Cumulants from short-range correlations and baryon number conservation at next-to-leading order

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We calculate the baryon number cumulants within acceptance with short-range correlations and global baryon number conservation in terms of cumulants in the whole system without baryon conservation. We extract leading and next-to-leading order terms of the large baryon number limit approximation. These approximations are checked to be very close to the exact results.

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I. INTRODUCTION

The phase diagram of the strongly interacting matter is not yet well explored. In particular, the search for the first-order phase transition between the hadronic matter and quark-gluon plasma, and the corresponding critical endpoint, which are predicted by the effective models, remains a big challenge in high-energy physics [1–4]. It is known that the fluctuations of conserved charges, e.g., baryon number, electric charge, and strangeness are sensitive to the relevant critical phenomena. Therefore, many theoretical projects, as well as experiments in relativistic heavy-ion collisions, have been established to study them [1,5–24].

Cumulants are commonly used to quantify these fluctuations because they naturally appear in statistical mechanics [10,25–32]. On the other hand, the factorial cumulants might be easier to interpret since they represent integrated multiparticle correlation functions [4,33–41]. However, both the cumulants and factorial cumulants are affected also by fluctuations unrelated to the phase transition, for instance, the impact parameter fluctuations and the conservation laws, e.g., the baryon number conservation [26,30,39,41–50].

In our previous paper [51], we derived analytically the baryon number factorial cumulant generating function in a finite acceptance, assuming short-range correlations and global baryon number conservation. We followed the subensemble acceptance method [52], recently applied to the van der Waals model [53]. Among other results, we calculated the factorial cumulants and cumulants within the limit of small short-range correlation strengths, α_k, and large baryon number, B. We also reproduced the relations between cumulants in a subsystem with all correlations and cumulants in the whole system without baryon conservation, initially obtained in Ref. [52].

The charge conservation effects become significant already at the acceptance of about 10% [45]. Therefore, it is important to understand the modifications due to baryon number conservation very well. However, the previous result [51,52] represents the approximation obtained in the thermodynamic limit; in particular, it means a large baryon number B for low-energy collisions. In heavy-ion collisions, B can be quite small, e.g., for small systems, such as Ar+Sc or Be+Be collisions at CERN SPS by the NA61/SHINE Collaboration [54], or even in peripheral collisions at the LHC or RHIC. Consequently, it is desired to obtain a more precise analytic result. In this paper, we extend this study and propose a method of obtaining the first correction to the cumulants in the large baryon number limit. We compare our approximate analytic results with the brute-force computations to verify the importance of the extracted correction. We find that this correction improves the results, especially for small baryon numbers. In our method, it is also possible to calculate even higher order corrections. Finally, we note that the strengths of the short-range correlations cannot assume arbitrary values.

In the next section, we show our method of extracting the baryon number cumulants assuming short-range correlations and global baryon number conservation. Then, we present the leading-order and next-to-leading order terms of cumulants in the subsystem with all correlations expanded in the large-B limit with respect to cumulants in the whole system without baryon conservation. This is our main result. In the fourth section, we show how our approximate analytic formulas work by comparison with the exact results. The alternative approach and the discussion on the limitations of α_k's originating from the probability theory can be found in the appendices.

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II. METHOD

A. Previous study

In our previous paper [51], we considered a system of fixed volume and some number of particles of one kind, say, baryons. We divided it into two subsystems (inside and

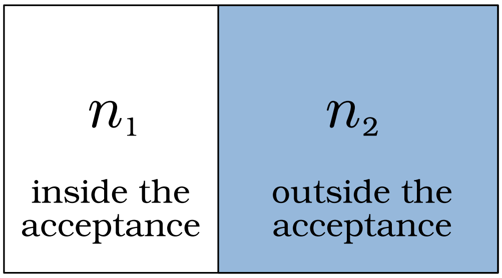

FIG. 1. The system is divided into two subsystems with n_1 particles in the first subsystem (inside the acceptance) and n_2 particles in the second one (outside the acceptance).

outside the acceptance, see Fig. 1) which can exchange particles. Let $P_1(n_1)$ and $P_2(n_2)$ be the probabilities that there are n_1 particles in the first subsystem and n_2 particles in the second one, respectively. The probability that there are n_1 particles in the first subsystem and n_2 particles in the second one is $P(n_1, n_2) = P_1(n_1)P_2(n_2)$ if there are no correlations between the two subsystems or (approximately) if there are only short-range correlations. Assuming the global baryon number conservation, this probability becomes

$$P_B(n_1, n_2) = A\ P_1(n_1)P_2(n_2)\delta_{n_1+n_2,B}, \tag{1}$$

where A is the normalization constant and B is the total baryon number. In this case, the probability that there are n_1 particles in the first subsystem (within acceptance) reads

$$P_B(n_1) = \sum_{n_2=0}^{\infty} P_B(n_1, n_2). \tag{2}$$

Then, we calculated the factorial cumulant generating function for the first subsystem with baryon number conservation:

$$G_{(1,B)}(z) = \ln\Bigg[\frac{A}{B!} \times \frac{d^B}{dx^B}\exp\left(\sum_{k=1}^{\infty}\frac{(xz-1)^k\hat{C}_k^{(1)}+(x-1)^k\hat{C}_k^{(2)}}{k!}\right)\Bigg|_{x=0}\Bigg], \tag{3}$$

where

$$\hat{C}_k^{(1)} = \langle n_1\rangle\alpha_k = f\langle N\rangle\alpha_k,$$
$$\hat{C}_k^{(2)} = \langle n_2\rangle\alpha_k = (1-f)\langle N\rangle\alpha_k \tag{4}$$

are the short-range factorial cumulants in the first and the second subsystems, respectively (see Refs. [4,51]), for the multiplicity distribution without global baryon conservation. Here $\langle N\rangle = \langle n_1\rangle + \langle n_2\rangle$ is the mean total number of particles in the system, $f = \langle n_1\rangle/\langle N\rangle$ is a fraction of particles in the first subsystem, and α_k describes the strength of k-particle short-range correlations ($\alpha_1 = 1$). We assumed that the total *average* number of particles $\langle N\rangle = \langle n_1\rangle + \langle n_2\rangle = B$. Introducing the global baryon number conservation further requires that the total number of particles $N = n_1 + n_2$ equals B in every event.

Using the factorial cumulant generating function (3), one can obtain the factorial cumulants in the first subsystem (within acceptance) with baryon number conservation:

$$\hat{C}_k^{(1,B)} = \frac{d^k}{dz^k}G_{(1,B)}(z)\bigg|_{z=1}. \tag{5}$$

We obtained [51] the analytic as well as approximate formulas for the factorial cumulants in the simple case of only two-particle short-range correlations. Then, we also derived the approximate factorial cumulants and cumulants in the limit of $B \to \infty$ assuming multiparticle short-range correlations. These results reproduced the findings of Ref. [52] obtained originally using a different approach.

B. Next-to-leading order correction

We extend the previous study and obtain the next-to-leading order terms of the expansion of the cumulants in the limit of $B \to \infty$. We derive the relevant expressions in two different ways. One method, presented in Appendix A, is based on analyzing the first few terms of the expansions, deducing the following terms, and then summing the infinite series. Here we present another method that is simpler.

In order to make the computations, we approximate Eq. (3) with Eq. (4) by

$$G_{(1,B)}(z) \approx \ln\Bigg[\frac{A}{B!}\frac{d^B}{dx^B}\Bigg[\exp\big((xz-1)fB + (x-1)\bar{f}B\big) \times \sum_{m=0}^{M}\frac{V^m}{m!}\Bigg]\Bigg|_{x=0}\Bigg], \tag{6}$$

where $\bar{f} = 1 - f$ and

$$V = \sum_{k=2}^{K}\left(\frac{(xz-1)^k}{k!}fB\alpha_k + \frac{(x-1)^k}{k!}\bar{f}B\alpha_k\right), \tag{7}$$

with M and K being the upper limits. Note that in Eq. (7) we allow for up to K-particle short-range correlations.

