Accounting for nonvanishing net-charge with unified balance functions

Claude Pruneau [1,*] Victor Gonzalez [1,†] Brian Hanley [1,‡] Ana Marin [2,§] and Sumit Basu [3,‖]

[1] *Department of Physics and Astronomy, Wayne State University, Detroit, Michigan 48201, USA*
[2] *Bundesministerium für Bildung und Forschung and GSI Helmholtzzentrum für Schwerionenforschung GmbH, Research Division and ExtreMe Matter Institute EMMI, Darmstadt, 64291 Germany*
[3] *Lund University, Department of Physics, Division of Particle Physics, Box 118, SE-221 00, Lund, Sweden*

(Received 21 September 2022; accepted 12 December 2022; published 5 January 2023)

The use of charge balance functions in heavy-ion collision studies was initially proposed as a probe of delayed hadronization and two-stage quark production in these collisions. It later emerged that general balance functions can also serve as a probe of the diffusivity of light quarks as well as the evolution of the systems formed in heavy-ion collisions. In this work, we reexamine the formulation of general balance functions and consider how to best define and measure these correlation functions in terms of differences of conditional densities of unlike-sign and like-sign particle pairs. We define general balance functions in terms of associated particle functions and show that these obey a simple sum rule. We additionally proceed to distinguish between balance functions expressed as differences of conditional densities valid irrespective of experimental acceptance boundaries and bound balance functions that explicitly account for the limited acceptance of experiments. General balance functions are additionally extended to accommodate strange, baryon, as well as charm and bottom quantum numbers based on the densities of these quantum numbers.

DOI: 10.1103/PhysRevC.107.014902

I. INTRODUCTION

Balance functions (BFs) were introduced in the study of heavy-ion collisions at the BNL Relativistic Heavy Ion Collider (RHIC) as a tool to investigate the evolution of particle production with collision centrality [1,2], and, more specifically, to seek evidence of delayed hadronization and two-stage quark emission in these collisions. More recently, it was also shown that BFs may serve as a probe of the diffusivity of light quarks [3,4] as well as the chemical evolution of the hot matter formed in *A*-*A* collisions [5,6]. The light quark diffusivity (LQD) was found to impact the shape and width of azimuthal projections of BFs: the larger the diffusivity is, the larger the BF become azimuthally ($\Delta\varphi$) as a result of light quark scatterings during the short lifetime of the dense QGP systems formed in heavy-ion collisions [3,4]. However, the shape of BFs is also influenced by a number of other phenomena, including the fraction of late (vs early) quark production determined by the temperature of the system [1,2,7], the presence of strong pressure gradients and the rapid transverse expansion of the QGP matter [8–10], quantum statistic effects (i.e., Hanbury-Brown–Twiss) [11], as well as feed down from resonance decays [5]. In spite of these caveats, measurements of general balance functions do provide a new and complementary approach towards the determination of viscous effects and the diffusivity of light quarks [12]. While studies of flow performed on the basis azimuthal multiparticle correlations are driven, in large collision systems, by the collision geometry and somewhat hampered by nonflow effects, the estimation of the diffusivity and viscous effects with balance functions is less dependent on knowledge of the collision geometry and relies explicitly on two-particle correlations and the impact of the medium on these correlations. Conclusions reached with the two approaches should thus yield mutually compatible values of these observables [12–14].

Figures 1(a) and 1(b) schematically represent the time evolution of the system temperature and the abundance of quarks and gluons commonly assumed to take place in collisions of heavy ions featuring a substantial quark gluon plasma (QGP) component and an extended isentropic expansion stage [1,15,16]. Strange (charm, bottom) quarks being heavier, their production shall preferentially occur at early times featuring the highest effective temperature whereas lighter up and down quarks can be abundantly produced at late stages of the collision (as well as early times) as the system hadronizes. The variable $\sqrt{s}$ represents the average effective collision energy of quarks and gluons at a given time during the collision. In locally thermalized system, $\sqrt{s}$ is determined by the effective temperature of the system [17,18]. As the temperature decreases, so does $\sqrt{s}$ and the particles created by collisions

*claude.pruneau@wayne.edu
†victor.gonzalez@cern.ch
‡bghanley@wayne.edu
§a.marin@gsi.de
‖sumit.basu@cern.ch

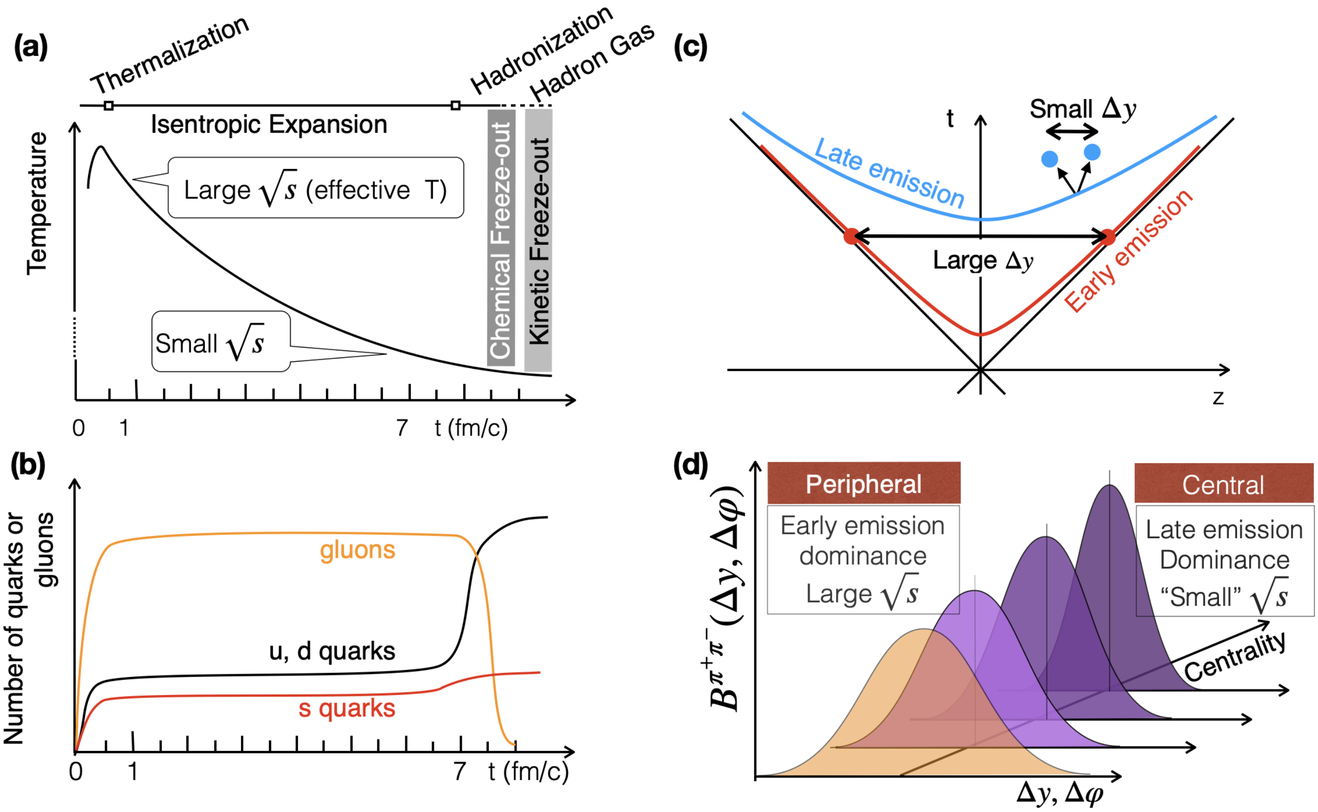

FIG. 1. (a) Schematic evolution of *A-A* collision system (effective) temperature vs time. The variable $\sqrt{s}$ here represents the average collision energy of quarks and gluons at a given time. In equilibrated systems, it is determined by the temperature and decreases as the system expands. (b) Schematic abundance of gluons and quarks vs time. (c) Schematic representation of early and late quark production and its impact of the relative rapidity of particle pairs. (d) Expected evolution of balance functions, plotted as functions of rapidity (pseudorapidity) and azimuth differences vs collision centrality.

accordingly feature smaller average longitudinal rapidity differences. Figure 1(c) schematically shows the relative effects of early and late emission of $q\bar{q}$ pairs on the rapidity difference of hadrons they eventually produce, whereas Fig. 1(d) qualitatively illustrates the evolution of the shape of balance functions on the collision centrality as a result of changes in the early-late quark emission dominance and the narrowing effect engendered by radial flow. Not shown are effects of scattering (diffusivity) of quarks and hadrons, which are expected to produce a broadening of the $\Delta\varphi$ width of balance functions [3,4]. In the absence of a QGP component or with a very-short-lived isentropic expansion stage, all particles would be produced at about the same time and average $\sqrt{s}$ and one would thus expect no substantial change of the balance function widths vs collision centrality. In the other extreme, i.e., if the system is fully thermalized, memory of the quark production time and mechanisms is lost, thereby resulting in very broad and featureless balance functions.

Precise determination of the diffusivity of light quarks and other properties shall require one properly controls and corrects measurements of the shape and integral of BFs. Indeed, determining the diffusivity of light quarks and other properties of the QGP will require careful comparison of high precision measurements with detailed calculations of the evolution of nuclear matter and its impact on the shape and strength of BFs [19]. It should additionally be noted that many heavy-ion models currently in use in the field, particularly those assuming grand-canonical particle production or a hydrodynamic expansion phase followed by Cooper-Fry particlization do not and cannot produce realistic balance functions. Further development and deployment of balance function measurements shall thus open the door to a better understanding of the microscopic plasma.

One must also note that the notion of charge balance function can readily be extended to general (charge) balance functions involving identified particle species as well as baryon number and strangeness balance functions, explicitly discussed for the first time in this work. It is thus important for theoreticians and experimentalists to clearly define the correlation functions known as *general balance functions* and agree on specific definitions and notations of the theoretical quantities being measured and their actual implementation in measurements. It is the purpose of this work to explore definition options and propose specific choices of formulations and notations of general balance functions as standards for use by the community. In this work, our goal is to assert what constitutes, theoretically, the most meaningful definition of general balance functions, and, experimentally, the best approach to measure them and their integrals.

