Particle-number fluctuations near the critical point of nuclear matter

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(Received 11 January 2023; accepted 25 January 2023; published 17 February 2023)

The equation of state with quantum statistics corrections is used for particle number fluctuations ω of isotopically symmetric nuclear matter with interparticle van der Waals and Skyrme local density interactions. The fluctuations, $\omega \propto 1/\mathcal{K}$, are analytically derived through the isothermal incompressibility $\mathcal{K}$ at first order over a small quantum-statistics parameter. Our approximate analytical results appear to be in good agreement with the results of accurate numerical calculations. These results are also close to those obtained by using more accurate Tolman and Rowlinson expansions of the incompressibility $\mathcal{K}$ near the critical point. A more general formula for fluctuations ω, improved at the critical point, was obtained for a finite particle number average $\langle N\rangle$ by neglecting, for simplicity, small quantum statistics effects. It is shown that for a large dimensionless parameter, $\alpha \propto \mathcal{K}^2\langle N\rangle/\mathcal{K}''$, where $\mathcal{K}''$ is the second derivative of the incompressibility $\mathcal{K}$ as function of the average particle density n, far from the critical point ($\alpha \gg 1$), one finds the traditional asymptote, $\omega \propto 1/\mathcal{K}$, for the fluctuations ω. For a small parameter, $\alpha \ll 1$, near the critical point, where $\mathcal{K} = 0$ and $\alpha = 0$, one obtains another asymptote of ω. These fluctuations, having a maximum near the critical point as function of the average density n, for finite values of $\langle N\rangle$ are finite and relatively small, in contrast to the results of the traditional calculations.

DOI: 10.1103/PhysRevC.107.024610

I. INTRODUCTION

Many works have been devoted to studying the properties of nuclear systems with strongly interacting particles; see, e.g., Refs. [1–8]. Realistic versions of the nuclear matter equation of state include both attractive and repulsive forces between particles. Thermodynamical behavior of this matter leads to the liquid-gas first-order phase transition which ends at the critical point; see Refs. [9–11] and, for special emphasis on finiteness of nuclear systems in multifragmetation reactions, Refs. [12,13]. Experimentally, the presence of the liquid-gas phase transition in nuclear matter was reported and then analyzed in numerous papers (see, e.g., Refs. [12–20]). Critical points in different systems of nuclear matter were studied in many theoretical works; see, e.g., recent Refs. [21–23]. These works, which are mainly based on the proposed van der Waals (vdW) and effective Skyrme local density (SLD) approaches for the equation of state accounting for quantum statistics (QvdW and QSLD) [21,24], were used to describe the properties of nuclear matter. Also, extensions for many-component systems, and applications to the fluctuation calculations (see, e.g., Refs. [25–31]), were suggested for different thermodynamical averages [9,13,32–43].

The role and size of the effects of quantum statistics was studied analytically for nuclear matter, and also for pure neutron and pure α-particle matter, in Refs. [44,45]. An analytical expression for dependence of the critical point parameters on the particle mass m, degeneracy factor g, and the QvdW and QSLD interaction constants a, b (or their matrices) for the vdW and those including γ for the SLD was derived in Refs. [44] and [45], respectively. In particular, the analytical approach of Ref. [44] was extended [45] to the effective simple SLD approach [22,46,47]; see also review articles [48]. This approach is related to the Skyrme forces through the potential part of the local energy-density functional. Our consideration was restricted to relatively small temperature, $T \lesssim 30$ MeV, and not too large particle density. On the other hand, the temperature T should be sufficiently large to satisfy the smallness of the quantum statistics parameter [9]. Within these restrictions, the number of nucleons can be determined by a conservation rule, and the chemical potential of such systems is regulated by the particle number density of nuclear matter. An extension of the formulation to fully relativistic hadron resonances in a gas system of baryons and antibarions with vdW two-body interactions was considered in Ref. [49]. Applications of this extended model to the net baryon number

fluctuations in relativistic nucleus-nucleus collisions was developed in Refs. [25–31,50]. We do not include the Coulomb forces and make no distinction between protons and neutrons (both these particles are referred to as nucleons). In addition, under these restrictions the nonrelativistic treatment becomes very accurate and is adopted in our studies.

In the present work we are going to apply the same analytical method as for derivations of the equation of state, taking into account the quantum statistics effects in terms of a few first corrections within the QvdW and QSLD models (see Refs. [44,45]), to analyze the particle number fluctuations near the critical point of nuclear matter; see also Ref. [51]. Different analytical and numerical approximations to these fluctuations might be helpful to determine ranges of their applicability. As shown, e.g., in Ref. [42], the expression of the particle number fluctuation, ω, in terms of the susceptibility, and then, through the incompressibility, $\omega \propto 1/\mathcal{K}$, was derived from the original definition for ω in terms of the moments of the Gibbs distribution, averaged over phase space variables, by assuming the smallness of the fluctuations. However, the limit of this traditional expression to the critical point, where $\mathcal{K} = 0$, is obviously divergent. Divergences of fluctuations ω near the critical point are incompatible with a more accurate equation of state, accounting for interparticle interaction, as a relation between statistically averaged characteristics of the desired system, which are determined up to these fluctuations [33,39,40,42]. The main critique of these divergences is that they are due to highly idealized assumptions (e.g., a mean-field approach up to particle correlations in the infinite system [10,11]). These assumptions were used in the derivations of fluctuations ω from the moments of the Gibbs distribution over particle number N in the grand canonical ensemble. The traditional expression for the fluctuation ($\omega \propto 1/\mathcal{K}$) in terms of the incompressibility $\mathcal{K}$ fails near the critical point. Some suggestions to overcome the divergence problem for the fluctuations, taking into account the particle correlations, can be found in Ref. [33]. For the vdW problem, one can find other specific semianalytical suggestions in Ref. [43]. We will apply another statistical approach [52,53] based on expansion of the free energy F over powers of a small difference of the particle number density ρ and its average n for a given temperature T. We are going to use explicitly the statistical Gibbs distribution averaged, however, in phase space for calculations of the particle number density dispersion and the corresponding fluctuation ω; see also Refs. [38–40]. As is well known (see, e.g., [38,40]), the expansion of F at second order leads to the traditional expression for the particle number fluctuations, $\omega \propto 1/\mathcal{K}$. Taking into account only fourth-order terms, and neglecting the second-order ones in a very close vicinity to the critical point, one has several improved results derived in Refs. [39,40]. We will take into account both fourth- and second-order terms and obtain a more general result for these fluctuations, accurately determining the two asymptotes far from and close to the critical point for a finite average particle number. In order to compare in detail a more general asymptote and the two above-mentioned asymptotes near the critical point, we take the phenomenological vdW and SLD density-dependent interactions as well-known examples. In this way, we will neglect the quantum statistics effects which are not very important for fluctuations, in contrast to the critical point calculations. Notice that the order parameter $\rho - n$ in our approach is essentially different from $\rho - n_c$, where n_c is the critical value of the average particle number density n, in the Landau local fluctuation theory. Therefore, in our mean-field approach, up to statistical correlations, it is possible to cross the critical point with finite fluctuations by changing a dimensionless parameter $\alpha \propto K^2\langle N\rangle/n^2TK''$ to zero. In this approach, the second derivative K'' of the isothermal incompressibility K is assumed to be not zero at the critical point ($K = 0$) for a finite average of the particle numbers N, $\langle N\rangle$. Otherwise, we will need to use an expansion of the free energy F up to high order terms. The parameter α is a measure of the effective distance from a critical point, which depends on the interparticle interaction and average particle number $\langle N\rangle$. Thus, we will have a transition from large effective parameter α of the traditional formula for the fluctuations ω to the Rowlinson formula [39] at small α, which is local near the critical point, within the Smoluchowski and Einstein fluctuation theory [52,53]. To some extent, that is a more general approach than the classical Landau fluctuation theory. The fluctuation calculations based on the statistically averaged level density obtained analytically in Refs. [54–57] will be adopted to determine the particle number fluctuations for finite nuclear systems in a forthcoming work.

The paper is organized as follows. In Sec. II we review some general relationships of the statistical physics used in our derivations. In Sec. III, the known analytical derivations and results for the classical fluctuations are presented following Refs. [9,38–40]. We apply them for the traditional analytical method of fluctuation calculations in terms of the incompressibility for the vdW and SLD interparticle interactions in Sec. IV. Then, in Sec. V, we present the improved derivations for the fluctuation calculations based on the statistically averaged Gibbs distribution. The same vdW and SLD interaction models are taken as simple exemplary cases. All obtained results are discussed in Sec. VI. We compare our analytical traditional calculations with those modified by Tolman [38] and Rowlinson [39], and with numerical calculations carried out by Gorenstein and his collaborators in Refs. [21,24,25]. The same parameters of simple phenomenological interactions, vdW and SLD, are used for the comparison between the results of Secs. IV and V. These results are summarized in Sec. VII. Some details of our derivations are presented in Appendices A–F. Pecularities of the classical fluctuations in terms of the first- and high-order susceptibilities and incompressibilities of nuclear matter are discussed in Appendices A and B, respectively. In Appendix C, the analytical results for the critical point are reviewed for the case of the QvdW and QSLD approaches taking into account the quantum statistics corrections to the vdW (Ref. [44,45]) and SLD [45] models. In Appendices D and E, following Ref. [38,39] we present some details of the derivations of the general improved fluctuation formula and its asymptote near the critical point, respectively. Appendix F is devoted to our approach following the Tolpygo classical fluctuation theory [40].

II. GENERAL POINTS

For calculations of classical fluctuations of the particle numbers, ω, within the grand canonical ensemble, one can start with the particle number average [9,42]

$$\langle N\rangle = \sum_N N \int W_{\rm eq}^{(N)}(\mathbf{q},\mathbf{p};T,\mu,V)d\Gamma_N. \quad (1)$$

Here, $W_{\rm eq}^{(N)}(\mathbf{q},\mathbf{p};T,\mu,V)$ is the Gibbs distribution function of phase space variables $\mathbf{q}$, $\mathbf{p}$; $d\Gamma_N = d\mathbf{q}d\mathbf{p}$ for a given particle number N (normalized as usually for a classical system). Other variables, T, μ, and V, are the temperature, chemical potential, and system volume in the grand canonical ensemble, respectively. The Gibbs probability distribution $W_{\rm eq}^{(N)}$ can be written in terms of the classical Hamiltonian $H_N(\mathbf{q},\mathbf{p})$ as

$$W_{\rm eq}^{(N)}(\mathbf{q},\mathbf{p};T,\mu,V) = \frac{1}{\mathcal{Z}(T,\mu,V)}\exp\{-[H_N(\mathbf{q},\mathbf{p})-\mu N]/T\}. \quad (2)$$

The Hamiltonian $H_N(\mathbf{q},\mathbf{p})$ is the basic part also for the normalization factor,

$$\mathcal{Z}(T,\mu,V) = \sum_N \int d\Gamma_N \exp\{-[H_N(\mathbf{q},\mathbf{p})-\mu N]/T\}. \quad (3)$$

Thus, the partition function $\mathcal{Z}(T,\mu,V)$ obeys the normalization condition for the distribution $W_{\rm eq}^{(N)}$. This distribution, averaged below over the phase space variables $\mathbf{q}$ and $\mathbf{p}$ for a given particle number N, will be denoted as $\overline{W}_{\rm eq}^{(N)}$ (the line above the quantity means averaging only over the phase space variables). In addition, the averaging over particle numbers N, as in Eq. (1), along with averaging over the phase space variables, will be shown by angle brackets, $\langle W_{\rm eq}^{(N)}\rangle$; see Eq. (2) and Ref. [42]. Using these notations, for the classical entropy $S(T,\mu,V)$ one has

$$S(T,\mu,V) = -\left\langle \ln W_{\rm eq}^{(N)}\right\rangle = \frac{1}{T}[\langle H\rangle - \mu\langle N\rangle + T\ln\mathcal{Z}(T,\mu,V)]. \quad (4)$$

We may now introduce the equilibrium thermodynamical potential, $\Omega(T,\mu,V)$, for the grand canonical ensemble with the help of the relationship

$$\Omega = F - \mu\langle H\rangle = U - TS - \mu\langle H\rangle, \quad U = \langle H\rangle. \quad (5)$$

The free energy $F(T,N,V)$ of the canonical ensemble is considered as function of the temperature T, particle number N, and system volume V,

$$F = U - TS. \quad (6)$$

Then, the chemical potential μ can be defined in terms of the free energy F, $\mu = (\partial F/\partial N)_T$ in the canonical ensemble. From Eq. (5) one finds the standard expression [9] for a thermodynamical potential Ω of the grand canonical ensemble in terms of the partition function $\mathcal{Z}$ [Eq. (3)],

$$\Omega(T,\mu,V) = -T\ln\mathcal{Z}(T,\mu,V). \quad (7)$$

For intensively large systems, one can consider a local density of the thermodynamic potential Ω per unit of volume V, i.e., the pressure $P(T,n)$ [9]. For such intensive systems in the grand canonical ensemble, one has $\Omega = -VP(T,n)$, where we can neglect the explicit volume dependence of the pressure, $P(T,n)$. This dependence is realized only through the averaged particle number density $n = N/V$. The equation of state, $P = P(T,n)$, can be found through the explicit expression for $P(T,n)$ as a function of temperature T and particle number density n. This takes place, e.g., if we can neglect the surface part of the system pressure (the capillary pressure due to the surface tension) with respect to its volume part.

III. CLASSICAL FLUCTUATIONS

As mentioned in the Introduction, the derivation of the expression for the particle number fluctuations ω in terms of the incompressibility $\mathcal{K}$ from the moments of a mean Gibbs distribution $\overline{W}_{\rm eq}^{(N)}$ is questionable near the critical point (CP), where $\mathcal{K} = 0$. Indeed, the assumption of smallness of ω in this derivation fails near the CP (see, e.g., Refs. [38–40]). Therefore, to clarify the behavior of fluctuations near the CP, one has to consider more accurately the derivations within the classical fluctuation theory [52,53] beginning from the dispersion (squared) of the particle number distribution:

$$D_N = \langle(\Delta N)^2\rangle = \langle N^2\rangle - \langle N\rangle^2, \quad (8)$$

where $\Delta N = N - \langle N\rangle$ is the deflection of the particle number N from its average $\langle N\rangle$. Averages $\langle N^\kappa\rangle$ are the mean κ moments of the distribution function $\overline{W}_{\rm eq}^{(N)}$ (see the previous section),

$$\langle N^\kappa\rangle = \int N^\kappa \overline{W}_{\rm eq}^{(N)} dN, \quad \int \overline{W}_{\rm eq}^{(N)} dN = 1, \quad \kappa = 1,2. \quad (9)$$

The particle number dispersion D_N [Eq. (8)] can be expressed in terms of these two moments, $\langle N\rangle$ and $\langle N^2\rangle$, of the Gibbs distribution $\overline{W}_{\rm eq}^{(N)}$, averaged over the phase space variables; see above. For intensive systems, it is convenient to calculate first the dispersion $\mathcal{D}_\rho$ of the particle number density fluctuations, e.g., in units of n^2 for the dimensionless reason,

$$\frac{\mathcal{D}_\rho}{n^2} = \frac{\langle(\Delta\rho)^2\rangle}{n^2} = \frac{\langle\rho^2\rangle - \langle\rho\rangle^2}{n^2}, \quad (10)$$

where $\Delta\rho = \rho - \langle\rho\rangle$ is the deflection of the particle number density ρ from its average $\langle\rho\rangle = n$ (see Ref. [38]).

We will discuss the normalization of the dispersion D_N in terms of the particle number fluctuations later. First, we will show that the dispersions (8) and (10) are significantly different by order of the power of $\langle N\rangle$, linear and quadratic, far and near the critical point, respectively.