To calculate the Bth derivative with respect to x in Eq. (6), we use the general Leibnitz formula as described in Ref. [51]. In this way, we evaluate the factorial cumulant generating function (6) and then the factorial cumulants (5). Having the factorial cumulants, we calculate the cumulants according to

$$\kappa_n = \sum_{k=1}^{n} S(n,k)\hat{C}_k, \tag{8}$$

where $S(n, k)$ is the Stirling number of the second kind [55].[1] For instance, the second cumulant reads

$$\kappa_2^{(1,B)} = \langle n_1\rangle + \hat{C}_2^{(1,B)}, \tag{9}$$

where $\hat{C}_1^{(1,B)} = \kappa_1^{(1,B)} = \langle n_1\rangle = fB$.

The global (both subsystems combined) short-range factorial cumulants, without the baryon number conservation, are

[1]See also Appendix A of Ref. [4] for explicit formulas for the first six cumulants.

defined as [compare with Eqs. (4)]

$$\hat{C}_n^{(G)} = B\alpha_n. \tag{10}$$

The global cumulants without the baryon number conservation, $\kappa_n^{(G)}$, are obtained by Eq. (8).

As shown in Ref. [51], the cumulants, $\kappa_n^{(1,B)}$, can be expressed as a power series in terms of B where the highest order term is linear in B. Namely,

$$\kappa_n^{(1,B)} \approx \underbrace{\kappa_n^{(1,B,\mathrm{LO})}}_{u_{n,1}B^1} + \underbrace{\kappa_n^{(1,B,\mathrm{NLO})}}_{u_{n,0}B^0} + \underbrace{\kappa_n^{(1,B,\mathrm{NNLO})}}_{u_{n,-1}B^{-1}} + \underbrace{\cdots}_{O(B^{-2})}, \tag{11}$$

where $\kappa_n^{(1,B,\mathrm{LO})}$, $\kappa_n^{(1,B,\mathrm{NLO})}$, and $\kappa_n^{(1,B,\mathrm{NNLO})}$ denote the leading-order, next-to-leading-order, and next-to-next-to-leading-order terms of the power series in B, respectively.

Let us focus on the second cumulant. Using the method presented in Appendix A, we deduced that in order to extract LO and NLO terms, it is convenient to multiply $\kappa_2^{(1,B)}$ by $(\kappa_2^{(G)})^2 = [B(1+\alpha_2)]^2$. We define

$$\widetilde{\kappa}_2^{(1,B)} = \kappa_2^{(1,B)}\left(\kappa_2^{(G)}\right)^2 = \kappa_2^{(1,B)}[B(1+\alpha_2)]^2. \tag{12}$$

Then, we expand $\widetilde{\kappa}_2^{(1,B)}$ into the power series in α_k up to the order of M, obtaining $\widetilde{\kappa}_2^{(1,B,\mathrm{ser})}$:

$$\widetilde{\kappa}_2^{(1,B)} \approx \widetilde{\kappa}_2^{(1,B,\mathrm{ser})} = u_{2,1}(1+\alpha_2)^2B^3 + u_{2,0}(1+\alpha_2)^2B^2 + \underbrace{\cdots}_{O(B)}. \tag{13}$$

The coefficients of the expansion are calculated as follows:

$$u_{2,1} = \frac{1}{(1+\alpha_2)^2}\lim_{B\to\infty}\frac{\widetilde{\kappa}_2^{(1,B,\mathrm{ser})}}{B^3},$$
$$u_{2,0} = \frac{1}{(1+\alpha_2)^2}\lim_{B\to\infty}\frac{\widetilde{\kappa}_2^{(1,B,\mathrm{ser})} - u_{2,1}(1+\alpha_2)^2B^3}{B^2}. \tag{14}$$

Clearly, it is possible to extract even higher terms in an analogous way. Using Eqs. (8) and (10), we express these coefficients in terms of the global short-range cumulants (without the baryon conservation), $\kappa_n^{(G)}$.

The same technique is applied to obtain the leading- and next-to-leading-order terms of $\kappa_3^{(1,B)}$. Namely, we multiply $\kappa_3^{(1,B)}$ by $(\kappa_2^{(G)})^2$.

It turns out that $\kappa_4^{(1,B)}$ needs to be multiplied by $(\kappa_2^{(G)})^4$. Namely,

$$\widetilde{\kappa}_4^{(1,B)} = \kappa_4^{(1,B)}\left(\kappa_2^{(G)}\right)^4 = \kappa_4^{(1,B)}[B(1+\alpha_2)]^4. \tag{15}$$

In this case, equations corresponding to Eqs. (13) and (14) read

$$\widetilde{\kappa}_4^{(1,B)} \approx \widetilde{\kappa}_4^{(1,B,\mathrm{ser})} = u_{4,1}(1+\alpha_2)^4B^5 + u_{4,0}(1+\alpha_2)^4B^4 + \underbrace{\cdots}_{O(B^3)}, \tag{16}$$

and

$$u_{4,1} = \frac{1}{(1+\alpha_2)^4}\lim_{B\to\infty}\frac{\widetilde{\kappa}_4^{(1,B,\mathrm{ser})}}{B^5},$$
$$u_{4,0} = \frac{1}{(1+\alpha_2)^4}\lim_{B\to\infty}\frac{\widetilde{\kappa}_4^{(1,B,\mathrm{ser})} - u_{4,1}(1+\alpha_2)^4B^5}{B^4}. \tag{17}$$

The results are computed using MATHEMATICA software [56].

III. RESULTS

As discussed above, the cumulants in the subsystem with the baryon number conservation and short-range correlations, $\kappa_n^{(1,B)}$, can be approximated by

$$\kappa_n^{(1,B)} \approx \kappa_n^{(1,B,\mathrm{LO})} + \kappa_n^{(1,B,\mathrm{NLO})} + \cdots, \tag{18}$$

where the leading-order (LO) term is linear in the baryon number, B, the next-to-leading-order (NLO) term is independent of B, and the next terms depend at most on B^{-1}. $\kappa_n^{(1,B,\mathrm{LO})}$ and $\kappa_n^{(1,B,\mathrm{NLO})}$ expressed in terms of the cumulants in the whole system without baryon conservation, $\kappa_m^{(G)}$, are given by

$$\kappa_1^{(1,B)} = fB = f\kappa_1^{(G)}, \tag{19}$$

$$\kappa_2^{(1,B,\mathrm{LO})} = \bar{f}f\kappa_2^{(G)}, \tag{20}$$

$$\kappa_2^{(1,B,\mathrm{NLO})} = \frac{1}{2}\bar{f}f\frac{\left(\kappa_3^{(G)}\right)^2 - \kappa_2^{(G)}\kappa_4^{(G)}}{\left(\kappa_2^{(G)}\right)^2}, \tag{21}$$

$$\kappa_3^{(1,B,\mathrm{LO})} = \bar{f}f(1-2f)\kappa_3^{(G)}, \tag{22}$$

$$\kappa_3^{(1,B,\mathrm{NLO})} = \frac{1}{2}f\bar{f}(1-2f)\frac{\kappa_3^{(G)}\kappa_4^{(G)} - \kappa_2^{(G)}\kappa_5^{(G)}}{\left(\kappa_2^{(G)}\right)^2}, \tag{23}$$

$$\kappa_4^{(1,B,\mathrm{LO})} = f\bar{f}\left[\kappa_4^{(G)} - 3f\bar{f}\left(\kappa_4^{(G)} + \frac{\left(\kappa_3^{(G)}\right)^2}{\kappa_2^{(G)}}\right)\right], \tag{24}$$

$$\kappa_4^{(1,B,\mathrm{NLO})} = \frac{1}{2}f\bar{f}\left\{\frac{\kappa_3^{(G)}\kappa_5^{(G)} - \kappa_2^{(G)}\kappa_6^{(G)}}{\left(\kappa_2^{(G)}\right)^2} + 3f\bar{f}\left[\frac{2\left(\kappa_3^{(G)}\right)^4 - 5\kappa_2^{(G)}\left(\kappa_3^{(G)}\right)^2\kappa_4^{(G)} + \left(\kappa_2^{(G)}\right)^2\kappa_3^{(G)}\kappa_5^{(G)}}{\left(\kappa_2^{(G)}\right)^4} + \frac{\left(\kappa_4^{(G)}\right)^2 + \kappa_2^{(G)}\kappa_6^{(G)}}{\left(\kappa_2^{(G)}\right)^2}\right]\right\}. \tag{25}$$

