

I rearrange the last relationship

$$Y(q)/q^{m-1} = \imath \sum_{k=-2N}^{2N} f_{m,k}(e^{-\imath qh} - 1)e^{-\imath qhk}$$

$$= \imath \sum_{k=-2N+1}^{2N} (f_{m,k-1} - f_{m,k})e^{-\imath qhk}$$

$$+ \imath(-f_{m,-2N})e^{\imath qh2N} + \imath(f_{m,2N})e^{-\imath qh(2N+1)}. \tag{19}$$

Thus, the left side of the expression is represented as a Fourier series on the interval $-\pi/h \leqslant q \leqslant \pi/h$:

$$f_{m,k-1} - f_{m,k} = -\frac{\imath h}{2\pi}\int_{-\pi/h}^{\pi/h} Y(q)\frac{e^{\imath qhk} dq}{q^{m-1}} \tag{20}$$

for $k = -2N, \ldots, 2N$. Recursive solving of Eq. (20) from $k = 2N + 1$ ($f_{m,2N+1} = 0$) gives

$$f_{m,k} = \frac{h}{\pi}\int_{-\pi/h}^{\pi/h}\left(\sum_{\nu=k+1}^{2N+1} e^{\imath qh\nu}\right)\frac{Y(q)dq}{q^{m-1}}$$

$$= -\frac{\imath h}{2\pi}\int_{-\pi/h}^{\pi/h}\frac{e^{\imath qh(k+1)}(1 - e^{\imath qh(2N-k+1)})}{(1 - e^{\imath qh})q^{m-1}} Y(q)dq. \tag{21}$$

The $F(x,y)$ is defined by $f_{m,k}$ $(m = 0, 1, \ldots, 2l)$, $k = 0, 1, \ldots, 2N$ from Eqs. (12) and (14) as

$$F(x,y) \approx \sum_{k,j=0}^{N} \Delta_k(x) F_{k,j} \Delta_j(y), \tag{22}$$

where

$$F_{k,j} = \sum_{n_1,n_2=0}^{l} \frac{(2l-n_1)!}{n_1!(l-n_1)!}\frac{(2l-n_2)!}{n_2!(l-n_2)!}(-2(k+0.25)h)^{n_1-l} \times(-2(j+0.25)h)^{n_2-l}\imath^{n_1+n_2} f_{2l-n_1-n_2,k+j} \quad (23)$$

$$= -\frac{\imath h}{2\pi}\int_{-\pi/h}^{\pi/h} h_l^+(q(k+0.25)h)\frac{e^{\imath qh}(1-e^{\imath qh(2N-k-j+1)})}{1-e^{\imath qh}} \times Y(q)h_l^+(q(j+0.25)h)qdq. \quad (24)$$

Thus, the range of known scattering data defines the value of h and, therefore, the inversion accuracy.

Equation (3) is solved by substituting

$$L(x,y) \approx \sum_{j=0}^{N} P_j(x)\Delta_j(y). \quad (25)$$

Substitution of Eqs. (22) and (25) into Eq. (3) and linear independence of the basis functions give

$$\sum_{m=0}^{N}\left(\delta_{jm} + \sum_{n=0}^{N}\left[\int_x^{max((m+0.25)h,(n+0.25)h)} \Delta_m(t)\Delta_n(t)dt\right]F_{n,j}\right)P_m(x) = -\sum_{k=0}^{N}\Delta_k(x)F_{k,j}. \quad (26)$$

It is convenient to define

$$\zeta_{nmp} = \int_{(p+0.25)h}^{max((m+0.25)h,(n+0.25)h)} \Delta_m(t)\Delta_n(t)dt = \frac{h}{6}(2\delta_{nm}(\delta_{np}+2\eta_{n\geqslant p+1}) + \delta_{n(m-1)}\eta_{n\geqslant p} + \delta_{n(m+1)}\eta_{m\geqslant p}), \quad (27)$$

where δ_{kp} are the Kronecker symbols δ_{kp}, and

$$\eta_a = \begin{cases} 1 & \text{if } a \text{ is true,} \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

Since $\Delta_k(hp) \equiv \delta_{kp}$, one finally gets a system of equations

$$\sum_{m=0}^{N}\left(\delta_{jm} + \sum_{n=p}^{N}\zeta_{nmp}F_{n,j}\right)P_{p,m} = -F_{p,j}, \quad (29)$$

for $P_k(h(p+0.25)) \equiv P_{p,k}$ $(p,k=0,\ldots,N)$ $(j,p=0,\ldots,N)$.