The angle brackets in Eq. (10) are defined by

$$\langle\rho^\kappa\rangle = \int_0^{\rho_{\rm up}} d\rho\rho^\kappa W(\rho), \quad \int_0^{\rho_{\rm up}} d\rho W(\rho) = 1, \quad (11)$$

where again $\kappa = 1,2$ and $\rho_{\rm up}$ is the upper limit of the integration over particle number density ρ. Notice that for the vdW interparticle interaction one has a restriction to the upper limit $\rho_{\rm up}$ of this integration by its value $1/(3b)$, where b is

the volume exclusion parameter [9,44,45], while $\rho_{\rm up} = \infty$ for the SLD interaction case. We will expand approximately ρ_{up} to the infinity in all following normalization integrals. The probability distribution function. The probability distribution function $W(\rho)$ in Eq. (11) can be approximated by [38]

$$W(\rho) = W^{(0)} \exp\left[-\frac{F(\rho) - F(n)}{T}\right], \tag{12}$$

where $W^{(0)}$ is the normalization constant,

$$W^{(0)} = \left\{\int_0^\infty d\rho \exp\left[-\frac{F(\rho) - F(n)}{T}\right]\right\}^{-1}. \tag{13}$$

We omit the temperature variable T in the free energy F because it is a constant in all our following derivations.

Following the ideas of Smoluchowski and Einstein (see Refs. [38,52,53]), for small fluctuations, the free energy $F(\rho)$ for an intensive system can be expanded in powers of a difference between the particle number density, ρ, and its statistical average, $\langle\rho\rangle = n$. Up to fourth-order terms for the fixed temperature T, one writes

$$F(\rho) = F(n) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial\rho^2}\right)_{\rho=n}(\rho - n)^2 + \frac{1}{24}\left(\frac{\partial^4 F}{\partial\rho^4}\right)_{\rho=n}(\rho - n)^4 + \cdots. \tag{14}$$

At fourth order, as will be used below, one writes

$$\Delta\{\rho\} \equiv F(\rho) - F(n) = \frac{1}{2}\left(\frac{\partial^2 F}{\partial\rho^2}\right)_{\rho=n}(\rho - n)^2 + \frac{1}{24}\left(\frac{\partial^4 F}{\partial\rho^4}\right)_{\rho=n}(\rho - n)^4. \tag{15}$$

We introduced here the functional $\Delta\{\rho\}$ of the particle number density ρ. The first- and third-order terms in these expansions can be put to zero because our system is considered to be at statistical equilibrium with a minimum of the free energy $F(\rho)$ and there are no external fields under consideration. We will assume also that the fourth-order terms dominate over high order terms. As shown in Refs. [10,11,13], a little more complexly but still analytically, sixth-order terms can be also taken into account for study of the tricritical point. Some appearing constants could be included in the normalization factor $W^{(0)}$; see Eqs. (12) and (13). Using the expansion of the free energy $F(\rho)$, Eq. (14) at fourth order, one can immediately rewrite the probability distribution (12) as

$$W_4(\rho) = W_4^{(0)} \exp\left[-\frac{F_2}{2T}\left((\rho - n)^2 + \frac{F_4}{12}(\rho - n)^4\right)\right], \tag{16}$$

where

$$W_4^{(0)} = \left\{\int_0^\infty d\rho \exp\left[-\frac{F_2}{2T}\left((\rho-n)^2+\frac{F_4}{12T}(\rho-n)^4\right)\right]\right\}^{-1}, \tag{17}$$

and F_2 and F_4 are derivatives of the free energy F given by Eq. (D3). Indeed, according to Eqs. (16) and (17), one has the normalization condition,

$$\int_0^\infty W_m(\rho) d\rho = 1, \quad m = 2, 4. \tag{18}$$

Therefore, assuming that the second-order correction in the expansion (14) for the free energy F is relatively large with respect to high order terms, one can neglect all other fourth- and high-order terms in Eqs. (16) and (17),

$$W_2(\rho) = W_2^{(0)} \exp\left[-\frac{F_2}{2T}(\rho - n)^2\right], \tag{19}$$

where

$$W_2^{(0)} = \left\{\int_0^\infty d\rho \exp\left[-\frac{F_2}{2T}(\rho - n)^2\right]\right\}^{-1}. \tag{20}$$

Thus, from Eq. (15) one finds[1]

$$\langle(\rho - n)^2\rangle = 2\frac{\langle\Delta_2\{\rho\}\rangle}{F_2}, \tag{21}$$

where $\Delta_2\{\rho\}$ is given by $\Delta\{\rho\}$, Eq. (15), at the second order,

$$\Delta_2\{\rho\} = \tfrac{1}{2}F_2(\rho - n)^2 \tag{22}$$

(see Ref. [38], where $A(x)$ is taken here as the free energy $F(\rho)$ for a given temperature T). Calculating independently the average of $(\rho - n)^2$ in the left-hand side (l.h.s.) of Eq. (21) by using approximately the probability distribution W_2 of the second order, Eq. (19), for the particle number density dispersion $\mathcal{D}_\rho^{(2)}$ one has

$$\mathcal{D}_\rho^{(2)} \equiv \langle(\rho - n)^2\rangle = \int_0^\infty (\rho - n)^2 W_2(\rho) d\rho; \tag{23}$$

see Eq. (18) for $m = 2$. Calculating analytically the integral in Eq. (23) at the second order [Eq. (19)] and comparing the result with the expression on right of Eq. (21), one obtains (see Ref. [38])

$$\langle\Delta\{\rho\}\rangle \approx T/2. \tag{24}$$

For the expressions of the derivatives of the free energy $F(\rho)$ in terms of the incompressibility $\mathcal{K}$ and its second derivative, one can use the well-known [9] relationship between the pressure $P(\rho)$ and free energy $F(\rho)$,

$$P(\rho) = -\frac{\partial F}{\partial V}, \tag{25}$$

where

$$-\frac{\partial F}{\partial V} = \frac{\rho^2}{\langle N\rangle}\frac{\partial F(\rho)}{\partial\rho}. \tag{26}$$

[1]Notice that the distribution W_2, Eq. (19), can be obtained also starting from the Gibbs expression, $\propto \exp(S)$, where S is the entropy (see Ref. [9]). Expanding the entropy near the statistical equilibrium, $(\partial S/\partial\rho)_{\rho=n} = 0$, one has a probability distribution of the same Gaussian form. We use here the definitions of the entropy S and free energy F, through the partition function $\mathcal{Z}$ [see Eq. (6)]. In addition, one can take into account that the high-order (second and higher) derivatives of the entropy S at the equilibrium are identical to those of the free energy F over the particle number density ρ for a constant temperature T, and all probability distributions are normalized to 1.

Differentiating the identity (25) over ρ and accounting also for the zero first derivative of F at the statistical equilibrium $\rho = n$ in expansion (14), one can express the second derivative of $F(\rho)$ over ρ at $\rho = n$ in terms of the incompressibility $\mathcal{K}$,

$$\left(\frac{\partial^2 F(\rho)}{\partial \rho^2}\right)_{\rho=n} = \frac{\langle N\rangle \mathcal{K}(n)}{n^2} \tag{27}$$

with

$$\mathcal{K}(n) = \left(\frac{\partial P(\rho)}{\partial \rho}\right)_{\rho=n}. \tag{28}$$

Notice that the pressure $P(\rho)$ is an intensive quantity which depends on particle number N or volume V only through the particle number density ρ in our system. Using Eqs. (24) and (27), from the particle number density dispersion $\mathcal{D}_\rho$, normalized by n^2, Eq. (10), at the second-order expansion of the free energy, $\mathcal{D}_\rho^{(2)}$, Eq. (22), and Eq. (19) for the probability distribution W_2, one obtains

$$\frac{\mathcal{D}_\rho^{(2)}}{n^2} = \frac{T}{\langle N\rangle \mathcal{K}}. \tag{29}$$

Transferring this expression into the particle number dispersion D_N, Eq. (8), normalized by $\langle N\rangle^2$, for an intensive system, one has

$$\frac{\mathcal{D}_\rho}{n^2} \approx \frac{D_N}{\langle N\rangle^2}. \tag{30}$$

Using finally the normalization of the dispersion D_N by $\langle N\rangle$ we arrive at the well-known [9,32,37–42] local expression for the fluctuations ω in terms of the local isothermal incompressibility, $\mathcal{K}(T, \rho)$:

$$\omega(T, n) \approx \frac{D_N}{\langle N\rangle} \approx \frac{T}{\mathcal{K}}, \quad \mathcal{K} = \left(\frac{\delta P}{\delta n}\right)_T, \tag{31}$$

where P is the pressure, $P(T, n)$, Eq. (25), and $P = P(T, n)$ is the equation of state in canonical variables.

Notice that this quadratic-approach result coincides with the well-known traditional result of the Landau theory of classical fluctuations [9]. Landau [9] normalized particle number dispersion, D_N, by $\langle N\rangle$. However, the result for the fluctuation ω, Eq, (31), $\propto 1/\mathcal{K}$, is divergent near the critical point, where $\mathcal{K} = 0$. In order to improve these fluctuation results we will expand the free energy F up to high-order terms in Eq. (14) (Sec. V). But first, in the next section, let us study the traditional result (31) in more details.

IV. PARTICLE NUMBER FLUCTUATIONS AND INCOMPRESSIBILITY

As shown in Sec. III and Appendices A and B, the fluctuations of particle numbers, ω [see Eq. (31)], can be expressed in terms of the isothermal incompressibility $\mathcal{K}$. We will compare the results obtained by employing different approximations near the critical point. Note that in the derivations of both formulas, Eqs. (A1) and (31), any fluctuations are assumed to be small. Nevertheless, we will compare the results obtained by employing different approximations near the critical point with the popular formulas given by Eqs. (A1) and (31), though the value of ω near the CP obtained by these formulas is expected to be large. In fact, it is still an open question whether the traditional formula (31) can be applied near the critical point in the density-temperature (n-T) plane.

For calculations of the incompressibility $\mathcal{K}$, one can use the so-called [24] quantum van der Waals (QvdW) or [22] quantum Skyrme local density (QSLD) interaction approaches; see equations of state (C1) or (C6), respectively. In this section, we will follow Refs. [44,45] at first order of the quantum statistics expansion (see Appendix C). Moreover, this analytical approach will be applied to the simplest uniform one-component intensive system of nucleons interacting through the repulsive and attractive effective forces. For this purpose, the incompressibility $\mathcal{K}$ will be considered as a linear and (slightly) nonlinear response of the pressure P to the particle number density n variations for the nucleon system at constant temperature T. Our purpose in this section is to derive analytical results for the fluctuations ω of particle numbers near the critical point (CP) within the QvdW and QSLD approaches at first order of the quantum statistics parameter; see Ref. [45].

For relatively small fluctuations of particle numbers, ω, one has Eq. (31). In Eq. (31), P is the pressure for the equation of state, which is given in the one-component QvdW and QSLD models for symmetric nucleons matter by Eqs. (C1) and (C6), respectively. Notice that we use a more general definition of the incompressibility $\mathcal{K}$ as the variation derivative of the pressure of the equation of state over the particle number density n at constant temperature, in contrast to its standard definition as the following first partial derivative of pressure (see Appendix A). With this approximation for the incompressibility, $\mathcal{K}_1$, one finds from Eq. (31) the expression for a magnitude of fluctuations:

$$\omega \approx \omega_1 = \frac{T}{\mathcal{K}_1}, \quad \mathcal{K}_1 = \left(\frac{\partial P}{\partial n}\right)_T. \tag{32}$$

Equation (31) can be first derived from Eq. (A1) in terms of the susceptibility χ (see Appendix A). As shown in Appendices A and B, using then linear variations for the chemical potential μ as function of the particle number density n, which are therefore valid for small fluctuations, one can derive Eq. (32). The value of this fluctuation, ω_1, diverges in the CP limit, $n \to n_c$ and $T \to T_c$. Therefore, it can be considered only on a finite [sufficiently large for applications of Eq. (32) but small for using expansion near the CP] distance from the CP. The incompressibility $\mathcal{K}$ in Eq. (31) as function of the density n and temperature T, can be expanded in power series near the critical point T_c, n_c over both variables T and n. The derivatives are evaluated here at the current point T, n within the precision of high-order terms. Up to second-order terms, one has

$$\begin{aligned}\omega \approx \omega_3 &= \frac{T}{\mathcal{K}_3},\\ \mathcal{K}_3 &= \left(\frac{\partial P}{\partial n}\right)_T + \left(\frac{\partial^2 P}{\partial n^2}\right)_T (n - n_c)\\ &\quad + \frac{\partial^2 P}{\partial n \partial T}(T - T_c) + \frac{1}{2}\left(\frac{\partial^3 P}{\partial n^3}\right)_T (n - n_c)^2.\end{aligned} \tag{33}$$

As suggested in Refs. [38,39], we use approximately the following definition, valid at the critical point:[2]

$$\left(\frac{\partial P}{\partial n}\right)_T = 0, \quad \left(\frac{\partial^2 P}{\partial n^2}\right)_T = 0. \tag{34}$$

Assuming (see Refs. [38,39]) then that the variations linear in temperature and quadratic in density are dominant over high- and low-order variations, one can define another approximation near the CP:

$$\omega \approx \omega_3^{(c)} = \frac{T}{\mathcal{K}_3^{(c)}}, \tag{35}$$

$$\mathcal{K}_3^{(c)} = \frac{\partial^2 P}{\partial n \partial T}(T - T_c) + \frac{1}{2}\left(\frac{\partial^3 P}{\partial n^3}\right)_T (n - n_c)^2. \tag{36}$$

As mentioned above, in these both approaches, Eqs. (33) and (35) [with Eq. (36)], to the variation of the incompressibility $\mathcal{K}$ and of the fluctuations ω, Eq. (31), all the derivatives are still taken at a current point n, T.

A. Fluctuations within the QvdW approach

For the Fermi statistics parameter in the quantum van der Waals (QvdW) model, one finds [44,45]

$$\delta \equiv \frac{\varepsilon}{1 - bn}, \quad \varepsilon = \frac{\hbar^3 \pi^{3/2} n}{2\, g\, (mT)^{3/2}}, \tag{37}$$

where m is the particle mass and g the system degeneracy. Substituting Eq. (C1) for the pressure $P_{\rm W}$ in the fluctuation ω_1, Eq. (32), one can now obtain ω_1 at the first order over a small quantum-statistics parameter δ, Eq. (37), in the explicit analytical form

$$\omega(T, n) \approx \omega_1 = [(1 + 2\delta)/(1 - nb)^2 - 2na/T]^{-1}, \tag{38}$$

where δ is given by Eq. (37), $\delta = \delta(T, n)$. Then, the behavior of $\omega(T, n)$, Eq. (38), near the critical point (T_c, n_c; see Eq. (C2) for the first-order analytical CP expressions [44,45]) will be studied within the QvdW model. Expanding now the incompressibility $\mathcal{K}$ in powers of the temperature difference $T - T_c$ [using the expression in the parentheses of Eq. (38) for the fluctuations ω_1], we calculate immediately derivatives of the pressure $P_{\rm W}$, Eq. (C1), over the density n at a current T, n point. With the help of the new variables

$$\tau \equiv T/T_c - 1, \quad \nu \equiv n/n_c - 1, \tag{39}$$

one can fix first $n = n_c$ ($\nu = 0$) and find the behavior of $\omega(T, n)$ as function of temperature T near the critical point. For this purpose, it is convenient to present $\delta(T, n)$, Eq. (37), as

$$\delta(T, n) = \delta[(1 + \tau)T_c, (1 + \nu)n_c]. \tag{40}$$

We will take now the limit of this expression at $\nu = 0$ and a small τ. In this case, $\nu = 0$, one can approximate $\delta(T, n)$ at the first-order expansion over τ by

$$\delta((1 + \tau)T_c, n_c) \approx \frac{\hbar^3 \pi^{3/2} n_c}{2g\,(mT_c)^{3/2}(1-\beta)}\left(1 - \frac{3}{2}\tau\right), \tag{41}$$

where $\beta = bn_c$.