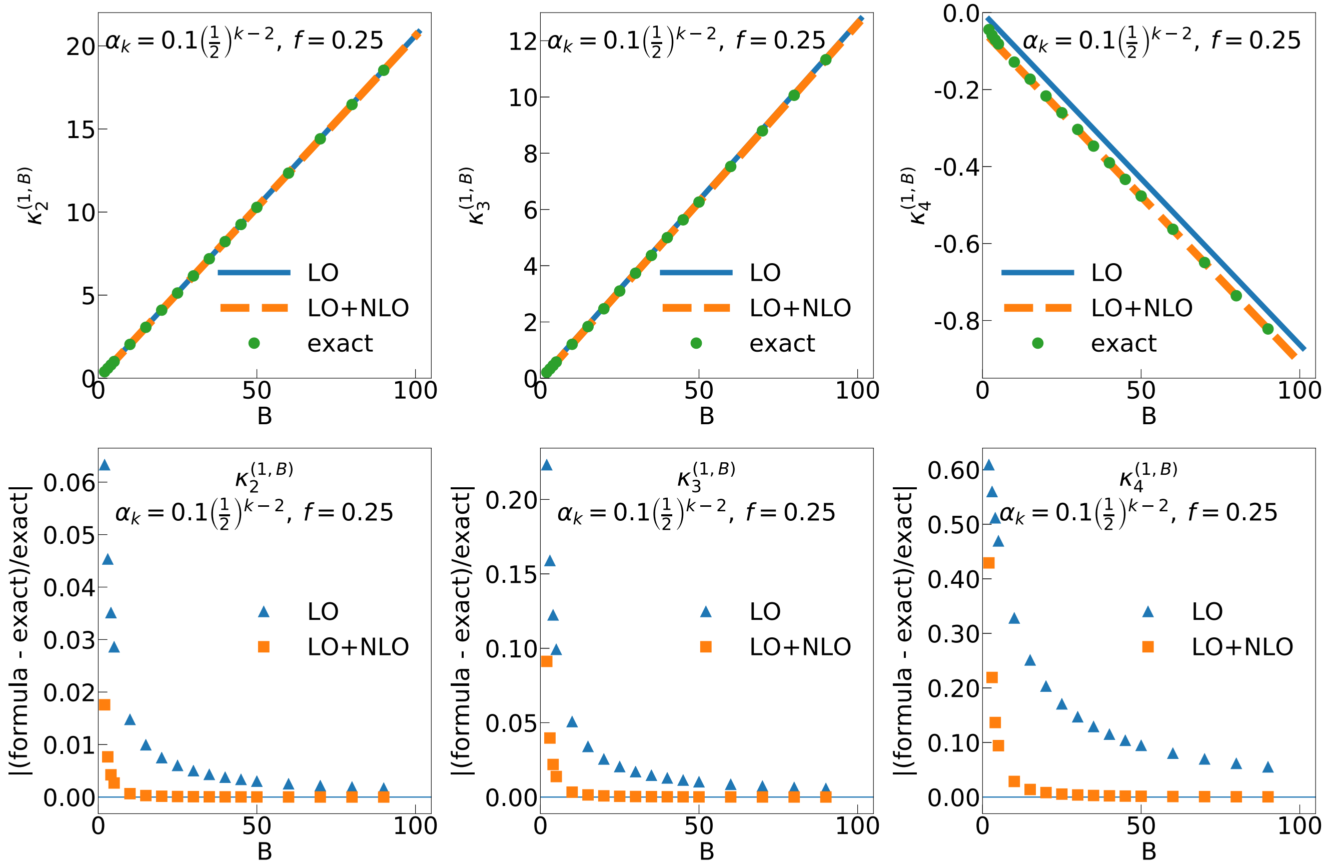

FIG. 2. Upper plots: The second (left), third (middle), and fourth (right) cumulant in the first subsystem (within acceptance) with the short-range correlations and baryon number conservation as a function of B. $\alpha_k = 0.1(\frac{1}{2})^{k-2}$, $k = 2, 3, ..., 6$, $\alpha_1 = 1$, $f = 0.25$. "LO" denotes the results obtained from the leading-order terms [Eqs. (20), (22), and (24)]. "LO+NLO" denotes the results obtained using the leading-order and next-to-leading-order terms [Eqs. (20), (22), (24) plus Eqs. (21), (23), and (25)]. The "exact" points denote the direct calculation from Eqs. (3)–(5). The exact results are presented for $B = 2, 3, 4, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70, 80, 90$. Lower plots: The relative error for each B. The "formula" can be a result of LO or LO+NLO. As seen, the use of the next-to-leading-order term improves the results significantly.

The leading-order terms, $\kappa_n^{(1,B,\mathrm{LO})}$, were already obtained in Ref. [51] and originally in Ref. [52]. The final results for $\kappa_2^{(1,B,NLO)}$ and $\kappa_3^{(1,B,NLO)}$ are obtained already for $M = 3$ and $K = 5$ [see Eqs. (6) and (7)]. We have checked (up to $M = 7$ and $K = 7$) that when increasing M or K the results remain unchanged.[2] The final results for $\kappa_4^{(1,B,NLO)}$ are obtained for $M = 5$ and $K = 6$. We have verified them up to $M = 6$ and $K = 7$ as well as up to $M = 7$ and $K = 6$. When further increasing M or K the computations become challenging.

We note that in the absence of the short-range correlations (global baryon number conservation being the only source of correlations), the global cumulants follow the Poisson distribution, $\kappa_m^{(G)} = \langle N \rangle = B$. In this case, all the presented NLO terms vanish and LO terms become the binomial distribution cumulants. This is in agreement with the results of Ref. [57] (without antibaryons).

IV. EXAMPLES

We calculate the cumulants for the selected values of the k-particle short-range correlation strengths, α_k's, the fraction of particle number in the first subsystem, f, and the baryon number, B. We do this in three different ways. First, we calculate the cumulants by a straightforward differentiation using Eqs. (3)–(5) (exact results).[3] Second, we calculate them using only the leading-order terms [Eqs. (20), (22), and (24)]. Finally, we calculate them by applying both the leading- and next-to-leading-order terms.

The k-particle short-range correlations are typically small and the higher order ones are expected to be smaller than the lower order ones. Thus, as an example, we study the case of $\alpha_k = 0.1(\frac{1}{2})^{k-2}$, $k = 2, 3, ..., 6$, $\alpha_1 = 1$, $f = 0.25$.[4] In Fig. 2,

[2]Note that increasing M gives the next terms of α_k power series expansion whereas increasing K allows for higher multiparticle correlations, e.g., $K = 6$ allows for up to six-particle short-range correlations.

[3]In this method, we calculate up to $B = 90$. Calculation of higher derivatives in Eq. (3) becomes challenging.

[4]As seen from Eqs. (19)–(25), the six-particle correlation is the highest order appearing in the LO and NLO terms of the first four cumulants.

we show the cumulants as a function of the baryon number, B, calculated in three ways and we also present the relative errors with respect to the exact ones. We see that including the NLO term gives a significantly better approximation of the exact results. We have also studied other values of parameters and the observations are consistent. We note that for $f = 0.5$, $\kappa_3^{(1,B)}$ vanishes for all values of B [58,59].

It is worth mentioning that α_k cannot be arbitrary. We discuss this issue in Appendix B.

