

Cluster mean-field description of α emission

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We show that the Hartree-Fock-Bogoliubov (HFB) method is able to describe experimental values of α decay widths by including a residual nucleon-nucleon surface Gaussian interaction (SGI) within the standard procedure used to calculate the nuclear mean field. We call this method the cluster HFB (CHFB) approach. In this way we correct the deficient asymptotic behavior of the corresponding single-particle wave functions generated by the standard mean field. The corrected mean field becomes a sum between the standard mean Woods-Saxon-like field and a cluster Gaussian component centered at the same radius as the SGI. Thus, we give a confirmation of the mean field plus cluster potential structure, which was assumed in our previous work on α -decay widths. Systematic calculations evidence the linear correlation between the SGI strength and fragmentation potential, allowing for reliable predictions concerning the half-lives of superheavy emitters.

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I. INTRODUCTION

From the very first theories of α -emission published by Gamow [1] and independently by Condon and Gurney [2], almost a century passed until α particles were experimentally observed on the surface of nuclei [3]. However, describing the formation of α particles on the surface of an atomic nucleus from two protons and two neutrons still remains a considerable theoretical challenge within the microscopic theory of α decay from heavy nuclei. This radioactive process is fundamental in explaining the dynamics of various exotic physical systems, like superheavy and highly unstable nuclei [4]. The estimations of absolute α -decay widths, where only one shell model configuration was considered, were smaller than the experimental data by several orders of magnitude [5,6]. The typical example is the decay process $^{212}\text{Po} \rightarrow ^{208}\text{Pb} + \alpha$, where two proton and two neutron orbitals were considered above the doubly magic ^{208}Pb . It was soon realized that by increasing the number of single-particle (sp) configurations the value of the decay width substantially increases [7,8]. But even if a very large number of shells was included in order to simulate the continuum part of the spectrum, the absolute decay width still deviated by more than one order of magnitude [9,10]. The reason why the absolute decay width increases with the number of configurations is due to the clustering of the nucleons forming α particles, implying the inclusion of high lying configurations in the formation process [11]. Even so, the calculated absolute decay widths still differed from experimental observations by at least one order of magnitude [12–15].

The phenomenological model used to solve this problem consists of representing the emission process through a cluster

moving in an attractive pocket-like potential located at the nuclear surface. Under the assumptions of the R -matrix theory [16], this model predicts an analytic linear dependence between the logarithm of the reduced width and the fragmentation potential, defined by the difference between the Coulomb barrier and Q value [17]. It remains valid for many strong emission processes, including proton radioactivity and heavy cluster decay [18]. This indicates that the representation we are seeking must be provided by an attractive potential like that “pocket” potential in addition to the standard Woods-Saxon plus spin-orbit mean field. Furthermore, it is interesting to point out the existence of an alternative description of clustering phenomena employing the nonlinear Schrödinger equation and solutions on quantum droplets [19].

The idea of extending the description of nuclear interactions beyond the mean field is not new, but in this work we show that the proposed potential is a consequence of the Hartree-Fock-Bogoliubov (HFB) approach, provided the usual nucleon-nucleon interaction is enhanced on the nuclear surface where the nuclear density decreases. The microscopic formalism to estimate the α -particle formation probability has been developed previously (see Refs. [12,20]), but for the completeness of the overall presentation we will briefly describe those features which are of interest for the present work.

II. THEORETICAL BACKGROUND

A. Surface Gaussian interaction (SGI)

The α -decay process

$$P(\text{parent}) \rightarrow D(\text{daughter}) + \alpha \quad (2.1)$$

is allowed when the energy release (Q value) is positive. This surplus is transformed into the relative kinetic energy of the α -core system $Q = \mu_\alpha v^2/2$, where μ_α is the reduced mass of the daughter- α system. α decay between ground states (g.s.) takes place for select few very light elements (for example, ^5He , ^5Li , ^8Be) and becomes much more prevalent in the region of the nuclear chart with $Z > 50$. The basic requirement to properly describe emission processes is that the basis wave functions follow a correct asymptotic behavior. It turns out that the asymptotic value of sp orbitals provided by the standard Woods-Saxon potential is too small to reproduce the experimental value of the α -decay width. A successful solution to this problem was proposed in Ref. [21], where the decaying state was described by a combination of a shell-model wave function plus a cluster component $\Phi = \Phi_{SM} + \Phi_{clus}$. The cluster component is expected to contain the high-lying shell model configurations, and the shell model component Φ_{SM} is evaluated within a major shell only. The cluster component Φ_{clus} is expanded in terms of shifted Gaussians and is used to diagonalize the residual two-body interaction. A similar method was recently applied to describe anomalous large $B(E1)$, $B(E2)$ values, and α -decay half-lives corresponding to transitions from states of ^{212}Po [22].

A different proposal was presented in Ref. [23], namely the use of a sum between a Woods-Saxon mean field and a Gaussian potential centered beyond the nuclear surface at $R_{cl} = 1.3(A_D^{1/3} + 4^{1/3})$ with a length parameter $b_{cl} = 1$ fm. This was used to generate sp orbitals able to properly describe the absolute value of α -decay widths from even-even emitters. A similar potential, but with a Woods-Saxon form factor multiplied by a Gaussian clustering correction was used in Ref. [24] to describe α clustering in some emitters above doubly magic nuclei.

Various nuclear collective states are described within a microscopic formalism by a residual interaction peaked on the nuclear surface. In particular, in this work we will describe two-particle (pp , nn) collective states formed by a nucleon-nucleon residual interaction enhanced on the nuclear surface. In doing this, we generalize the well-known surface delta interaction (SDI) in the form of the surface Gaussian interaction (SGI)

$$v_{SGI}(\mathbf{r}_\tau, \mathbf{R}_\tau) = v_{\text{rel}}(r_\tau)v_{\text{c.m.}}(R_\tau) \\ = -v_0 \exp\left(-\frac{|\mathbf{r}_\tau|^2}{b_{\text{rel}}^2}\right) \exp\left(-\frac{(|\mathbf{R}_\tau| - R_0)^2}{b_{\text{c.m.}}^2}\right) \quad (2.2)$$

given here in terms of the the relative and c.m. coordinates $\mathbf{r} = \mathbf{r}_{1\tau} - \mathbf{r}_{2\tau}$, $\mathbf{R}_\tau = (\mathbf{r}_{1\tau} + \mathbf{r}_{2\tau})/2$. We will add this component to the standard nucleon-nucleon interaction $v_{\text{rel}}(r_\tau)$, given by the usual Gaussian shape

$$v(r_\tau, R_\tau) = -v_0 \exp\left(-\frac{r_\tau^2}{b_{\text{rel}}^2}\right) \\ \times \left[1 + x_c \exp\left(-\frac{(R_\tau - R_0)^2}{b_{\text{c.m.}}^2}\right)\right], \quad (2.3)$$

where $r_\tau = |\mathbf{r}_\tau|$, $R_\tau = |\mathbf{R}_\tau|$, and x_c plays the role of the mixing residual strength, common for protons and neutrons.