The critical point (CP) of the liquid-gas phase transition satisfies the well-known equations (34) (see Ref. [9]). Using now Eq. (38) and the first of these CP equations near the CP, one finds ($\nu = 0$)

$$\omega \approx \omega_1^{(c)}((1 + \tau)T_c, n_c) = \frac{T_c n_c}{P_c} G_{{\rm W},\tau}\, \tau^{-1}, \tag{42}$$

where

$$G_{{\rm W}\tau} \approx \frac{P_c}{T_c n_c}\frac{(1 - \beta)^2}{1 - \delta_c}, \quad \delta_c = \delta(T_c, n_c). \tag{43}$$

Taking the exclusion-volume parameter b from Eq. (C5) and the results for T_c, n_c, and P_c for nucleon matter ($g = 4$, $m = 938$ MeV) from Table I, one finally obtains $G_{{\rm W},\tau} \approx 0.29$. This value is only slightly different from that of $G_{{\rm W},\tau} \approx 0.26$, obtained in Ref. [25]. For the case of the classical vdW model ($\delta_c = 0$), one arrives at the well-known result $G_{{\rm W},\tau} = 1/6$.

Similarly, using Eq. (41), for the fluctuations $\omega(T, n)$, Eq. (38), at the second-order expansion over ν, for the constant $T = T_c$ ($\tau = 0$), one finds

$$\delta(T_c, (1 + \nu)n_c) \approx \frac{\hbar^3 \pi^{3/2}\, n_c}{2g(mT_c)^{3/2}(1 - \beta)}\left(1 + \frac{\nu}{1 - \beta} + \frac{\nu^2 \beta}{(1 - \beta)^2}\right). \tag{44}$$

Finally, using the fluctuations ω_1 [Eq. (38)] and both CP equations of Eq. (34) near the CP, one arrives for $\tau = 0$ at

$$\omega \approx \omega_1^{(c)}(T_c, (1 + \nu)n_c) = \frac{T_c n_c}{P_c}\, G_{{\rm W},\nu}\, \nu^{-2}, \tag{45}$$

where

$$G_{{\rm W},\nu} \approx \frac{P_c}{T_c n_c}\frac{(1 - \beta)^4}{3\beta[2\delta_c(1 + \beta) + \beta]} \approx 0.33. \tag{46}$$

The last number was obtained by using Eq. (C5) and Table I. For the case of the classical vdW approach ($\delta_c = 0$), one has from Eq. (46) that $G_{{\rm W},\nu} = 2/9$, which is the same as that shown in Ref. [24].

TABLE I. Results for the CP parameters of the vdW model [Eq. (C4)] (second column) and of the QvdW at first order over the quantum statistics parameter δ [Eqs. (C2) and (37)] (third column) for symmetric nuclear matter; see Eq. (C5) for vdW parameters. Numerical results obtained within the accurate QvdW model in Ref. [24] are shown in the fourth column.

Critical point parameter	vdW Eq. (C4)	First order Eq. (C2)	Numerical full QvdW
T_c (MeV)	29.2	19.0	19.7
n_c (fm^{-3})	0.100	0.065	0.079
P_c (MeV fm^{-3})	1.09	0.48	0.56

[2]The CP is assumed to be of the simplest second-order, in contrast to a high-order CP when high-order derivatives become also zero.

B. Fluctuations within the QSLD approach

Substituting the pressure $P_{\rm Sk}$, Eq. (C6), onto Eq. (32) for the QSLD fluctuation $\omega_{{\rm Sk},1}$ at the first order over a small quantum-statistics parameter ε [see Eq. (37)], one obtains $\omega_{{\rm Sk},1}$, also in explicit analytical form,

$$\omega_{\rm Sk}(T,n) \approx \omega^{(1)}_{{\rm Sk},1} = \left[1+2\varepsilon-2\frac{a_{\rm Sk}n}{T} + b_{\rm Sk}(\gamma+1)(\gamma+2)\frac{n^{\gamma+1}}{T}\right]^{-1}, \quad (47)$$

where $\varepsilon=\varepsilon(T,n)$ is the quantum-statistics parameter [see Eq. (37)]. Other QSLD interaction parameters, $a_{\rm Sk}$, $b_{\rm Sk}$, and γ, are defined in Eq. (C9). For the classical (zero) SLD approximation to the fluctuations, $\omega^{(0)}_{\rm Sk}$, one finds from Eq. (47)

$$\omega^{(0)}_{\rm Sk}(T,n) \approx \omega^{(0)}_{{\rm Sk},1} = \left\{1-2\frac{a_{\rm Sk}n^{(0)}_{\rm Sk}}{T^{(0)}_{\rm Sk}} + b_{\rm Sk}(\gamma+1)(\gamma+2)\frac{\left[n^{(0)}_{\rm Sk}\right]^{\gamma+1}}{T^{(0)}_{\rm Sk}}\right\}^{-1}. \quad (48)$$

As in Sec. IV A, the fluctuations $\omega_{\rm Sk}(T,n)$, Eq. (47), near the critical point $T^{(1)}_{{\rm Sk},c}$, $n^{(1)}_{{\rm Sk},c}$ (see Eq. (C7) for the first-order analytical CP expressions [44,45]) will be derived within the QSLD approach. Expanding now the incompressibility $\mathcal{K}$ in terms of powers of the temperature difference $T-T_c$ [see the expression in parentheses of Eq. (47)], for the fluctuations $\omega_{{\rm Sk},1}$ with the help of the new variables [Eq. (39)], one can, again, fix first n at $n=n_c$ ($\nu=0$). In this way, one finds the behavior of $\omega_{\rm Sk}(T,n)$ as function of temperature T near the critical point. The quantum-statistics parameter $\varepsilon(T,n)$ for the QSLD approach plays the same role as $\delta(T,n)$ for the QvdW fluctuation calculations. We have similar expressions for them [see Eqs. (40), (41), and (44)] where we only need to replace δ by ε. Using now Eq. (47), and the first equation in Eq. (34), at first order over τ near the CP, one finally obtains ($\nu=0$)

$$\omega_{\rm Sk} \approx \frac{T_c n_c}{P_c} G_{{\rm Sk},\tau}\tau^{-1}, \quad (49)$$

with

$$G_{{\rm Sk},\tau} \approx \frac{P_c}{T_c n_c}\frac{1}{1-\varepsilon_c} \approx 0.32, \quad (50)$$

where $\varepsilon_c=\varepsilon(T_c,n_c)$; see Eq. (37). In these evaluations, we used the first-order critical temperature $T^{(1)}_{\rm Sk}$ and particle number density $n^{(1)}_{\rm Sk}$, from Table II for nucleon matter ($g=4$ and $m=938$ MeV), which were obtained in Ref. [45]. For the case of the classical SLD model ($\varepsilon_c=0$), one arrives at $G_{{\rm Sk},\tau}=0.27$.

Similarly, we use Eq. (40). Replacing then δ by ε for the fluctuations $\omega_{\rm Sk}(T,n)$, Eq. (47), in the second-order expansion over ν we apply both CP equations in Eq. (34) near the CP at constant $T=T_c$ ($\tau=0$). Finally, for the fluctuations

TABLE II. Results for the CP parameters of symmetric nuclear matter in the quantum-statistics Skyrme local-density (QSLD) model; second, third, and fourth columns are given for $\gamma=1/6$ in the three upper lines and $\gamma=1$ in the three bottom lines as shown in the fifth column; see Eq. (C9) and Ref. [45]. Zeroth- and first-order results for the CP values [Eqs. (C8) and (C7)] are shown in the second and third columns, respectively. Numerical results obtained within the accurate QSLD model in Ref. [22] are shown in the fourth column.

Critical point parameter	Zeroth order Eq. (C8)	First order Eq. (C7)	Numerical full QSMF	γ
$T_{{\rm Sk},c}$ (MeV)	20.1	15.1	15.3	1/6
$n_{{\rm Sk},c}$ (fm^{-3})	0.060	0.047	0.048	
$P_{{\rm Sk},c}$ (MeV fm^{-3})	0.325	0.194		
$T_{{\rm Sk},c}$ (MeV)	25.9	21.2	21.3	1
$n_{{\rm Sk},c}$ (fm^{-3})	0.065	0.059	0.059	
$P_{{\rm Sk},c}$ (MeV fm^{-3})	0.560	0.421		

$\omega_{\rm Sk}$ [Eq. (47)] at $\tau=0$, one approximately arrives at

$$\omega_{\rm Sk} \approx \frac{T_c n_c}{P_c} G_{{\rm Sk},\nu}\nu^{-2}, \quad (51)$$

where

$$G_{{\rm Sk},\nu} \approx \frac{P_c}{T_c n_c}\frac{2T_c}{\gamma(\gamma+1)(\gamma+2)b_{\rm Sk}n_c^{\gamma+1}} \approx 0.47. \quad (52)$$

In the last estimate, we used Eq. (C9) for the SLD parameters and Table II at $\gamma=1/6$. Notice that the first order in expansion of the inverse fluctuation ω^{-1} for $T=T_c$ over ν disappears because of the second equation for the critical point in Eq. (34). For the case of the classical SLD model ($\delta_c=0$), one finds from Eq. (52) a slightly different value, $G_{{\rm Sk},\nu}\approx 0.48$.

As the QvdW and QSLD fluctuations $\omega(T,n)$, Eqs. (38) and (47) respectively, are functions of the two variables T and n, one needs to introduce the two-dimensional critical-order index, 1 and 2. The first component is related to the fluctuation change along the T axis and the second one is along the n axis of the T,n range. Another characteristic of the critical point (T_c,n_c) in the T-n plane is the two-dimensional fluctuation-slope coefficient $\{G_{{\rm W},\tau},G_{{\rm W},\nu}\}\approx\{0.29,0.33\}$ for the QvdW and $\{G_{{\rm Sk},\tau},G_{{\rm Sk},\nu}\}\approx\{0.32,0.47\}$ for the QSLD approach. For the QvdW case, the slopes $\{G_{{\rm W},\tau},G_{{\rm W},\nu}\}$ depend on the vdW interaction parameters through the critical values T_c and n_c (and therefore P_c) and explicitly through the exclusion-volume constant b. In the case of the QSLD approach, $\{G_{{\rm Sk},\tau},G_{{\rm Sk},\nu}\}$, one obtains their dependence on the interaction parameters ($a_{\rm Sk}$, $b_{\rm Sk}$, and γ) for $G_{{\rm Sk},\tau}$ [Eq. (50)] only through the critical temperature T_c and density n_c while for $G_{{\rm Sk},\nu}$ one finds also the explicit dependence on $b_{\rm Sk}$, and γ. Notice also that the temperature, $\propto 1/\tau$ [Eqs. (42) and (49)], and the density, $\propto 1/\nu^2$ [Eqs. (45) and (51)], fluctuation dependence near the CP can be seen also from Eq. (36) for $\omega^{(c)}_3$. In the derivations of fluctuations $\omega^{(c)}_1$ and $\omega^{(c)}_{{\rm Sk},1}$ [Eqs. (42) and (45) and Eqs. (49) and (51), respectively] and $\omega^{(c)}_3$ [Eq. (36)] for the QvdW model we used Eq. (34) for the CP before

the CP limit, in contrast to the ω_1 [Eqs. (38) and (47)] and ω_3 [Eq. (33)] approximations. Thus, similarly, one obtains qualitatively the same properties of the fluctuations near the CP for the QSLD approach as for the QvdW model.

V. IMPROVED CALCULATIONS OF THE FLUCTUATIONS NEAR THE CRITICAL POINT

As mentioned in Introduction, we need to improve the calculations of the particle number (density) fluctuations to avoid their divergence near the critical point having zero incompressibility, $\mathcal{K} = 0$, as seen from Eq. (31). In order to clarify the behavior of fluctuations near the critical point, one has to expand the free energy $F(\rho)$ up to high-order terms in expansion (14) beyond the second-order approach considered in Sec. III. These high-order terms are needed because the second derivative of the free energy is zero at the CP. Assuming that the fourth derivative of the free energy is not zero at the critical point, one can stop the expansion (14) at the fourth-order term; see Eqs. (15) for $\Delta\{\rho\}$ and (16) for the probability distribution W_4. As shown in Appendix D, for the dispersion $\mathcal{D}_\rho$ at fourth order, Eq. (15), one obtains a more general expression, valid also at the critical point,

$$\frac{\mathcal{D}_\rho}{n^2} = \frac{\mathcal{A}}{2n^2}\left(\sqrt{1+\frac{4\langle\tilde{\Delta}\{\rho\}\rangle}{\alpha}}-1\right). \tag{53}$$

According to Eqs. (D2) and (D9), α and $\mathcal{A}$ are given by

$$\alpha = \frac{6\langle N\rangle\mathcal{K}^2}{n^2 T\mathcal{K}''}, \quad \mathcal{A} = \frac{12\mathcal{K}}{\mathcal{K}''}. \tag{54}$$

The statistical average of the dimensionless free energy difference $\tilde{\Delta}\{\rho\} = \Delta\{\rho\}/T$, Eq. (15), $\langle\tilde{\Delta}\{\rho\}\rangle$ can be largely approximated within the mean field approach by Eqs. (D16), (D17), and (D18). We introduced also the dimensionless parameter α, Eq. (D9), which is a measure of the effective distance from the critical point. Indeed, for large α, far from the critical point, one finds asymptotically, from a more general equation (53) for the dimensionless dispersion $\mathcal{D}_\rho$, the following limit:

$$\frac{\mathcal{D}_\rho}{n^2} \to \frac{\mathcal{D}_\rho^{(2)}}{n^2} = \frac{T}{\langle N\rangle\mathcal{K}}, \quad \alpha \gg 1. \tag{55}$$

The superscript m in $\mathcal{D}_\rho^{(m)}$ means the mth-order term of the expansion of $\mathcal{D}_\rho$, Eq. (15). In particular, for the second-order term of this expansion one has $m = 2$. The particle number dispersion D_N, Eq. (8), can be evaluated from the following approximate relationship:

$$\frac{D_N}{\langle N\rangle^2} = \frac{\mathcal{D}_\rho}{n^2}. \tag{56}$$

Using, then, Eqs. (53) and (56), one obtains

$$D_N \approx \frac{\mathcal{A}\langle N\rangle^2}{2n^2}\left(\sqrt{1+\frac{4\langle\tilde{\Delta}\{\rho\}\rangle}{\alpha}}-1\right). \tag{57}$$

For small α, near the critical point, up to small corrections of high order in powers of $1/\langle N\rangle$, from Eqs. (56) and (57) one approximately arrives at another known limit [39]:

$$D_N \to \frac{\mathcal{D}_\rho^{(4)}\langle N\rangle^2}{n^2} = \langle N\rangle^{3/2}\sqrt{\frac{6T}{n^2\mathcal{K}''}}, \quad \alpha \ll 1. \tag{58}$$

The supercript $m = 4$ in $\mathcal{D}_\rho^{(4)}$ shows the fourth-order term of the same expansion (15). With the expressions (54) for α and $\mathcal{A}$ in terms of the incompressibility $\mathcal{K}$ and its second derivative $\mathcal{K}''$, one can rewrite Eq. (53) for the particle number dispersion in a more explicit way. Taking, for instance, the normalization of the particle number dispersion D_N [see Eq. (57)] as in Eq. (31), for the sake of comparison, one obtains

$$\omega = D_N/\langle N\rangle \tag{59}$$

$$\approx \frac{6\langle N\rangle\mathcal{K}}{n^2\mathcal{K}''}\left(\sqrt{1+\frac{2\langle\tilde{\Delta}\{\rho\}\rangle n^2 T\mathcal{K}''}{3\langle N\rangle\,\mathcal{K}^2}}-1\right); \tag{60}$$

see Eqs. (D16)–(D18) for expressions for $\langle\tilde{\Delta}_F\rangle$ as functions of α. According to Eqs. (55) and (58), one finds the limits to the expressions for the two asymptotical traditional ($\alpha \gg 1$) and improved ($\alpha \ll 1$) dispersions in a more explicit form:

$$\omega = \frac{D_N}{\langle N\rangle} \to \frac{\langle N\rangle\mathcal{D}_\rho^{(2)}}{n^2} = \frac{T}{\mathcal{K}}, \quad \alpha \gg 1, \tag{61}$$

$$\omega = \frac{D_N}{\langle N\rangle} \to \frac{\langle N\rangle\mathcal{D}_\rho^{(4)}}{n^2} = \sqrt{\frac{6\langle N\rangle T}{n^2\mathcal{K}''}}, \quad \alpha \ll 1. \tag{62}$$

Both these limits, far from the critical point ($\alpha \gg 1$) and near the CP ($\alpha \ll 1$), are well known; see Refs. [9] and [39] (Appendix E), respectively, also Ref. [40] (Appendix F). The limit for $\alpha \ll 1$ in Eq. (62) to the CP is the same as in Ref. [39] if we neglect the second-order term and keep only the fourth-order component in derivations of Sec. III and Appendix D; see more details in Appendix E.