V. COMMENTS AND SUMMARY

In this paper, we have extended the results of Ref. [51]. We have presented the method of obtaining successive terms of the large-B limit expansion of the baryon number cumulants in the subsystem with the short-range correlations and global baryon number conservation. We have expressed them by the cumulants in the whole system without baryon number conservation. The newly obtained next-to-leading-order terms have improved the approximation as seen in Sec. IV. These terms might be important for small colliding nuclei such as Ar+Sc or Be+Be collisions at CERN SPS by the NA61/SHINE Collaboration or even peripheral collisions of large nuclei. Moreover, our technique of calculating the cumulants can be extended to even smaller B by computing even higher order corrections. We believe that this progress will be helpful in the efforts to explore the QCD phase diagram with baryon number fluctuations measurements.

In this calculation, we have neglected antibaryons, which makes our results applicable to lower collision energies. We have presented baryon number cumulants, whereas typically in the experiments the proton (net-proton) number cumulants are measured because it is more challenging to detect neutrons.

An interesting but rather challenging extension of this method would be to take into account also antibaryons, as well as other long-range correlations.

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APPENDIX A: THE SERIES EXPANSION APPROACH

Here we present another way of obtaining the leading- and next-to-leading-order terms of the B power series expansion of the cumulants.

We obtain the factorial cumulants, $\hat{C}_n^{(1,B)}$, in the same way as in Sec. II B. Then, we expand them into the α_k power series up to the order of M. As shown in Ref. [51], $\hat{C}_n^{(1,B)}$ can be written as a power series in terms of B,

$$\hat{C}_n^{(1,B)} \approx \hat{C}_n^{(1,B,\mathrm{ser})} = \underbrace{\hat{C}_n^{(1,B,\mathrm{LO})}}_{v_{n,1}B^1} + \underbrace{\hat{C}_n^{(1,B,\mathrm{NLO})}}_{v_{n,0}B^0} + \underbrace{\hat{C}_n^{(1,B,\mathrm{NNLO})}}_{v_{n,-1}B^{-1}} + \underbrace{\cdots}_{O(B^{-2})}, \tag{A1}$$

where $\hat{C}_n^{(1,B,\mathrm{LO})}$, $\hat{C}_n^{(1,B,\mathrm{NLO})}$, and $\hat{C}_n^{(1,B,\mathrm{NNLO})}$ denote the leading-, next-to-leading-, and next-to-next-to-leading-order terms. The coefficients, $v_{n,1}$ and $v_{n,0}$, are calculated as follows:

$$v_{n,1} = \lim_{B\to\infty} \frac{\hat{C}_n^{(1,B,\mathrm{ser})}}{B}, \qquad v_{n,0} = \lim_{B\to\infty} \left(\hat{C}_n^{(1,B,\mathrm{ser})} - v_{n,1}B\right). \tag{A2}$$

The subsequent coefficients can be obtained in a similar way. Note that the leading-order terms, $\hat{C}_n^{(1,B,\mathrm{LO})}$, were presented in Ref. [51].

We calculate the LO and NLO terms of cumulants from factorial cumulants using Eq. (8). For instance,

$$\begin{aligned}\kappa_3^{(1,B)} &= \langle n_1\rangle + 3\hat{C}_2^{(1,B)} + \hat{C}_3^{(1,B)}\\ &\approx fB + 3\left(\hat{C}_2^{(1,B,\mathrm{LO})} + \hat{C}_2^{(1,B,\mathrm{NLO})}\right) + \left(\hat{C}_3^{(1,B,\mathrm{LO})} + \hat{C}_3^{(1,B,\mathrm{NLO})}\right)\\ &= \underbrace{\left(fB + 3\hat{C}_2^{(1,B,\mathrm{LO})} + \hat{C}_3^{(1,B,\mathrm{LO})}\right)}_{\kappa_3^{(1,B,\mathrm{LO})}} + \underbrace{\left(3\hat{C}_2^{(1,B,\mathrm{NLO})} + \hat{C}_3^{(1,B,\mathrm{NLO})}\right)}_{\kappa_3^{(1,B,\mathrm{NLO})}},\end{aligned} \tag{A3}$$

where $\hat{C}_1^{(1,B)} = \kappa_1^{(1,B)} = \langle n_1\rangle = fB$.

Next, we express $\kappa_n^{(1,B,\mathrm{LO})}$ and $\kappa_n^{(1,B,\mathrm{NLO})}$ by the global (in the whole system) factorial cumulants, $\hat{C}_n^{(G)}$, with short-range correlations but without baryon number conservation, defined in Eq. (10), and then by the global cumulants without the baryon number conservation, $\kappa_n^{(G)}$.

For the Poisson distribution, all the cumulants $\kappa_n^{(G)} = \langle N\rangle = B$. Therefore,

$$\bar{\kappa}_n^{(G)} = \frac{\kappa_n^{(G)}}{B} - 1 \tag{A4}$$

describes the deviation from the Poisson distribution. We expect it to be small (for example, $\bar{\kappa}_2^{(G)} = \alpha_2$) and thus we express $\kappa_n^{(1,B,\mathrm{LO})}$ and $\kappa_n^{(1,B,\mathrm{NLO})}$ in terms of $\bar{\kappa}_m^{(G)}$.

We note that when using Eqs. (6) and (7), the final results for $\kappa_1^{(1,B)}$, $\kappa_2^{(1,B,\mathrm{LO})}$, and $\kappa_3^{(1,B,\mathrm{LO})}$ are obtained already for $M=1$ and $K=3$, and an increase of M and K does not modify the results. In other cases, we obtain more terms when increasing M. However, we can deduce empirically the series in $\bar{\kappa}_2^{(G)}$ using the first few terms and confirm them by increasing M and K in Eqs. (6) and (7). We have performed the computations up to $M=8$, $K=6$, and also up to $M=4$, $K=8$. By summing the series up to infinity, we derive the finite analytic formulas for LO and NLO terms of the cumulants, and we obtain the same results as given by Eqs. (19)–(25).

1. $\kappa_2^{(1,B)}$ and $\kappa_3^{(1,B)}$

The leading-order terms of $\kappa_2^{(1,B)}$ and $\kappa_3^{(1,B)}$ read [51,52]

$$\kappa_2^{(1,B,\mathrm{LO})} = \bar{f} f \kappa_2^{(G)}, \quad \kappa_3^{(1,B,\mathrm{LO})} = \bar{f} f (1-2f)\kappa_3^{(G)}. \tag{A5}$$

The NLO term of $\kappa_2^{(1,B)}$ is given by the power series in $\bar{\kappa}_2^{(G)}$. This is understandable since in the process of computation we expand the factorial cumulants into power series about $\alpha_k = 0$ up to order M (see Sec. II B). The approximated NLO term of the second cumulant (with $M=8$) is given by

$$\begin{aligned}
\kappa_2^{(1,B,\mathrm{NLO})} \approx \tfrac{1}{2} f\bar{f}\big[&-\bar{\kappa}_2^{(G)} + 2\big(\bar{\kappa}_2^{(G)}\big)^2 - 3\big(\bar{\kappa}_2^{(G)}\big)^3 + 4\big(\bar{\kappa}_2^{(G)}\big)^4 - 5\big(\bar{\kappa}_2^{(G)}\big)^5 + 6\big(\bar{\kappa}_2^{(G)}\big)^6 - 7\big(\bar{\kappa}_2^{(G)}\big)^7 + 8\big(\bar{\kappa}_2^{(G)}\big)^8 \\
&+ 2\bar{\kappa}_3^{(G)}\big(1 - 2\bar{\kappa}_2^{(G)} + 3\big(\bar{\kappa}_2^{(G)}\big)^2 - 4\big(\bar{\kappa}_2^{(G)}\big)^3 + 5\big(\bar{\kappa}_2^{(G)}\big)^4 - 6\big(\bar{\kappa}_2^{(G)}\big)^5 + 7\big(\bar{\kappa}_2^{(G)}\big)^6 - 8\big(\bar{\kappa}_2^{(G)}\big)^7\big) \\
&+ (\bar{\kappa}_3^{(G)})^2\big(1 - 2\bar{\kappa}_2^{(G)} + 3\big(\bar{\kappa}_2^{(G)}\big)^2 - 4\big(\bar{\kappa}_2^{(G)}\big)^3 + 5\big(\bar{\kappa}_2^{(G)}\big)^4 - 6\big(\bar{\kappa}_2^{(G)}\big)^5 + 7\big(\bar{\kappa}_2^{(G)}\big)^6\big) \\
&+ \bar{\kappa}_4^{(G)}\big(-1 + \bar{\kappa}_2^{(G)} - \big(\bar{\kappa}_2^{(G)}\big)^2 + \big(\bar{\kappa}_2^{(G)}\big)^3 - \big(\bar{\kappa}_2^{(G)}\big)^4 + \big(\bar{\kappa}_2^{(G)}\big)^5 - \big(\bar{\kappa}_2^{(G)}\big)^6 + \big(\bar{\kappa}_2^{(G)}\big)^7\big)\big].
\end{aligned} \tag{A6}$$