Notations for the different components involved in the elaboration of BFs are defined in Sec. II, whereas the notion of general balance functions is introduced in Sec. III based on integral quantities. The notion of general balance function is extended to correlation functions of pairs of identified particle species in Sec. IV. Charge conservation and the presence of net-charge imply a BF sum rule discussed in Sec. V. This naturally leads to extensions involving baryon and strangeness balance functions in Sec. VI. Experimental considerations, involving, in particular, measurements of balance function in difference coordinates, e.g., Δy and $\Delta\varphi$, are discussed in Sec. IX. We also briefly discuss in Sec. X the connection between balance functions and the $\nu_{\rm dyn}^{+-}$ observable [20]. This work is summarized in Sec. XI.

II. NOTATION AND DEFINITIONS

Herein, the identity of particle species (e.g., π^+, K^+, etc.) is represented with Greek letters α, β, etc., and their respective antiparticles (e.g., π^-, K^-, etc.) with barred letters $\bar{\alpha}$, $\bar{\beta}$.

Single and pair densities of species α and β are denoted and defined according to

$$\rho_1^{\alpha}(y_1,\varphi_1,p_{\rm T,1}) \equiv \frac{d^3N_1^{\alpha}}{dy_1 d\varphi_1 dp_{\rm T,1}}, \tag{1}$$

$$\rho_2^{\alpha\beta}(y_1,\varphi_1,p_{\rm T,1},y_2,\varphi_2,p_{\rm T,2}) \equiv \frac{d^6N_2^{\alpha\beta}}{dy_1 d\varphi_1 dp_{\rm T,1} dy_2 d\varphi_2 dp_{\rm T,2}}, \tag{2}$$

where N_1^{α} and $N_2^{\alpha\beta}$ respectively represent numbers of particles of species α and pairs of particles of species α and β. Variables y_1, φ_1, $p_{\rm T,1}$ and y_2, φ_2, $p_{\rm T,2}$ are the rapidity, azimuth, and transverse momentum of particles of species α and β, respectively.

The average number of particles of species α measured per event and within an acceptance Ω is

$$\langle N_1^{\alpha}\rangle = \int_{\Omega} \rho_1^{\alpha}(y,\varphi,p_{\rm T})dy d\varphi dp_{\rm T} = V\bar{\rho}_1, \tag{3}$$

where $V = \int_{\Omega} dy d\varphi dp_{\rm T}$ is the selected/accepted phase space volume and $\bar{\rho}_1^{\alpha}$ the average density across this volume. Similarly, the average number of *pairs* of particles of species α and β, measured within Ω, is given by

$$\begin{aligned}\langle N_2^{\alpha\beta}\rangle &= \langle N_1^{\alpha}(N_1^{\beta}-\delta_{\alpha\beta})\rangle \\ &= \int_{\Omega} dy_1 d\varphi_1 dp_{\rm T,1} \int_{\Omega} dy_2 d\varphi_2 dp_{\rm T,2}\, \rho_2^{\alpha\beta} \\ &\quad \times (y_1,\varphi_1,p_{\rm T,1},y_2,\varphi_2,p_{\rm T,2}).\end{aligned} \tag{4}$$

In the following, if a particular variable, e.g., $p_{\rm T}$, is omitted from the expression of densities, it is assumed to be integrated across the fiducial acceptance of the detector. For instance,

$$\rho_1^{\alpha}(y_1,\varphi_1) = \int_{\Omega_1} dp_{\rm T,1}\rho_1^{\alpha}(y_1,\varphi_1,p_{\rm T,1}), \tag{5}$$

$$\begin{aligned}\rho_2^{\alpha\beta}(y_1,\varphi_1,y_2,\varphi_2) &= \int_{\Omega_1} dp_{\rm T,1} \int_{\Omega_2} dp_{\rm T,2}\, \rho_2^{\alpha\beta} \\ &\quad \times (y_1,\varphi_1,p_{\rm T,1},y_2,\varphi_2,p_{\rm T,2}),\end{aligned} \tag{6}$$

where Ω, with $i = 1, 2$, represents the $p_{\rm T}$ acceptance of particles of type α and β, respectively. For the sake of simplicity, the rapidity acceptance of the measurement shall be assumed herewith to be the same for all particles species: $-y_0 \leqslant y < y_0$.

The averages $\langle N_1^{\alpha}\rangle$ and $\langle N_2^{\alpha\beta}\rangle$ correspond to first and second factorial moments and are hereafter denoted f_1^{α} and $f_2^{\alpha\beta}$, respectively [21]. It is also useful to consider first- and second-order factorial cumulants, F_1^{α} and $F_2^{\alpha\beta}$, computed respectively as

$$F_1^{\alpha} = f_1^{\alpha}, \tag{7}$$

$$F_2^{\alpha\beta} = f_2^{\alpha\beta} - f_1^{\alpha}f_1^{\beta}, \tag{8}$$

as well as normalized second-order cumulants $R_2^{\alpha\beta}$ defined as

$$R_2^{\alpha\beta} = \frac{F_2^{\alpha\beta}}{F_1^{\alpha}F_1^{\beta}} = \frac{\langle N_2^{\alpha\beta}\rangle}{\langle N_1^{\alpha}\rangle\langle N_1^{\beta}\rangle} - 1, \tag{9}$$

where it is implicitly assumed that all integral quantities are determined within the measurement acceptance Ω.

III. INTEGRAL BALANCE FUNCTIONS

We first consider the definition of general balance functions (BF) based on integral quantities. Rather than defining BFs based on combinations of $+-$, $-+$, $++$, and $--$ particle pairs as in Ref. [22], we "split" the definition to consider $+-$ and $-+$ pairs relative to $--$ and $++$ pairs, respectively. The two definitions should evidently be equivalent for symmetric collision systems. Experimentally, however, instrumental artifacts may induce artificial differences between $+-$ and $-+$ pairs and it is thus of interest to explicitly verify that the two definitions yield the same value thereby enabling validation of experimental calibrations and correction methods [23].

Hereafter, we shall use the notation $I^{\alpha\bar{\beta}}$ for integral balance functions, which correspond, as we shall see, to integrals across the measurement acceptance of differential balance functions denoted $B^{\alpha\bar{\beta}}(y_1, y_2)$ defined in Sec. IV.

Let us tentatively define general charge (integral) balance functions according to

$$I^{\alpha\bar{\beta}} = \frac{\langle N_2^{\alpha\bar{\beta}}\rangle}{\langle N_1^{\bar{\beta}}\rangle} - \frac{\langle N_2^{\bar{\alpha}\bar{\beta}}\rangle}{\langle N_1^{\bar{\beta}}\rangle}, \tag{10}$$

$$I^{\bar{\alpha}\beta} = \frac{\langle N_2^{\bar{\alpha}\beta}\rangle}{\langle N_1^{\beta}\rangle} - \frac{\langle N_2^{\alpha\beta}\rangle}{\langle N_1^{\beta}\rangle}, \tag{11}$$

which should give us a measure of how many particles of type α ($\bar{\alpha}$) balance each "trigger" particle $\bar{\beta}$ (β). One straightforwardly verifies these expressions converge to unity for $\alpha = +$, $\bar{\beta} = -$ and $\bar{\alpha} = -$, $\beta = +$, i.e.,

$$I^{+-} \to 1, \tag{12}$$

$$I^{-+} \to 1, \tag{13}$$

in the ideal limit of a 4π detection system with full $p_{\rm T}$ coverage and collisions involving a vanishing net-charge Q, e.g., $p\bar{p}$ collisions. Indeed, for $\alpha, \beta = +$ and $\bar{\alpha}, \bar{\beta} = -$, by virtue of charge conservation, the creation of a particle of type $\alpha = +$ must be accompanied by the production of a particle of type

$\bar{\alpha} = -$. If the number of such pair creations (i.e., number of sources) is N_s in a given event, then the *total* number of singles and pairs are

$$N_1^+ = N_s, \tag{14}$$
$$N_1^- = N_s, \tag{15}$$
$$N_2^{+-} = N_s^2, \tag{16}$$
$$N_2^{-+} = N_s^2, \tag{17}$$
$$N_2^{++} = N_s(N_s - 1), \tag{18}$$
$$N_2^{--} = N_s(N_s - 1). \tag{19}$$

The expressions (10) and(11) computed over the full 4π acceptance (all rapidities and transverse momenta) thus indeed converge to unity:

$$I^{-+}(4\pi) = I^{+-}(4\pi) = \frac{\langle N_s^2\rangle}{\langle N_s\rangle} - \frac{\langle N_s^2 - N_s\rangle}{\langle N_s\rangle} = 1. \tag{20}$$

However, the above definitions (10) and (11), do not account for the presence, *ab initio*, of a nonvanishing net-charge Q. For instance, pp collisions feature $Q = 2$ *ab initio* and given the electric charge is a conserved quantity, the event-wise single and pair yields shall be

$$N_1^+ = N_s + Q, \tag{21}$$
$$N_1^- = N_s, \tag{22}$$
$$N_2^{+-} = (N_s + Q)N_s, \tag{23}$$
$$N_2^{-+} = N_s(N_s + Q), \tag{24}$$
$$N_2^{++} = (N_s + Q)(N_s + Q - 1), \tag{25}$$
$$N_2^{--} = N_s(N_s - 1), \tag{26}$$

in each event. The definitions (10) and (11) thus yield

$$I^{+-}(4\pi) = \frac{\langle (N_s + Q)N_s\rangle}{\langle N_s\rangle} - \frac{\langle N_s(N_s + -1)\rangle}{\langle N_s\rangle} = 1 + Q, \tag{27}$$

$$I^{-+}(4\pi) = \frac{\langle (N_s + Q)N_s\rangle}{\langle N_s + Q\rangle} - \frac{\langle (N_s + Q)(N_s + Q - 1)\rangle}{\langle N_s + Q\rangle} = 1 - Q, \tag{28}$$

where the notation (4π) indicates that the integral is computed in full angular and $p_{\rm T}$ acceptance. The presence of the terms Q and $-Q$ in the above two equations results from charge conservation and the initial net-charge Q. It implies, for instance, that the integral of the $p\bar{p}$ BF measured in pp collisions could amount to $+3$ or -1 depending on trigger species. Similarly, Pb–Pb collisions could yield BF integral amounting to $1 + (82 + 82) = 165$ or $1 - (82 + 82) = -163$. Evidently, the impact of the nonvanishing net-charge should be less important in the central rapidity region when the beam rapidity is very large (e.g., LHC energies) but could be significant at low RHIC Beam Energy Scan energies that involve beam rapidities of order four or smaller. The presence of nonvanishing net-charge may then confuse the interpretation of balance functions and their integrals. It is then convenient to eliminate this dependence and modify the definition of integral balance functions (10) and (11) according to

$$I^{\alpha\bar{\beta}} \equiv \frac{\langle N_2^{\alpha\bar{\beta}}\rangle}{\langle N_1^{\bar{\beta}}\rangle} - \frac{\langle N_2^{\bar{\alpha}\bar{\beta}}\rangle}{\langle N_1^{\bar{\beta}}\rangle} - \left(\langle N_1^{\alpha}\rangle - \langle N_1^{\bar{\alpha}}\rangle\right), \tag{29}$$

$$I^{\bar{\alpha}\beta} \equiv \frac{\langle N_2^{\bar{\alpha}\beta}\rangle}{\langle N_1^{\beta}\rangle} - \frac{\langle N_2^{\alpha\beta}\rangle}{\langle N_1^{\beta}\rangle} + \left(\langle N_1^{\alpha}\rangle - \langle N_1^{\bar{\alpha}}\rangle\right), \tag{30}$$

which shall, by construction, yield $I^{+-} \to 1$, $I^{-+} \to 1$ in the full 4π- and $p_{\rm T}$-acceptance limit.