B. Cluster Hartree-Fock-Bogoliubov approach

The mean field can be generated by diagonalizing the HFB equations [25]

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + \Gamma^{(dir)}(\mathbf{r})\right]\psi_{am}(\mathbf{r}) \\ + \int d\mathbf{r}' \Gamma^{(exc)}(\mathbf{r}, \mathbf{r}')\psi_{am}(\mathbf{r}') = \epsilon_a \psi_{am}(\mathbf{r}), \quad (2.4)$$

depending upon direct and exchange potentials

$$\Gamma^{(dir)}(\mathbf{r}_\tau) = \int d\mathbf{r}'_\tau v(\mathbf{r}_\tau, \mathbf{r}'_\tau)\rho(\mathbf{r}'_\tau), \\ \Gamma^{(exc)}(\mathbf{r}_\tau, \mathbf{r}'_\tau) = -v(\mathbf{r}_\tau, \mathbf{r}'_\tau)\rho(\mathbf{r}_\tau, \mathbf{r}'_\tau), \\ \tau = p, n \quad (2.5)$$

in terms of densities

$$\rho(\mathbf{r}_\tau) = \sum_{a=1}^{n_\tau} V_{\tau a}^2 \sum_{m=-j_a}^{j_a} |\psi_{am}(\mathbf{r}_\tau)|^2, \\ \rho(\mathbf{r}_\tau, \mathbf{r}'_\tau) = \sum_{a=1}^{n_\tau} V_{\tau a}^2 \sum_{m=-j_a}^{j_a} \psi_{am}^*(\mathbf{r}'_\tau)\psi_{am}(\mathbf{r}_\tau). \quad (2.6)$$

We use the standard plus surface residual potential (2.3) and we call this procedure the cluster HFB (CHFB) approach. This clustered mean field describes the dynamics of proton and neutron quasiparticle pairs. The amplitudes $U_{\tau a}$, $V_{\tau a}$ are given by the quasiparticle creation operator written in terms of the particle operators

$$\alpha_{am_a}^\dagger = U_a c_{am_a}^\dagger + V_a c_{a-m_a}, \quad (2.7)$$

where $a = (\tau_a \epsilon_a l_a j_a)$. They satisfy the standard system of gap equations

$$\Delta_a = \sum_{b=1}^{n_\tau} G_0(ab)\Omega_b U_b V_b = v_0 \sum_{b=1}^{n_\tau} G_0^{(0)}(ab) \frac{\Omega_b \Delta_b}{2E_b}, \\ a = 1, 2, \dots, n_\tau, \quad (2.8)$$

where n_τ is the number of considered sp levels and

$$\Omega_b = \frac{1}{2} \hat{j}_b^2 = j_b + \frac{1}{2}. \quad (2.9)$$

The monopole pairing interaction is given by

$$G_0(ab) = -\frac{4}{\widehat{j_a j_b}} \langle aa; 0 | v | bb; 0 \rangle = v_0 G_0^{(0)}(ab). \quad (2.10)$$

In Appendix A we estimate the matrix elements of this interaction for the wave functions provided by the diagonalization of the mean field. The amplitudes

$$\begin{pmatrix} U_a \\ V_a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \pm \frac{\epsilon_a - \lambda_\tau}{E_a} \end{pmatrix}^{\frac{1}{2}}, \quad \tau = p, n \quad (2.11)$$

are defined in terms of the quasiparticle energy

$$E_a = \sqrt{(\epsilon_a - \lambda_\tau)^2 + \Delta_a^2}, \quad \tau = p, n, \quad (2.12)$$

where λ_τ are Lagrange multipliers accounting for the conservation of the number of particles. We solve the system (2.8) by

looking for an effective strength of the pairing interaction v_0 required to obtain the experimental value of the gap parameter at the Fermi level. It can be approximated by the well-known ansatz

$$\Delta_{a_F} = \Delta_{\text{exp}} \sim \frac{12}{\sqrt{A}} \text{ MeV}. \quad (2.13)$$

In Appendix B we show that the CHFB procedure predicts a mean field potential of the form

$$V_{MF}(r_\tau) = V_0(r_\tau) + V_{cl}(r_\tau), \quad \tau = p, n. \quad (2.14)$$

V_0 describes the standard mean field close to the Woods-Saxon shape. It has a somewhat involved expression following from computational details that are not crucial for the physics of this discussion. These details are described in Appendix B and the expression for the potential is given in Eq. (B12) in terms of other quantities defined and computed there. V_{cl} is also described in detail in the same Appendix, but it can be written in Gaussian form

$$V_{cl}(r_\tau) = A_{cl} \exp \left[- \left(\frac{r_\tau - R_{cl}}{b_{cl}} \right)^2 \right]. \quad (2.15)$$

The cluster parameters can be derived analytically for a step-function density in terms of original sp interaction parameters (2.3), with a proof being outlined in Appendix B leading to Eq. (B15). R_0 is parametrized in Eq. (B14). The length parameters b_{rel} , $b_{\text{c.m.}}$, and b_{cl} characterize the corresponding Gaussians found in the structure of the potential (2.14). Their values are once again discussed in Appendix B and shown to be those in Eq. (B15):

$$R_{cl} = R_0, \quad b_{cl} = \sqrt{2} b_{\text{c.m.}} = b_{\text{rel}} / \sqrt{2}. \quad (2.16)$$

Our numerical analysis has shown that the realistic sp densities (2.6) provide results that are very close to the above analytic approximations. The inclusion of the SGI residual interaction in simultaneously solving the mean field (2.4) and pairing equations (2.8) is a procedure going beyond the mean field approach [26]. In our case it describes collective pp and nn pair states entering the structure of the α particle. pn pairing generally has a very small contribution to α decay from heavy nuclei [27] and is therefore neglected here. Thus, we can justify on microscopic grounds the use of a similar potential in Ref. [23].

C. Decay width for deformed nuclei

A very good approximation of the total decay width connecting the g.s. of deformed even-even nuclei is given by the following factorization [16,20,28]:

$$\Gamma = \Gamma_0 D(\beta_2) \quad (2.17)$$

between the monopole decay width

$$\Gamma_0 = \hbar v \left[\frac{R \mathcal{F}_0(R)}{G_0(\chi, \rho)} \right]^2, \quad (2.18)$$

where R is the α -core center of mass (c.m.) radius, and the deformation factor

$$D(\beta_2) = \sum_L \exp \left[-2 \frac{L(L+1)}{\chi} \sqrt{\frac{\chi}{\rho} - 1} \right] \mathcal{K}_{L0}^2(\beta_2) \quad (2.19)$$

induced by the Coulomb field characterized by the quadrupole deformation β_2 . Here, $G_0(\chi, \rho)$ is the monopole irregular Coulomb function depending upon the Coulomb parameter $\chi = 4Z_D e^2 / (\hbar v)$ and reduced radius $\rho = \kappa R$, where $\hbar \kappa = \mu_\alpha v$ is the linear momentum and

$$\mathcal{K}_{LL'}(\beta_2) = \int_{-1}^1 Y_{L0}(x) e^{\beta_2 B P_2(x)} Y_{L'0}(x) dx, \quad (2.20)$$

$$B \equiv \frac{2}{5} \chi \beta_2 \left(2 - \frac{\rho}{\chi} \right) \sqrt{\frac{5}{4\pi} \frac{\rho}{\chi} \left(1 - \frac{\rho}{\chi} \right)}$$

defines the Fröman propagator matrix [20,29]. Higher order multipoles of the nuclear shape are important in the description of the α -emission spectrum, particularly when transitions to excited states are involved. However, one can still obtain good results when restricting the analysis only to the quadrupole moment. For a more detailed discussion and comparison of these methods, one can see Ref. [18], and references indicated therein.

D. Formation amplitude

In the framework outlined above, the α -particle formation amplitude can be calculated within a spherical approach. It is given by the following overlap integral [20]:

$$\mathcal{F}_0(R) = \langle \Psi_P | \Psi_D \Psi_\alpha \rangle, \quad (2.21)$$

where Ψ_P , Ψ_D , and Ψ_α are the wave functions of the parent, daughter, and α particle, respectively. The above relation is a good approximation beyond the geometrical touching configuration, where the α -core antisymmetrization becomes less important. It is convenient to write the formation amplitude by using a harmonic oscillator (ho) representation since then all integrals can be performed analytically. Thus, the wave function diagonalizing the mean field (MF) can be written

$$\psi_{\tau \ell j m}(x) = \langle x | \psi_{\tau \ell j m} \rangle = \mathcal{R}_{\tau \ell j}(r) \mathcal{Y}_{jm}^{(\ell \frac{1}{2})}(\hat{r}, s), \quad (2.22)$$

where $x = (\mathbf{r}, s)$, in terms of the radial MF wave function and spin-orbit harmonics, respectively,

$$\mathcal{R}_{\tau \ell j}(r) = \sum_n d_{\tau \ell j}^n \mathcal{R}_{nl}^{(\beta)}(r), \quad (2.23)$$

$$\mathcal{Y}_{jm}^{(\ell \frac{1}{2})}(\hat{r}, s) = [i^\ell Y_\ell(\hat{r}) \otimes \chi_{\frac{1}{2}}(s)].$$