Solution of Eq. (29) gives $P_{p,k}$. Next, one should calculate potential values at points $r = hp$ $(p=0,\ldots,N)$ from Eq. (7) by some finite difference formula.

I tested the developed approach by restoring the potential function $V(r) = -3\exp(-3r/2)$ from the corresponding scattering data. Results are presented in Figs. 2 and 3, where $h = 0.04$, $R = 4$. The input S matrix was calculated at points shown in Fig. 2 up to $q = 8$. The S matrix was interpolated by a quadratic spline in the range $0 < q < 8$. The S matrix was approximated as asymptotic $S(q) \approx \exp(-2i\alpha/q)$ for $q > 8$, where α was calculated at $q = 8$.

III. ENERGY-INDEPENDENT COMPLEX PARTIAL POTENTIALS

Two-particle relativistic potential models can be represented in a nonrelativistic form [44]. Thus, the applicability of methods for solving the inverse problem is not limited only to nonrelativistic quantum mechanics. As we showed earlier [19], modern data of the NN partial-wave analysis up to $E_{\rm lab} = 3$ GeV can be described by the energy-independent complex partial potential for $^1S_0$ single wave at least.

After analyzing the Marchenko theory [7,8], including for the case of $l > 0$ [9], I found that Eqs. (2)–(7) apply to nonunitary S matrices describing absorption. In this case, the absorbing partial S matrix should be defined as

$$S(q) = \begin{cases} S_u(q) + S_n(q) & \text{for} \quad q > 0, \\ S_u^+(-q) - S_n^+(-q) & \text{for} \quad q < 0, \end{cases} \quad (30)$$

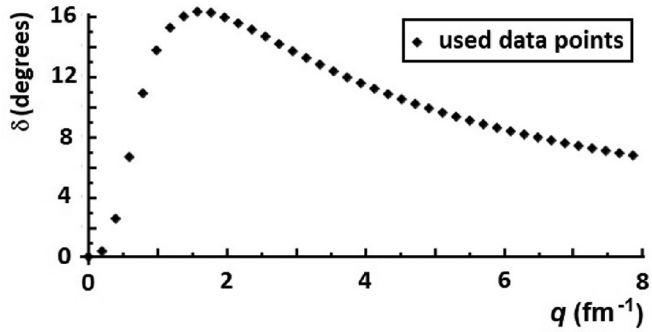

FIG. 2. Data used to reconstruct $V(r) = V_0\exp(-ar)$, where $V_0 = -3$ fm$^{-2} = -124.5$ MeV, $a = 1.5$ fm^{-1}. Angular orbital momentum $l = 1$. Units correspond to the NN system.

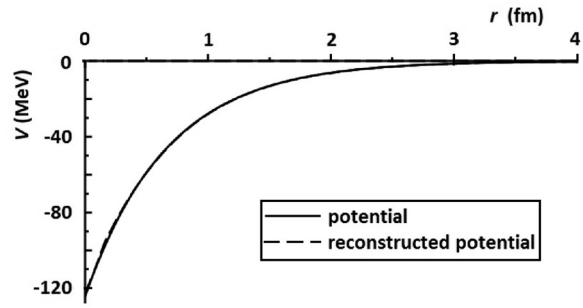

FIG. 3. Initial and reconstructed potentials: $V(r) = V_0\exp(-ar)$, where $V_0 = -3$ fm$^{-2} = -124.5$ MeV, $a = 1.5$ fm^{-1}. Units correspond to the NN system.