Experimentally, a full acceptance is not achievable, and one might be limited to, e.g., $-y_0 \leqslant y < y_0$ and $p_{\rm T,min} \leqslant p_{\rm T} < p_{\rm T,max}$, with full azimuthal acceptance.[1] One straightforwardly verifies that the balance functions (29) and (30) computed within such limited acceptance Ω shall be smaller than unity. It is also useful to note that $I^{\alpha\bar{\beta}}$ and $I^{\bar{\alpha}\beta}$ can also be expressed in terms of integral cumulants and normalized integral cumulants (sometimes called reduced cumulants) according to

$$I^{\alpha\bar{\beta}} = F_1^{\alpha}R_2^{\alpha\bar{\beta}} - F_1^{\bar{\alpha}}R_2^{\bar{\alpha}\bar{\beta}}, \tag{31}$$
$$I^{\bar{\alpha}\beta} = F_1^{\bar{\alpha}}R_2^{\bar{\alpha}\beta} - F_1^{\alpha}R_2^{\alpha\beta}. \tag{32}$$

Hereafter, we shall denote the arithmetic average of $I^{\alpha\bar{\beta}}$ and $I^{\bar{\alpha}\beta}$ as $I^{\alpha\beta,s}$:

$$I^{\alpha\beta,s} = \tfrac{1}{2}(I^{\alpha\bar{\beta}} + I^{\bar{\alpha}\beta}). \tag{33}$$

It is evidently clear, by virtue of Eqs. (27) and (28), that $I^{\alpha\beta,s}$ shall converge to unity in the full 4π- and $p_{\rm T}$-acceptance limit, irrespective of the net-charge Q of the system.

IV. DIFFERENTIAL BALANCE FUNCTIONS

With the definitions (29) and (30) in hand, we consider the formulation of balance function based on conditional densities $\rho_2^{\alpha|\beta}(y_1|y_2)$ [1] computed according to

$$\rho_2^{\alpha|\beta}(y_1|y_2) = \frac{\rho_2^{\alpha\beta}(y_1, y_2)}{\rho_1^{\beta}(y_2)}. \tag{34}$$

By construction, $\rho_2^{\alpha|\beta}(y_1|y_2)$ amounts to the density of a species α at y_1 given a particle of species β is detected at y_2.[2] To simplify the discussion, we first neglect the net-charge Q and write differential balance function according to

$$B^{\alpha|\bar{\beta}}(y_1|y_2) = \rho_2^{\alpha|\bar{\beta}}(y_1|y_2) - \rho_2^{\bar{\alpha}|\bar{\beta}}(y_1|y_2) \tag{35}$$

$$= \frac{\rho_2^{\alpha\bar{\beta}}(y_1, y_2)}{\rho_1^{\bar{\beta}}(y_2)} - \frac{\rho_2^{\bar{\alpha}\bar{\beta}}(y_1, y_2)}{\rho_1^{\bar{\beta}}(y_2)}, \tag{36}$$

which is to be considered a function of y_1 *only* since y_2 is "given." Particle $\bar{\beta}$, found at y_2, is considered the "trigger" particle whereas particles α and $\bar{\alpha}$, detected at y_1, are called "associated" particles. While it is intuitively tempting to think

[1] The discussion is formulated in terms of particle rapidities but readily also applies to pseudorapidities

[2] Hereafter, for simplicity and without loss of generality, densities and balance functions are written as functions of rapidity only.

of the function $B^{\alpha|\bar{\beta}}(y_1|y_2)$ as the density of particles of type α at y_1 *given* a particle of type $\bar{\beta}$ is found at y_2, one must acknowledge that $B^{\alpha|\bar{\beta}}(y_1|y_2)$ can in fact be negative across some fraction of the domain y_1 and thus does not amount, strictly speaking, to a particle density.

Considering once again the basic case of an inclusive charge balance function, e.g., $\alpha = \beta = +$, one writes

$$B^{+|-}(y_1|y_2) = \frac{\rho_2^{+-}(y_1, y_2)}{\rho_1^-(y_2)} - \frac{\rho_2^{--}(y_1, y_2)}{\rho_1^-(y_2)}. \quad (37)$$

Since y_2 is given, one can then proceed to integrate $B^{+|-}(y_1|y_2)$ over y_1. The production of a negatively charged particle must be accompanied by the production of a positively charged one somewhere in phase space. The integral of the balance function $B^{+|-}(y_1|y_2)$, denoted

$$I^{+|-}(y_2|\Omega) \equiv \int_\Omega dy_1 B^{+|-}(y_1|y_2), \quad (38)$$

thus converges to unity, by construction, in the 4π- and full p_T-acceptance limit:

$$\lim_{\Omega\to 4\pi} I^{+|-}(y_2|\Omega) \to 1. \quad (39)$$

Evidently, in that limit, $I^{+|-}(y_2|\Omega)$ has the same value for all y_2. However, for a given y_2 and a finite acceptance Ω : $-y_0 \leqslant y < y_0$, the integral $I^{+|-}(y_2)$ shall in general depend on y_0. Consider that, if the given value is $y_2 = 0$ and the acceptance of the measurement is symmetric $-y_0 \leqslant y < y_0$, it is obviously easier to "catch" the balancing partner than if the given position is $y_2 = y_0$. Indeed, in that case, balancing partners can only be found on "one side" whereas for $y_2 = 0$, balancing partners can be found on two sides. One thus concludes the integral $I^{+|-}(y_2)$ is a function of y_2 which depends on the shape and width of $B^{+|-}(y_1|y_2)$. It thus makes sense to consider the *average* of $I^{+|-}(y_2)$ across the acceptance Ω of the measurement:[3]

$$\bar{I}^{+-}(y_0) \equiv \int_{-y_0}^{y_0} dy_2 P_1^-(y_2) I^{+|-}(y_2), \quad (40)$$

where $P_1^-(y_2)$ represents the probability of finding the first particle at y_2. Clearly, this probability is

$$P_1^-(y_2) = \frac{1}{\langle N_1^-\rangle(y_0)} \rho_1^-(y_2), \quad (41)$$

where $\langle N_1^-\rangle(y_0) = \int_{-y_0}^{y_0} \rho_1^-(y)dy$, and which, by construction, satisfies $\int_{-y_0}^{y_0} dy_2 P_1^-(y_2) = 1$. The dependence of $\langle N_1^\pm\rangle$ on the longitudinal acceptance y_0 is further omitted for simplicity. The average sought for is thus

$$\bar{I}^{+-} = \frac{1}{\langle N_1^-\rangle} \int_{-y_0}^{y_0} dy_2 \int_{-y_0}^{y_0} dy_1 [\rho_2^{+-}(y_1, y_2) - \rho_2^{--}(y_1, y_2)]$$
$$= \frac{1}{\langle N_1^-\rangle} [\langle N_2^{+-}\rangle - \langle N_2^{--}\rangle], \quad (42)$$

[3]For simplicity, we assume a symmetric acceptance $-y_0 \leqslant y < y_0$.

which is identical in form to Eq. (10) for $\alpha = +, \bar{\beta} = -$ when $Q = 0$. It thus becomes natural to define the BF as a joint function of y_1 and y_2 according to

$$B^{+-}(y_1, y_2|y_0) = \frac{1}{\langle N_1^-\rangle}[\rho_2^{+-}(y_1, y_2) - \rho_2^{--}(y_1, y_2)], \quad (43)$$

the integral of which yields Eq. (10). The same reasoning, repeated for $B^{-+}(y_1, y_2|y_0)$, yields

$$B^{-+}(y_1, y_2|y_0) = \frac{1}{\langle N_1^+\rangle}[\rho_2^{-+}(y_1, y_2) - \rho_2^{++}(y_1, y_2)]. \quad (44)$$

The expression (43) was derived based on Eq. (37) and thus neglects the presence of a nonvanishing net-charge Q. For $Q \neq 0$, integration of $B^{+|-}(y_1|y_2)$ over the full phase space $\Omega \to 4\pi$ shall then yield $1 + Q$ rather than 1. However, note that, by definition, integration of the difference $\rho_1^+(y) - \rho_1^-(y)$ yields the net-charge Q. To obtain a balance function definition that integrates to 1, even in the presence of $Q \neq 0$, it thus suffices to subtract this difference from Eq. (37). Repeating the same reasoning for $B^{-|+}(y_1|y_2)$, one thus proceed to define charge balance functions according to

$$B^{+|-}(y_1|y_2) = \frac{\rho_2^{+-}(y_1, y_2)}{\rho_1^-(y_2)} - \frac{\rho_2^{--}(y_1, y_2)}{\rho_1^-(y_2)} - [\rho_1^+(y_1) - \rho_1^-(y_1)], \quad (45)$$

$$B^{-|+}(y_1|y_2) = \frac{\rho_2^{-+}(y_1, y_2)}{\rho_1^+(y_2)} - \frac{\rho_2^{++}(y_1, y_2)}{\rho_1^+(y_2)} + [\rho_1^+(y_1) - \rho_1^-(y_1)]. \quad (46)$$