Here, $\mathcal{R}_{nl}^{(\beta)}(r)$ denotes the spherical ho wave function depending upon the ho size parameter $\beta = M_N \omega / \hbar$. The formation amplitude becomes [20]

$$\mathcal{F}_0(R) = \sum_{N_\alpha} \mathcal{W}_{N_\alpha} \mathcal{R}_{N_\alpha 0}^{(4\beta)}(R) \equiv \sum_{N_\alpha} \mathcal{F}_{N_\alpha 0}(R), \quad (2.24)$$

where N_α is the ho radial quantum number corresponding to the α -particle motion with angular momentum $L_\alpha = 0$. The

$\mathcal{W}$ coefficients are given by the following superposition:

$$\mathcal{W}_{N_\alpha} = 8 \sum_{n_\alpha N_p N_n} \mathcal{G}_{N_p} \mathcal{G}_{N_n} \times \langle n_\alpha, 0; N_\alpha, 0; 0 | N_p, 0; N_n, 0; 0 \rangle \mathcal{I}_{n_\alpha 0}^{(\beta\beta_\alpha)}, \quad (2.25)$$

where the bra-ket product is the standard Talmi-Moshinsky (TM) recoupling coefficient connecting the pp and nn pairs to α -particle coordinates. Here, $\mathcal{I}$ is the overlap integral between the ho sp components $\mathcal{R}_{n_\alpha 0}^{(\beta)}$ and the α -particle wave function $\mathcal{R}_{00}^{(\beta_\alpha)}$. The quantity $\mathcal{G}_{N_p}$ ($\mathcal{G}_{N_n}$) contains only proton (neutron) degrees of freedom

$$\mathcal{G}_{N_\tau} = \sum_{n_1 n_2 l j} \mathcal{B}_\tau(n_1 l j n_2 l j; 0) \times \left\langle (ll)0 \left(\frac{1}{2} \frac{1}{2} \right) 0; 0 \left| \left(l \frac{1}{2} \right) j \left(l \frac{1}{2} \right) j; 0 \right\rangle \times \sum_{n_\tau} \langle n_\tau 0 N_\tau 0; 0 n_1 l n_2 l; 0 \rangle \mathcal{I}_{n_\tau 0}^{(\beta\beta_\alpha)}, \quad (2.26)$$

where the bra-ket in the second line denotes the jj - LS recoupling coefficient and the $\mathcal{B}$ coefficient contains the nuclear structure information

$$\mathcal{B}_\tau(n_1 l j n_2 l j; 0) = \frac{\hat{j}}{\sqrt{2}} U_{\tau \ell l j} V_{\tau \ell l j} d_{\tau \ell l j}^{n_1} d_{\tau \ell l j}^{n_2}. \quad (2.27)$$

Equation (2.25) contains products of quantities which depend only on proton or neutron degrees of freedom.

III. NUMERICAL APPLICATION

The formation of an α cluster is a collective process, less sensitive to specific details connected to the sp level structure. It turns out that the essential part of the sp mean field for decay processes is given by distances beyond the geometrical touching radius

$$R_c = 1.2(A_D^{1/3} + A_\alpha^{1/3}). \quad (3.1)$$

A. Mean field shape

In Fig. 1 we plotted the proton CHFB potential of Eq. (2.14) calculated for ^{242}Pu (dashed line) and Woods-Saxon potential with universal parametrization [30–32] plus SGI residual interaction (solid line), satisfying the conditions (2.16). The residual strength x_c has the value of 19 MeV which reproduces the observed α -decay width. The overall effect obtained is the formation of pocket-like potential structures centered on the nuclear surface which favor nucleon clustering. One notices that both versions give practically the same results concerning the estimate of the decay width beyond the geometrical touching radius $R_c = 9.38$ fm.

For this reason we performed our analysis by using a Woods-Saxon sp potential with universal parametrization plus a residual SGI, satisfying the conditions (2.16) predicted by the CHFB formalism. We considered the standard value of the nucleon-nucleon radius $b_{\text{rel}} = 2$ fm and a slightly larger radius than the touching radius $R_0 = 1.275(A_D^{1/3} + A_\alpha^{1/3})$, corresponding to a small percent of the equilibrium nuclear

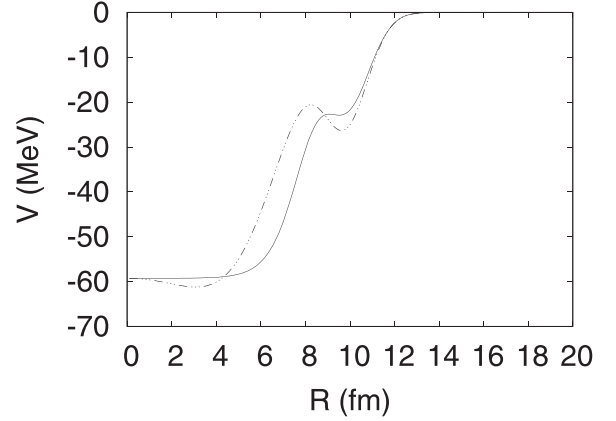


FIG. 1. Proton HFB mean field plus SGI interaction (dashed line) and WS plus SGI potential (solid line) in the case of ^{242}Pu .

density, as predicted by the nuclear matter calculations of the α -clustering transition. This value is known as the Mott density for the α formation [33,34]. Thus, the only free parameter of the model is the strength x_c of the SGI and it was adjusted to reproduce experimental decay widths.

B. Pairing strength systematics

We analyzed superfluid even-even α emitters ranging from rare earths to actinides and superheavy nuclei with experimental data available at the ENSDF [35].

The main nuclear structure ingredients enter the $\mathcal{B}$ coefficients (2.27). They are given by the expansion coefficients of the sp orbitals in terms of ho components and Bardeen-Cooper-Schrieffer (BCS) amplitudes depending upon the strength of the pairing interaction. Therefore, we began with the analysis of this strength v_0 by using the systematics of the pairing gap. Panel (a) of Fig. 2 shows the pairing interaction strength versus the mass number across the nuclear chart for the case of no residual interaction ($x_c = 0$). Similarly, panel (b) shows the same plot compared with the case of α emitters having their values of x_c taken from the decay systematics. What is observed in the first case is a significant increase of the pairing strength for small mass numbers. This behavior is consistent with a recent microscopic description of two-proton emitters [36], where a value $v_0 \sim 45$ MeV was obtained in free space in order to reproduce the experimental value of a simultaneous two-proton decay width. Notice that the mean value for α emitters with $A > 150$ is of ≈ 9 MeV for the pp and nn pairing strengths, respectively. Turning on the residual interaction, these values go to roughly ≈ 10 MeV, so they do not change significantly.

C. Analysis of the plateau condition

As we already mentioned, the clustering process takes place on the nuclear surface, where the low density favors the formation of α particles. According to Eq. (2.24) the formation amplitude is a coherent superposition of four-body radial ho functions multiplied by $\mathcal{W}$ coefficients, plotted in panel (a) of Fig. 3 for the decay of ^{242}Pu . By a dot-dashed

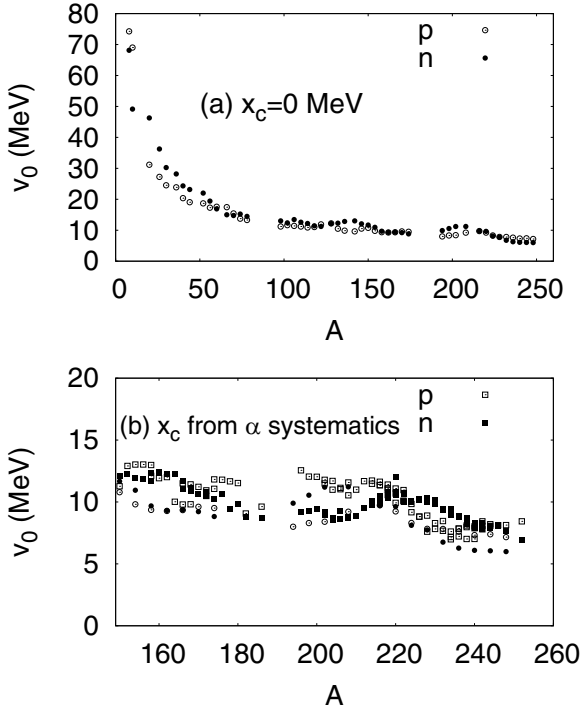


FIG. 2. (a) shows the pairing interaction strength versus mass number across the nuclear chart for $x_c = 0$. (b) compares the case above (circles) with that of α emitters (squares) having x_c taken from the decay systematics.