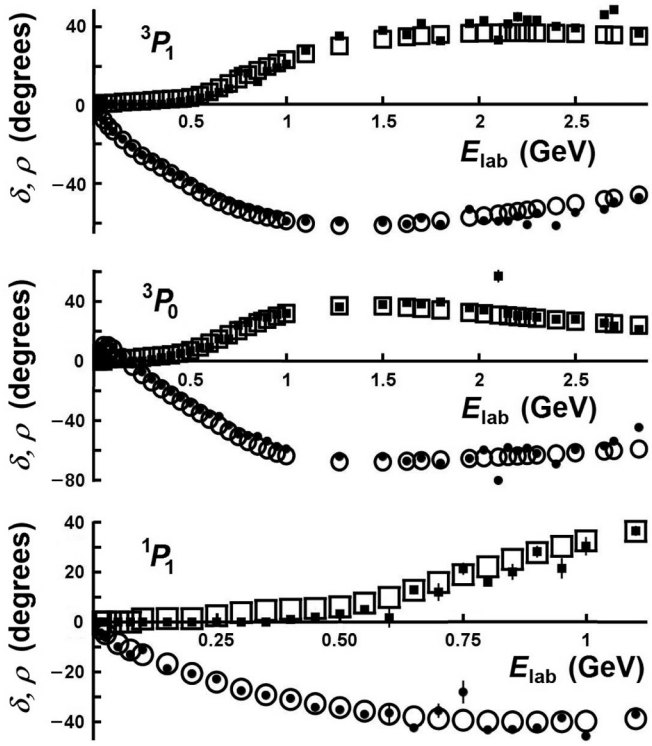

FIG. 4. Data used to reconstruct P-wave partial potentials are phase shifts δ (solid circles) and inelasticity parameters ρ (solid squares) (from Refs. [42,43]). Circles and squares stand for δ and ρ correspondingly calculated from the reconstructed partial potentials.

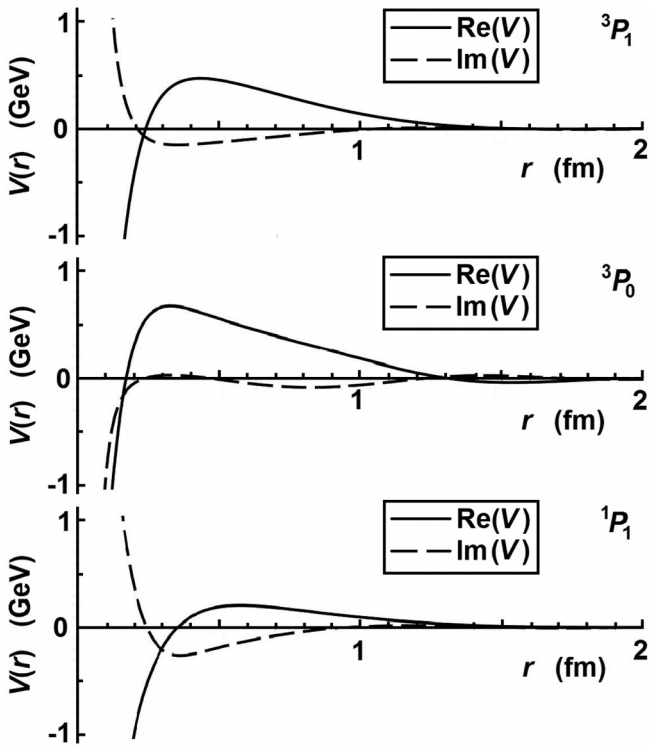

FIG. 5. Real and imaginary parts of the reconstructed partial potentials.

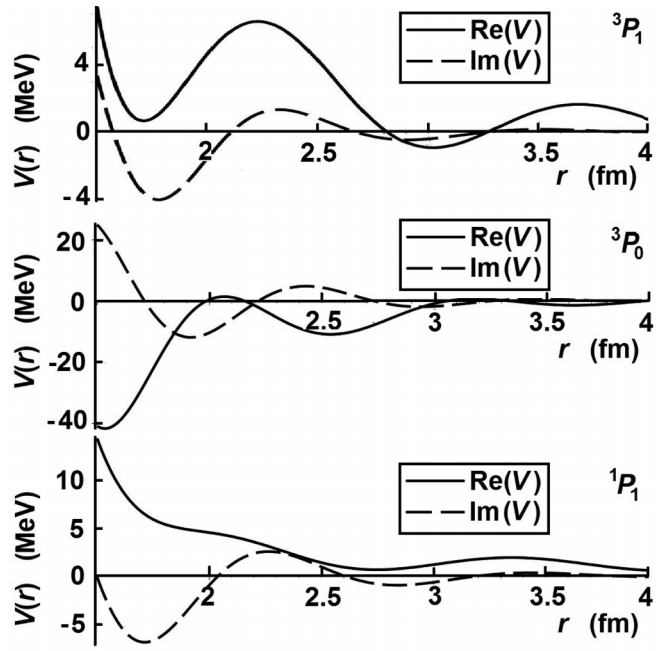

FIG. 6. Real and imaginary parts of the reconstructed partial potentials.

where superscript + means Hermitian conjugation. For $q > 0$, one should define

$$S_u(q) = e^{2\imath\delta(q)}, \quad S_n(q) = -\sin^2(\rho(q))e^{2\imath\delta(q)}, \tag{31}$$

where $\delta(q)$ and $\rho(q)$ are phase shift and inelasticity parameter correspondingly. With an S matrix defined in this way, Eqs. (24)–(29) remain valid and allow calculating local and energy-independent OP from data Eq. (6) with an absorptive S matrix.