These expressions are defined at a given value of y_2 and must thus be averaged over the acceptance Ω to yield a BF defined for all values of y_1 and y_2. Proceeding as above, one takes the averages of $B^{+|-}(y_1|y_2)$ and $B^{-|+}(y_1|y_2)$ weighed by the probabilities $P_1^\alpha(y_2) = \rho_1^\alpha(y_2)/\langle N_1^\alpha\rangle$ of finding a particle of type α at y_2, for $\alpha = -, +$, respectively. This yields "bound" balance functions

$$B^{+-}(y_1, y_2|y_0) = \frac{1}{\langle N_1^-\rangle}[\rho_2^{+-}(y_1, y_2) - \rho_2^{--}(y_1, y_2) - \rho_1^+(y_1)\rho_1^-(y_2) + \rho_1^-(y_1)\rho_1^-(y_2)], \quad (47)$$

$$B^{-+}(y_1, y_2|y_0) = \frac{1}{\langle N_1^+\rangle}[\rho_2^{-+}(y_1, y_2) - \rho_2^{++}(y_1, y_2) - \rho_1^-(y_1)\rho_1^+(y_2) + \rho_1^+(y_1)\rho_1^+(y_2)]. \quad (48)$$

By construction, these integrate to unity in the 4π (full p_T coverage) limit even in the presence of a nonvanishing net-charge, i.e., $Q \neq 0$. Noting the presence of terms of the form $\rho_2^{\alpha\beta} - \rho_1^\alpha\rho_1^\beta$, it is convenient to write the above expressions as

$$B^{+-}(y_1, y_2|y_0) = \frac{1}{\langle N_1^-\rangle}[C_2^{+-}(y_1, y_2) - C_2^{--}(y_1, y_2)], \quad (49)$$

$$B^{-+}(y_1, y_2|y_0) = \frac{1}{\langle N_1^+\rangle}[C_2^{-+}(y_1, y_2) - C_2^{++}(y_1, y_2)], \quad (50)$$

where we introduced the differential correlation functions $C_2^{\alpha\beta}$ defined according to

$$C_2^{\alpha\beta}(y_1, y_2) = \rho_2^{\alpha\beta}(y_1, y_2) - \rho_1^\alpha(y_1)\rho_1^\beta(y_2). \quad (51)$$

The expressions Eqs. (45) and (46) were defined for charge balance functions but their structure does not limit their applicability to inclusive measurements and we show in Sec. V they obey a simple sum rule which also conserves charges and accounts for the net-charge of the system. It is thus appropriate to introduce general balance functions according to

$$B^{\alpha|\bar{\beta}}(y_1|y_2) = A_2^{\alpha|\bar{\beta}}(y_1|y_2) - A_2^{\bar{\alpha}|\bar{\beta}}(y_1|y_2), \tag{52}$$

$$B^{\bar{\alpha}|\beta}(y_1|y_2) = A_2^{\bar{\alpha}|\beta}(y_1|y_2) - A_2^{\alpha|\beta}(y_1|y_2), \tag{53}$$

where we introduced single "associated" particle functions according to

$$A_2^{\alpha|\beta}(y_1|y_2) = \frac{C_2^{\alpha\beta}(y_1|y_2)}{\rho_1^{\beta}(y_2)} = \frac{\rho_2^{\alpha\beta}(y_1, y_2)}{\rho_1^{\beta}(y_2)} - \rho_1^{\alpha}(y_1). \tag{54}$$

It should first be noted that $A_2^{\alpha|\bar{\beta}}(y_1|y_2)$ and $B^{\alpha|\bar{\beta}}(y_1|y_2)$ are single-particle and single-variable functions, given the rapidity y_2 is considered given and thus not a free variable in the context of the definitions in Eqs. (52)–(54). Additionally, by construction, and in the absence of correlations, the density $\rho_2^{\alpha\beta}(y_1, y_2)$ factorizes according to

$$\rho_2^{\alpha\beta}(y_1, y_2) = \rho_1^{\alpha}(y_1)\rho_1^{\beta}(y_2). \tag{55}$$

The associated particle function $A_2^{\alpha|\beta}(y_1|y_2)$ then vanishes, by definition, for independent-particle emission (i.e., no correlations). However, in the presence of correlations, the pair density $\rho_2^{\alpha\beta}(y_1, y_2)$ may be larger or smaller than $\rho_1^{\alpha}(y_1)\rho_1^{\beta}(y_2)$ over some kinematic domain of y_1 and y_2. The function $A_2^{\alpha|\beta}(y_1|y_2)$ may then be positive, negative, or null across some portions of the acceptance. It is similarly straightforward to observe that the balance functions may also be negative or null across some portions of the acceptance. As such, neither $A_2^{\alpha|\beta}(y_1|y_2)$ nor $B^{\alpha|\beta}(y_1|y_2)$ can be considered single-particle densities. It should be additionally noted that the shape and strength of $A_2^{\alpha|\beta}(y_1|y_2)$ and thus $B^{\alpha|\beta}(y_1|y_2)$ may depend strongly on y_2. For instance, at rapidity y_2 near the beam rapidity y_B, one expects the particle production to be largely dominated by the fragmentation of the beam components whereas at central rapidity ($y \approx 0$ in a collider mode), particle production is determined by large $\sqrt{s}$ processes. The widths and shapes of BFs are thus indeed expected to vary appreciably with the selected rapidity y_2.

Experimentally, measurements of (general) balance functions are restricted to finite ranges of rapidity, transverse momentum, as well as, in some cases, azimuth. The general balance functions (52) and (53) must then be "averaged" for the position of the trigger particle: y_2, $p_{T,2}$, and φ_2. Repeating the steps leading to Eqs. (49) and (50), one gets the bound general balance functions defined according to

$$B^{\alpha\bar{\beta}}(y_1, y_2|\Omega) = \frac{1}{\langle N_1^{\bar{\beta}}\rangle}\left[C_2^{\alpha\bar{\beta}}(y_1, y_2) - C_2^{\bar{\alpha}\bar{\beta}}(y_1, y_2)\right], \tag{56}$$

$$B^{\bar{\alpha}\beta}(y_1, y_2|\Omega) = \frac{1}{\langle N_1^{\beta}\rangle}\left[C_2^{\bar{\alpha}\beta}(y_1, y_2) - C_2^{\alpha\beta}(y_1, y_2)\right], \tag{57}$$

which are applicable to same, $\alpha = \beta$, or mixed, $\alpha \neq \beta$, particle species, each carrying a single unit of charge.

It is worth mentioning that Eqs. (56) and (57) are not applicable to physical systems involving multiply charged particles, i.e., when particles of type α, β may be multicharge species, such as Δ^{++} or ^{4}He, and so on. In such cases, one must replace the single and pair particle densities, ρ_1^{α} and $\rho_2^{\alpha\beta}$, by single and pair electric charge densities defined according to

$$\rho_{e1}^{\alpha} = n_e^{\alpha}\rho_1^{\alpha}, \tag{58}$$

$$\rho_{e2}^{\alpha\beta} = n_e^{\alpha}n_e^{\beta}\rho_2^{\alpha\beta}, \tag{59}$$

where n_e^{α} and n_e^{β} represent the number of elementary charges of species α and β, respectively. Correspondingly, for cases where α and β correspond to specific particle species, Eqs. (47) and (48) transform to

$$B^{\alpha\bar{\beta}}(y_1, y_2|\Omega) = \frac{n_e^{\alpha}}{\langle N_1^{\bar{\beta}}\rangle}\left[\rho_2^{\alpha\bar{\beta}}(y_1, y_2) - \rho_2^{\bar{\alpha}\bar{\beta}}(y_1, y_2) - \rho_1^{\alpha}(y_1)\rho_1^{\bar{\beta}}(y_2) + \rho_1^{\bar{\alpha}}(y_1)\rho_1^{\bar{\beta}}(y_2)\right], \tag{60}$$

$$B^{\bar{\alpha}\beta}(y_1, y_2|\Omega) = \frac{n_e^{\alpha}}{\langle N_1^{\beta}\rangle}\left[\rho_2^{\bar{\alpha}\beta}(y_1, y_2) - \rho_2^{\alpha\beta}(y_1, y_2) - \rho_1^{\bar{\alpha}}(y_1)\rho_1^{\beta}(y_2) + \rho_1^{\alpha}(y_1)\rho_1^{\beta}(y_2)\right]. \tag{61}$$

When particles of type α, β include different species with different number of elementary charges, e.g., $+$, $++$, and $+++$, the single and pair electric charge densities shall then be defined according to

$$\rho_{e1}^{\alpha} = \sum_{\gamma} n_e^{\gamma}\rho_1^{\gamma}, \tag{62}$$

$$\rho_{e2}^{\alpha\beta} = \sum_{\gamma}\sum_{\mu} n_e^{\gamma}n_e^{\mu}\rho_2^{\gamma\mu}, \tag{63}$$

where γ (μ) refers to particles of type α (β) with an elementary charge of n_e^{γ} (n_e^{μ}) units. Equations (47) and (48) then transform to

$$B^{\alpha\bar{\beta}}(y_1, y_2|\Omega) = \frac{1}{\sum_{\bar{\mu}} n_e^{\bar{\mu}}\langle N_1^{\bar{\mu}}\rangle}\left[\rho_{e2}^{\alpha\bar{\beta}}(y_1, y_2) - \rho_{e2}^{\bar{\alpha}\bar{\beta}}(y_1, y_2) - \rho_{e1}^{\alpha}(y_1)\rho_{e1}^{\bar{\beta}}(y_2) + \rho_{e1}^{\bar{\alpha}}(y_1)\rho_{e1}^{\bar{\beta}}(y_2)\right], \tag{64}$$

$$B^{\bar{\alpha}\beta}(y_1, y_2|\Omega) = \frac{1}{\sum_{\mu} n_e^{\mu}\langle N_1^{\mu}\rangle}\left[\rho_{e2}^{\bar{\alpha}\beta}(y_1, y_2) - \rho_{e2}^{\alpha\beta}(y_1, y_2) - \rho_{e1}^{\bar{\alpha}}(y_1)\rho_{e1}^{\beta}(y_2) + \rho_{e1}^{\alpha}(y_1)\rho_{e1}^{\beta}(y_2)\right]. \tag{65}$$

V. BALANCE FUNCTIONS SUM RULES

Can the notion of balance function duly apply to mixed species of particle? Do the definitions (52) and (53) properly account for charge conservation and the charge of the system?