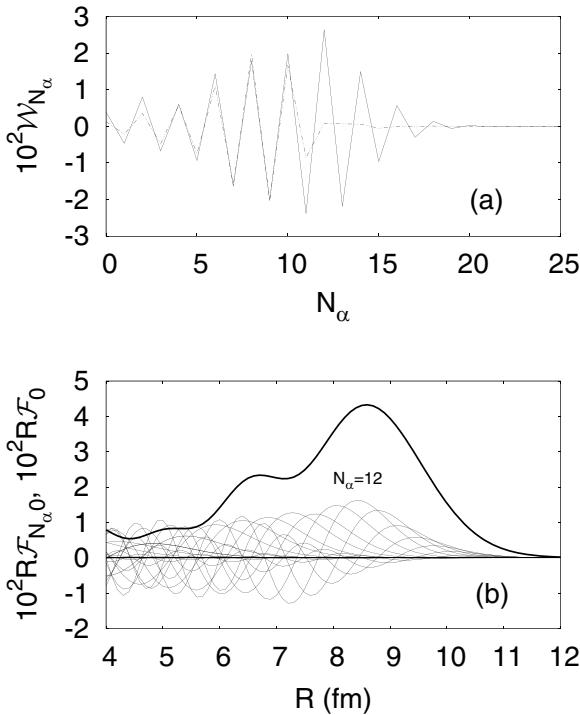


FIG. 3. (a) $\mathcal{W}$ coefficients (2.25) versus the quartet radial quantum number N_α in the absence of SGI interaction (dot-dashed line) and for $x_c = 19$ MeV (solid line) corresponding to the decay of ^{242}Pu . (b) The radial components of the α -formation amplitude (2.24) (thin solid lines) and the total value (thick solid line).

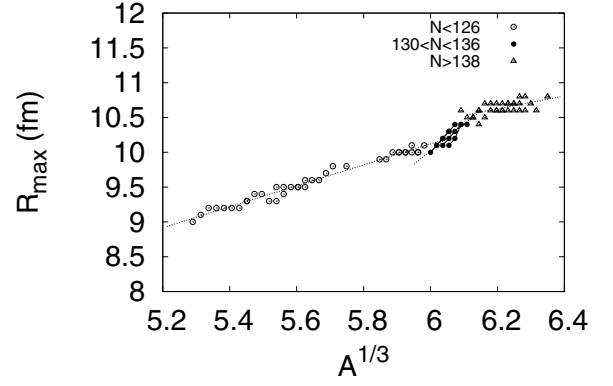


FIG. 4. Radius corresponding to the peak of the α -particle wave function versus parent mass number to the power $\frac{1}{3}$.

line the $\mathcal{W}$ -coefficients are given corresponding to the absence of the residual SGI interaction ($x_c = 0$), while the solid line denotes the case reproducing the experimental decay width, namely $x_c = 19$ MeV. One notices the occurrence of large components with $N_\alpha > 10$ in the latter case. In spite of the staggered character of these coefficients, the products with ho functions $\mathcal{F}_{N_\alpha 0}(R)$ plotted in panel (b) have a coherent behavior. They give the maximum of the summed formation amplitude $\mathcal{F}_0(R)$, plotted in the same panel by a thicker line. Its maximal value corresponds to the largest component with the c.m. radial quantum number $N_\alpha = 12$.

Figure 4 shows the systematics for the radius corresponding to the maximal value of the α -particle formation amplitude versus the parent mass number to the power $\frac{1}{3}$. One observes three regions of linear correlations, corresponding to the neutron numbers $N < 126$ (empty circles), $130 \leq N \leq 136$ (filled circles), and $N \geq 138$ (empty triangles).

The first and third regions are in fact quite similar in behavior, with the second region bridging them. The separation between the second and third regions becomes unambiguous if one looks at Fig. 6 to be discussed in Sec. III D. It is interesting to observe that the second region is comprised of Rn, Ra, Th, and U isotopes, the lightest one being ^{216}Rn while the heaviest nucleus is ^{228}U . These two configurations of nucleons can be imagined as a ^{208}Pb core coupled to a number of α particles of 2 and 5, respectively, with all other nuclei in between having a number of nucleons compatible with arrangements consisting of a ^{208}Pb core, a number of 2–4 α particles and an additional number of 1–3 pp or nn pairs. We are not stating that this is indeed an accurate physical picture, but it does tie further into the discussion of Sec. III D and Fig. 7, where the data pertaining to this region suggest enhanced clustering features due to the small number of nucleons found above the closed shells of ^{208}Pb . In any case, the slope, intercept, and standard deviation following for a basic linear fit of the data for each region are given in Table I.

The calculated decay width (2.17) should not depend upon the radius beyond the nuclear surface, thus satisfying the so-called plateau condition, due to the fact that in a phenomenological approach both internal $R\mathcal{F}_0$ and external $G_0(R)$ functions satisfy the same Schrödinger equation. Our case is that of a semimicroscopic approach. The internal

TABLE I. Systematics of peak radius versus mass number to the power $\frac{1}{3}$.

region	a	b	σ
$N < 126$	1.503	1.102	0.052
$130 \leq N \leq 136$	3.850	-13.082	0.060
$N \geq 138$	0.932	4.845	0.072

formation amplitude in Eq. (2.18) is provided by a microscopic method, while the external wave function satisfies the Coulomb equation and therefore the plateau condition is not automatically satisfied.

In order to check to what extent the plateau condition is satisfied we analyzed the behavior of the calculated decay width for different values of the residual strength. The result is shown in Fig. 5 as a function of radius in the case of the parent nucleus ^{242}Pu . The results are shown for two different types of calculations. Panel (a) is for the computation without the Fröman correction, while panel (b) shows the results corrected for the nuclear deformation within the Fröman approximation. x_c ranges between 5–19 MeV with smaller values corresponding to broader plateaus in the logarithm of the decay widths ratio. One observes that the theoretical calculations converge to the observed value with increasing values of x_c . Furthermore, the calculations corrected for nuclear deformation make a better estimate of the decay width by a factor of roughly 5 over the spherical calculation for a given value of the residual

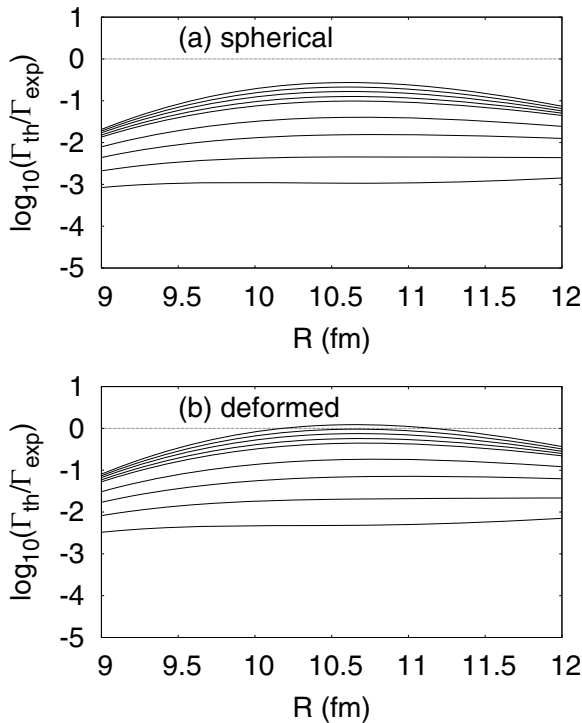


FIG. 5. Logarithm of the ratio between the theoretical and experimental decay width versus radius in the case of spherical (a) and deformed (b) calculations. The nucleus is ^{242}Pu and x_c ranges between 5–19 MeV, with the smaller values corresponding to a wider plateau.