Here we recognize the series

$$\kappa_2^{(1,B,\mathrm{NLO})} \approx \frac{1}{2} f\bar{f}\left[\sum_{n=1}^{N=8} n\big(-\bar{\kappa}_2^{(G)}\big)^n + 2\bar{\kappa}_3^{(G)}\sum_{n=1}^{N=8} n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1} + (\bar{\kappa}_3^{(G)})^2\sum_{n=1}^{N=7} n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1} - \bar{\kappa}_4^{(G)}\sum_{n=1}^{N=8}\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}\right]. \tag{A7}$$

Then, we assume that the next terms follow this pattern and we sum up with $N \to \infty$. We obtain

$$\begin{aligned}
\kappa_2^{(1,B,\mathrm{NLO})} &= \frac{1}{2}\bar{f} f\left[-\frac{\bar{\kappa}_2^{(G)}}{\big(1+\bar{\kappa}_2^{(G)}\big)^2} + \frac{2\bar{\kappa}_3^{(G)}}{\big(1+\bar{\kappa}_2^{(G)}\big)^2} + \frac{\big(\bar{\kappa}_3^{(G)}\big)^2}{\big(1+\bar{\kappa}_2^{(G)}\big)^2} - \frac{\bar{\kappa}_4^{(G)}}{1+\bar{\kappa}_2^{(G)}}\right] \\
&= -\frac{1}{2}\bar{f} f\frac{\bar{\kappa}_2^{(G)} - 2\bar{\kappa}_3^{(G)} + \bar{\kappa}_4^{(G)}\big(1+\bar{\kappa}_2^{(G)}\big) - (\bar{\kappa}_3^{(G)})^2}{\big(1+\bar{\kappa}_2^{(G)}\big)^2}.
\end{aligned} \tag{A8}$$

We substitute back Eq. (A4) and we obtain Eq. (21).

The case of the NLO term of $\kappa_3^{(1,B)}$ is similar. The approximated result (with $M=8$) reads

$$\begin{aligned}
\kappa_3^{(1,B,\mathrm{NLO})} \approx \frac{1}{2} f\bar{f}(1-2f)\Bigg[&\sum_{n=1}^{N=8} n\big(-\bar{\kappa}_2^{(G)}\big)^n + \bar{\kappa}_3^{(G)}\sum_{n=1}^{N=8} n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1} + \bar{\kappa}_4^{(G)}\sum_{n=1}^{N=8} n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1} \\
&+ \bar{\kappa}_3^{(G)}\bar{\kappa}_4^{(G)}\sum_{n=1}^{N=7} n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1} - \bar{\kappa}_5^{(G)}\sum_{n=1}^{N=8}\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}\Bigg].
\end{aligned} \tag{A9}$$

By letting $N \to \infty$, we have

$$\begin{aligned}
\kappa_3^{(1,B,\mathrm{NLO})} &= \frac{1}{2} f\bar{f}(1-2f)\left[-\frac{\bar{\kappa}_2^{(G)}}{\big(1+\bar{\kappa}_2^{(G)}\big)^2} + \frac{\bar{\kappa}_3^{(G)}}{\big(1+\bar{\kappa}_2^{(G)}\big)^2} + \frac{\bar{\kappa}_4^{(G)}}{\big(1+\bar{\kappa}_2^{(G)}\big)^2} + \frac{\bar{\kappa}_3^{(G)}\bar{\kappa}_4^{(G)}}{\big(1+\bar{\kappa}_2^{(G)}\big)^2} - \frac{\bar{\kappa}_5^{(G)}}{1+\bar{\kappa}_2^{(G)}}\right] \\
&= -\frac{1}{2} f\bar{f}(1-2f)\frac{\bar{\kappa}_2^{(G)} - \bar{\kappa}_3^{(G)} - \bar{\kappa}_4^{(G)} + \bar{\kappa}_5^{(G)}\big(1+\bar{\kappa}_2^{(G)}\big) - \bar{\kappa}_3^{(G)}\bar{\kappa}_4^{(G)}}{\big(1+\bar{\kappa}_2^{(G)}\big)^2}.
\end{aligned} \tag{A10}$$

Using Eq. (A4), we obtain Eq. (23).

2. $\kappa_4^{(1,B)}$

a. Leading-order term

In the case of $\kappa_4^{(1,B)}$, the series appear already in the LO term. The approximated result (with $M = 8$) is

$$\kappa_4^{(1,B,\mathrm{LO})} \approx f\bar{f}B\left[1-6f\bar{f}+(1-3f\bar{f})\bar{\kappa}_4^{(G)}-3f\bar{f}\left(\sum_{n=1}^{N=8}\left(-\bar{\kappa}_2^{(G)}\right)^n+2\bar{\kappa}_3^{(G)}\sum_{n=1}^{N=8}\left(-\bar{\kappa}_2^{(G)}\right)^{n-1}+\left(\bar{\kappa}_3^{(G)}\right)^2\sum_{n=1}^{N=7}\left(-\bar{\kappa}_2^{(G)}\right)^{n-1}\right)\right]. \quad \text{(A11)}$$

After applying $N \to \infty$,

$$\begin{aligned}\kappa_4^{(1,B,\mathrm{LO})} &= f\bar{f}B\left[1-6f\bar{f}+(1-3f\bar{f})\bar{\kappa}_4^{(G)}-3f\bar{f}\left(-\frac{\bar{\kappa}_2^{(G)}}{1+\bar{\kappa}_2^{(G)}}+\frac{2\bar{\kappa}_3^{(G)}}{1+\bar{\kappa}_2^{(G)}}+\frac{\left(\bar{\kappa}_3^{(G)}\right)^2}{1+\bar{\kappa}_2^{(G)}}\right)\right]\\ &= f\bar{f}B\left[1-6f\bar{f}+(1-3f\bar{f})\bar{\kappa}_4^{(G)}-3f\bar{f}\frac{\left(\bar{\kappa}_3^{(G)}\right)^2+2\bar{\kappa}_3^{(G)}-\bar{\kappa}_2^{(G)}}{1+\bar{\kappa}_2^{(G)}}\right]. \end{aligned} \quad \text{(A12)}$$

Then, using Eq. (A4), we obtain Eq. (24) which is exactly in agreement with the results of Refs. [51,52]. This confirms that our method of series expansion works.