Figure 1 shows the dependences of the generalized expression (53) (solid line "1") for the dispersion $\mathcal{D}_\rho$ in the units, explained in the caption, as a function of the critical parameter α. The dashed ("2" and "3") and dotted ("4" and "5") lines present the asymptotes for $\alpha \gg 1$ and $\alpha \ll 1$, for the main term and its first correction, respectively:

$$\mathcal{D}_\rho \to \frac{n^2}{c_4^{1/2}}\frac{1-1/(2\alpha)}{2\sqrt{\alpha}}, \quad \alpha \gg 1, \tag{63}$$

$$\mathcal{D}_\rho \to \frac{n^2}{c_4^{1/2}}(1/2 - q\sqrt{\alpha}), \quad \alpha \ll 1. \tag{64}$$

where

$$q = 2\frac{[\Gamma(1/4)+\Gamma(3/4)]\Gamma(5/4)}{4\Gamma(1/4)\Gamma(5/4)} \approx 0.331, \tag{65}$$

and $\Gamma(x)$ is the standard gamma function. As seen from this figure, lines "2" and "5" show the well-known asymptotic results, Eqs. (55) and (58), in Refs. [38,39]. The convergence is seen even better if we take into account also the first corrections to these main components of the asymptotes [see Eqs. (63) and (64)]. We formally prolonged them analytically to other values of α far away from the limit boundaries, where we have main asymptotes [Eqs. (55) and (58)]. The reason is to find the values of α where one finds their convergence to a more general formula (53) far from ($\alpha \gg 1$) and near ($\alpha \ll 1$) the critical point, relatively. As seen from this figure,

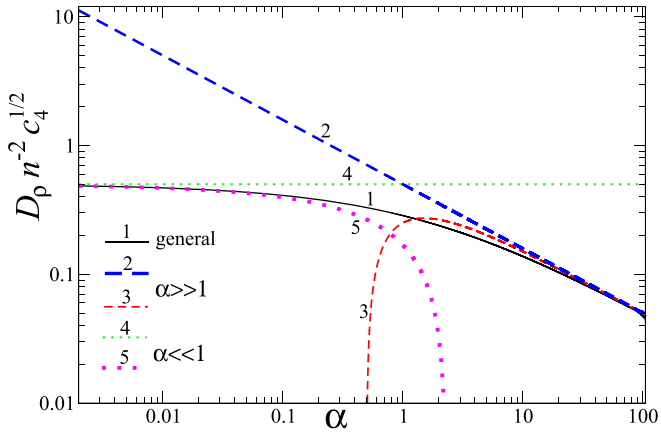

FIG. 1. The particle number density dispersion, $\mathcal{D}_\rho$, normalized by the factor $n^2/\sqrt{c_4}$, Eq. (F3) with (D3), as a function of the parameter α, Eq. (D9) [in Eq. (54)], for a symmetric nucleon system. Solid line "1" shows the generalized formula (53). Dashed lines (rare "2" and frequent "3") show asymptotes (the main term and that with the first correction) in expansion over $1/\alpha$) at $\alpha \gg 1$, valid far from the critical point, respectively; see Eq. (63). Dotted lines (frequent "4" and rare "5") present the opposite asymptotes (the main constant term, 1/2, and that with the first correction in expansion over $\sqrt{\alpha}$), Eq. (64), respectively.

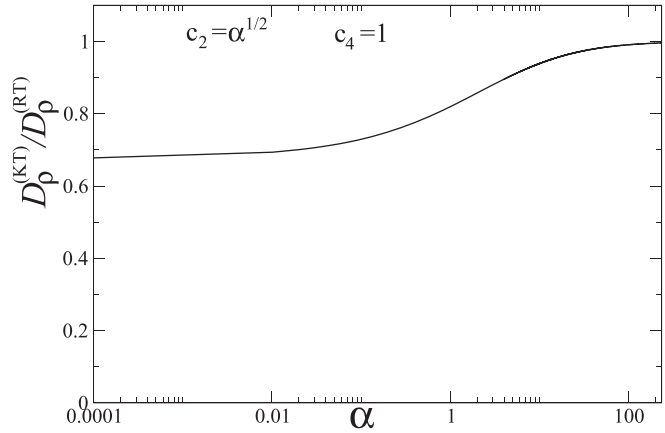

FIG. 2. Ratio of the particle number density dispersion, $\mathcal{D}_\rho^{(KT)}$, Eq. (F1) with Eq. (F2), in the K. Tolpygo (KT) approach (Appendix F) to that of $\mathcal{D}_\rho^{(RT)}$, Eq. (53), in the R. Tolman (RT) approach (Appendix D and Fig. 1) as a function of the same parameter α, Eq. (D9), or Eq. (F5) for $c_4 = 1$ and $c_2 = \sqrt{\alpha}$ for the same system as that of Fig. 1.

one can see their convergence at the ends of the shown interval of α. In some sense, the formula (53), derived in Appendix D in the mean field approach [10] neglecting density-density correlations, is more "universal" than the traditional fluctuation formula (31), as an analytical transition of the result for fluctuations ω from that at the critical point ($\alpha = 0$) to the result of Eq. (31). The tradition formula (31) is valid in fact far from the critical point at $\alpha \gg 1$ at any large but finite particle number average $\langle N\rangle$. We emphasize that this transition is presented independently of the specific effective interactions. Thus, from this figure, one can evaluate the values of α, as a measure of the distance from the critical point, for which one can use the asymptotes (61) and (62).

The expressions (60) and, in particular, (62) can be finite at the critical point if the second derivative of the incompressibility $\mathcal{K}$, $\mathcal{K}''$, is not zero. As noticed in Refs. [39,40], the result, Eq. (62), for some intensive systems agrees better with the experimental data on opalescence than the traditional Eq. (31). However, we should note that the approaches used in Refs. [38,39] (Appendices D and E) and in Ref. [40] (Appendix F) with approximately the same probability distribution W_4, Eq. (16), for calculations of the statistical averaged dispersion, or variance $\langle(\rho - n)^2\rangle$, are somewhat different. As shown in Appendix D for the derivations by the Tolman approach [38], the statistical consistency condition (D7) for the quantity $(\rho - n)^2$ in average, $\langle(\rho - n)^2\rangle$, and the mean field approach neglecting density-density correlations, with the expansion (14), is essentially used, in contrast to the Tolpygo approach [40]; see Appendix F. This explains a difference in analytical results for the dispersion D_ρ in Appendices D and F. Therefore, the limits of the variance D_ρ in both compared approaches, to the critical point ($\alpha \to 0$), are different by a constant; cf. Eqs. (58) and (F6) (see Fig. 2). As seen from Fig. 2, the limit for large α is the same for these compared approaches. Another peculiarity of our approach to the classical fluctuation theory in both discussed versions (see Appendices D and F) is the basic one-parameter analytical transition over the effective distance α from the CP, in contrast to the two-parameter analytical transition over $c_2 \propto F_2$ and $c_4 \propto F_4$, separately. Both these approaches are remarkable in showing that for the mean field approach (up to the correlations above a mean field) for the finite particle number average $\langle N\rangle$ of a nuclear matter piece is finite everywhere including the critical point, in contrast to the traditional divergent result, $\omega = T/\mathcal{K}$. Notice also that in this way we may consider high-order critical points by taking into account high-order terms in the expansion (15) for the free energy. It is clear also how to extend Appendix D to the fluctuation results, accounting for even more important effects of density-density correlations.

So far we did not need to specify the interactions which are presented here only in terms of the pressure of the equation of state through the incompressibility and its second derivatives. Notice also that, for relatively large temperatures T and small mean particle-number densities n, the quantum statistics parameter ε [Eq. (37)] is small. Therefore, in this part of the T-n plane, in contrast to the calculations of the critical points, for simplicity one can neglect the quantum statistics effects in the pressure for approximate evaluations of the fluctuation ω. Indeed, as shown in the previous section, the fluctuations ω within the QvdW and QSLD models do not depend much on these effects. Therefore, we will first consider more accurate calculations, near the CP, of the fluctuations ω in terms of the same vdW and SLD pressures of the corresponding equations of state, neglecting small quantum statistics corrections [9,44,45].

Substituting now the pressure for the vdW equation of state (C1) at $\delta = 0$ into Eq. (62), valid near the CP, one obtains

$$\omega = \frac{(1 - bn)^2\sqrt{\langle N\rangle}}{bn} \quad (\alpha \to 0)\ \text{vdW}. \tag{66}$$

Notice that this result is independent of temperature T and of the attractive vdW constant a but depends on the product of the particle number density n times the repulsive exclusion-volume interaction constant b. It is not the case for the SLD interparticle interaction. As expected, the value of ω at the critical point (C4) is finite (of the order of or smaller than $\sqrt{N}$), for a finite average number of particles, $\langle N\rangle$. More accurately, this value of the fluctuations, ω, is $4\sqrt{\langle N\rangle}/3$. Notice that this value is a little larger than that in Ref. [40] because of reasons explained above in this section; see Appendices D and F. Substituting the SLD equation of state (C6) at $\varepsilon = 0$ into Eq. (62), one arrives at

$$\omega = \sqrt{\frac{6T\langle N\rangle}{\gamma(\gamma+1)(\gamma+2)bn^{\gamma+1}}} \quad (\alpha \to 0)\ \text{SLD}. \tag{67}$$

As seen from this expression, the SLD fluctuations depend on the temperature T and interaction constants b and γ, but are independent of the attractive interaction constant a as in the vdW case. For the value at the critical point, one obtains also finite results of the order of $\sqrt{N}$, namely, $1.45\sqrt{\langle N\rangle}$ for $\gamma = 1/6$ and $0.44\sqrt{\langle N\rangle}$ for $\gamma = 1$ for a given value of the particle number average $\langle N\rangle$. Thus, in contrast to the traditional expression (31), for the fluctuations ω, Eqs. (66) and (67), valid in the limit to the CP, depend on the mean particle number $\langle N\rangle$ by a factor which is proportional to the value of $\sqrt{\langle N\rangle}$.

VI. DISCUSSION OF THE RESULTS

Figure 3 shows the particle number fluctuations $\omega(T, n)$ as a function of the dimensionless temperature T/T_c versus density n/n_c variables for symmetric nuclear matter by the traditional calculations employing Eq. (31). The zeroth-order approximation using Eq. (C3) [vdW (a)] and the first-order [QvdW (b), Eq. (38)] approach within the quantum statistics expansion over δ are shown in these contour plots. The contour plot of Fig. 3(b) presents the calculations of fluctuations ω_1 [Eq. (38)] without using an expansion over a distance from the critical point. As seen from Fig. 3 [cf. panels (a) and (b)], the quantum statistics effects in fluctuations ω are small, as demonstrated by their numerical values. Note that we excluded a large shift of the critical point by choosing the scaling CP units. Then, the panels (a) and (b) become qualitatively very similar. As a function of the density n, the ω_1 contour plot (b) is approximately symmetric with respect to the CP. The vdW contour plot (b) is only a little asymmetric far from the CP. As functions of the temperature T, both plots [(a) and (b)] are similar but very asymmetric with respect to $T = T_c$. Therefore, they are shown only above the critical point, $T > T_c$. Huge values of the fluctuations near the critical point are shown by white regions. Contour plots for fluctuations ω at a few next high orders in the quantum statistics expansion over δ are visually almost the same as for the first order and, therefore, are not shown in Fig. 3.

Figure 4 presents a comparison between fluctuations ω using different approximations [within Eq. (31)] near the CP, separately, as functions of the mean density n, $T = T_c$, panel (a), and temperature T, $n = n_c$, panel (b), both with a better

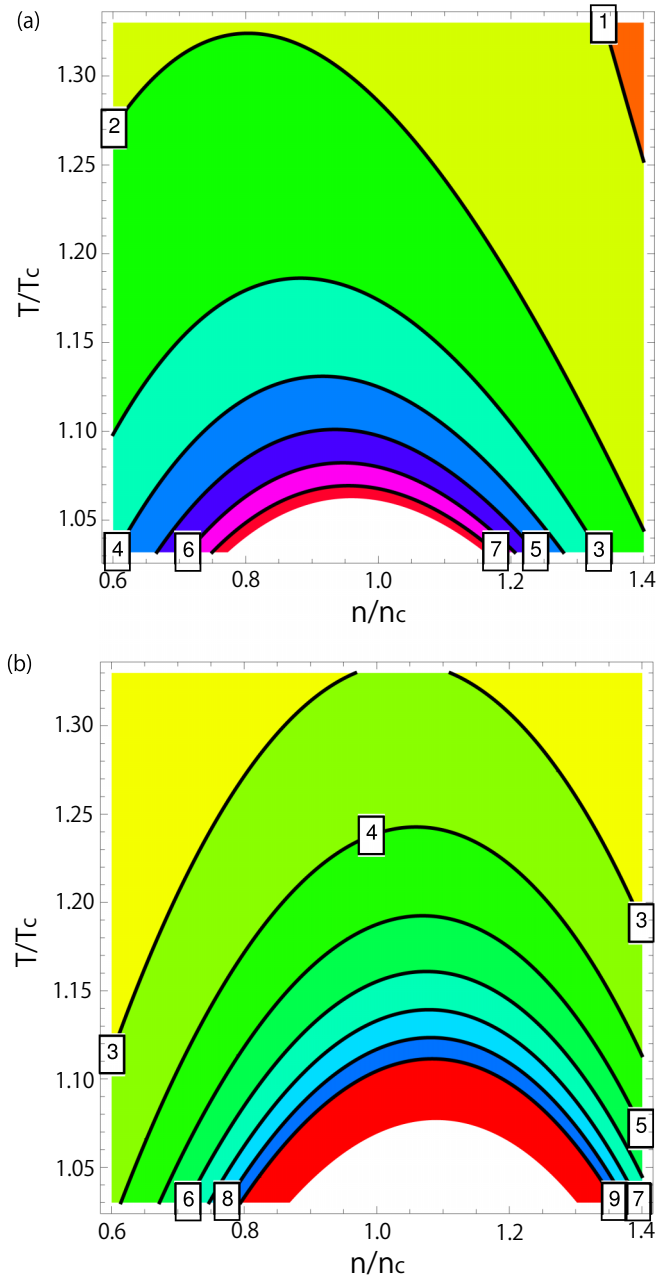

FIG. 3. Contour plots for the QvdW approximations to the particle number fluctuations ω as functions of the averaged density n and temperature T (in units of the corresponding critical values T_c and n_c) near the critical point. The zero approximation, vdW (a), Eq. (31) with the vdW pressure [Eq. (C3)], and the first-order QvdW approach (b) in the quantum statistics expansion over a small parameter δ [Eq. (38)]. Numbers in white squares at lines of constant fluctuations $\omega(T, n)$ show their values.