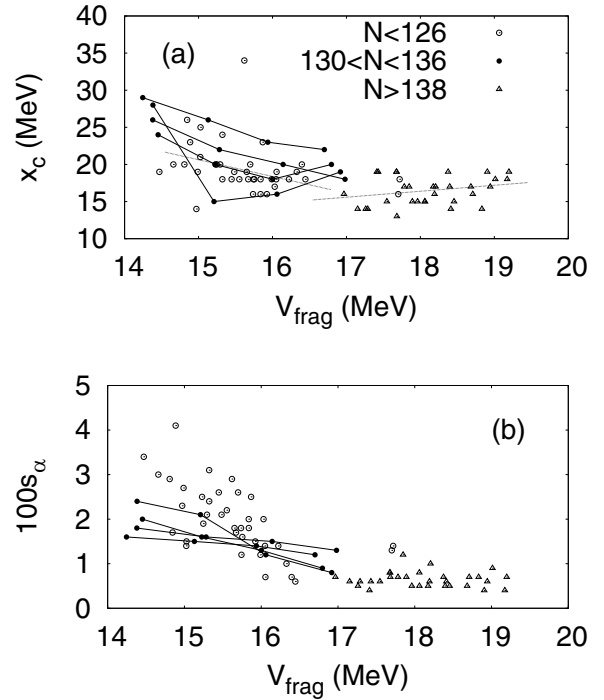


FIG. 6. Residual interaction strength (a) and α -decay spectroscopic factor (b) versus the fragmentation potential.

strength. This underlines once again the importance of nuclear deformation in the barrier penetration process. The approximate plateau condition is established at a little over 10 fm, that is about 1 fm beyond the geometrical contact radius. It is important to stress that we determined the strength x_c reproducing the experimental decay width by using the following condition:

$$\left\langle \log_{10} \frac{\Gamma_{th}^{def}(R)}{\Gamma_{exp}} \right\rangle = 0, \quad (3.2)$$

where the mean value is considered in the interval of ± 1 fm around the radius R_{max} where the maximal value is reached.

D. Decay width systematics

Systematic calculations of α -decay widths are presented in Fig. 6, namely the dependence of the residual interaction strength (panel a) and spectroscopic factor (panel b)

$$s_\alpha = \int_0^\infty |R\mathcal{F}_0(R)|^2 dR, \quad (3.3)$$

on the fragmentation potential as suggested by the phenomenological systematics of Ref. [18]. One observes once again two major trends with a transition region in the same neutron ranges as found in the systematics of Fig. 4. For the first and third regions, the slope, intercept and standard deviation are given in Table II.

It is interesting to note that this phenomenon is reminiscent of a very similar feature found in proton emission. There, the proton-decay spectroscopic factor exhibits two trends around the charge number $Z = 68$ where both shape-coexistence phenomena and an abrupt change from oblate to prolate

TABLE II. Systematics of the residual interaction strength versus fragmentation potential.

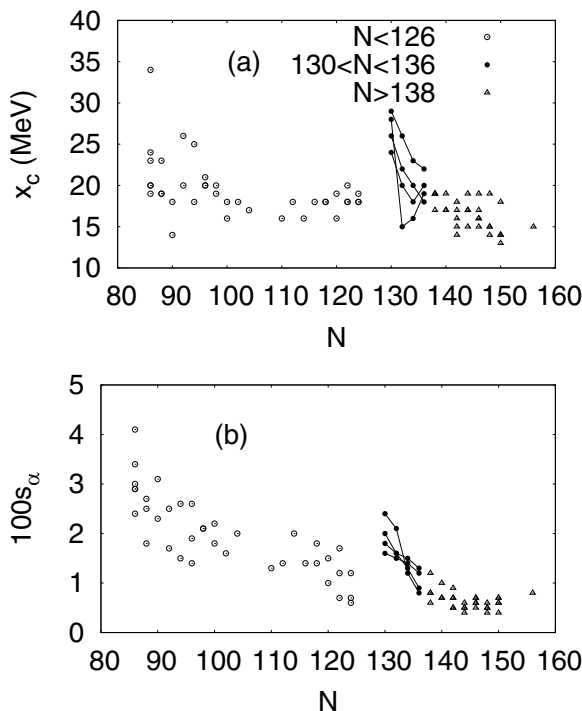
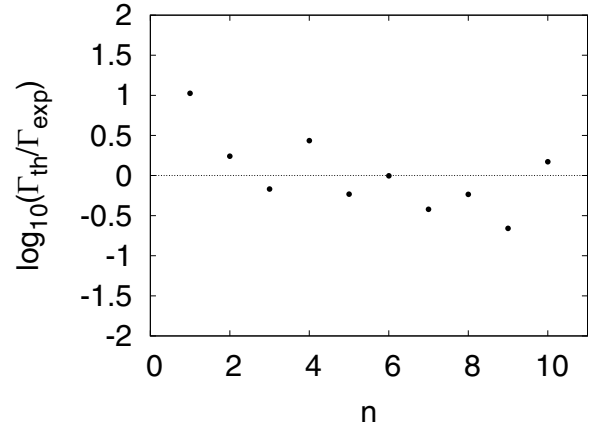
region	a	b	σ
$N < 126$	-2.248	54.351	2.070
$N \geq 138$	0.804	1.919	1.762

deformations are observed [18,37]. However, as noted previously, in the case of α -decay clustering phenomena play a very important role in the dynamics of this particular transition. This is seen in Fig. 7 where the residual interaction strength (panel a) and spectroscopic factor (panel b) are plotted versus the neutron number. One observes the typical behavior of large clustering near closed shells followed by a decreasing trend.

In phenomenological studies of the α -spectrum fine structure using a monopole plus quadrupole-quadrupole (QQ) interaction, the coupling strength of the QQ component behaves in an analogous manner and is proportional to the reduced width, thereby acting as a measure of clustering on the nuclear surface [38].

E. Predictions for superheavy emitters

In order to test the predictive power of the model, we have used the systematics of Table II to calculate the decay widths of known even-even superheavy emitters. The results are shown in Fig. 8, namely the logarithm of the ratio between the calculated and experimental widths function of the index number of Table III. In spite of the somewhat large scattering of data for actinides in the range $N \geq 138$, one

FIG. 7. Residual interaction strength (a) and α -decay spectroscopic factor (b) versus the neutron number.FIG. 8. Logarithm of the ratio for the predicted to experimental decay width versus index number for even-even superheavy α emitters.

observes an overall good agreement between the calculated and experimental values for the decay widths of superheavy emitters, usually within a factor of 3. This is quite reasonable in the context of the experimental uncertainties involved in these measurements. The last column of Table III contains the quantity ϵ , namely the ratio of the largest recorded uncertainty in the total measured half-life relative to the recommended value tabulated in Ref. [35] at the time of this work. In contrast, similar experimental uncertainties in the region of the actinides where the relevant data are fitted tend to be smaller, of the order of $\approx 1\%$ or less.

Of particular interest is the case of the parent nucleus $^{266}_{106}\text{Sg}$. Not only are the reported uncertainties in the total half-life quite large, but the α -decay branching ratio itself is currently recommended only as an estimated lower bound of $\% \alpha \geq 18.0$. Perhaps the order of magnitude discrepancy between the α -decay width following from these reported values and our calculation is indicative of a measurement that can be improved.