b. Next-to-leading-order term

Now we focus on the first correction (next-to-leading-order) term of $\kappa_4^{(1,B)}$. Here, we obtained a much more complicated series (we also show an approximated result with $M = 8$):

$$\frac{\kappa_4^{(1,B,\mathrm{NLO})}}{-\frac{1}{2}f\bar{f}} \approx \left(\bar{\kappa}_2^{(G)}-2\left(\bar{\kappa}_2^{(G)}\right)^2+3\left(\bar{\kappa}_2^{(G)}\right)^3-4\left(\bar{\kappa}_2^{(G)}\right)^4+5\left(\bar{\kappa}_2^{(G)}\right)^5-6\left(\bar{\kappa}_2^{(G)}\right)^6+7\left(\bar{\kappa}_2^{(G)}\right)^7-8\left(\bar{\kappa}_2^{(G)}\right)^8\right) \quad \text{(A13a)}$$

$$+f\bar{f}\left(-6\bar{\kappa}_2^{(G)}+9\left(\bar{\kappa}_2^{(G)}\right)^2-3\left(\bar{\kappa}_2^{(G)}\right)^3-18\left(\bar{\kappa}_2^{(G)}\right)^4+60\left(\bar{\kappa}_2^{(G)}\right)^5-129\left(\bar{\kappa}_2^{(G)}\right)^6+231\left(\bar{\kappa}_2^{(G)}\right)^7-372\left(\bar{\kappa}_2^{(G)}\right)^8\right) \quad \text{(A13b)}$$

$$+\bar{\kappa}_3^{(G)}\left(-1+2\bar{\kappa}_2^{(G)}-3\left(\bar{\kappa}_2^{(G)}\right)^2+4\left(\bar{\kappa}_2^{(G)}\right)^3-5\left(\bar{\kappa}_2^{(G)}\right)^4+6\left(\bar{\kappa}_2^{(G)}\right)^5-7\left(\bar{\kappa}_2^{(G)}\right)^6+8\left(\bar{\kappa}_2^{(G)}\right)^7\right) \quad \text{(A13c)}$$

$$+f\bar{f}\bar{\kappa}_3^{(G)}\left(3+12\bar{\kappa}_2^{(G)}-69\left(\bar{\kappa}_2^{(G)}\right)^2+192\left(\bar{\kappa}_2^{(G)}\right)^3-405\left(\bar{\kappa}_2^{(G)}\right)^4+732\left(\bar{\kappa}_2^{(G)}\right)^5-1197\left(\bar{\kappa}_2^{(G)}\right)^6+1824\left(\bar{\kappa}_2^{(G)}\right)^7\right) \quad \text{(A13d)}$$

$$+f\bar{f}\left(\bar{\kappa}_3^{(G)}\right)^2\left(-21+99\bar{\kappa}_2^{(G)}-270\left(\bar{\kappa}_2^{(G)}\right)^2+570\left(\bar{\kappa}_2^{(G)}\right)^3-1035\left(\bar{\kappa}_2^{(G)}\right)^4+1701\left(\bar{\kappa}_2^{(G)}\right)^5-2604\left(\bar{\kappa}_2^{(G)}\right)^6\right) \quad \text{(A13e)}$$

$$+f\bar{f}\left(\bar{\kappa}_3^{(G)}\right)^3\left(-24+96\bar{\kappa}_2^{(G)}-240\left(\bar{\kappa}_2^{(G)}\right)^2+480\left(\bar{\kappa}_2^{(G)}\right)^3-840\left(\bar{\kappa}_2^{(G)}\right)^4+1344\left(\bar{\kappa}_2^{(G)}\right)^5\right) \quad \text{(A13f)}$$

$$+f\bar{f}\left(\bar{\kappa}_3^{(G)}\right)^4\left(-6+24\bar{\kappa}_2^{(G)}-60\left(\bar{\kappa}_2^{(G)}\right)^2+120\left(\bar{\kappa}_2^{(G)}\right)^3-210\left(\bar{\kappa}_2^{(G)}\right)^4\right) \quad \text{(A13g)}$$

$$+f\bar{f}\bar{\kappa}_4^{(G)}\left(9-33\bar{\kappa}_2^{(G)}+72\left(\bar{\kappa}_2^{(G)}\right)^2-126\left(\bar{\kappa}_2^{(G)}\right)^3+195\left(\bar{\kappa}_2^{(G)}\right)^4-279\left(\bar{\kappa}_2^{(G)}\right)^5+378\left(\bar{\kappa}_2^{(G)}\right)^6-492\left(\bar{\kappa}_2^{(G)}\right)^7\right) \quad \text{(A13h)}$$

$$+f\bar{f}\bar{\kappa}_3^{(G)}\bar{\kappa}_4^{(G)}\left(30-90\bar{\kappa}_2^{(G)}+180\left(\bar{\kappa}_2^{(G)}\right)^2-300\left(\bar{\kappa}_2^{(G)}\right)^3+450\left(\bar{\kappa}_2^{(G)}\right)^4-630\left(\bar{\kappa}_2^{(G)}\right)^5+840\left(\bar{\kappa}_2^{(G)}\right)^6\right) \quad \text{(A13i)}$$

$$+f\bar{f}\left(\bar{\kappa}_3^{(G)}\right)^2\bar{\kappa}_4^{(G)}\left(15-45\bar{\kappa}_2^{(G)}+90\left(\bar{\kappa}_2^{(G)}\right)^2-150\left(\bar{\kappa}_2^{(G)}\right)^3+225\left(\bar{\kappa}_2^{(G)}\right)^4-315\left(\bar{\kappa}_2^{(G)}\right)^5\right) \quad \text{(A13j)}$$

$$+f\bar{f}(\bar{\kappa}_4^{(G)})^2\left(-3+6\bar{\kappa}_2^{(G)}-9\left(\bar{\kappa}_2^{(G)}\right)^2+12\left(\bar{\kappa}_2^{(G)}\right)^3-15\left(\bar{\kappa}_2^{(G)}\right)^4+18\left(\bar{\kappa}_2^{(G)}\right)^5-21\left(\bar{\kappa}_2^{(G)}\right)^6\right) \quad \text{(A13k)}$$

$$+(1+3f\bar{f})\bar{\kappa}_5^{(G)}\left(-1+2\bar{\kappa}_2^{(G)}-3\left(\bar{\kappa}_2^{(G)}\right)^2+4\left(\bar{\kappa}_2^{(G)}\right)^3-5\left(\bar{\kappa}_2^{(G)}\right)^4+6\left(\bar{\kappa}_2^{(G)}\right)^5-7\left(\bar{\kappa}_2^{(G)}\right)^6+8\left(\bar{\kappa}_2^{(G)}\right)^7\right) \quad \text{(A13l)}$$

$$+(1+3f\bar{f})\bar{\kappa}_3^{(G)}\bar{\kappa}_5^{(G)}\left(-1+2\bar{\kappa}_2^{(G)}-3\left(\bar{\kappa}_2^{(G)}\right)^2+4\left(\bar{\kappa}_2^{(G)}\right)^3-5\left(\bar{\kappa}_2^{(G)}\right)^4+6\left(\bar{\kappa}_2^{(G)}\right)^5-7\left(\bar{\kappa}_2^{(G)}\right)^6\right) \quad \text{(A13m)}$$

$$+(1-3f\bar{f})\bar{\kappa}_6^{(G)}\left(1-\bar{\kappa}_2^{(G)}+\left(\bar{\kappa}_2^{(G)}\right)^2-\left(\bar{\kappa}_2^{(G)}\right)^3+\left(\bar{\kappa}_2^{(G)}\right)^4-\left(\bar{\kappa}_2^{(G)}\right)^5+\left(\bar{\kappa}_2^{(G)}\right)^6-\left(\bar{\kappa}_2^{(G)}\right)^7\right), \quad \text{(A13n)}$$

where for readability we divide $\kappa_4^{(1,B,\mathrm{NLO})}$ by $(-\frac{1}{2}f\bar{f})$.

Now we address the series one by one, assuming, as before that the remaining terms of the series (up to infinity) follow the same patterns. For reference, we number the series in Eq. (A13) by the line letters.

Series a, c, k, l, m, n. Some of the series are easy to calculate:

$$\mathrm{a}: -\sum_{n=1}^{\infty} n\left(-\bar{\kappa}_2^{(G)}\right)^n = \frac{\bar{\kappa}_2^{(G)}}{\left(1+\bar{\kappa}_2^{(G)}\right)^2}, \quad \text{(A14)}$$

$$\mathrm{c}: -\sum_{n=1}^{\infty} n\left(-\bar{\kappa}_2^{(G)}\right)^{n-1} = -\frac{1}{\left(1+\bar{\kappa}_2^{(G)}\right)^2}, \quad \text{(A15)}$$

$$\mathrm{n}: \sum_{n=0}^{\infty}\left(-\bar{\kappa}_2^{(G)}\right)^n = \frac{1}{1+\bar{\kappa}_2^{(G)}}. \quad \text{(A16)}$$

l and m are the same as c. k is just 3 times c.