resolution (see Sec. IV A). Huge bumps near the CP in the fluctuations ω_1 [solid line "1", Eq. (38)] are shown in both panels of this figure. Similar bumps appear near the CP for fluctuation ω_3 as those in Eq. (33) for ω_1, which is not shown therefore in Fig. 4 for simplicity: These two approaches, ω_1 and ω_3, near the CP converge to each other in the limit to the CP with decreasing distance from the CP. A divergence of

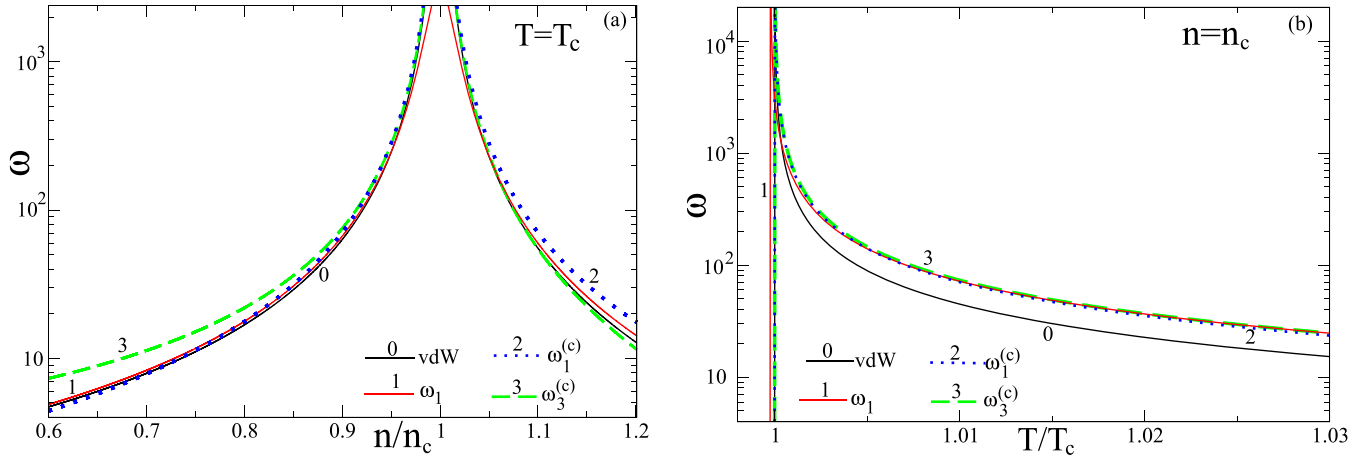

FIG. 4. Fluctuations of the particle numbers, ω, for a nucleon system as function of the mean particle-number density n (a) and of the temperature T (b) in units of critical values n_c and T_c, respectively. Solid black lines "0" show the zeroth order (vdW), and other lines present different approximations with the first quantum-statistics correction; solid red lines "1" are the results of calculations by Eq. (38) for ω_1; dotted blue lines "2" show Eqs. (45) in (a) and (42) in (b) for $\omega_1^{(c)}$; dashed green lines "3" are given by Eq. (36) for $\omega_3^{(c)}$.

the fluctuations [see Eq. (31) for ω] at the CP peak is seen explicitly in the $\omega_1^{(c)}$ "2" curves [dashed blue, Eqs. (45) in Fig. 3(a) and (42) in Fig. 3(b)], as well as in the $\omega_3^{(c)}$ "3' lines' [long-dashed green, Eq. (36)]. They explicitly diverge at the CP as in the standard vdW approach (thin black solid line). Notice that the lines "2" and "3" converge to each other better, the smaller the distance is from the CP. This is naturally in good agreement with the analytical arguments based on Eq. (36), and in line with the arguments given in Sec. IV A. Such an agreement becomes essentially worse with increasing distance from the CP. Both the "2" and "3" curves have a similar divergent behavior because in calculations of both curves we neglected first- and second-order derivatives of the isothermal incompressibilities over density n near the critical point, Eq. (34). A huge sharp bump in the density ($T = T_c$) (a) and, even much sharper, in the temperature ($n = n_c$) (b) dependence for different approximations are largely in agreement. This agrees also with the accurate numerical calculations [21] using the same formula (31) for the fluctuations ω at the incompressibility $\mathcal{K}$, close to zero in the CP limit, $\mathcal{K} \to 0$. As seen from Fig. 3, differences between the position of this bump and CP values for the temperature dependence (b) are more pronounced in contrast to the density function (a). But, in fact, these differences are relatively very small within errors of the derivations (see also Fig. 3). Note also that the density n behavior (a) is largely symmetric with respect to the CP, in contrast to a very asymmetric temperature T dependence (b). This is seen also in the contour plots of Fig. 3 where we show T ranges only above the CP.

Notice that it is obviously impossible to realize practically the conditions for validity of the considered approximations to the fluctuations ω calculated in terms of the incompressibility $\mathcal{K}$ by Eq. (31) in the limit to the CP. We have to involve more and more terms of expansion of the variation derivative of the incompressibility $\mathcal{K}$ [Eq. (31)] over a distance from the CP. On the way to the CP, one has to stop at small but finite distance from the CP where a huge bump appears. The considered variations fail because they become smaller or of the order of next derivatives contributions in the expansion of the incompressibility $\mathcal{K}$ in the denominator of the fluctuations ω, Eq. (31), beyond Eq. (33); see Refs. [38–40]. As mentioned above, one may find also arguments for validity of the derivations of Eqs. (31) [or Eq. (A1)] for the fluctuations ω through the derivatives of the thermodynamic averages (pressure or particle number density) in Refs. [9,33,38–40,42,43]. As emphasized in these works, large values of relative fluctuations are in contradiction with the basic assumptions of statistical physics because thermodynamic averages, defined up to their fluctuations, become meaningless [9,33,38,39,42,43]. According to the assumptions in these derivations (see Secs. III and IV), we should have an opposite tendency, namely, that the relative fluctuations ω must be small, in particular near the critical point. Therefore, more accurate calculations of the particle number fluctuations in terms of the statistically averaged Gibbs distribution over particle numbers should be considered in a very close range near the critical point of nuclear matter.

In order to compare with the traditional calculations of Sec. IV (Figs. 3 and 4), we will discuss now the fluctuations within a more general theory (see Sec. V) for the same two simple examples of the vdW and SLD approaches to the interparticle interactions but neglecting small quantum corrections. We will discuss then the asymptotic approximations to the generalized formula (53) for the particle number fluctuations ω far from and close to the critical point [Eqs. (61) and (62)].

Figure 5 shows the particle number fluctuations ω [Eq. (60), solid lines] divided by constant $\langle N\rangle$, $\omega/\langle N\rangle = D_N/\langle N\rangle^2$, where D_N is the dispersion for the vdW interparticle interaction parameters, critical temperature ($T = T_c$), and several typical particle numbers averages. Their asymptotes, Eq. (61), for large α are shown at the same values of the particle number average $\langle N\rangle$ (short double lines mean an interruption of the lines to simplify the presentation of the figure). See also Fig. 6 for the critical parameter α as function of the particle number density n/n_c at the critical value of the temperature $T = T_c$ and the same set of values of $\langle N\rangle$.

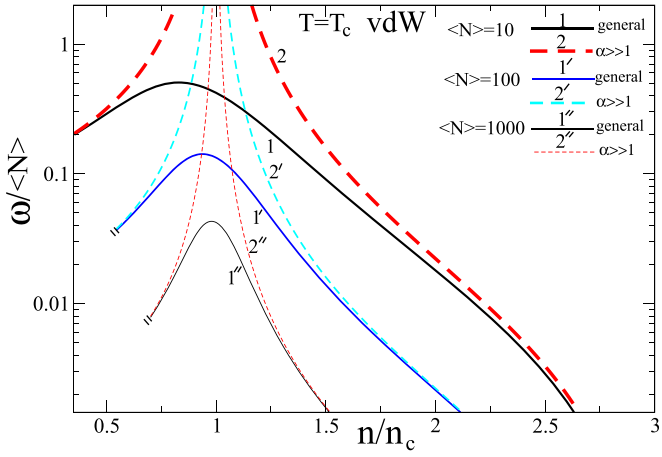

FIG. 5. The particle number fluctuations ω [Eq. (60), solid lines], divided by constant $\langle N\rangle$, i.e., the dispersion D_N normalized by $\langle N\rangle^2$, are shown as functions of the average particle number density n (in units of its critical value) at the critical temperature $T = T_c$ for a symmetric nucleon system with the vdW effective interaction at different particle number averages $\langle N\rangle$. Dashed lines present the corresponding traditional asymptote, Eq. (61) ($\alpha \gg 1$). The particle number averages $\langle N\rangle = 10$ ("1" and "2"), 100 (the same but with primes), and 1000 (with double primes) are taken as typical examples. Solid lines "1", "1′", and "1″" are obtained by the generalized formula (53); and dashed lines "2", "2′", and "2″" show the traditional asymptote (61) in the same units. In order to compare with the traditional approach, the parameters of the vdW effective interactions are given by Eq. (C5) [see Eq. (C2) and Table I for the critical values].

Dashed lines are the traditional approach (61) for $\alpha \gg 1$, valid far from the critical point (Fig. 5). This (traditional) approach is related to the second-order power expansion of the free energy $F(\rho)$ over difference $\langle\rho\rangle - n$ in Eq. (14) for the fixed temperature T at the critical point, $T = T_c$ (see Sec. III). The dashed lines present the second-order asymptote of the

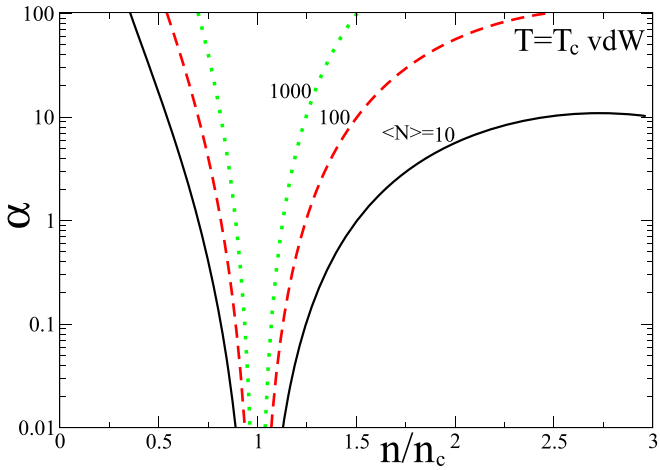

FIG. 6. The parameter α, Eq. (D9), as function of the particle number average n in the critical value units n_c at the critical temperature $T = T_c$ for the vdW interaction, and at the same values of the particle number averages $\langle N\rangle$.

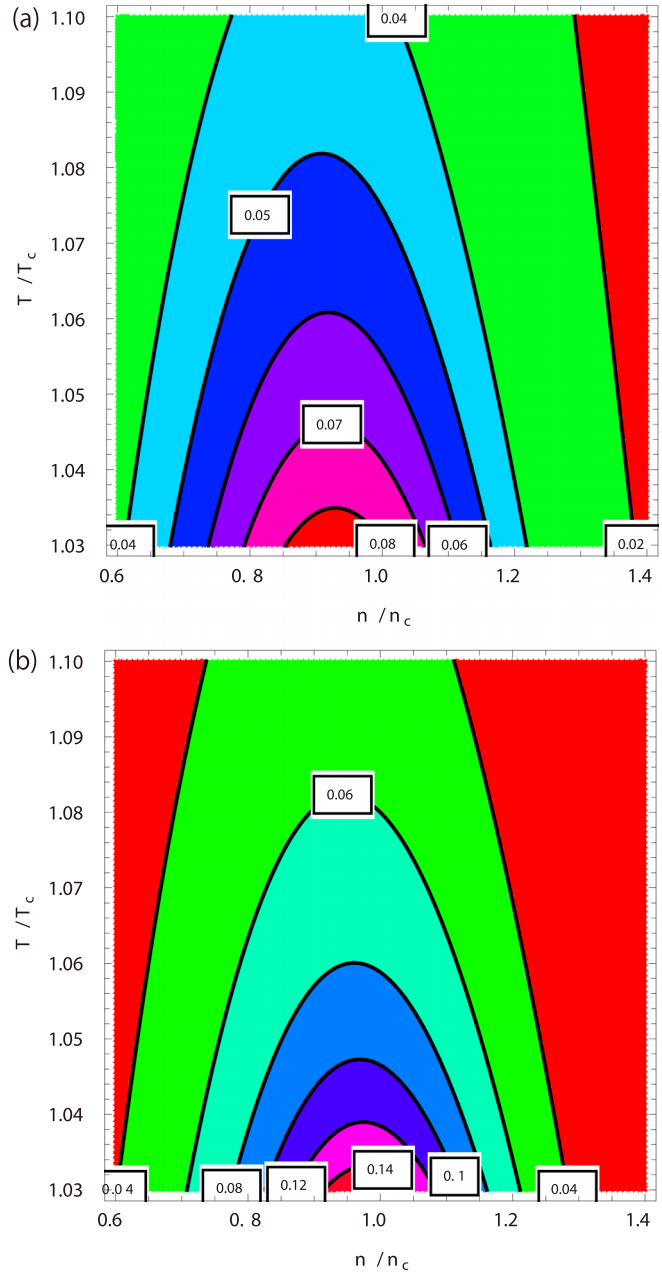

FIG. 7. Contour plots for the improved calculations of the fluctuations ω [see Eqs. (60) (top) and (62) (bottom)], both divided by the particle number average $\langle N\rangle$, as functions of particle number density n and temperature T in their critical values' units. The interval of n/n_c is the same as in Fig. 3. Slightly smaller temperatures T/T_c are taken to see more details near the critical point. For example, we use $\langle N\rangle = 100$ in these plots.

generalized formula (53) at $\alpha \gg 1$. This traditional result is the same as that of Eqs. (31) and (61) for the fluctuations ω, shown in Figs. 3(a) and 4(a), as a curve for the pure vdW approach neglecting the quantum effects but with another normalization. The normalization of the dispersion D_N in Fig. 5 is taken as $\langle N\rangle^2$ for a uniform comparison at different effective distances α from the CP. There is clearly seen a divergence of this asymptotic ($\alpha \gg 1$; see dashed lines) approach at the

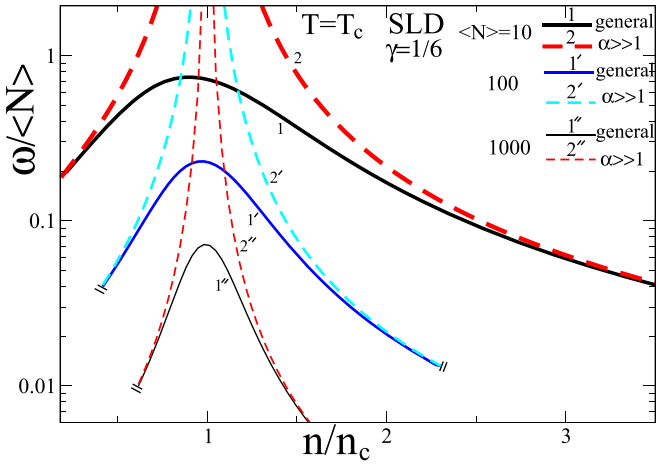

FIG. 8. The same as in Fig. 5 but for the SLD effective interaction with the same parameters [Eq. (C9) and critical values of Eq. (C7) and Table II] as in Sec. IV for $\gamma = 1/6$.