IV. RESULTS AND CONCLUSIONS

Extending our previous results [19], I analyzed modern single P-wave NN phase shift data up to 3 GeV [42,43] (SW16, single-energy solutions, Fig. 4).

The phase shift and inelasticity parameter data were smoothed for $q > 3\,\mathrm{fm}^{-1}$ by the following functions, correspondingly:

$$\delta(q) \sim \sum_{k=3}^{8} A_k/q^k, \quad \rho(q) \sim \sum_{k=0}^{9} B_k/q^k, \tag{32}$$

TABLE I. Coefficients of Eq. (32) for the $^3P_0$ wave.

	j	3	4	5	6	7
$\delta_1 = \delta_2$ (rad) :	A_j (fm^j)	−566	9050	−88500	379000	−551000
δ_3 (rad) :	A_j (fm^j)	−1520	16000	−69600	143000	−113000
	j	0	1	2	3	4	5
ρ_1 (rad) :	B_j (fm^j)	0.1	17.7	−394	3180	−10300	11600
$\rho_2 = \rho_3$ (rad) :	B_j (fm^j)	0	12.2	−283	2450	−8210	9380

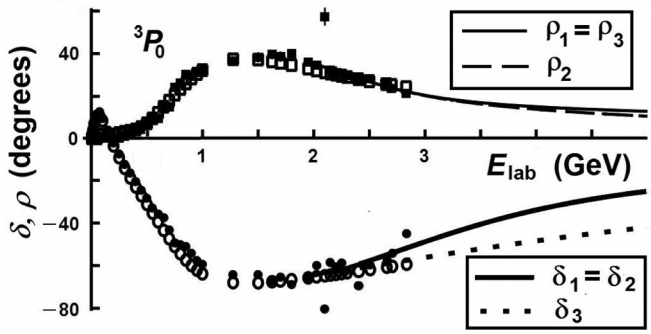

FIG. 7. Different asymptotics of phase shift and inelasticity parameter for the $^3P_0$ wave.

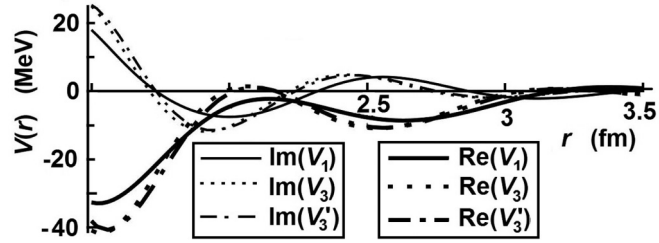

FIG. 9. $^3P_0$ partial potentials reconstructed from data and different asymptotics shown in Fig. 7 (a pair $\{\delta_i, \rho_i\}$ corresponds to V_i; V_3 was reconstructed with $h = 0.02$ fm, and V'_3 was reconstructed with $h = 0.04$ fm).

where the coefficients were fitted by the least-squares method. The SW16 data and asymptotics (32) above $q > 3$ fm^{-1} were used to calculate coefficients of Eqs. (24) with $h = 0.02$ fm corresponding to $q_{\max} \approx 157.08$ fm^{-1}.

The used data are described by energy-independent complex partial potentials (Figs. 5 and 6). Thus, this study presents an IP solution algorithm for finite range potentials (fixed-l, single partial wave inversion, in frames of the Marchenko theory).

I investigated the numerical stability of the developed method changing asymptotics of the $^3P_0$ S matrix. Table I displays three sets of parameters describing the data, but giving different S-matrix asymptotics (Fig. 7). The potential is found to be fairly stable under reasonable variations of the data asymptotics. The range $r < 0.5$ fm shows the most significant changes of the potential (Figs. 8 and 9). As demonstrated in Fig. 9, increasing the value of h from 0.02 fm to 0.04 fm has a negligible effect on the potential for $r > 0.3$ fm, where the finite-difference errors are small. For $r < 0.3$ fm, the optimal value of h is 0.02 fm.