TABLE III. Predictions for superheavy even-even α emitters. Deformation parameters are taken from [39]. Uncertainties relative to the recommended value of the total half-life are taken from the maximal values tabulated in Ref. [35] at the time of this writing.

n	Nucleus	β_2	Q MeV	V_{frag} MeV	$\log_{10} \Gamma_{\text{exp}}$ MeV	$\log_{10} \frac{\Gamma_{\text{th}}}{\Gamma_{\text{exp}}}$	ϵ %
1	$^{266}_{106}\text{Sg}$	0.230	8.762 51	17.603	-23.420	1.024	95
2	$^{264}_{108}\text{Hs}$	0.229	10.591 20	16.332	-18.545	0.242	—
3	$^{266}_{108}\text{Hs}$	0.230	10.335 20	16.537	-18.703	-0.168	5
4	$^{270}_{108}\text{Hs}$	0.231	9.300 7	17.470	-21.896	0.435	6
5	$^{270}_{110}\text{Ds}$	0.221	11.200 50	16.075	-17.341	-0.232	35
6	$^{286}_{114}\text{Fl}$	-0.096	10.345 60	17.528	-20.943	-0.003	24
7	$^{288}_{114}\text{Fl}$	0.053	10.090 70	17.733	-21.244	-0.420	22
8	$^{290}_{116}\text{Lv}$	0.072	11.000 80	17.270	-19.517	-0.234	4.2
9	$^{292}_{116}\text{Lv}$	-0.070	10.800 70	17.420	-19.597	-0.658	5
10	$^{294}_{118}\text{Og}$	-0.087	11.810 60	16.855	-18.596	0.172	76

IV. CONCLUSIONS

We have used the HFB mean field plus a residual nucleon-nucleon SGI in order to describe α clustering in even-even nuclei. We call this method the cluster HFB (CHFB) approach.

We have shown that the shape of the resulting mean field is close to the Woods-Saxon potential with universal parametrization plus a Gaussian clustering correction with parameters determined by the residual nucleon-nucleon SGI. The strength of the residual interaction was chosen to reproduce experimentally observed decay widths. We have shown that the residual strength evaluated in this way is linearly correlated with the fragmentation potential which is in agreement with the behavior of the α -particle preformation inferred from phenomenological theories. We have evidenced two such regions of linear correlation for emitters in the range between rare earths and actinides, the transition between the two regimes corresponding to the well-known high clustering found in the region above ^{208}Pb . The predictive power of the model was tested by estimating the half-lives of superheavy α emitters, with good agreement being found with the experimental widths.

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APPENDIX A: MATRIX ELEMENTS OF THE NUCLEON-NUCLEON INTERACTION

The pairing function is given by

$$\Psi_{abJM}(x_1, x_2) = \delta_{ab} \delta_{J0} \delta_{M0} [\psi_a(x_1) \otimes \psi_a(x_2)]_{00}. \quad (\text{A1})$$

We first expand each sp wave function in terms of ho components

$$\begin{aligned} \Psi_{aa00}(x_1, x_2) &= \sum_{n_a n'_a} d_a^{n_a} d_a^{n'_a} \Phi_{aa00}^{n_a n'_a}(x_1, x_2), \\ \Phi_{aa00}^{n_a n'_a}(x_1, x_2) &\equiv \left[\left(\phi_{n_a l_a}^{(\beta)}(\mathbf{r}_1) \otimes \chi_{\frac{1}{2}}(s_1) \right)_{j_a} \right. \\ &\quad \left. \otimes \left(\phi_{n'_a l'_a}^{(\beta)}(\mathbf{r}_2) \otimes \chi_{\frac{1}{2}}(s_2) \right)_{j_a} \right]_{00}, \end{aligned} \quad (\text{A2})$$

and then we change from the jj to the LS coupling scheme where one considers the spin singlet component. Finally we change the radial part by using the Talmi-Moshinsky transformation from absolute to relative and cm coordinates through the notation $|\Phi_{aa00}^{n_a n'_a}\rangle \equiv |n_a l_a j_a n'_a l'_a j_a\rangle$,

$$\begin{aligned} \langle \Psi_{aa;0}(x_1, x_2) | v(\mathbf{r}, \mathbf{R}) | \Psi_{bb;0}(x_1, x_2) \rangle &\equiv \langle aa; 0 | V | bb; 0 \rangle = \sum_{n_a n'_a n_b n'_b} d_a^{n_a} d_a^{n'_a} d_b^{n_b} d_b^{n'_b} \langle n_a l_a j_a n'_a l'_a j_a | v(\mathbf{r}, \mathbf{R}) | n_b l_b j_b n'_b l'_b j_b \rangle, \\ \langle n_a l_a j_a n'_a l'_a j_a | v(\mathbf{r}, \mathbf{R}) | n_b l_b j_b n'_b l'_b j_b \rangle &\equiv \left\langle \left(l_a l_a \right) 0 \left(\frac{1}{2} \frac{1}{2} \right) 0; 0 \left| \left(l_a \frac{1}{2} \right) j_a \left(l'_a \frac{1}{2} \right) j'_a; 0 \right| \left(l_b l_b \right) 0 \left(\frac{1}{2} \frac{1}{2} \right) 0; 0 \left| \left(l_b \frac{1}{2} \right) j_b \left(l'_b \frac{1}{2} \right) j'_b; 0 \right| \right\rangle \\ &\times \sum_{lL} \sum_N \langle n l N L; 0 | n_a l_a n'_a l'_a; 0 \rangle \sum_{N'} \langle n' l' N' L; 0 | n_b l_b n'_b l'_b; 0 \rangle \langle \mathcal{R}_{nl}^{(\beta/2)}(r) | v_{\text{rel}}(r) | \mathcal{R}_{n'l}^{(\beta/2)}(r) \rangle \\ &\times \langle \mathcal{R}_{NL}^{(2\beta)}(R) | v_{\text{c.m.}}(R) | \mathcal{R}_{N'L}^{(2\beta)}(R) \rangle, \end{aligned} \quad (\text{A3})$$

where

$$2(n_a + n'_a + l_a) = 2(n + N) + l + L, \quad 2(n_b + n'_b + l_b) = 2(n' + N') + l + L. \quad (\text{A4})$$

For a potential depending only on the relative coordinate like the spin singlet gaussian interaction

$$v_{\text{rel}}(r) = -v_0 \exp\left(-\frac{r^2}{b_{\text{rel}}^2}\right), \quad (\text{A5})$$

the main building block becomes diagonal in N .

APPENDIX B: MEAN FIELD POTENTIAL

We calculate the direct and exchange potentials (2.5) depending on the densities (2.6). As we have already shown, the spherical approach is accurate enough for the evaluation of the α -particle formation amplitude. Therefore the first density in Eq. (2.6) can be estimated in terms of the spherical sp wave functions summed on spin projections

$$\begin{aligned} |\psi_a(\mathbf{r})|^2 &= \sum_{m=-j}^j |\psi_{am}(\mathbf{r})|^2 = \mathcal{R}_{\tau \ell j}^2(r) \sum_{m=-j}^j [\mathcal{Y}_{jm}^{(l \frac{1}{2})}(\hat{\mathbf{r}}, s)]^\dagger \mathcal{Y}_{jm}^{(l \frac{1}{2})}(\hat{\mathbf{r}}, s) \\ &= \frac{1}{4\pi} \mathcal{R}_{\tau \ell j}^2(r) \left[(2j+1) + \sum_{L>0}^{2j} (2L+1) \sum_{m=-j}^j C_{m0m}^{jLj} C_{\frac{1}{2}0\frac{1}{2}}^{jLj} P_L(\hat{\mathbf{r}}) \right] \end{aligned} \quad (\text{B1})$$

and satisfying the normalization rule

$$\int |\psi_a(\mathbf{r})|^2 d\mathbf{r} = 2j + 1. \quad (\text{B2})$$

As such, the density can be expanded as

$$\begin{aligned} \rho(r, \cos \theta) &= \rho_0(r) + \sum_{L>0}^{2j} \rho_L(r) P_L(\cos \theta), \quad \rho_0(r) \equiv \frac{1}{4\pi} \sum_a (2j_a + 1) V_a^2 \mathcal{R}_a^2(r), \\ \rho_L(r) &\equiv \frac{1}{4\pi} \sum_a V_a^2 \mathcal{R}_a^2(r) \sum_{L>0}^{2j_a} (2L + 1) C_{\frac{1}{2}0\frac{1}{2}}^{j_a L j_a} \sum_{m=-j_a}^{j_a} C_{m0m}^{j_a L j_a}. \end{aligned} \quad (\text{B3})$$