The other series are less obvious.

Series i and j. We begin with i and denote the coefficient by a_n.

$$\mathrm{i}:\ 30\big(1-3\bar{\kappa}_2^{(G)}+6\big(\bar{\kappa}_2^{(G)}\big)^2-10\big(\bar{\kappa}_2^{(G)}\big)^3+15\big(\bar{\kappa}_2^{(G)}\big)^4-21\big(\bar{\kappa}_2^{(G)}\big)^5+28\big(\bar{\kappa}_2^{(G)}\big)^6+\cdots\big)=30\sum_{n=1}^{\infty}a_n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}. \tag{A17}$$

Note that $a_n=\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$ and each a_n is a sum of the arithmetic sequence.

Therefore,

$$\mathrm{i}:\ 30\sum_{n=1}^{\infty}\frac{n(n+1)}{2}\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}=\frac{30}{\big(1+\bar{\kappa}_2^{(G)}\big)^3}. \tag{A18}$$

Similarly in j:

$$\mathrm{j}:\ 15\sum_{n=1}^{\infty}\frac{n(n+1)}{2}\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}=\frac{15}{\big(1+\bar{\kappa}_2^{(G)}\big)^3}. \tag{A19}$$

Series h. For h, the situation is similar:

$$\mathrm{h}:\ 3\big(3-11\bar{\kappa}_2^{(G)}+24\big(\bar{\kappa}_2^{(G)}\big)^2-42\big(\bar{\kappa}_2^{(G)}\big)^3+65\big(\bar{\kappa}_2^{(G)}\big)^4-93\big(\bar{\kappa}_2^{(G)}\big)^5+126\big(\bar{\kappa}_2^{(G)}\big)^6-164\big(\bar{\kappa}_2^{(G)}\big)^7+\cdots\big)=3\sum_{n=1}^{\infty}a_n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}, \tag{A20}$$

where we deduce how to obtain the subsequent terms by observing that

$a_1=3$,

$a_2=3+(3+5\times 1)=11$,

$a_3=3+(3+5\times 1)+(3+5\times 2)=24$,

...

$a_n=3+(3+5\times 1)+(3+5\times 2)+\cdots+(3+5(n-1))$.

a_n is a sum of n terms which we call b_m: $a_n=\sum_{m=1}^{n}b_m$, where $b_m=3+5(m-1)$. Thus,

$$a_n=\sum_{m=1}^{n}b_m=\sum_{m=1}^{n}[3+5(m-1)]=\frac{(5n+1)n}{2}, \tag{A21}$$

$$\mathrm{h}:\ 3\sum_{n=1}^{\infty}\frac{n(5n+1)}{2}\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}=-\frac{3\big(2\bar{\kappa}_2^{(G)}-3\big)}{\big(1+\bar{\kappa}_2^{(G)}\big)^3}. \tag{A22}$$

Series f and g. The series in line f has one more level of complexity:

$$\mathrm{f}:\ -24\big(1-4\bar{\kappa}_2^{(G)}+10\big(\bar{\kappa}_2^{(G)}\big)^2-20\big(\bar{\kappa}_2^{(G)}\big)^3+35\big(\bar{\kappa}_2^{(G)}\big)^4-56\big(\bar{\kappa}_2^{(G)}\big)^5+\cdots\big)=-24\sum_{n=1}^{\infty}a_n\big(-\bar{\kappa}_2^{(G)}\big)^{n-1}, \tag{A23}$$

where

$a_1=1$,

$a_2=1+(1+2)=4$,

$a_3=1+(1+2)+(1+2+3)=10$,

$a_4=1+(1+2)+(1+2+3)+(1+2+3+4)=20$,

...

$a_n=1+(1+2)+\cdots+(1+2+\cdots+n)$.

Again, $a_n=\sum_{m=1}^{n}b_m$, where $b_m=1+2+\cdots+m=\frac{m(m+1)}{2}$. Then,

$$a_n=\sum_{m=1}^{n}b_m=\sum_{m=1}^{n}\frac{m(m+1)}{2}=\frac{n(n+1)(n+2)}{6}. \tag{A24}$$

Eventually,

$$\text{f}: \quad -24\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)}{6}\left(-\bar{\kappa}_2^{(G)}\right)^{n-1} = -\frac{24}{\left(1+\bar{\kappa}_2^{(G)}\right)^4}. \tag{A25}$$

The same series appears in g:

$$\text{g}: \quad -6\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)}{6}\left(-\bar{\kappa}_2^{(G)}\right)^{n-1} = -\frac{6}{\left(1+\bar{\kappa}_2^{(G)}\right)^4}. \tag{A26}$$

Series e. e is similar:

$$\text{e}: -3\left(7-33\bar{\kappa}_2^{(G)}+90\left(\bar{\kappa}_2^{(G)}\right)^2-190\left(\bar{\kappa}_2^{(G)}\right)^3+345\left(\bar{\kappa}_2^{(G)}\right)^4-567\left(\bar{\kappa}_2^{(G)}\right)^5+868\left(\bar{\kappa}_2^{(G)}\right)^6+\cdots\right) = -3\sum_{n=1}^{\infty}a_n\left(-\bar{\kappa}_2^{(G)}\right)^{n-1}, \tag{A27}$$

where

$a_1 = 7,$
$a_2 = 7 + [7 + (7 + 12)] = 33,$
$a_3 = 7 + [7 + (7 + 12)] + [7 + (7 + 12) + (7 + 2 \times 12)] = 90,$
$a_4 = 7 + [7 + (7 + 12)] + [7 + (7 + 12) + (7 + 2 \times 12)] + [7 + (7 + 12) + (7 + 2 \times 12) + (7 + 3 \times 12)] = 190,$
$a_n = 7 + [7 + (7 + 12)] + \cdots + [7 + (7 + 12) + (7 + 2 \times 12) + \cdots + (7 + (n - 1) \times 12)].$

We denote it as $a_n = \sum_{m=1}^{n} b_m$, where $b_m = \sum_{k=1}^{m}[7 + (k - 1)12] = (6m + 1)m$. Thus,

$$a_n = \sum_{m=1}^{n} b_m = \sum_{m=1}^{n}(6m+1)m = \frac{n(n+1)(4n+3)}{2}, \tag{A28}$$

$$\text{e}: \ -3\sum_{n=1}^{\infty}\frac{n(n+1)(4n+3)}{2}\left(-\bar{\kappa}_2^{(G)}\right)^{n-1} = \frac{3\left(5\bar{\kappa}_2^{(G)}-7\right)}{\left(1+\bar{\kappa}_2^{(G)}\right)^4}. \tag{A29}$$

Series d. Now we focus on d:

$$\begin{aligned}\text{d}: \ &-1(-3)+2\left(6\bar{\kappa}_2^{(G)}\right)-3\left[23\left(\bar{\kappa}_2^{(G)}\right)^2\right]+4\left[48\left(\bar{\kappa}_2^{(G)}\right)^3\right]-5\left[81\left(\bar{\kappa}_2^{(G)}\right)^4\right]+6\left[122\left(\bar{\kappa}_2^{(G)}\right)^5\right]\\ &-7\left[171\left(\bar{\kappa}_2^{(G)}\right)^6\right]+8\left[228\left(\bar{\kappa}_2^{(G)}\right)^7\right]+\cdots = -\sum_{n=1}^{\infty}na_n\left(-\bar{\kappa}_2^{(G)}\right)^{n-1},\end{aligned} \tag{A30}$$

where

$a_1 = -3,$
$a_2 = -3 + 9 = 6,$
$a_3 = -3 + 9 + (9 + 8) = 23,$
$a_4 = -3 + 9 + (9 + 8) + (9 + 2 \times 8) = 48,$
$a_5 = -3 + 9 + (9 + 8) + (9 + 2 \times 8) + (9 + 3 \times 8) = 81,$
...
$a_n = -3 + 9 + (9 + 8) + (9 + 2 \times 8) + \cdots + [9 + (n - 2) \times 8].$