The phase shifts of elastic NN scattering are slowly varying smooth functions of energy, suggesting that the quasimacroscopic geometric potential model is applicable [25]. I assume that developed method enables the reconstruction of the corresponding partial NN potentials. Both real and imaginary parts of the constructed potentials are oscillatory for $r > 1.5$ fm (Figs. 6). The oscillatory behavior is stable with respect to reasonable transformations of the unknown asymptotics of the S matrix (Figs. 7 and 9). The imaginary parts of the potentials change sign by oscillating. Potential with a negative imaginary part is absorptive, while a positive imaginary part of potential leads to an increase in the outgoing particle flux. The imaginary parts of the obtained potentials oscillate changing sign, however, the method of their construction ensures absorption above the inelastic threshold. For energies below the inelastic threshold, the S matrix is almost unitary, with slight deviations from exact unity that can be attributed to computational errors. Thus, one cannot interpret the regions of the positive signs of imaginary parts of potentials as regions of particle production. The origin and nonartificial nature of the oscillations in optical potentials are discussed in Ref. [33]. As noted earlier by Fernández-Soler and Ruiz Arriola [45], "short-wavelength fluctuations/oscillations are inherent to the maximum energy or CM momentum being fixed for the phase shift". In this conclusion, they refer to the results of our calculations [16]. Then, we did not pay attention to this feature of the IP solution, and we considered the oscillations an artifact of the used IP solution in which the potential is explicitly, algebraically expressible in terms of Hankel functions. Reconstructed $^1S_0$ NN partial potential [19] shows that short-wavelength oscillations may indeed be a manifestation of a physical phenomenon. There are also similar oscillations in the reconstructed P-wave partial potentials (Fig. 6). Microscopic calculations also predict such oscillations [46].

The present study demonstrates that IP solutions yield optical model energy-independent NN partial potentials not only in the $^1S_0$ NN wave, but also in other single NN partial waves. The optical model partial NN potentials with a repulsive core does not necessarily exhibit a strong energy dependence up to 3 GeV, as previously stated in [45]. Instead of the soft repulsive core that was reconstructed in our model for the real part of the $^1S_0$ NN partial potential, the barriers at about 0.2–1.3 fm were reconstructed for the real parts of the P-waves partial potentials.[1]

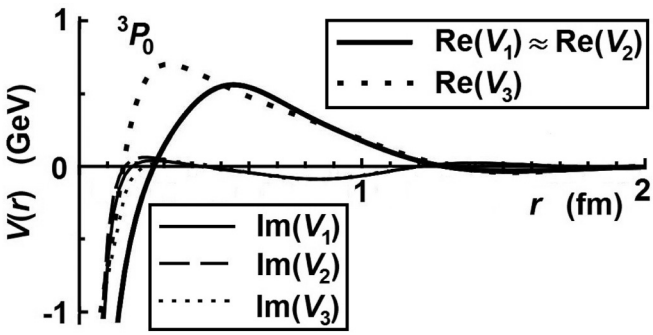

FIG. 8. $^3P_0$ partial potentials reconstructed from data and different asymptotics shown in Fig. 7 (a pair $\{\delta_j, \rho_j\}$ corresponds to V_j).

[1]The reconstructed P-wave NN complex potentials may be requested from the author in the FORTRAN code.