One expects, for instance, that the emission of a negative pion π^- shall be balanced by the production of a positive (+ve) particle. Such a +ve particle could of course be a π^+, but it does not have to be. Indeed, balancing the charge of the π^- can be accomplished, in part, via the production of a K^+, a proton p, or some other positively charged particle. In general, particles with masses larger than the mass of the proton tend to decay into either π^+, K^+, p, or some positive weakly decaying particle. Such weak decays may lead to the production of π^+, K^+, p, or positrons e^+. The balance function $B^{+|\pi^-}$, which loosely speaking corresponds to the "probability" of finding a charge balancing partner to the π^- shall thus amount to the sum of balance functions $B^{\alpha|\pi^-}$ that involve particle of type α charge balancing the π^-:

$$B^{+|\pi^-}(y_1|y_2) = \sum_\alpha B^{\alpha|\pi^-}(y_1|y_2), \tag{66}$$

where the sum on α spans all particle species that potentially balance the production of a π^-. It should be noted that, even though $B^{+|\pi^-}$ is not necessarily non-negative, and thus strictly speaking not a probability, the above sum-rule applies nonetheless, as we demonstrate below. Evidently, if the sum rule applies to the "theoretical" balance functions $B^{+|\pi^-}(y_1|y_2)$, it shall apply also, by virtue of its derivation, to the bound (experimental) functions $B^{+\pi^-}(y_1, y_2)$. We show, in the next paragraph, that the sum rule, Eq. (66), does apply, by construction, to any other types of positive (negative) particle species β ($\bar{\beta}$):

$$B^{+|\bar{\beta}}(y_1|y_2) = \sum_\alpha B^{\alpha|\bar{\beta}}(y_1|y_2), \tag{67}$$

$$B^{-|\beta}(y_1|y_2) = \sum_\alpha B^{\bar{\alpha}|\beta}(y_1|y_2). \tag{68}$$

Such balance functions sum rules have already been considered in the context of net proton number fluctuations for a system with vanishing net-charge [23] but are here extended to include the presence of nonvanishing net-charge in a collision system.

In the remainder of this section, which can be omitted in a first reading, we show that the definitions (45) and (46) and the charge conservation limit (39) imply

$$\lim_{\Omega\to 4\pi} I^{+|\bar{\beta}}(y_2) = \int dy_1 B^{+|\bar{\beta}}(y_1|y_2) \to 1, \tag{69}$$

$$\lim_{\Omega\to 4\pi} I^{-|\beta}(y_2) = \int dy_1 B^{-|\beta}(y_1|y_2) \to 1. \tag{70}$$

The definitions (45) and (46) thus not only account for charge conservation but also properly handle the presence of net-charge. The derivation is carried out for $B^{+|\bar{\beta}}(y_1|y_2)$ but evidently trivially applies to $B^{-|\beta}(y_1|y_2)$.

The derivation of the sum rule (67), based on the definition (45), is accomplished by partitioning the single and pair densities according to

$$\rho_1^+ = \sum_\alpha \rho_1^\alpha, \tag{71}$$

$$\rho_1^- = \sum_\alpha \rho_1^{\bar{\alpha}}, \tag{72}$$

$$\rho_2^{+-} = \sum_\beta \sum_\alpha \rho_2^{\alpha\bar{\beta}}, \tag{73}$$

$$\rho_2^{--} = \sum_\beta \sum_\alpha \rho_2^{\bar{\alpha}\bar{\beta}}, \tag{74}$$

where the sums span all species or antispecies as appropriate, and arguments y_1 and y_2 are omitted to simplify the notation. The integral I^{+-}, computed in the full acceptance limit, may then be written as

$$1 = \int dy_1 B^{+-}(y_1|y_2) \tag{75}$$

$$= \int dy_1 \left\{ \frac{(\rho_2^{+-} - \rho_2^{--})}{\rho_1^-} - \rho_1^+ + \rho_1^- \right\}. \tag{76}$$

Inserting the decompositions (71)–(74), one gets

$$1 = \int dy_1 \frac{1}{\sum_\beta \rho_1^{\bar{\beta}}} \left[\sum_\beta \sum_\alpha \left(\rho_2^{\alpha\bar{\beta}} - \rho_2^{\bar{\alpha}\bar{\beta}}\right) \right] - \sum_\alpha \left(\rho_1^\alpha - \rho_1^{\bar{\alpha}}\right). \tag{77}$$

Multiplying the first term within brackets by $1 = \rho_1^{\bar{\beta}}/\rho_1^{\bar{\beta}}$ and the second term by $1 = \sum_\beta \rho_1^{\bar{\beta}} / \sum_\beta \rho_1^{\bar{\beta}}$, and rearranging the sums, one obtains

$$1 = \int dy_1 \frac{1}{\sum_\beta \rho_1^{\bar{\beta}}} \left\{ \sum_\beta \sum_\alpha \left[\frac{\rho_1^{\bar{\beta}}\left(\rho_2^{\alpha\bar{\beta}} - \rho_2^{\bar{\alpha}\bar{\beta}}\right)}{\rho_1^{\bar{\beta}}} - \rho_1^{\bar{\beta}}\left(\rho_1^\alpha - \rho_1^{\bar{\alpha}}\right) \right] \right\}. \tag{78}$$

Extracting $\rho_1^{\bar{\beta}}$ from the sum $\sum_\alpha$, one gets

$$1 = \int dy_1 \frac{1}{\sum_\beta \rho_1^{\bar{\beta}}} \left\{ \sum_\beta \rho_1^{\bar{\beta}} \sum_\alpha \left[\frac{\left(\rho_2^{\alpha\bar{\beta}} - \rho_2^{\bar{\alpha}\bar{\beta}}\right)}{\rho_1^{\bar{\beta}}} - \left(\rho_1^\alpha - \rho_1^{\bar{\alpha}}\right) \right] \right\}, \tag{79}$$

in which one identifies the expression within the square brackets as $B^{\alpha\bar{\beta}}(y_1|y_2)$. Swapping the order of the sum and the integral, one finally gets

$$1 = \sum_\beta \frac{\rho_1^{\bar{\beta}}}{\sum_{\beta'} \rho_1^{\bar{\beta}'}} \int dy_1 \left\{ \sum_\alpha B^{\alpha\bar{\beta}} \right\}, \tag{80}$$

which is true, in general, i.e., for any number of partitions α and β if and only if

$$1 = \int dy_1 \sum_\alpha B^{\alpha\bar{\beta}} = \sum_\alpha \int dy_1 B^{\alpha\bar{\beta}}. \tag{81}$$

The sum $\sum_\alpha B^{\alpha\bar{\beta}}$, which spans all +ve species, thus indeed integrates to 1 and the sum rule is proven. Experimentally, in a limited acceptance, this sum still corresponds to $B^{+|\bar{\beta}}(y_1|y_2)$ but the functions do not integrate to unity: the components $B^{\alpha\bar{\beta}}$ partition the sum $B^{+|\bar{\beta}}(y_1|y_2)$ and their contribution to this sum is a function of the size of the acceptance and the specific processes that lead to the joint production of species α and $\bar{\beta}$.

VI. BARYON NUMBER AND STRANGENESS BALANCE FUNCTIONS

The notion of balance function is readily extended to baryon, strangeness, and charm balance functions. One must, however, account for the baryon number, strangeness number, or charm carried by the particles.

The baryon number of elementary hadrons is nominally confined to a minimal set of values $(-1, 0, 1)$ and hadrons with a null baryon number (i.e., mesons) are to be ignored in the computation of baryon balance functions. The computation of baryon balance functions shall then nominally be restricted to hadrons with baryons number with $B = 1$ and antibaryons with $B = -1$. However, it is well known that baryons produced in heavy-ion collisions may bind to form light nuclei (e.g., ^{2}H, ^{3}He, ^{4}He, and their respective antinuclei). Such $B = A$ and $B = -A$ baryons and antibaryons should thus nominally be included in the computation and measurements of baryon balance functions. However, the production of light-nuclei and antilight-nuclei at central rapidities is a relatively rare occurrence. Nuclei and antinuclei may then likely be neglected, at least in a first approximation, in the computation of baryon number balance functions.

Nominally, sum rules of the form (67) and (68) should apply to baryon balance functions. Unfortunately, the detection of neutrons remains a significant challenge at collider energies. Contributions of the form $B^{\mathrm{n}|\bar{\beta}}$, where $\bar{\beta}$ represent a specific antibaryon (e.g., antiproton), shall thus be hard to assess. However, partial balance functions $B^{p|\bar{p}}$, $B^{\mathrm{n}|\bar{\mathrm{p}}}$, $B^{\Lambda|\bar{\mathrm{p}}}$, $B^{\Sigma|\bar{\mathrm{p}}}$, $B^{\Xi|\bar{\mathrm{p}}}$, and $B^{\Omega|\bar{\mathrm{p}}}$ should nearly exhaust balancing contributions to the production of $\bar{p}$. The balance function sum rule (67) shall then enable estimation of $B^{\mathrm{n}|\bar{\mathrm{p}}}$, which, in turn, could be used to estimate cumulants of the neutron fluctuations [24].

The situation with strangeness balance functions is readily more complicated. First, one notes that multiply strange baryons, $s > 1$, and antistrange baryons, $s < -1$, may be produced in elementary-particle or nucleus-nucleus collisions. Accounting for the produced strangeness (or antistrangeness) must then be based on strangeness densities rather than number densities. Assuming the labels α and β identify specific (unique) species, we define single- and pair-strangeness densities according to

$$\rho_{s1}^{\alpha} = n_s^{\alpha}\rho_1^{\alpha}, \tag{82}$$

$$\rho_{s2}^{\alpha\beta} = n_s^{\alpha} n_s^{\beta}\rho_2^{\alpha\beta}, \tag{83}$$

in which n_s^{α} and n_s^{β} are the number of strange quarks (antiquarks) in particles of type α and β, respectively. If the definitions of the labels α and β each span several particle species, then one must sum across these species as in Eqs. (62) and (63) defined for electric charges to obtain single and pair densities.

Strangeness (unbound) balance functions can then be nominally computed as

$$B^{\alpha|\bar{\beta}}(y_1|y_2) = \tilde{A}_2^{\alpha|\bar{\beta}}(y_1|y_2) - \tilde{A}_2^{\bar{\alpha}|\bar{\beta}}(y_1|y_2), \tag{84}$$

$$B^{\bar{\alpha}|\beta}(y_1|y_2) = \tilde{A}_2^{\bar{\alpha}|\beta}(y_1|y_2) - \tilde{A}_2^{\alpha|\beta}(y_1|y_2), \tag{85}$$

where we introduced *strange* "associated" particle functions according to

$$\tilde{A}_2^{\alpha|\beta}(y_1|y_2) = \frac{\rho_{s2}^{\alpha|\beta}(y_1, y_2)}{\rho_{s1}^{\beta}(y_2)} - \rho_{s1}^{\alpha}(y_1). \tag{86}$$

The second and more fundamental difficulty arises from the kaon sector. Nominally, particle production yields charged kaons, $K^{\pm}$, as well as neutral kaons, K^0, and antikaons, $\bar{\mathrm{K}}^0$. K^0 and $\bar{\mathrm{K}}^0$ are, however, known to readily mix and yield weak eigenstates K_s^0 and K_l^0. The strangeness number of K_s^0 and K_l^0 is undefined (e.g., it is neither positive nor negative). It is thus not possible to include the contributions of K^0 and $\bar{\mathrm{K}}^0$ in balance functions to account for the production of strange and antistrange quarks. Strange BFs shall thus be forever blind to the production of these two particles, which experimentally materialize as either K_s^0 or K_l^0. Measurements of strange balance functions in heavy-ion collisions remain nonetheless of great interest given the production of s or $\bar{s}$ quarks is generally thought to feature a time evolution distinct of that of lighter quarks [1]. Quantitative comparisons of strange and charge balance functions may then enable better understanding and modeling of the collision dynamics and the properties of the QGP formed in *A*-*A* collisions.