We denote it as $a_n = -3 + \sum_{m=1}^{n-1} b_m$, where $b_m = 9 + (m - 1)8$. So,

$$a_n = -3 + \sum_{m=1}^{n-1} b_m = -3 + \sum_{m=1}^{n-1}[9 + (m-1)8] = 4n^2 - 3n - 4, \tag{A31}$$

$$\text{d}: \ -\sum_{n=1}^{\infty}n(4n^2-3n-4)\left(-\bar{\kappa}_2^{(G)}\right)^{n-1} = -\frac{3\left(\left(\bar{\kappa}_2^{(G)}\right)^2-8\bar{\kappa}_2^{(G)}-1\right)}{\left(1+\bar{\kappa}_2^{(G)}\right)^4}. \tag{A32}$$

Series b. The last one is b:

$$\begin{aligned}\text{b}: \ &1\left(-6\bar{\kappa}_2^{(G)}\right)-2\left(-\frac{9}{2}\left(\bar{\kappa}_2^{(G)}\right)^2\right)+3\left(-1\left(\bar{\kappa}_2^{(G)}\right)^3\right)-4\left(\frac{9}{2}\left(\bar{\kappa}_2^{(G)}\right)^4\right)+5\left(12\left(\bar{\kappa}_2^{(G)}\right)^5\right)-6\left(\frac{43}{2}\left(\bar{\kappa}_2^{(G)}\right)^6\right)+7\left(33\left(\bar{\kappa}_2^{(G)}\right)^7\right)\\ &-8\left(\frac{93}{2}\left(\bar{\kappa}_2^{(G)}\right)^8\right)+\cdots = -\sum_{n=1}^{\infty}na_n\left(-\bar{\kappa}_2^{(G)}\right)^{n},\end{aligned} \tag{A33}$$

where

$a_1 = -6,$

$a_2 = -6 + \frac{3}{2} = -\frac{9}{2},$

$a_3 = -6 + \frac{3}{2} + (\frac{3}{2} + 2) = -1,$

$a_4 = -6 + \frac{3}{2} + (\frac{3}{2} + 2) + (\frac{3}{2} + 2 \times 2) = \frac{9}{2},$

$a_5 = -6 + \frac{3}{2} + (\frac{3}{2} + 2) + (\frac{3}{2} + 2 \times 2) + (\frac{3}{2} + 3 \times 2) = 12,$

...

$a_n = -6 + \frac{3}{2} + (\frac{3}{2} + 2) + (\frac{3}{2} + 2 \times 2) + \cdots + (\frac{3}{2} + (n-2) \times 2).$

We denote it as: $a_n = -6 + \sum_{m=1}^{n-1} b_m$, where $b_m = \frac{3}{2} + (m-1)2$. So,

$$a_n = -6 + \sum_{m=1}^{n-1} b_m = -6 + \sum_{m=1}^{n-1} \left[\frac{3}{2} + (m-1)2\right] = \frac{2n^2 - 3n - 11}{2}, \quad \text{(A34)}$$

$$\text{b}: \; -\sum_{n=1}^{\infty} \frac{n(2n^2 - 3n - 11)}{2} \left(-\bar{\kappa}_2^{(G)}\right)^n = -\frac{3\bar{\kappa}_2^{(G)}\left(\left(\bar{\kappa}_2^{(G)}\right)^2 + 5\bar{\kappa}_2^{(G)} + 2\right)}{\left(1 + \bar{\kappa}_2^{(G)}\right)^4}. \quad \text{(A35)}$$

Final formula for $\kappa_4^{(1,B,NLO)}$. We plug in all these results into Eq. (A13). Using Eq. (A4), we obtain Eq. (25).

APPENDIX B: LIMITS ON α_k

In this Appendix, we discuss the limits on the values of the short-range correlation coefficients, α_k.

1. Probability distribution

First, we focus on the discrete probability distribution itself. We straightforwardly differentiate the probability generating function, $H(z) = \sum_{n=0}^{\infty} P(n) z^n$. We use the facts that $H(z) = e^{G(z)}$ and $G(z) = \sum_{k=1}^{\infty} \frac{(z-1)^k}{k!} \hat{C}_k$, where $G(z)$ is the factorial cumulant generating function. Therefore, the multiplicity probability distribution is given by

$$P(m) = \frac{1}{m!} \frac{d^m}{dz^m} \left[\exp\left(\sum_{k=1}^{\infty} \frac{(z-1)^k}{k!} \hat{C}_k \right) \right] \Bigg|_{z=0}. \quad \text{(B1)}$$

In our case, $\hat{C}_k$ is given by Eq. (10). Clearly, $P(m)$ must satisfy the condition $0 \leqslant P(m) \leqslant 1$ for all m. This is the crucial test for the validity of the set of values of α_k's.[5]

2. Central moments

The kth central moment is defined as $\mu_k = \langle (x - \langle x \rangle)^k \rangle$.[6] Obviously, the even central moments have to be greater than or equal to 0.

First of all, the variance, $\mu_2 = \kappa_2 = \sigma^2 = \langle (n - \langle n \rangle)^2 \rangle \geqslant 0$. Using the definition of the factorial cumulants and assuming the short-range correlations (10), we obtain

$$\hat{C}_2 = \frac{d^2 G(z)}{dz^2}\bigg|_{z=1} = -\langle n \rangle^2 + \langle n(n-1) \rangle = \alpha_2 \langle n \rangle. \quad \text{(B2)}$$

Therefore,

$$\langle n \rangle (\alpha_2 + 1) = \langle n^2 \rangle - \langle n \rangle^2 = \sigma^2 \geqslant 0. \quad \text{(B3)}$$

This puts the lower limit on α_2:

$$\alpha_2 \geqslant -1. \quad \text{(B4)}$$

A similar discussion applies to the fourth and sixth central moments resulting in more complicated relations between α_k's and $\langle n \rangle$.

3. Kurtosis-skewness inequality

There exists an inequality between kurtosis, K, and skewness, S [60]:[7]

$$K \geqslant S^2 + 1, \quad \text{(B5)}$$

or in terms of the central moments,

$$\frac{\mu_4}{\mu_2^2} \geqslant \frac{\mu_3^2}{\mu_2^3} + 1, \quad \text{(B6)}$$

or in terms of cumulants,

$$\frac{\kappa_4}{\kappa_2^2} + 2 \geqslant \frac{\kappa_3^2}{\kappa_2^3}. \quad \text{(B7)}$$

This condition also gives nontrivial relations between α_k's.

[5]In practice, we assume that $\alpha_k \neq 0$ for finite k, e.g., $k \leqslant 6$.

[6]It is straightforward to check that $\mu_2 = \kappa_2$, $\mu_3 = \kappa_3$, $\mu_4 = \kappa_4 + 3\kappa_2^2$, $\mu_5 = \kappa_5 + 10\kappa_3\kappa_2$, and $\mu_6 = \kappa_6 + 15\kappa_4\kappa_2 + 10\kappa_3^2 + 15\kappa_2^3$.

[7]This inequality can be justified quite easily. Here we follow Ref. [61]. Suppose x is a random variable from the distribution with mean $\langle x \rangle$ and standard deviation σ. Let $y = \frac{x - \langle x \rangle}{\sigma}$. Clearly, $\langle y \rangle = 0$, $\sigma_y^2 = \langle y^2 \rangle = 1$. We use the Cauchy-Schwartz inequality for probability theory, $\langle ab \rangle^2 \leqslant \langle a^2 \rangle \langle b^2 \rangle$, where a and b are the random variables. Let $a = y$, $b = y^2 - 1$. This brings us to $\langle y^4 \rangle \geqslant \langle y^3 \rangle^2 + 1$, being equivalent to (B6).

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