Notice that the direct part of the potential with $\mathbf{r} \equiv \mathbf{r}_\tau$ is evaluated

$$\Gamma^{(dir)}(\mathbf{r}) = \int d\mathbf{r}' v(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') = V_{MF}(r, b_{\text{rel}}, \infty, 0) + x_c V_{MF}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) \quad (\text{B4})$$

as a sum of two terms, namely a standard mean field potential given by the relative inter-nucleon interaction and a term given by the SGI internucleon interaction (2.2). The general expression of the mean field is obtained through the following integral, where the major contribution is due to the monopole density term:

$$\begin{aligned} V_{MF}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) &= -v_0 \exp \left[-\left(\frac{r}{b_{\text{rel}}} \right)^2 - \left(\frac{r - 2R_0}{2b_{\text{c.m.}}} \right)^2 \right] I(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0), \\ I(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) &\equiv \int d\mathbf{r}' \exp \left[-\frac{r'^2 - 2rr' \cos \theta}{b_{\text{rel}}^2} - \frac{r'^2 + 2rr' - 4r'R_0}{(2b_{\text{c.m.}})^2} \right] \rho(\mathbf{r}') \\ &= 2\pi \int_0^\infty r'^2 dr' \exp \left[-\frac{r'^2}{b_{\text{rel}}^2} - \frac{r'^2 + 2rr' - 4r'R_0}{(2b_{\text{c.m.}})^2} \right] \int_{-1}^1 d\cos \theta \exp \left[\frac{2rr' \cos \theta}{b_{\text{rel}}^2} \right] \rho(r', \cos \theta) \\ &\approx \frac{b_{\text{rel}}^2}{4r} \int_0^\infty r' dr' \exp \left[-\frac{r'^2}{b_{\text{rel}}^2} - \frac{r'^2 + 2rr' - 4r'R_0}{(2b_{\text{c.m.}})^2} \right] \\ &\quad \times \left[\exp \left(\frac{2rr'}{b_{\text{rel}}^2} \right) - \exp \left(-\frac{2rr'}{b_{\text{rel}}^2} \right) \right] \sum_a (2j_a + 1) V_a^2 \mathcal{R}_a^2(r') \equiv \sum_a (2j_a + 1) I_a(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0). \end{aligned} \quad (\text{B5})$$

Let us stress on the fact that the above general mean field expression has a Woods-Saxon plus a Gaussian shape centered around R_0 given by the integral I . By replacing the monopole density with its mean value

$$\rho_0(r') = \sum_a (2j_a + 1) V_a^2 \mathcal{R}_a^2(r') \rightarrow \frac{N_\tau}{R_\tau} \Theta(R_\tau - r'), \quad (\text{B6})$$

where R_τ is the equivalent radius of the constant density distribution, one obtains the integral in terms of the erf function

$$I(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) = I^{(+)}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) - I^{(-)}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0), \quad (\text{B7})$$

where

$$\begin{aligned} I^{(\pm)}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) &\equiv \frac{N_\tau}{R_\tau} \frac{b_{\text{rel}}^2}{4r} \int_0^{R_\tau} r' dr' \exp \left[-\frac{r'^2}{b_{\text{rel}}^2} - \frac{r'^2 + 2rr' - 4r'R_0}{(2b_{\text{c.m.}})^2} \pm \frac{2rr'}{b_{\text{rel}}^2} \right] \\ &= \frac{N_\tau}{R_\tau} \frac{b_{\text{rel}}^2}{4r} \left\{ \frac{\sqrt{\pi} b^{(\pm)}}{4a^{\frac{3}{2}}} \exp \left(\frac{(b^{(\pm)})^2}{4a} \right) \left[\text{erf} \left(\frac{2aR_\tau - b^{(\pm)}}{2\sqrt{a}} \right) + \text{erf} \left(\frac{b^{(\pm)}}{2\sqrt{a}} \right) \right] \right. \\ &\quad \left. - \frac{1}{2a} [\exp(R_\tau(b^{(\pm)} - aR_\tau)) - 1] \right\} \end{aligned} \quad (\text{B8})$$

with

$$a = \frac{1}{b_{\text{rel}}^2} + \frac{1}{(2b_{\text{c.m.}})^2}, \quad b^{(\pm)} = \pm \frac{2r}{b_{\text{rel}}^2} + \frac{4R_0 - 2r}{(2b_{\text{c.m.}})^2}. \quad (\text{B9})$$

Using the obvious notation

$$I(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) \equiv I_0(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) + I_{cl}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0), \quad (\text{B10})$$

where the first term contains erf functions and the second one exponentials, we can express the potential (B5) as

$$V_{MF}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) = V_0(r) + V_{cl}(r), \quad (\text{B11})$$

where

$$\begin{aligned}
 V_0(r) &= -v_0 \exp \left[-\left(\frac{r}{b_{\text{rel}}} \right)^2 - \left(\frac{r-2R_0}{2b_{\text{c.m.}}} \right)^2 \right] I_0(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0), \\
 V_{cl}(r) &= -v_0 \exp \left[-\left(\frac{r}{b_{\text{rel}}} \right)^2 - \left(\frac{r-2R_0}{2b_{\text{c.m.}}} \right)^2 \right] I_{cl}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) \\
 &\equiv A_{cl}^{(-)} \exp \left[-\left(\frac{r-R_{cl}^{(-)}}{b_{cl}} \right)^2 \right] - A_{cl}^{(+)} \exp \left[-\left(\frac{r-R_{cl}^{(+)}}{b_{cl}} \right)^2 \right]
 \end{aligned} \tag{B12}$$

with

$$\begin{aligned}
 A_{cl}^{(\pm)} &= -v_0 \frac{N_\tau}{R_\tau} \frac{b_{\text{rel}}^2}{2a} \frac{1}{4r} \exp \left[\frac{R_{cl}^{(\pm)2} - R_\tau^2}{b_{cl}^2} - \frac{R_0^2 - R_0 R_\tau}{b_{\text{c.m.}}^2} \right], \\
 R_{cl}^{(-)} &= \frac{2R_0 b_{\text{rel}}^2}{(2b_{\text{c.m.}})^2 + b_{\text{rel}}^2} - R_\tau = R_0 \left(\frac{2}{y+1} - r_\tau \right), \\
 R_{cl}^{(+)} &= \frac{2R_0 b_{\text{rel}}^2}{(2b_{\text{c.m.}})^2 + b_{\text{rel}}^2} + R_\tau \left[\frac{(2b_{\text{c.m.}})^2 - b_{\text{rel}}^2}{(2b_{\text{c.m.}})^2 + b_{\text{rel}}^2} \right] = R_0 \left(\frac{2}{y+1} + r_\tau \frac{y-1}{y+1} \right), \\
 b_{cl}^2 &= \frac{(2b_{\text{c.m.}})^2 b_{\text{rel}}^2}{(2b_{\text{c.m.}})^2 + b_{\text{rel}}^2} = \frac{(2b_{\text{c.m.}})^2}{y+1}, \quad y \equiv \frac{(2b_{\text{c.m.}})^2}{b_{\text{rel}}^2}, \quad r_\tau \equiv \frac{R_\tau}{R_0} = 0.75.
 \end{aligned} \tag{B13}$$

We used the systematic rules

$$R_\tau = 1.2A_D^{1/3}, \quad R_0 = 1.6A^{1/3}. \tag{B14}$$

R_τ is the equivalent radius of the constant density distribution used in the approximation of the density found in Eq. (B6). The simple parametrization given here, equivalent to the usual spherical nuclear saturation radius, was found to be valid for all the emitters studied in this work. R_0 is parametrized here in terms of A , the mass number of the parent nucleus. The resulting value is slightly beyond that of the geometrical touching radius of Eq. (3.1) and is equivalent with the parametrization given in the main text at the end of Sec. III A. The value was chosen for its universal validity across the calculations performed in this work.