Clearly, the notion of balance function can also be applied to charmness or bottomness. Recent measurements have shown that measurements of correlation functions of charmed hadrons are in fact possible but it remains to be established whether such observations can be formulated as genuine charm balance functions [25–28].

The existence of a gluon dominated phase at very early time of the evolution of *A*-*A* collisions could provide significant insights and help distinguish the light and heavy quark evolution dynamics. Light quarks are more likely to be produced late in collisions. The light hadrons they form are thus accordingly less sensitive to early time dynamics. By contrast, the production of heavy quarks (strange, charm, bottom) requires higher $\sqrt{s}$ elementary collisions and is thus likely limited to early times. One expects that charm and bottom quarks being the heaviest, their production should be limited to very early times. Balance functions of open charm (bottom) particles should then reflect early time production and possibly heavy quark scattering within the QGP. However, given the mass of charm and bottom quarks are considerably heavier than those of strange, up, and down quarks, they should be subjected to smaller diffusivity effects [3]. The balance function of charm might be then truly representative of early time collisions and one might expect a gradation of sensitivity to early times, that of charm and bottom being the largest, followed by strangeness, and much less sensitivity from the lighter u and d quarks.

VII. BALANCE FUNCTIONS AND NORMALIZED CORRELATION FUNCTIONS

Rather than conducting measurements of balance functions (and their integral) in terms of densities $\rho_2^{\alpha\beta}(y_1, y_2)$, it is also of interest to consider measurements based on normalized differential two-particle cumulants defined according to

$$R_2^{\alpha\beta}(y_1, y_2) \equiv \frac{C_2^{\alpha\beta}(y_1, y_2)}{\rho_1^{\alpha}(y_1)\rho_1^{\beta}(y_2)} = \frac{\rho_2^{\alpha\beta}(y_1, y_2)}{\rho_1^{\alpha}(y_1)\rho_1^{\beta}(y_2)} - 1, \quad (87)$$

where the functions $C_2^{\alpha\beta}(y_1, y_2)$ are defined by Eq. (51). Analyses in terms of such normalized cumulants are of particular interest, experimentally, because this observable is robust against particle losses (efficiency) and thus, nominally, reduces the need for complicated efficiency correction procedures. The (unbound) balance functions (45) and (46) may then be written as

$$B^{\alpha|\bar{\beta}}(y_1|y_2) = \rho_1^{\alpha}(y_1)R_2^{\alpha\bar{\beta}}(y_1|y_2) - \rho_1^{\bar{\alpha}}(y_1)R_2^{\bar{\alpha}\bar{\beta}}(y_1|y_2), \quad (88)$$

$$B^{\bar{\alpha}|\beta}(y_1|y_2) = \rho_1^{\bar{\alpha}}(y_1)R_2^{\bar{\alpha}\beta}(y_1|y_2) - \rho_1^{\alpha}(y_1)R_2^{\alpha\beta}(y_1|y_2), \quad (89)$$

in which the normalized correlation functions R_2 are written with arguments of the form $(y_1|y_2)$ to emphasize they are functions of y_1 given a value y_2. However, given a specific acceptance Ω, one can operationally define symmetric balance functions, i.e., functions of two parameters y_1 and y_2 by averaging the integral of $B^{\alpha|\bar{\beta}}(y_1|y_2)$ and $B^{\bar{\alpha}|\beta}(y_1|y_2)$ across the acceptance of y_2. This is achieved by averaging the integrals across the acceptance by weighing them with the probability to measure specific values of y_2. Proceeding as in Sec. IV, one then obtains bounded balance functions of the form

$$B^{\alpha\bar{\beta}}(y_1, y_2) = \frac{1}{\langle N_1^{\bar{\beta}}\rangle}\big[\rho_1^{\alpha}(y_1)\rho_1^{\bar{\beta}}(y_2)R_2^{\alpha\bar{\beta}}(y_1, y_2) - \rho_1^{\bar{\alpha}}(y_1)\rho_1^{\bar{\beta}}(y_2)R_2^{\bar{\alpha}\bar{\beta}}(y_1, y_2)\big], \quad (90)$$

$$B^{\bar{\alpha}\beta}(y_1, y_2) = \frac{1}{\langle N_1^{\beta}\rangle}\big[\rho_1^{\bar{\alpha}}(y_1)\rho_1^{\beta}(y_2)R_2^{\bar{\alpha}\beta}(y_1, y_2) - \rho_1^{\alpha}(y_1)\rho_1^{\beta}(y_2)R_2^{\alpha\beta}(y_1, y_2)\big], \quad (91)$$

in which the R_2 are now written with arguments of the form (y_1, y_2) to indicate they they are indeed functions of two parameters.

VIII. BALANCE FUNCTIONS VS INVARIANT MOMENTUM

The particle pair separation in momentum space is nominally determined by the energy of the process that produces a particular correlated pair. However, transport processes such as longitudinal and radial flow may affect the separation measured in term of angular separation, e.g., azimuth angle pair separation, $\Delta\varphi$. The shape and strength of balance functions thus measured are influenced by both production and transport processes. To reduce this causal ambiguity, it may then be advantageous to carry out the BF measurements in terms of a relative momentum invariant, $P_{\rm inv}$, which is primarily determined by production processes and less affected by transport phenomena such as radial or longitudinal collective flow. To this end, Pratt *et al.* proposed BF measurements shall be carried in terms of particle pairs relative four-momentum computed in the reference frame of the two-particle center of mass according to [29]

$$q^{\mu} = \big(p_a^{\mu} - p_b^{\mu}\big) - P^{\mu}\frac{P\cdot(p_a - p_b)}{P^2} = \big(p_a^{\mu} - p_b^{\mu}\big) - P^{\mu}\frac{m_a^2 - m_b^2}{s}, \quad (92)$$

in which $\mu = 0, x, y, z$, P is the total four-momentum of the two particles $P^{\mu} = p_a^{\mu} + p_b^{\mu}$, and the invariant $\sqrt{s} = [(p_a + p_b)^2]^{1/2}$ represents the center-of-mass (COM) energy of the pair. The square of the invariant momentum difference of the particles computed in the pair COM frame is

$$P_{\rm inv}^2 = -q^2 = -(p_a - p_b)^2 + \frac{\big(m_a^2 - m_b^2\big)^2}{P^2}. \quad (93)$$

Denoting the two-particle transverse momentum, $P_{\rm T} = (P_x^2 + P_y^2)^{1/2}$, it is convenient, as suggested by Pratt *et al.* [29], to define three projections of the relative momentum according to

$$P_{\rm long} = \frac{1}{\sqrt{s + P_{\rm T}^2}}(P_0 q_z - P_z q_0), \quad (94)$$

$$P_{\rm side} = \frac{P_x q_y - P_y q_y}{P_{\rm T}}, \quad (95)$$

$$P_{\rm out} = \sqrt{\frac{s}{s + P_{\rm T}^2}}\frac{P_x q_x + P_y q_y}{P_{\rm T}}, \quad (96)$$

and such that

$$P_{\rm inv}^2 = P_{\rm long}^2 + P_{\rm side}^2 + P_{\rm out}^2. \quad (97)$$

As illustrated in Fig. 2, $\vec{P}_{\rm long}$ is the pair momentum difference along the beam axis (longitudinal separation), $\vec{P}_{\rm out}$ is along the two-particle transverse momentum $\vec{P}_{\rm T}$ (outwards separation), and $\vec{P}_{\rm side}$ points in the sidewards direction, i.e., in a direction perpendicular to both $P_{\rm long}$ and $P_{\rm out}$. The pair density in terms of $P_{\rm long}$, $P_{\rm out}$, $P_{\rm side}$ is

$$\rho_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side}) = \frac{d^3N_2^{\alpha\beta}}{dP_{\rm long}\, dP_{\rm out}\, dP_{\rm side}}. \quad (98)$$

Following a similar reasoning as that leading to Eqs. (56) and (57), general balance functions may be written:

$$B^{\alpha\bar{\beta}}(P_{\rm long}, P_{\rm out}, P_{\rm side}|\Omega) = \frac{1}{\langle N_1^{\bar{\beta}}\rangle}\big[C_2^{\alpha\bar{\beta}}(P_{\rm long}, P_{\rm out}, P_{\rm side}) - C_2^{\bar{\alpha}\bar{\beta}}(P_{\rm long}, P_{\rm out}, P_{\rm side})\big], \quad (99)$$

$$B^{\bar{\alpha}\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side}|\Omega) = \frac{1}{\langle N_1^{\beta}\rangle}\big[C_2^{\bar{\alpha}\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side}) - C_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side})\big], \quad (100)$$

in which

$$C_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side}) = \rho_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side}) - \big[\rho_1^{\alpha}\rho_1^{\beta}\big](P_{\rm long}, P_{\rm out}, P_{\rm side}), \quad (101)$$