[1] J. M. Sparenberg and D. Baye, Phys. Rev. C **55**, 2175 (1997).
[2] D. Baye and J. M. Sparenberg, J. Phys. A: Math. Gen. **37**, 10223 (2004).
[3] A. Pupasov, B. F. Samsonov, J. M. Sparenberg, and D. Baye, Phys. Rev. Lett. **106**, 152301 (2011).
[4] V. I. Kukulin and R. S. Mackintosh, J. Phys. G: Nucl. Part. Phys. **30**, R1 (2004).
[5] R. S. Mackintosh, Scholarpedia **7**, 12032 (2012); http://www.scholarpedia.org/article/Inverse_scattering:_applications_in_nuclear_physics#honnef.
[6] B. M. Levitan, *Generalized Translation Operators and Some of Their Applications* (Israel Program for Scientific Translations, Jerusalem, 1964).
[7] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach, New York, 1963).
[8] V. A. Marchenko, *Sturm-Liouville Operators and Their Applications* (Naukova Dumka, Kiev, 1977).
[9] M. Blazek, Mate.-Fys. Casopis **13**, 147 (1963); M. Blažek, Commun. Math. Phys. **3**, 282 (1966).
[10] M. G. Krein, DAN SSSR **105**, 433 (1955).
[11] B. M. Levitan, *Sturm-Liouville Inverse Problems* (Nauka, Moscow, 1984).
[12] R. G. Newton, *Scattering Theory of Waves and Particles* (Springer, New York, 1982).
[13] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer, New York, 1989).
[14] H. V. von Geramb and H. Kohlhoff, in *Quantum Inversion Theory and Applications*, Lecture Notes in Physics, Vol. 427, edited by H. V. von Geramb (Springer Verlag, Berlin, 1994), pp. 285–313.
[15] H. Kohlhoff and H. V. von Geramb, in *Quantum Inversion Theory and Applications*, Lecture Notes in Physics, Vol. 427, edited by H. V. von Geramb (Springer Verlag, Berlin, 1994), pp. 314–341.
[16] N. A. Khokhlov and V. A. Knyr, Phys. Rev. C **73**, 024004 (2006).
[17] N. A. Khokhlov, V. A. Knyr, and V. G. Neudatchin, Phys. Rev. C **75**, 064001 (2007).
[18] N. A. Khokhlov, arXiv:2102.01464.
[19] N. A. Khokhlov and L. I. Studenikina, Phys. Rev. C **104**, 014001 (2021).
[20] M. Burrows, R. B. Baker, Ch. Elster, S. P. Weppner, K. D. Launey, P. Maris, and G. Popa, Phys. Rev. C **102**, 034606 (2020).
[21] M. Vorabbi, P. Finelli, and C. Giusti, Phys. Rev. C **98**, 064602 (2018).
[22] M. Vorabbi, P. Finelli, and C. Giusti, Phys. Rev. C **96**, 044001 (2017).
[23] H. Guo, H. Liang, Y. Xu, Y. Han, Q. Shen, C. Cai, and T. Ye, Phys. Rev. C **95**, 034614 (2017).
[24] M. Vorabbi, P. Finelli, and C. Giusti, Phys. Rev. C **93**, 034619 (2016).
[25] A. Funk, H. V. von Geramb, and K. A. Amos, Phys. Rev. C **64**, 054003 (2001).
[26] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[27] Z. Ahmed, J. Phys. A: Math. Gen. **39**, 9965 (2006).
[28] N. Bebiano, J. da Providência, S. Nishiyama, and J. P. da Providência, J. Math. Phys. **61**, 082106 (2020).
[29] W.-Y. Ai, C. M. Bender, and S. Sarkar, Phys. Rev. D **106**, 125016 (2022).
[30] J. R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions* (John Wiley & Sons, Inc., New York, 1972).
[31] M. L. Goldberger, *Collision Theory* (John Wiley & Sons, Inc., New York, 1967).
[32] A. Papastylianos, S. A. Sofianos, H. Fiedeldey, and E. O. Alt, Phys. Rev. C **42**, 142 (1990).
[33] E. O. Alt, L. L. Howell, M. Rauh, and S. A. Sofianos, Phys. Rev. C **49**, 176 (1994).
[34] K. Maleknejad and M. Tavassoli Kajani, Appl. Math. Comput. **145**, 623 (2003).
[35] K. Maleknejad and Y. Mahmoudi, Appl. Math. Comput. **149**, 799 (2004).
[36] B. Asady, M. Tavassoli Kajani, A. Hadi Vencheh, and A. Heydari, Appl. Math. Comput. **160**, 517 (2005).
[37] K. Maleknejad and F. Mirzaee, Appl. Math. Comput. **160**, 579 (2005).
[38] S. Yousefi and M. Razaghi, Math. Comput. Simul. **70**, 1 (2005).
[39] E. Babolian and A. Shahsavaran, J. Comput. Appl. Math. **225**, 87 (2009).
[40] C. Cattania and A. Kudreyko, Appl. Math. Comput. **215**, 4164 (2010).
[41] H. A. Antosiewicz, in *Bessel Functions of Fractional Order* in *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, edited by Milton Abramowitz and Irene A. Stegun (U.S. Government Printing Office, Washington, DC, 1972), Vol. 55, p. 435.
[42] R. A. Arndt, I. I. Strakovsky, and R. L. Workman, Phys. Rev. C **62**, 034005 (2000).
[43] R. A. Arndt, W. J. Briscoe, R. L. Workman, and I. I. Strakovsky, http://gwdac.phys.gwu.edu/.
[44] B. D. Keister and W. N. Polyzou, Adv. Nucl. Phys. **20**, 225 (1991).
[45] P. Fernandez-Soler and E. Ruiz Arriola, Phys. Rev. C **96**, 014004 (2017).
[46] K. A. Wendt, R. J. Furnstahl, and S. Ramanan, Phys. Rev. C **86**, 014003 (2012).