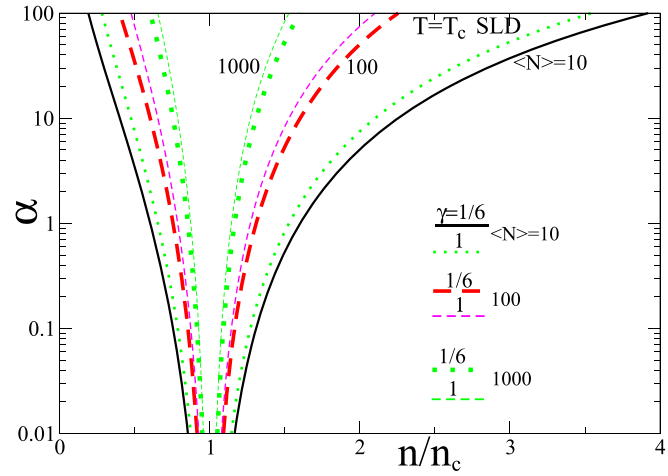

FIG. 10. The same as in Fig. 6 but for the SLD interaction with $\gamma = 1/6$ and 1.

critical point as in Fig. 3(a). As seen from Fig. 5, one obtains a maximum of the finite small value near the critical density value, $n \approx n_c$. This maximum in the dependence on the particle number density n, $\omega/\langle N\rangle \approx D_N/\langle N\rangle^2$, near the critical temperature $T \approx T_c$, monotonically decreases rapidly with increasing particle number average $\langle N\rangle$, in contrast to an increasing behavior of the dispersion D_N [Eq. (8)]. Notice that our analytical calculations shown in Fig. 5 are in a qualitative agreement with the numerical results presented in Fig. 11 of Ref. [11]. These results were obtained by using the numerical statistical percolation model of the phase transitions [58]. We should only take into account that the second variance in Ref. [11] is related to the dispersion D_N, i.e., the fluctuation $\omega/\langle N\rangle$ in Fig. 5, multiplied by $\langle N\rangle^2$.

Figure 7 presents contour plots for the fluctuations ω over the particle number average $\langle N\rangle$, i.e., the quantity $\omega/\langle N\rangle$. In the upper plot (a) we show the improved results of calculations, according to Eq. (60), while in the bottom plot (b) we consider the limit of Eq. (60), Eq. (62), at $\alpha \ll 1$. As seen

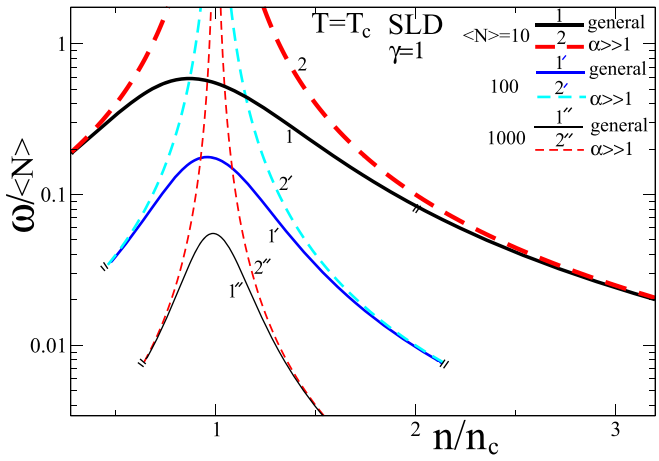

FIG. 9. The same as in Fig. 8 but for the SLD interaction with the parameters of Eqs. (C9) and (C7) at $\gamma = 1$ (Table II).

from these two plots, the results are similar in both panels near the critical point. We find in both plots a final maximum at a finite particle number average $\langle N\rangle$, in contrast to another limit result, Eq. (61), shown in Fig. 3(a). It is convenient to normalize the fluctuations as $D_N/\langle N\rangle^2$ because the dispersion D_N is of the order of $\langle N\rangle^2$ near the critical point ($\alpha \ll 1$). This is in contrast to the results for the fluctuations, valid far from the critical point ($\alpha \gg 1$) where D_N is of the order of $\langle N\rangle$, as usual in the standard statistical physics [9].

Figures 8 and 9 show qualitatively the same fluctuations, $\omega/\langle N\rangle$, as in Fig. 5, but for the SLD interaction with parameters $\gamma = 1/6$ and $\gamma = 1$, respectively; see also Fig. 10 for the critical parameter α as function of the particle number density n/n_c for the SLD case at both values of γ. The difference between the vdW and SLD cases is only in a slightly greater asymmetry of the vdW curves and their small deflections of the maxima from the critical point. We may note also slightly larger values at maxima in Figs. 8 and 9, as compared with the vdW results presented in Fig. 5 for the same particle number averages $\langle N\rangle$. The same qualitative agreement with the results of Ref. [11] was found as for the vdW interparticle interaction, mentioned above.

VII. SUMMARY

The generalized particle number fluctuations ω are derived for an isotopically symmetric nuclear matter within the Smoluchowski Einstein statistical theory. This more general result is obtained by using the fourth-order expansion of the free energy $F(\rho)$ over small difference of the particle number density ρ from its average n, and including the second-order terms. Thus, we found the fluctuation ω as a function of the dimensionless parameter $\alpha \propto \mathcal{K}^2\langle N\rangle/n^2 T\mathcal{K}''$, where $\mathcal{K}$ and $\mathcal{K}''$ are the isothermal incompressibility and its second derivative at a given temperature T. In the limit of large α, $\alpha \gg 1$, we derived the traditional asymptotic expression for the fluctuations ω, $\omega \propto 1/\mathcal{K}$. This result is equivalent to that obtained early by the second-order power expansion of the free energy $F(\rho)$ over the particle number density difference

$\rho - n$, where n is the average of ρ. For small values of α near the critical point, $\alpha \ll 1$, one finds another known finite asymptotic expression of ω. This expression is improved locally near this point at finite particle number averages $\langle N\rangle$. Such an asymptote was derived early by using the fourth-order power expansion of the free energy $F(\rho)$ over small $\rho - n$ but neglecting the second-order term which is zero at the critical point. We found that the values of α determine the effective distances from the critical point where one can apply these well-known asymptotes. These results are obtained for any interparticle interactions. In addition, these two asymptotes were studied in detail by using the specific vdW and SLD interactions as simple examples.

Equations of state obtained within the quantum van der Waals (QvdW) and Skyrme local density (QSLD) approaches were used to study analytically the particle-number fluctuation ω, first by the traditional calculations. These analytical calculations were performed in terms of the isothermal incompressibility $\mathcal{K}$, $\omega \propto 1/\mathcal{K}$, in the vicinity of the critical point in isotopically symmetric nuclear matter. The expressions for the fluctuations ω are obtained accounting for the leading first-order corrections using the quantum statistics expansion over the small parameter $\propto \varepsilon \propto \hbar^3 n/g(mT)^{3/2}$ in the QvdW model and that (ε) in the QSLD model. A simple and explicit dependence of the particle number fluctuations ω on the system parameters, such as the particle mass m, degeneracy factor g, and interaction parameters a and b for the QvdW and $a_{\rm Sk}$, $b_{\rm Sk}$, and γ for the QSLD approaches, is demonstrated at the first order of this expansion. Such an analytical dependence on the particle mass m and degeneracy factor g is absent within the classical vdW and SLD approximations. The quantum correction effects, which are quite significant to obtain the CP parameters of the nucleon matter, appear to be small for the fluctuations ω. They lead to a notable asymmetry of the $\omega(T, n)$ values in the T-n plane as function of temperature T for both discussed models. In this respect, the temperature dependence of the fluctuations ω is especially pronounced for all these approximations.

We derived the analytical expressions for the fluctuations ω in terms of the incompressibility $\mathcal{K}$ near the critical point as functions of the distances from the CP, in units of $T_{\rm c}$ and $n_{\rm c}$. For the temperature T behavior of the fluctuations ω at constant critical density, $n = n_{\rm c}$, one obtains $\omega \propto (T - T_{\rm c})^{-1}$ with the critical index -1 for the order parameter $T - T_c$ of the Landau theory of phase transitions. The particle number density n dependence of ω at $T = T_{\rm c}$ has another critical index -2, $\omega \propto (n - n_{\rm c})^{-2}$, for the order parameter $n - n_c$. The temperature behavior of the fluctuations ω was obtained to be qualitatively the same for the QvdW and QSLD approaches but with slightly different slope coefficients. They are in good agreement with more accurate numerical calculations for the QvdW case. To our knowledge, there are no numerical results for the fluctuation slope constant in the QSLD case. The QvdW density dependence near the CP is essentially different from that of the SLD model by the slope coefficient. This is in contrast to the slope coefficients in temperature dependence of the fluctuations ω. We found good qualitative and quantitative agreement between these analytical results and those accounting for a high-order derivative expansion near the critical point which were suggested by Tolman and Rowlinson.

In line with the accurate traditional numerical calculations of the particle number fluctuations ω in terms of the incompressibility $\mathcal{K}$, $\omega \propto 1/\mathcal{K}$, we found analytically an expected huge bump near the critical point. The obvious reason is the divergence in the zero incompressibility limit, $\mathcal{K} \to 0$, at the CP, for all compared approaches to the incompressibility $\mathcal{K}$. The results are similar to those of the approximate first-order analytical and more accurate numerical calculations realized with and without using the expansion of the incompressibility near the CP at zeroth- (vdW or SLD) and first-order (QvdW or QSLD) approaches over a small parameter of the quantum statistics, respectively. Several leading high-order derivative approximations to the incompressibility $\mathcal{K}$ were analyzed near the critical point. The convergence of the simplest explicitly given analytical results for the fluctuations ω near the critical point to their approximations, suggested by Tolman and Rowlinson in Refs. [38,39], was found for the isothermal incompressibility $\mathcal{K}$. This is expected because, as is well known, the traditional calculations of particle number fluctuations ω in terms of the incompressibility diverge at the critical point for infinite nuclear matter. Therefore, these results cannot be applicable in a close distance from the CP. They lead to indetermination of the corresponding averaged particle numbers, which are defined up to their fluctuations, in the equation of state. The well-known reason is that the derivation of these particle number fluctuations ω in terms of the isothermal susceptibility, or the incompressibility $\mathcal{K}$, from the original definition through the moments of the Gibbs distribution over particle numbers in the grand canonical ensemble fails if fluctuations are not small. This is common for any used interparticle (vdW and SLD) interactions. These results of the fluctuation calculations are weakly dependent on the quantum statistics corrections.

We analyzed the particle number fluctuations ω, improved near the critical point, for a finite particle-number piece of nuclear matter. We took into account the additional fourth-order terms in expansion of the free energy $F(\rho)$ in powers of small difference between the density ρ and its average n beyond the quadratic approximation of the traditional classical-fluctuations theory, but along with the quadratic terms. Using the vdW and SLD interparticle interaction approaches and neglecting small quantum statistical effects, we obtained analytically finite values of the particle number fluctuations ω near the critical point at any finite particle number average $\langle N\rangle$. This is in contrast to the traditional divergent calculations in terms of the incompressibility or particle number density susceptibility. As shown in our calculations, the fluctuations ω, divided by the particle number averaging $\langle N\rangle$, have a relatively small finite maximum near the critical point. This maximum of the particle number fluctuations, $\omega/\langle N\rangle$, decreases with increasing particle number average $\langle N\rangle$, having the zero limit when $\langle N\rangle$ goes to the infinite. For the dispersion (or variance) D_N, one respectively finds the increasing dependence on $\langle N\rangle$, in agreement with the numerical results obtained earlier by the percolation model of phase transitions. A range of the critical point vicinity, where the traditional ($\alpha \gg 1$) results for fluctuations, $\omega \propto 1/\mathcal{K}$, cannot be applied,

decreases with increasing particle number average $\langle N\rangle$. The transition range between the two asymptotes, $\alpha \gg 1$ and $\alpha \ll 1$, is smaller the larger the value of $\langle N\rangle$ is for significantly large values of $\langle N\rangle$.

As perspectives, we are going to develop the Smoluchovski Einshtein method to higher-order expansions over the order parameter $\rho - n$ than the present fourth-order approach for studying the phase transitions. We will study also the fluctuations near the critical point in terms of moments of the statistical level density by using another alternative microscopic-macroscopic approach for finite Fermi systems [54–57]. The improved saddle-point method [59–65] for analytical calculations of the inverse Laplace integrals for the level density near the critical point will be used to remove the divergences. Then, we will calculate the level-density moments averages over the particle number and other variables by using the initial definition for the corresponding statistical fluctuations. Our derivations within the vdW and SLD forces can be straightforwardly extended to other types of interparticle interactions, in particular, to more general and more realistic statistical nuclear approaches. In particular, our derivations might be extended to account for the isotopic proton-neutron asymmetry. We believe that our results of interest also to shed more light on the reasons for the experimental opalescence phenomenon data.

ACKNOWLEDGMENTS

We gratefully thank M. I. Gorenstein and A. S. Sanzhur for many fruitful discussions and suggestions, as well D. V. Anchishkin, A. Bonasera, J. Natowitz, R. V. Poberezhnyuk, and V. Vovchenko for many useful discussions. We thank also very much A. Haensel for important help with the computer facilities used in our calculations. S.N.F., A.G.M. and U.V.G. acknowledge support in part by the budget program "Support for the development of priority areas of scientific researches," a project of the Academy of Sciences of Ukraine (Code 6541230, No. 0122U000848). A.G.M. acknowledges also support in part by the U.S .Department of Energy under Grant No. DE-FG03-93ER-40773.

APPENDIX A: FLUCTUATIONS AND SUSCEPTIBILITY

According to Ref. [42], taking the variations of both sides of Eq. (1) over μ with the help of Eqs. (2) and (3) and changing the order of the integrations over the phase space Γ and derivative over the chemical potential μ, for small first-order variations $\delta\mu$ in μ, for the particle number fluctuations $D_N/\langle N\rangle$, where $D_N = \langle(\Delta N)^2\rangle$ is the particle number dispersion normalized to $\langle N\rangle$, one obtains

$$\omega(T,n) = \frac{T\chi}{n}, \quad \chi = \left(\frac{\delta n}{\delta\mu}\right)_T, \tag{A1}$$

where $n = n(T,\mu)$ is the particle number density average in the grand canonical ensemble. Notice that this result is the same as that found in Ref. [38] [Eqs. (8) and (10)] in the second-order approximation of the Smoluchovski and Einstein fluctuation theory (Sec. III). Evaluating this linear response χ far from the critical point as $(\delta n/\delta\mu)_T \sim n/\mu$, one finds small fluctuations $\omega(T,n) \sim T/\mu$ if $T/\mu \ll 1$, i.e., for relatively small temperatures.

In Eq. (A1), the variation derivative is the isothermal susceptibility χ. Assuming, again, small relative fluctuations with respect to the average particle number, at the linear (first-order) variations, we can restrict ourselves to the linear response function (linear susceptibility),

$$\chi^{(1)} = (\partial n/\partial\mu)_T. \tag{A2}$$

Within this linear approximation, one has explicitly

$$\omega \approx \omega^{(1)}(T,n) = \frac{T}{n}\left(\frac{\partial n}{\partial\mu}\right)_T. \tag{A3}$$

The linear response χ [Eqs. (A1)–(A3)] diverges at the critical point, in contrast to its derivations. As shown in Appendix B under the same condition of small fluctuations, one obtains from Eq. (A1) [in particular, from Eq. (A3)] the well-known expression (31) [or Eq. (32)] for the fluctuations, normalized to $\langle N\rangle$, in terms of the isothermal incompressibility $\mathcal{K}$ [9,32,37–42].