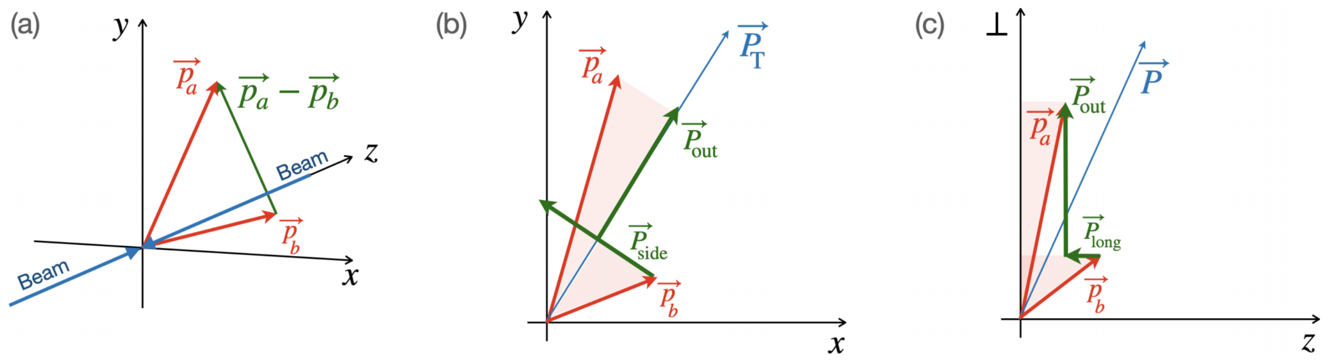

FIG. 2. Schematic diagram of the pair differences $P_{\rm long}$, $P_{\rm out}$, $P_{\rm side}$ defined based on the particle momenta $\vec{p}_a$ and $\vec{p}_b$ with respect to the beam axis and the total pair momentum $\vec{P}$ introduced in the text.

where the notation $[\rho_1^\alpha\rho_1^\beta](P_{\rm long}, P_{\rm out}, P_{\rm side})$ stands for

$$\begin{aligned}
&\left[\rho_1^\alpha\rho_1^\beta\right](P_{\rm long}, P_{\rm out}, P_{\rm side})\\
&= \int_{-y_0}^{y_0} \rho_1^\alpha(y_1,\phi_1,p_{{\rm T},1})\rho_1^\alpha(y_2,\phi_2,p_{{\rm T},2})\\
&\quad\times \delta(P_{\rm long} - f_{\rm long}(y_1,\phi_1,p_{{\rm T},1},y_2,\phi_2,p_{{\rm T},2}))\\
&\quad\times \delta(P_{\rm out} - f_{\rm out}(y_1,\phi_1,p_{{\rm T},1},y_2,\phi_2,p_{{\rm T},2}))\\
&\quad\times \delta(P_{\rm side} - f_{\rm side}(y_1,\phi_1,p_{{\rm T},1},y_2,\phi_2,p_{{\rm T},2}))\\
&\quad\times dy_1 d\phi_1 dp_{{\rm T},1} dy_2 d\phi_2 dp_{{\rm T},2},
\end{aligned} \tag{102}$$

in which functions $f_{\rm long}$, $f_{\rm out}$, $f_{\rm side}$ map variables y_1, ϕ_1, $p_{{\rm T},1}$, y_2, ϕ_2, $p_{{\rm T},2}$ onto $P_{\rm long}$, $P_{\rm out}$, $P_{\rm side}$ according to Eqs. (92)–(96).

The determination of BFs based on Eqs. (99) and (100) requires that measured-pair yields $N_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side}|\Omega)$ be fully corrected for efficiency losses to obtain densities $\rho_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side})$ and correlation functions $C_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side})$. Alternatively, experimentally, it may be preferable to compute the BFs in terms of normalized cumulants

$$R_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side}) = \frac{C_2^{\alpha\beta}(P_{\rm long}, P_{\rm out}, P_{\rm side})}{\left[\rho_1^\alpha\rho_1^\beta\right](P_{\rm long}, P_{\rm out}, P_{\rm side})}, \tag{103}$$

because these are approximately robust against particle (efficiency) losses.

IX. ACCEPTANCE AVERAGING OF THE BALANCE FUNCTION

At RHIC and LHC, the systems produced in *A*-*A* collisions feature large longitudinal and transverse pressure gradients. It is then of interest to carry measurements as a function of differences $\Delta y = y_1 - y_2$ and $\Delta\varphi = \varphi_1 - \varphi_2$ simultaneously. The realization of such measurements in individual particle coordinates requires the handling of four-dimensional (4D) histograms. Even when using a relatively small number of bins along each dimension, one ends up, computationally, with very large objects that may challenge the capacity of computing nodes used for the data analysis. One additionally also faces a statistical accuracy challenge: the measured pairs are spread across a vast number of bins and it may become difficult to achieve sufficient statistical accuracy across the entire phase space. It is then often desirable to *ab initio* reduce the dimensionality of the measurement by projecting this 4D space onto a two-dimensional (2D) space Δy vs $\Delta\varphi$. One must then consider how such projections impact the balance functions B and their integrals $I^{\alpha|\bar{\beta}}$ in measurements featuring a limited acceptance $-y_0 \leqslant y < y_0$.

To carry out computations in Δy and $\Delta\varphi$ coordinates, one first considers the transformations

$$y_1, y_2 \to \Delta y \equiv y_1 - y_2, \quad \bar{y} \equiv (y_1 + y_2)/2, \tag{104}$$

$$\varphi_1, \varphi_2 \to \Delta\varphi \equiv \varphi_1 - \varphi_2, \quad \bar{\varphi} \equiv (\varphi_1 + \varphi_2)/2, \tag{105}$$

which both feature a Jacobian $J = 1$. Densities $\rho_2^{\alpha\beta}(y_1, y_2, \varphi_1, \varphi_2)$ thence transform to $\rho_2^{\alpha\beta}(\Delta y, \bar{y}, \Delta\varphi, \bar{\varphi})$ according to

$$\begin{aligned}
&\rho_2^{\alpha\beta}(\Delta y, \bar{y}, \Delta\varphi, \bar{\varphi})\\
&\quad= \int dy_1 \int dy_2 \rho_2^{\alpha\beta}(y_1, y_2, \varphi_1, \varphi_2)\delta\\
&\qquad\times (\Delta y - y_1 + y_2)\delta(\bar{y} - (y_1 + y_2)/2.0)\\
&\qquad\times \delta(\Delta\varphi - \varphi_1 + \varphi_2)\delta(\bar{\varphi} - (\varphi_1 + \varphi_2)/2.0).
\end{aligned} \tag{106}$$

Measurements of $B^{\alpha|\beta}(\Delta y, \Delta\varphi)$ can be carried out as simple projections of the 4D space spanned by y_1, φ_1, y_2, φ_2 or averages across the acceptances $\bar{y} = (y_1 + y_2)/2$ and $\bar{\varphi} = (\varphi_1 + \varphi_2)/2$. Obtaining simple projections is trivial, given it suffices to fill histograms of the two densities in terms of the Δy and $\Delta\varphi$ coordinates, e.g.,

$$\rho_2^{\alpha\beta}(\Delta y) \equiv \int_\Omega d\bar{y}\rho_2^{\alpha\beta}(\Delta y, \bar{y}). \tag{107}$$

However, such projections emphasize small values of Δy, e.g., $\Delta y \approx 0$, of the two-particle phase space at the expense of regions with $\Delta y \approx 2y_0$ near the edge of the acceptance. It is thus advantageous to consider averages across the $\bar{y}$ acceptance as follows:

$$\bar{\rho}_2^{\alpha\beta}(\Delta y) \equiv \frac{1}{\Omega(\Delta y)}\int_\Omega d\bar{y}\rho_2^{\alpha\beta}(\Delta y, \bar{y}) = \frac{1}{\Omega(\Delta y)}\rho_2^{\alpha\beta}(\Delta y), \tag{108}$$

where the overbar in $\bar{\rho}$ represents the averaging across $\bar{y}$ and $\Omega(\Delta y)$ is the width of the acceptance in $\bar{y}$ at the given Δy. For

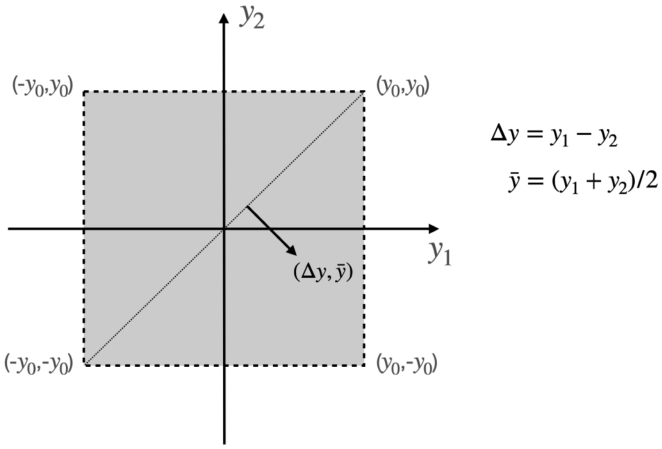

FIG. 3. Definition of the pair acceptance used in the definition of bound balance functions.

a square and symmetric two-particle acceptance, $-y_0 \leqslant y_1, y_2 < y_0$, as illustrated in Fig. 3, the value of $\Omega(\Delta y)$ amounts to

$$\Omega(\Delta y) = 2y_0 - |\Delta y|. \tag{109}$$

The function $\Omega(\Delta y)$ is often called the acceptance factor. It should be clear, however, that its use does not constitute an acceptance "correction" but involves acceptance averaging along $\bar{y}$.

Projections of balance functions $B^{\alpha|\bar{\beta}}(y_1, y_2)$ onto Δy are carried in the same way, and one distinguishes straight and acceptance averaged projections denoted

$$B^{\alpha|\bar{\beta}}(\Delta y) \equiv \int_\Omega d\bar{y} B^{\alpha|\bar{\beta}}(\Delta y, \bar{y}), \tag{110}$$

$$\bar{B}^{\alpha|\bar{\beta}}(\Delta y) \equiv \frac{1}{\Omega(\Delta y)} B^{\alpha|\bar{\beta}}(\Delta y), \tag{111}$$

respectively, with similarly formed expressions for $B^{\bar{\alpha}|\beta}(\Delta y)$ and $\bar{B}^{\bar{\alpha}|\beta}(\Delta y_1)$. Evidently, these expressions can be used to compute balance functions based on correlation functions, e.g., $C_2^{\bar{\alpha}|\beta}(\Delta y)$, given by Eqs. (49) and (50), or normalized cumulants, represented in Eqs. (90) and (91). By construction, integrals of $B^{\alpha|\bar{\beta}}(\Delta y)$, $B^{\bar{\alpha}|\beta}(\Delta y)$, yield results identical to those obtained with densities and correlation functions $C_2^{\bar{\alpha}|\beta}(y_1, y_2)$. However, integrals of acceptance averaged balance functions $\bar{B}^{\alpha|\bar{\beta}}(\Delta y)$, $\bar{B}^{\bar{\alpha}|\beta}(\Delta y)$ do not given they feature the acceptance factor $\Omega(\Delta y)$ in their definition. Balance function integrals can nonetheless be recovered by inserting this acceptance factor explicitly in the BF integral as follows:

$$I^{\alpha|\bar{\beta}}(\Omega) = \int_\Omega \Omega(\Delta y) \bar{B}^{\alpha|\bar{\beta}}(\Delta y) d\Delta y. \tag{112}$$