A special case is given by $y = 1$, i.e., $2b_{\text{c.m.}} = b_{\text{rel}}$, leading to the following values:

$$R_{cl}^{(-)} = R_0(1 - r_\tau) \ll R_0, \quad R_{cl}^{(+)} = R_0, \quad b_{cl}^2 = 2b_{\text{c.m.}}^2. \tag{B15}$$

At $r = R_0$, these give

$$A_{cl}^{(+)} \approx -v_0 \frac{N_\tau}{R_\tau} \frac{b_{\text{rel}}^4}{16R_0} \exp \left[-\frac{(R_0 - R_\tau)^2}{2b_{\text{c.m.}}^2} \right]. \tag{B16}$$

We can rewrite the direct part of the mean field (B5) as the following summation:

$$V_{MF}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) = \sum_a (2j_a + 1) V_{MF}^{(a)}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0), \tag{B17}$$

in terms of the general function

$$V_{MF}^{(a)}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) \equiv -v_0 \exp \left[-\left(\frac{r}{b_{\text{rel}}} \right)^2 - \left(\frac{r-2R_0}{2b_{\text{c.m.}}} \right)^2 \right] I_a(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0). \tag{B18}$$

Concerning the exchange part one obtains for the first monopole leading term the following expression:

$$\begin{aligned}
 \int d\mathbf{r}' \Gamma^{(exc)}(\mathbf{r}, \mathbf{r}') \psi_{am}(\mathbf{r}') &= - \int d\mathbf{r}' v(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}, \mathbf{r}') \psi_{am}(\mathbf{r}') = - \int d\mathbf{r}' v(\mathbf{r}, \mathbf{r}') \sum_b V_b^2 \sum_{\mu=-j_b}^{j_b} \psi_{b\mu}(\mathbf{r}') \psi_{b\mu}(\mathbf{r}) \psi_{am}(\mathbf{r}') \\
 &= - \int d\mathbf{r}' v(\mathbf{r}, \mathbf{r}') \sum_b V_b^2 \sum_{\mu=-j_b}^{j_b} \mathcal{R}_b(r') \mathcal{Y}_{j_b\mu}^\dagger(\hat{r}') \mathcal{R}_b(r) \mathcal{Y}_{j_b\mu}(\hat{r}) \mathcal{R}_a(r') \mathcal{Y}_{j_a m}(\hat{r}') \\
 &\approx - \int d\mathbf{r}' v(\mathbf{r}, \mathbf{r}') V_a^2 \frac{1}{4\pi} \mathcal{R}_a^2(r') \mathcal{R}_a(r) \mathcal{Y}_{j_a m}(\hat{r}) = V_{MF}^{(a)}(r, b_{\text{rel}}, b_{\text{c.m.}}, R_0) \psi_a(\mathbf{r}),
 \end{aligned} \tag{B19}$$

where we notice a smaller contribution given by only one a th direct mean field term (B18).

- [1] G. Gamow, *Z. Phys.* **51**, 204 (1928).
- [2] R. W. Gurney and E. U. Condon, *Nature (London)* **122**, 439 (1928).
- [3] J. Tanaka *et al.*, *Science* **371**, 260 (2021).
- [4] Y. K. Gambhir, A. Bhagwat, and M. Gupta, *Ann. Phys. (NY)* **320**, 429 (2005).
- [5] H. J. Mang, *Phys. Rev.* **119**, 1069 (1960).
- [6] A. Săndulescu, *Nucl. Phys.* **37**, 332 (1962).
- [7] V. G. Soloviev, *Phys. Lett.* **1**, 202 (1962).
- [8] H. J. Mang, *Annu. Rev. Nucl. Sci.* **14**, 1 (1964); J. K. Poggenburg, H. J. Mang, and J. O. Rasmussen, *Phys. Rev.* **181**, 1697 (1969).
- [9] T. Fliessbach, H. J. Mang, and J. O. Rasmussen, *Phys. Rev. C* **13**, 1318 (1976).
- [10] I. Tonozuka and A. Arima, *Nucl. Phys. A* **323**, 45 (1979).
- [11] F. A. Janouch and R. J. Liotta, *Phys. Rev. C* **27**, 896 (1983).
- [12] D. S. Delion, A. Insolia, and R. J. Liotta, *Phys. Rev. C* **46**, 1346 (1992); **49**, 3024 (1994).
- [13] D. S. Delion and J. Suhonen, *Phys. Rev. C* **61**, 024304 (2000).
- [14] S. M. Lenzi, O. Dragun, E. E. Maqueda, R. J. Liotta, and T. Vertse, *Phys. Rev. C* **48**, 1463 (1993).
- [15] R. Id Betan and W. Nazarewicz, *Phys. Rev. C* **86**, 034338 (2012).
- [16] A. M. Lane and R. G. Thomas, *Rev. Mod. Phys.* **30**, 257 (1958).
- [17] D. S. Delion, *Phys. Rev. C* **80**, 024310 (2009).
- [18] A. Dumitrescu and D. S. Delion, *At. Data Nucl. Data Tables* **145**, 101501 (2022).
- [19] A. S. Carstea and A. Ludu, *Phys. Rev. Res.* **3**, 033054 (2021).
- [20] D. S. Delion, *Theory of Particle and Cluster Emission* (Springer-Verlag, Berlin, 2010).
- [21] K. Varga, R. G. Lovas, and R. J. Liotta, *Phys. Rev. Lett.* **69**, 37 (1992); *Nucl. Phys. A* **550**, 421 (1992).
- [22] D. S. Delion, R. J. Liotta, P. Schuck, A. Astier, and M.-G. Porquet, *Phys. Rev. C* **85**, 064306 (2012).
- [23] D. S. Delion and R. J. Liotta, *Phys. Rev. C* **87**, 041302(R) (2013).
- [24] D. Bai, Z. Ren, and G. Röpke, *Phys. Rev. C* **99**, 034305 (2019).
- [25] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin, 1980).
- [26] P. Schuck, D. S. Delion, J. Dukelsky, M. Jemai, E. Litvinova, G. Röpke, and M. Tohyama, *Phys. Rep.* **929**, 1 (2021).
- [27] V. V. Baran and D. S. Delion, *Phys. Rev. C* **94**, 034319 (2016).
- [28] S. Ghinescu and D. S. Delion, *J. Phys. G: Nucl. Part. Phys.* **48**, 105108 (2021).
- [29] P. O. Fröman, *Mat. Fys. Skr. Dan. Vid. Selsk.* **1**, 3 (1957).
- [30] S. Cwiok, J. Dudek, W. Nazarewicz, W. Skalski, and J. Werner, *Comput. Phys. Commun.* **46**, 379 (1987).
- [31] J. Dudek, Z. Szymanski, and T. Werner, *Phys. Rev. C* **23**, 920 (1981).
- [32] J. Dudek, Z. Szymanski, T. Werner, A. Faessler, and C. Lima, *Phys. Rev. C* **26**, 1712 (1982).
- [33] G. Röpke, A. Schnell, P. Schuck, and P. Nozieres, *Phys. Rev. Lett.* **80**, 3177 (1998).
- [34] A. Tohsaki, H. Horiuchi, P. Schuck, and G. Röpke, *Rev. Mod. Phys.* **89**, 011002 (2017).
- [35] Evaluated Nuclear Structure Data Files of the Brookhaven National Laboratory, <http://www.nndc.bnl.gov/ensdfl/>.
- [36] S. A. Ghinescu and D. S. Delion, *Phys. Rev. C* **106**, 034602 (2022).
- [37] D. S. Delion and A. Dumitrescu, *Phys. Rev. C* **103**, 054325 (2021).
- [38] D. S. Delion and A. Dumitrescu, *Phys. Rev. C* **87**, 044314 (2013).
- [39] P. Möller, R. J. Nix, W. D. Myers, and W. Swiatecki, *At. Data Nucl. Data Tables* **59**, 185 (1995).