Let us consider variations of the relationship (1) over the chemical potential μ, taking into account high-order variations, for instance second-order ones. We will still take these variations at constant temperature T, i.e., consider nonlinear (second-order) isothermal susceptibility $\chi^{(2)}$. Equation (A1) is valid for any order of the variation derivative (nonlinear susceptibility), but now one can specify it for the second-order fluctuations $\omega^{(2)}$. Taking immediately the variations over μ up to the second order at $T = \text{const}$ in Eq. (1), one obtains high (second) order corrections to Eq. (A3). Equation (A3) is named usually the second cumulant of the averaged Gibbs distribution function, Eq. (2), averaged over the phase space. The dispersion $\delta^{(2)}(\langle N\rangle)$, taking into account up to the third cumulant moment of the averaged Gibbs distribution, takes the form

$$\frac{T}{\langle N\rangle}\delta^{(2)}(\langle N\rangle) = \omega^{(1)}(\delta\mu)^1 + \frac{1}{2T}\omega^{(2)}(\delta\mu)^2 + \cdots, \tag{A4}$$

where $\omega^{(2)}$ is the so-called kurtosis. It can be normalized by $\langle N^2\rangle$, in analogy with $\omega^{(1)}$, Eq. (A3) (see Ref. [24]): $\omega^{(2)} = (\langle N^3\rangle - \langle N\rangle^3)/\langle N^2\rangle$. Similarly, one can obtain the third-order moment (or third cumulant) of the averaged Gibbs distribution, Eq. (2). This third-order moment is coming from the third-order variations of the average $\langle N\rangle$, Eq. (1), over the chemical potential μ, and so on. This allows us to go beyond the restrictions of the first-order cumulant fluctuations $\omega^{(1)}$, shown explicitly in Eq. (A3). Namely, this is beyond the first variation derivative for the susceptibility χ: linear susceptibility $\chi^{(1)}$, Eq. (A2). The expression (A1) for the fluctuation ω of the particle number is more general. However, it is still singular exactly at the CP where the linear susceptibility $\chi^{(1)}$ (A2) is infinity in the sum (A4).

APPENDIX B: DERIVATIONS OF THE CLASSICAL PARTICLE-NUMBER FLUCTUATIONS

Within the canonical ensemble, one can use the free energy $F(T,N,V)$, Eq. (6), as a characteristic thermodynamic function of the temperature T, particle number N, and volume V.

Assuming the thermodynamic limit condition for our infinite system, one can express F in terms of that per particle [9],

$$F(T,N,V) = Nf(T,\tilde{v}), \tag{B1}$$

where $\tilde{v}$ is the volume per particle,

$$\tilde{v} = \frac{1}{n}, \quad n = N/V. \tag{B2}$$

For the pressure P and chemical potential μ, one has

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = -\left(\frac{\partial f}{\partial \tilde{v}}\right)_T \tag{B3}$$

and

$$\mu = \left(\frac{\partial F}{\partial N}\right)_T = f - \frac{1}{n}\left(\frac{\partial f}{\partial \tilde{v}}\right)_T. \tag{B4}$$

Taking the first variation of Eq. (B4) over particle number density n through the relationship (B2), one obtains

$$\delta\mu = \frac{1}{n^3}\left(\frac{\partial^2 f}{\partial \tilde{v}^2}\right)_T \delta n. \tag{B5}$$

Therefore, one finds

$$\left(\frac{\partial n}{\partial \mu}\right)_T = \frac{n^3}{(\partial^2 f/\partial \tilde{v}^2)_T}. \tag{B6}$$

According to Eq. (A3) and Eqs. (B6), (B3), and (B2), one arrives at Eq. (31).

Note that the same result can be obtained more easily by using the Jacobian (linear) transformations [9]

$$\left(\frac{\partial n}{\partial \mu}\right)_T = \frac{D(n,T)}{D(\mu,T)} = \frac{1}{D(\mu,T)/D(n,T)} \tag{B7}$$

and

$$n = \left(\frac{\partial P}{\partial \mu}\right)_T = \frac{D(P,T)}{D(\mu,T)}. \tag{B8}$$

Therefore, substituting Eqs. (B7) and (B8) into Eq. (A3) for the particle number fluctuations ω, one can carry out cancellation in ratios of the denominator by using the Jacobian properties. Finally, once again one obtains Eq. (31).

Note that these derivations, based on the first derivative transformations, fail near the critical point because of the divergence of fluctuations due to zeros in the denominators. Therefore, strictly speaking, Eq. (31) cannot be used in the close vicinity of the critical point [see Eq. (34)], in contrast to the fluctuation formula; see, e.g., Eq. (60) obtained in Sec. V from the moments of the averaged Gibbs distribution.

APPENDIX C: ANALYTICAL CRITICAL-POINT RESULTS WITHIN THE QVDW AND QSLD MODELS

1. The van der Waals model with quantum-statistics corrections

Following Refs. [44,45], we introduce a small quantum statistics parameter δ of expansion of the pressure $P(T,n)$, accounting for the vdW interaction in terms of the vdW attractive parameter a and repulsive exclusion-volume parameter b. For the Fermi statistics, one has Eq. (37) for δ. Up to the first leading quantum statistics corrections over δ to the vdW model, one has

$$P_{\rm W}(T,n) = \frac{nT}{1-bn}[1+\delta+O(\delta^2)] - a\,n^2. \tag{C1}$$

It was shown in Refs. [44,45] that at small δ the expansion of the pressure $P_{\rm W}(T,n)$ over powers of δ becomes rapidly convergent to the accurate results for sufficiently large temperature T and small particle-number density n. Therefore, even the first-order terms provide already a good approximation. The first quantum-statistics corrections in Eq. (C1) increase with the particle number density n and decrease with the increase of the system temperature T, particle mass m, and degeneracy factor g. A new feature of quantum statistics effects in the system of particles with the vdW interaction is the additional factor $(1-bn)^{-1}$ in the correction δ [Eq. (37)] with respect to the ideal gas case. Thus, the quantum statistics effects become stronger due to the repulsive interaction between particles.

The first-order equation of state [Eq. (C1)] within the quantum vdW (QvdW) model describes the corresponding liquid-gas phase transition. The critical point (CP) of this transition satisfies the equations of (34) [9]. Using Eq. (C1) in the first approximation over δ, one derives from Eq. (34) the system of two equations for the CP parameters n_c and T_c at the same corresponding order. The solutions of this system in the same first-order approximation over δ have the form

$$\begin{aligned} T_c^{(1)} &\cong T_c^{(0)}(1-2\delta_0), \\ n_c^{(1)} &\cong n_c^{(0)}(1-2\delta_0). \end{aligned} \tag{C2}$$

In Eq. (C2), the values $T_c^{(0)}$ and $n_c^{(0)}$ are the CP parameters of the classical vdW model with the pressure [Eq. (C1) at $\delta = 0$]

$$P_{\rm W}^{(0)}(T,n) = \frac{nT}{1-bn} - a\,n^2. \tag{C3}$$

These CP values are the zero-order approximation in the QvdW, $\delta = 0$:

$$\begin{aligned} T_c^{(0)} &= \frac{8a}{27b} \cong 29.2\ \text{MeV}, \quad n_c^{(0)} = \frac{1}{3b} \cong 0.100\ \text{fm}^{-3}, \\ P_c^{(0)} &= \frac{a}{27b^2} \cong 1.09\ \text{MeV fm}^{-3}. \end{aligned} \tag{C4}$$

The constants a and b of the QvdW model, $a > 0$ and $b > 0$, are responsible for attractive and repulsive interactions between particles, respectively. We will compare our analytical first-order results for fluctuations with those of more accurate numerical calculations [25–31]. Therefore, as in Refs. [21,22,24,44,45], we fix the model parameters a and b using the ground state properties of isotopic symmetric nuclear matter (see, e.g., Ref. [1]): at $T = 0$ and $n = n_0 = 0.16\ \text{fm}^{-3}$, one requires $P = 0$ and the binding energy per nucleon $\varepsilon(T = 0, n = n_0)/n_0 = -16$ MeV. From the above requirements, one finds

$$a = 329.8\ \text{MeVfm}^3, \quad b = 3.35\ \text{fm}^3. \tag{C5}$$

The parameter δ_0 in Eq. (C2) is given by Eq. (37), taken at the CP of the zero-order approximation (C4), i.e., at $n = n_c^{(0)}$ and $T = T_c^{(0)}$, $\delta_0 = \delta(T = T_c^{(0)}, n = n_c^{(0)})$. Substituting Eq. (C2) for the results of the corresponding critical temperature, $T_c^{(1)}$,

and density, $n_c^{(1)}$, into the equation of state [Eq. (C1)], at a given perturbation order, one can calculate the CP pressure $P_c^{(1)}$ at the same order, $P_c^{(1)} = P_{\rm W}(T = T_c^{(1)}, n = n_c^{(1)})$ [Eq. (C1)]. Notice that the temperature $T_c^{(1)}$ and density $n_c^{(1)}$ are decreased for Fermi statistics with respect to $T_c^{(0)}$ and $n_c^{(0)}$, in contrast to the opposite behavior for Bose particles.

2. The Skyrme local-density model with quantum statistics corrections

The pressure function of the quantum-statistics Skyrme local-density (QSLD) model [22], after some transformations, can be presented as [45]

$$P_{\rm Sk}(T, n) = nT(1 + \varepsilon) - a_{\rm Sk} n^2 + b_{\rm Sk} n^{\gamma+2}, \tag{C6}$$

where $a_{\rm Sk}$, $b_{\rm Sk}$, and γ are interaction constants of the QSLD parametrization [22].

Within the QSLD approach, one can consider the critical points for a first-order liquid-gas phase transition, for instance, for pure nucleon matter. The critical point (CP) for the QSLD model obeys the same equation (34) but with the quantum-statistics Skyrme local-density pressure, $P = P_{\rm Sk}(T, n)$ [Eq. (C6)]. Solving the system of equations (34) with the equation of state (C6) in the first-order approximation over ε, Eq. (37), one obtains [45]

$$\begin{aligned} T_{{\rm Sk},c}^{(1)} &\cong T_{{\rm Sk},c}^{(0)}(1 - 2\varepsilon_0), \\ n_{{\rm Sk},c}^{(1)} &\cong n_{{\rm Sk},c}^{(0)}\left(1 - \frac{2\varepsilon_0}{\gamma + 1}\right). \end{aligned} \tag{C7}$$

In Eq. (C7), the temperature $T_{{\rm Sk},c}^{(0)}$ and density $n_{{\rm Sk},c}^{(0)}$ are the solutions of equations [see Eq. (34) with the QSLD pressure (C6)] at zero-order perturbation, $\varepsilon = 0$:

$$\begin{aligned} T_{{\rm Sk},c}^{(0)} &= \frac{2\gamma a_{\rm Sk} n_{{\rm Sk},c}^{(0)}}{\gamma + 1}, \\ n_{{\rm Sk},c}^{(0)} &= \left[\frac{2a_{\rm Sk}}{b_{\rm Sk}(\gamma + 1)(\gamma + 2)}\right]^{1/\gamma}; \end{aligned} \tag{C8}$$

see also Ref. [47] where another Skyrme parametrization for the critical temperature and particle number density at zero quantum statistics corrections was used. For the parameters $a_{\rm Sk}$ and $b_{\rm Sk}$ of Skyrme parametrization, the degeneracy for nucleon system, $g = 4$, and $m = 938$ MeV, one has [22]

$$\begin{aligned} a_{\rm Sk} &= 1.167\ \text{GeV fm}^3, \\ b_{\rm Sk} &= 1.475\ \text{GeV fm}^{3+3\gamma}, \quad \gamma = 1/6, \\ a_{\rm Sk} &= 0.399\ \text{GeV fm}^3, \\ b_{\rm Sk} &= 2.049\ \text{GeV fm}^{3+3\gamma}, \quad \gamma = 1. \end{aligned} \tag{C9}$$

The QSLD parameters are chosen by fitting the properties of one component (in our case, nucleons) at the temperature $T = 0$.

The value ε_0 in Eq. (C7) is defined by Eq. (37) for ε at $T = T_{{\rm Sk},c}^{(0)}$ and $n = n_{{\rm Sk},c}^{(0)}$ [Eq. (C8)]. For the CP pressure at $\varepsilon = 0$, one finds from Eqs. (C6) and (C8),

$$P_{{\rm Sk},c}^{(0)} = n_{{\rm Sk},c}^{(0)} T_{{\rm Sk},c}^{(0)} - a_{\rm Sk}\left[n_{{\rm Sk},c}^{(0)}\right]^2 + b_{\rm Sk}\left[n_{{\rm Sk},c}^{(0)}\right]^{\gamma+2}. \tag{C10}$$

The first-order pressure, $P_{{\rm Sk},c}^{(1)}$, can be straightforwardly calculated from Eq. (C6) using the expressions for $T_{{\rm Sk},c}^{(1)}$ and $n_{{\rm Sk},c}^{(1)}$ [Eq. (C7)], $P_{{\rm Sk},c}^{(1)} = P_{\rm Sk}(T = T_{{\rm Sk},c}^{(1)}, n = n_{{\rm Sk},c}^{(1)})$ [Eq. (C6)].

APPENDIX D: MORE ACCURATE IMPROVED FLUCTUATIONS

Taking into account the quadratic term in the expansion (14) along with the second-order term, the quantity $(\rho - n)^2$, which we are going to average, should be statistically consistent with the expansion (14) up to fourth order terms in the mean field approximation [10,38]. Using the denotation $x = \rho - n$ for shortness, for x^2 one has [see Eq. (15)]

$$x^4 + \mathcal{A}x^2 - \mathcal{B}\{\rho\} = 0, \tag{D1}$$

where

$$\mathcal{A} = 12F_2/F_4, \quad \mathcal{B}\{\rho\} = 24\Delta\{\rho\}/F_4. \tag{D2}$$

Here, F_2 and F_4 are the derivatives of the free energy F over the density ρ:

$$F_m = (\partial^m F/\partial\rho^m)_{\rho=n}, \quad m = 2, 4, \tag{D3}$$

and $\Delta\{\rho\}$ is given by Eq. (15). Equation (D1) is a complicated self-consistent transcendent identity for x because the last term $\mathcal{B}\{\rho\}$ depends on $x = \rho - n$ in a cumbersome way through Eqs. (D2) and (15). Taking the statistical average over the Gibbs distribution $W_{\rm eq}^{(N)}$, Eq. (2), in Eq. (D1) term by term, one has

$$\langle x^4\rangle + \mathcal{A}\langle x^2\rangle - \langle\mathcal{B}\{\rho\}\rangle = 0, \tag{D4}$$

where

$$\langle\mathcal{B}\{\rho\}\rangle = 24\langle\Delta\{\rho\}\rangle/F_4. \tag{D5}$$

The angle brackets have the same meaning as in Sec. II, including averaging over the phase space **p** and **q**, and over the particle numbers N. Expanding $\langle x^4\rangle$ over the statistical correlations, one can present $\langle x^4\rangle$ in terms of the square $\langle x^2\rangle^2$ and a density-density correlation term:

$$\langle x^4\rangle = (\langle x^2\rangle)^2 + \text{corr. term}, \tag{D6}$$

where corr. term $= \langle x^4\rangle - \langle x^2\rangle^2$ is the density-density correlation term. In the mean field approximation, we may neglect this small density-density correlation term in Eq. (D6) because it is due to the residue interaction above a mean field.