X. BALANCE FUNCTIONS AND THE $\nu_{\rm dyn}$ OBSERVABLE

The $\nu_{\rm dyn}$ observable was initially developed and used for the study of net-charge fluctuations [20]. As such, it corresponds to the "dynamical" or nonstatistical components of net-charge fluctuations. It can, however, also be used for the study of the relative abundance fluctuations of particles species α and β. In that context, it is most succinctly written as a combination of normalized cumulants, according to

$$\nu_{\rm dyn}^{\alpha\beta} = R_2^{\alpha\alpha} + R_2^{\beta\beta} - 2R_2^{\alpha\beta}, \tag{113}$$

with $R_2^{\alpha\beta}$ correlators defined and computed according to Eq. (9). In the context of studies of net-charge fluctuations within the acceptance $\Omega : -y_0 \leqslant y < y_0$, the above reduces to

$$\nu_{\rm dyn}^{+-}(\Omega) = R_2^{++}(\Omega) + R_2^{--}(\Omega) - 2R_2^{+-}(\Omega), \tag{114}$$

with

$$R_2^{++}(\Omega) = \frac{\langle N_+(N_+ - 1)\rangle}{\langle N_+\rangle^2} - 1, \tag{115}$$

$$R_2^{--}(\Omega) = \frac{\langle N_-(N_- - 1)\rangle}{\langle N_-\rangle^2} - 1, \tag{116}$$

$$R_2^{+-}(\Omega) = \frac{\langle N_+ N_-\rangle}{\langle N_+\rangle\langle N_-\rangle} - 1, \tag{117}$$

in which $\langle N_+(N_+ - 1)\rangle$, $\langle N_-(N_- - 1)\rangle$, and $\langle N_+ N_-\rangle$ correspond, respectively, to average number of positive particle pairs $++$, average number of negative particle pairs $--$, and average number of unlike sign pairs $+-$ detected with the acceptance $\Omega : -y_0 \leqslant y < y_0$.

We next verify that the above expression for $\nu_{\rm dyn}^{+-}(\Omega)$ is approximately equal to charge BFs computed with Eqs. (29) and (30). To this end, we write BF integrals $I(\Omega)$ according to

$$I(\Omega) = \tfrac{1}{2}[F_1^+ R_2^{+-} + F_1^- R_2^{-+} - F_1^- R_2^{--} - F_1^+ R_2^{++}]. \tag{118}$$

Defining $\omega(\Omega) = F_1^-/F_1^+$, and acknowledging that $R_2^{+-} = R_2^{-+}$, we divide the above expression by $-F_1^+/2$ and get

$$-\frac{2I^s}{F_1^+} = [\omega R_2^{--} + R_2^{++} - (1+\omega)R_2^{+-}], \tag{119}$$

where we have omitted the dependence on Ω to simplify the notation. This expression reduces to $-\nu_{\rm dyn}^{+-}$, given by Eq. (114), in the limit $\omega(\Omega) \to 1$ is approximately valid at high collision energy for light particles. Denoting the total average charged particle multiplicity $\langle N\rangle \equiv F_1^+ + F_1^- = \langle N_1^+\rangle + \langle N_1^-\rangle$, one thus recovers the known result

$$I^s = -\frac{\langle N\rangle}{4}\nu_{\rm dyn}^{+-}, \tag{120}$$

valid in that limit [20]. It is important to note that, at energies of the CERN Super Proton Synchrotron (SPS) and RHIC, or even at LHC energy, the limit $\langle N_1^+\rangle = \langle N_1^-\rangle$ is not perfectly achieved. The precision of the approximation (120), predicated on $\omega(\Omega) \to 1$, must thus be explicitly verified, relative to the correct expression (119).

XI. SUMMARY

We examined the nominal definition of general charge balance function [1,2] and found that it is advantageous to define two complementary balance functions based on differences of conditional densities of like-sign and unlike-sign pairs of particles. We first proceeded to define integral balance functions and showed that, in order to account for a system's charge, the balance functions must include a term equal to the difference of positively and negatively charged particle multiplicities. We next showed that differential balance functions $B^{+|-}$ and $B^{-|+}$ defined from differences of conditional densities can also properly account for the system's net-charge provided one adds the difference of positive and negative densities to their definitions. We further showed that such charge balance functions can be generalized to any combinations of species α and β. We showed, in particular, that such general balance functions also account for finite net-charge of the collision system being considered provided they include the density difference $\rho_1^{\alpha}(y) - \rho_1^{\bar{\alpha}}(y)$. We derived the simple sum rules (67), (68), and (81) that show that the sum of BFs of particle pairs $\alpha|\bar{\beta}$ feature an integral across the full phase space that converges to unity.

Additionally, we also showed charge BFs can be straightforwardly extended to baryon, strangeness, and charm BFs provided one accounts for the baryon, strangeness, and charm density rather than the particle density. As such, general balance functions could provide a path to a better and deeper understanding of the evolution of systems formed in *pp*, *p*-*A*, and *A*-*A* collisions. Moreover, although not explicitly discussed in this work, it is clear that measurements of balance functions within jets could potentially also yield a better understanding of the structure of jets and their modification in *A*-*A* collisions relative to those observed in *pp* collisions.

Finally, we derived expressions for bounded balance functions, i.e., balance functions measured in a specific acceptance, based on either densities $\rho_2^{\alpha\beta}$ or normalized correlation functions $R_2^{\alpha\beta}$. We showed that balance functions based on difference variables Δy and $\Delta\varphi$ may be computed as straight projections from 4D space $\{y_1, \varphi_1, y_2, \varphi_2\}$ or as weighted averages across the pair rapidity average $\bar{y} = (y_1 + y_2)/2$. We also derived a general formula that connects the integral of charge balance functions and the $\nu_{\rm dyn}^{+-}$ observable.

We have shown that general BFs $B^{+|-}$ and $B^{-|+}$ must include the density difference $\rho_1^{\alpha}(y) - \rho_1^{\bar{\alpha}}(y)$ to yield integrals that properly account for the net-charge of the collision system considered. But given ratios of particle and antiparticle yields tend towards unity in the central rapidity region, at top RHIC energy and at LHC, one may wonder, however, whether the inclusion of this term is absolutely essential and whether measurements based on the nominal conditional density difference would constitute reasonable approximations of the correct results. We have also shown that measurements of general balance functions based on $R_2^{\alpha\beta}$ may be carried out based on various experimentally driven approximations. The impact of the omission of the density difference and $R_2^{\alpha\beta}$ based approximations shall be explored in detail in future works.

ACKNOWLEDGMENTS

The authors thank Dr. Igor Altsybeev, Dr. Peter Christianssen, Dr. Scott Pratt, and Dr. Sergei Voloshin for insightful discussions and their suggestions. S.B. acknowledges the support of the Swedish Research Council (VR) and the Knut and Alice Wallenberg Foundation. This work was also supported in part by the United States Department of Energy, Office of Nuclear Physics (DOE NP), United States of America, under Grant No. DE-FG02-92ER40713.

[1] S. A. Bass, P. Danielewicz, and S. Pratt, Phys. Rev. Lett. **85**, 2689 (2000).
[2] S. Jeon and S. Pratt, Phys. Rev. C **65**, 044902 (2002).
[3] S. Pratt and C. Plumberg, Phys. Rev. C **102**, 044909 (2020).
[4] S. Pratt and C. Plumberg, Phys. Rev. C **104**, 014906 (2021).
[5] S. Pratt, W. P. McCormack, and C. Ratti, Phys. Rev. C **92**, 064905 (2015).
[6] S. Pratt and C. Plumberg, Phys. Rev. C **99**, 044916 (2019).
[7] S. Basu, S. Chatterjee, R. Chatterjee, T. K. Nayak, and B. K. Nandi, Phys. Rev. C **94**, 044901 (2016).
[8] C. A. Pruneau, S. Gavin, and S. A. Voloshin, Nucl. Phys. A **802**, 107 (2008).
[9] A. Bialas, Phys. Lett. B **579**, 31 (2004).
[10] S. Basu, S. Thakur, T. K. Nayak, and C. A. Pruneau, J. Phys. G **48**, 025103 (2021).
[11] S. Pratt and K. Martirosova, Phys. Rev. C **105**, 054906 (2022).
[12] S. Acharya *et al.* (ALICE Collaboration), Phys. Lett. B **833**, 137338 (2022).
[13] L. Adamczyk *et al.* (STAR Collaboration), Phys. Rev. C **94**, 024909 (2016).
[14] J. Adams *et al.* (STAR Collaboration), Phys. Rev. Lett. **90**, 172301 (2003).
[15] F. Scardina, M. Colonna, S. Plumari, and V. Greco, Phys. Lett. B **724**, 296 (2013).
[16] K. Fukushima, Rep. Prog. Phys. **80**, 022301 (2017).
[17] F. G. Gardim, G. Giacalone, M. Luzum, and J.-Y. Ollitrault, Nat. Phys. **16**, 615 (2020).
[18] S. Basu, R. Chatterjee, B. K. Nandi, and T. K. Nayak, Springer Proc. Phys. **174**, 189 (2016).
[19] S. Basu, V. Gonzalez, J. Pan, A. Knospe, A. Marin, C. Markert, and C. Pruneau, Phys. Rev. C **104**, 064902 (2021).
[20] C. Pruneau, S. Gavin, and S. Voloshin, Phys. Rev. C **66**, 044904 (2002).
[21] C. A. Pruneau, *Data Analysis Techniques for Physical Scientists* (Cambridge University Press, Cambridge, 2017).
[22] S. Pratt, Nucl. Phys. A **698**, 531 (2002).
[23] C. A. Pruneau, Phys. Rev. C **100**, 034905 (2019).
[24] P. Braun-Munzinger, A. Rustamov, and J. Stachel, The role of the local conservation laws in fluctuations of conserved charges, arXiv:1907.03032 (2019).
[25] S. Basu, P. Christiansen, A. Ohlson, and D. Silvermyr, Eur. Phys. J. C **81**, 1024 (2021).
[26] R. Vogt, Phys. Rev. C **98**, 034907 (2018).
[27] R. Aaij *et al.* (LHCb Collaboration), J. High Energy Phys. 06 (2012) 141; 03 (2014) 108.
[28] J. Adolfsson *et al.*, Eur. Phys. J. A **56**, 288 (2020).
[29] S. Pratt and S. Cheng, Phys. Rev. C **68**, 014907 (2003).