It seems that we do not need to take care of the identities (D1) and (D4) in the case when we might be able to solve analytically exactly our problem with the Gibbs averaging, accounting for statistical correlations in all orders above the mean field approximation. However, simplifying our statistical problem by using this mean field approach, one should take care of executing still these identities approximately with the statistical accuracy of the mean field approximation, i.e., after neglecting correlation terms of Eq. (D6). Thus, at the zero-order approximation over these correlations, from Eq. (D4) one finds the approximately closed equation of the consistency condition (D1), taken in average, for $\langle x^2\rangle$ with an accuracy up to such correlations:

$$\langle x^2\rangle^2 + \mathcal{A}\langle x^2\rangle - \langle\mathcal{B}\{\rho\}\rangle = 0. \tag{D7}$$

This equation optimizes our statistical errors when we use the mean field approximation W_4, Eq. (16), to the Gibbs distribution $W_{\rm eq}^{(N)}$, Eq. (2), in evaluations of $\langle\Delta\{\rho\}\rangle$; see Eq. (15) and Refs. [10,38]. Solving Eq. (D7) with respect to $\langle x^2\rangle$, for a real positive solution, one obtains

$$\langle x^2\rangle \approx \frac{\mathcal{A}}{2}\left(\sqrt{1+\frac{4\langle\mathcal{B}\rangle}{\mathcal{A}^2}}-1\right) = \frac{\mathcal{A}}{2}\left(\sqrt{1+\frac{4\langle\tilde{\Delta}\{\rho\}\rangle}{\alpha}}-1\right), \tag{D8}$$

where $\tilde{\Delta}\{\rho\} = \Delta\{\rho\}/T$ is a dimensionless quantity.

It was convenient and constructive in Eq. (D8) to re-write the variance $\langle x^2\rangle$ by introducing explicitly the critical dimensionless parameter $\alpha \propto F_2^2/F_4T$:

$$\alpha = \frac{6(F_2)^2}{TF_4}. \tag{D9}$$

Another dimensionless parameter is $c_4 \propto F_4/T$. These two parameters α and c_4 were introduced instead of the original parameters F_2 and F_4 of the potential difference $\Delta\{\rho\}$ [Eq. (15)]. Then, the constant $\mathcal{A}$ in Eq. (D8) can be expressed in terms of α, Eq. (D9), and c_4 as

$$\mathcal{A} = \sqrt{\frac{\alpha}{c_4}}, \quad c_4 = \frac{F_4}{24T}. \tag{D10}$$

It is helpful also to use the obvious relationship [see Eqs. (D9) and (D10)]

$$\frac{\mathcal{A}^2}{\langle\mathcal{B}(\rho)\rangle} = \frac{\alpha}{\langle\tilde{\Delta}\{\rho\}\rangle}. \tag{D11}$$

Obviously, at the critical point one has $\alpha = 0$ because $F_2 \propto \mathcal{K} = 0$ if F_4 is assumed to be relatively finite, $F_4 \geqslant \text{const} > 0$. For small parameter α, one has effectively a small distance from the critical point while for large α one finds a large distance from the CP in the averaged particle number density n for a given temperature T. Thus, α is a dimensionless effective measure of the distance from the CP in the density-temperature plane.

So far in this Appendix, the angle brackets were defined as the statistical averaging with the general Gibbs distribution $W_{\rm eq}^{(N)}$ of the grand canonical ensemble; see Eq. (2) for $W_{\rm eq}^{(N)}$. In order to evaluate now approximately the average of the dimensionless potential variation $\langle\tilde{\Delta}\{\rho\}\rangle$ which appears in Eq. (D8), we will use the average statistical distribution function W_4 as a good approximation to the averaged $W_{\rm eq}^{(N)}$, within the mean field approach. Then, for the statistical average of $\Delta\{\rho\}$ [Eq. (15)], $\langle\Delta\{\rho\}\rangle$, one approximately has (see Ref. [38] and Secs. III and V)

$$\langle\Delta\{\rho\}\rangle = \langle\Delta_2\{\rho\}\rangle + \langle\Delta_4\{\rho\}\rangle, \tag{D12}$$

where

$$\langle\Delta_2\{\rho\}\rangle = \frac{F_2}{2}\int_0^\infty (\rho-n)^2 W_4 d\rho, \quad \langle\Delta_4\{\rho\}\rangle = \frac{F_4}{24}\int_0^\infty (\rho-n)^4 W_4 d\rho, \tag{D13}$$

In Eq. (D13), W_4 is the probability distribution given by Eq. (16) with the normalization condition (18). With Eq. (15), from Eq. (D12), one writes

$$\langle\Delta_2\{\rho\}\rangle = \frac{F_2}{2}\frac{\int_0^\infty x^2 dx \exp\left[-\frac{1}{2T}(F_2x^2+F_4x^4/12)\right]}{\int_0^\infty dx \exp\left[-\frac{1}{2T}(F_2x^2+F_4x^4/12)\right]} \tag{D14}$$

and

$$\langle\Delta_4\{\rho\}\rangle = \frac{F_4}{24}\frac{\int_0^\infty x^4 dx \exp\left[-\frac{1}{2T}\left(F_2x^2+F_4x^4/12\right)\right]}{\int_0^\infty dx \exp\left[-\frac{1}{2T}(F_2x^2+F_4x^4/12)\right]}, \tag{D15}$$

where $x = \rho - n$, as above.

Using Eqs. (D12), (D14), and (D15) for calculations of the average of the dimensionless potential difference $\tilde{\Delta}\{\rho\}$, one finds more explicit expressions in terms of the modified Bessel functions:

$$\langle\tilde{\Delta}\{\rho\}\rangle = \langle\tilde{\Delta}_2\{\rho\}\rangle + \langle\tilde{\Delta}_4\{\rho\}\rangle, \tag{D16}$$

where

$$\langle\tilde{\Delta}_2\{\rho\}\rangle \equiv \frac{\langle\Delta_2\{\rho\}\rangle}{T} = \frac{\pi}{4\sqrt{2}}\Big\{(\alpha+4)I_{1/4}\Big(\frac{\alpha}{8}\Big) - \alpha I_{-1/4}\Big(\frac{\alpha}{8}\Big) - \alpha\Big[I_{3/4}\Big(\frac{\alpha}{8}\Big) - I_{5/4}\Big(\frac{\alpha}{8}\Big)\Big]\Big\}\Big/K_{1/4}\Big(\frac{\alpha}{8}\Big) \tag{D17}$$

and

$$\langle\tilde{\Delta}_4\{\rho\}\rangle \equiv \frac{\langle\Delta_4\{\rho\}\rangle}{T} = \frac{1}{8}\Big[(\alpha+2)K_{1/4}\Big(\frac{\alpha}{8}\Big) - \alpha K_{3/4}\Big(\frac{\alpha}{8}\Big)\Big]\Big/K_{1/4}\Big(\frac{\alpha}{8}\Big). \tag{D18}$$

Here, $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of the order ν [$K_\nu(z)$ is named also the MacDonald Bessel function]. For $\alpha \gg 1$, far from the critical point, one obtains $\langle\tilde{\Delta}\{\rho\}\rangle \approx \langle\Delta_2\rangle/T \approx 1/2$; see Eq. (24). In the case $\alpha \ll 1$, near the CP, one obtains $\langle\tilde{\Delta}\{\rho\}\rangle \approx \langle\Delta_4\rangle/T \approx 1/4$; see also Eq. (E6) in the next Appendix.

Dividing by n^2 the left and final right sides of Eq. (D8), one arrives at the dimensionless particle-density fluctuations, Eq. (10); see also Eq. (53). Differentiating the relationship (25) between the pressure $P(\rho)$ and free energy $F(\rho)$ over ρ, and using the conditions of the statistical equilibrium, one finds the relationships

$$F_2 = \frac{\langle N\rangle\mathcal{K}}{n^2}, \quad F_4 = \frac{\langle N\rangle\mathcal{K}''}{n^2}. \tag{D19}$$

They are useful in the derivations of Sec. V; see Eq. (54), and asymptotes (55) for $\alpha \gg 1$ and (58) for $\alpha \ll 1$, neglecting small corrections of high order in powers of $1/\langle N\rangle$. In principle, we may take into account the density-density correlations by using the standard iteration procedure. However, to calculate the correlation term in Eq. (D6) at any given order we have to specify the interparticle interaction.

APPENDIX E: ASYMPTOTICAL FOURTH-ORDER IMPROVED FLUCTUATIONS

It is useful to present briefly the derivation of the limit $\alpha \ll 1$ neglecting the second-order term of the free energy expansion at the very beginning [38,39]. In this case, for the free energy expansion, one has from Eq. (15)

$$\Delta_4^{(4)}\{\rho\} \equiv F(\rho) - F(n) = \frac{1}{24}\left(\frac{\partial^4 F}{\partial \rho^4}\right)_{\rho=n}(\rho - n)^4. \tag{E1}$$

As shown in Appendix D, in the mean field approximation, i.e., at the zero-order density-density correlations, one finds from Eqs. (D7) and (D5)

$$\langle(\rho - n)^4\rangle \approx \langle(\rho - n)^2\rangle^2 \approx \frac{24\langle\Delta_4^{(4)}\{\rho\}\rangle}{(\partial^4 F/\partial\rho^4)_{\rho=n}}, \tag{E2}$$

where $\Delta_4^{(4)}\{\rho\}$ is given by Eq. (E1), at the fourth order under the assumption of neglecting the second-order term. Notice that the angle brackets in Eq. (E2) have the same meaning as in Eqs. (D4)–(D6).

For the evaluation of average $\langle\Delta_4^{(4)}\rangle$ in the last equation in (E2), with good accuracy within the mean field approximation, one can use the probability distribution $W_{\rm eq}^{(N)} \approx W_4^{(4)}$, valid namely in the mean-field approximation,

$$W_4^{(4)}(\rho) = W_4^{(4),0}\exp\left[-\frac{F_4}{24T}(\rho - n)^4)\right], \tag{E3}$$

where

$$W_4^{(4),0} = \left\{\int_0^\infty d\rho \exp\left[-\frac{F_4}{24T}(\rho - n)^4)\right]\right\}^{-1}; \tag{E4}$$

see Eq. (16) without the second-order term. Therefore, as in Appendix D, one has

$$\left\langle\Delta_4^{(4)}\{\rho\}\right\rangle \equiv \frac{F_4}{24}\langle(\rho - n)^4\rangle \approx \frac{F_4}{24}\int_0^\infty (\rho - n)^4 W_4^{(4)} d\rho, \tag{E5}$$

where $W_4^{(4)}$ is the normalized probability distribution of the fourth order with zero second-order term, Eq. (E3) ($\int_0^\infty W_4^{(4)} d\rho = 1$). Calculating now analytically integral in Eq. (E5), and comparing the result with the expression on very right of Eq. (E2), one obtains

$$\left\langle\Delta_4^{(4)}\{\rho\}\right\rangle \approx \frac{T}{4}. \tag{E6}$$

Differentiating over ρ the relationship (25) between the pressure $P(\rho)$ and free energy $F(\rho)$ for a constant temperature T, similarly as for the second order case, one can express the fourth derivative of $F(\rho)$ over ρ at $\rho = n$ in terms of the second derivative of the incompressibility $\mathcal{K}$,

$$\left(\frac{\partial^4 F(\rho)}{\partial\rho^4}\right)_{\rho=n} = \frac{\langle N\rangle\mathcal{K}''(n)}{n^2}, \quad \text{with}$$
$$\mathcal{K}''(n) = \left(\frac{\partial^3 P(\rho)}{\partial\rho^3}\right)_{\rho=n}, \tag{E7}$$

where P is the pressure $P(T, \rho)$, Eq. (25), and $P = P(T, n)$ is the equation of state in canonical variables. Using Eqs. (E6) and (E7), from the particle number density dispersion $\mathcal{D}_\rho$ normalized by n^2, Eq. (10), at the fourth-order expansion of the free energy (taking again zero for the second-order term) $\mathcal{D}_4^{(4)}$, with the probability distribution $W_4^{(4)}$, Eq. (E3), i.e., in the mean field (zero-order correlations) approximation, one naturally obtains the same limit as given in Eq. (58). Employing finally the same normalization of the dispersion D_N by $\langle N\rangle$, in order to compare with Eq. (31), we arrive at the expression (62), derived early in Ref. [39].

APPENDIX F: OTHER IMPROVED APPROACH TO THE PARTICLE NUMBER FLUCTUATIONS

Following Ref. [40] we assume in fact the mean field approximation $W_4(\rho)$, Eq. (16), for the Gibbs distribution averaged in the phase space and particle numbers from the very beginning, everywhere in the calculations of particle number fluctuations. Finally, for calculations of the dispersion (variance) $D_\rho = \langle(\rho - n)^2\rangle$, one obtains

$$D_\rho = \langle(\rho - n)^2\rangle = \mathcal{M}_2/\mathcal{M}_0, \tag{F1}$$

where

$$\mathcal{M}_m(c_2, c_4) = \int_0^\infty d\rho(\rho - n)^m \times \exp[-c_2(\rho - n)^2 - c_4(\rho - n)^4] \approx 2\int_0^\infty dx x^m \times \exp(-c_2 x^2 - c_4 x^4), \quad m = 0, 2, \tag{F2}$$

$$c_2 = \frac{F_2}{2T}, \quad c_4 = \frac{F_4}{24T}; \tag{F3}$$

see Eq. (D3) for the derivatives F_m of the free energy F. From Eq. (F2) one obtains the explicit expressions for the moments of the distribution function, $\mathcal{M}_m$, in terms of the MacDonald Bessel functions $K_\nu(\alpha/8)$,

$$\mathcal{M}_2 = \tfrac{1}{8}(c_2/c_4)^{3/2}\exp(-\alpha/8) \times [K_{3/4}(\alpha/8) - K_{1/4}(\alpha/8)],$$
$$\mathcal{M}_0 = \tfrac{1}{2}(c_2/c_4)^{1/2}\exp(\alpha/8)K_{1/4}(\alpha/8), \tag{F4}$$

where

$$\alpha = c_2^2/c_4 \tag{F5}$$

[see Eqs. (D9) and (F3)].

1. The limit case $c_2 = 0$

Taking the limit $c_2 \to 0$ to the critical point, from Eqs. (F1) with Eq. (F4) for the moments $\mathcal{M}_m$, one obtains the dispersion:

$$\mathcal{D}_\rho = \frac{\Gamma(3/4)}{\Gamma(1/4)}\frac{1}{\sqrt{c_4}} = 0.338\sqrt{\frac{24T}{F_4}} = 0.676\sqrt{\frac{6Tn^2}{\langle N\rangle\mathcal{K}''}}. \tag{F6}$$

For the normalized dispersion, D_ρ/n^2, one finally finds

$$\frac{\mathcal{D}_\rho}{n^2} = 0.676\sqrt{\frac{6T}{\langle N\rangle n^2\mathcal{K}''}}. \tag{F7}$$

The constant in front of the square root is smaller than that in Eq. (58). For the particle number fluctuation ω, Eq. (59), at the critical point, from Eq. (F6) one approximately finds

$$\omega = 1.66\sqrt{\frac{T\langle N\rangle}{n^2\mathcal{K}''}}. \tag{F8}$$

2. The limit case $c_4 = 0$

Taking the limit $c_4 \to 0$, from Eqs. (F1) with Eq. (F4) for the moments $\mathcal{M}_m$, one obtains

$$D_\rho = \frac{1}{2c_2}\frac{T}{F_2} = \frac{Tn^2}{\langle N\rangle\mathcal{K}}. \tag{F9}$$

Similarly as in the previous subsection of this Appendix, for the particle number fluctuation ω, Eq. (59), from Eq. (F9) one approximately finds

$$\omega = T/\mathcal{K}. \tag{F10}$$

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