Green's function knockout formalism

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(Received 26 June 2022; revised 14 November 2022; accepted 14 December 2022; published 6 January 2023)

Knockout nuclear reactions, in which a nucleon is removed from a nucleus as a result of the collision with another nucleus, have been widely used as an experimental tool, both to populate isotopes further removed from stability and to obtain information about the single-particle nature of the nuclear spectrum. In order to fully exploit the experimental information, theory is needed for the description of both the structure of the nuclei involved, and the dynamics associated with the nucleon removal mechanisms. The standard approach, using theoretical shell-model spectroscopic factors for the structure description coupled with an eikonal model of reaction, has been successful when used in the context of the removal of valence nucleons in nuclei close to stability. However, it has been argued that the reaction theory might need to be revisited in the case of exotic nuclei, more specifically for highly asymmetric nuclei in which the deficient species (neutrons or protons) is being removed. We present here a new formalism for the nucleon-removal and -addition reaction through knockout and transfer reactions, that treats consistently structure and reaction properties using dispersive optical potentials. In particular, our formalism includes the dynamical effects associated with the removal of a nucleon from the projectile, which might explain the long standing puzzle of the quenching of spectroscopic factors in nuclei with extreme neutrons-to-protons ratios.

DOI: 10.1103/PhysRevC.107.014607

I. INTRODUCTION

Radioactive ion beams (RIBs) facilities are transforming the field of low-energy nuclear physics by setting short-lived, exotic isotopes within experimental reach. The availability of new experimental data has been matched by theoretical efforts towards the description of nuclear systems away from the stability valley, and, more generally, towards an understanding of nuclear structure in an exotic context [1]. The corresponding paradigm shift in the theory of nuclear structure has to be complemented by a revision of nuclear reaction theory, needed for the description of the experiments in which radioactive ions are involved. Within this context, a considerable effort has been devoted recently to the description of reactions with weakly-bound nuclei (see, e.g., [2–5], see also [6] and references therein).

However, the study of neutron-rich (respectively, proton-rich) raises a complementary question about the behavior of the deeply-bound protons (respectively, neutrons) belonging to the same nucleus. This question has been highlighted in publications by Gade and collaborators [7–9], in which they present a review of results of one-neutron (respectively, one-proton) knockout experiments expressed as a function of the difference $\Delta S = S_n - S_p$ between the neutron S_n and proton separation S_p energies (respectively, $\Delta S = S_p - S_n$). In this work, they arrive at the puzzling conclusion that theory is unable to account for as much as 80% of the quenching of single-particle strength when the knocked out particle belongs to the deficient species in systems with a large value of ΔS. Several authors have suggested that nuclear structure calculations might fail to fully account for short-range correlations between neutrons and protons in highly asymmetric systems, leading to an overestimation of the single-particle content of the states populated in knockout reactions [10,11].

On the other hand, the fact that the strong dependence on ΔS of the spectroscopic factor quenching is not observed in transfer and quasifree scattering experiments has led some authors to suggest that the issue might be in the theory associated with the description of the reaction process in the case of knockout experiments [12–16] (see Ref. [17] for a recent review). Knockout experiments are often described within the eikonal model [18,19], assumed to be valid for high beam energies. The sudden and the core spectator approximations, in which the nuclear degrees of freedom are frozen during the collision process, are used to describe the one-nucleon removal from the projectile. The core spectator approximation is based on the assumption that the characteristic decay times of the core states populated in the nucleon removal process are large compared to the collision time. This seems reasonable when the nucleon is removed from a state not too far away from the Fermi energy on a stable nucleus, since the narrow associated energy width is small, the corresponding damping (decay) time being therefore rather large.

However, it has been well known since knockout experiments were performed in the early 1960s (see, e.g., [20]) that hole states associated with the removal of deeply-bound

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nucleons have much larger widths, sometimes of the order of tens of MeV. In other words, hole states resulting from the removal of a deeply-bound nucleon can decay into complicated many-body nuclear states rather quickly, in times of the order of 10^{-23}–10^{-22} s, which is comparable to the short collision times associated with fast knockout experiments. This damping process results in dissipative effects associated with the dynamics of hole states in nuclei, not taken into account within the core spectator approximation. These effects have already been estimated in Ref. [21] within the intranuclear cascade approach, showing that they can excite the core above the particle emission threshold, leading to particle evaporation and a loss of flux in the outgoing channel that could account for the observed spectroscopic factor quenching. This discussion applies to the cases in which the residual core is the only species detected in the experiment, such as in most standard knockout reactions.

The above parlance highlights the general need to integrate in a consistent theoretical framework the structure and dynamics of many nucleons in a nucleus, and the description of reactions used to study them in an experimental context. An attempt to account for the dynamics between a nucleus and a transferred nucleon has already been implemented in the the Green's function transfer (GFT) formalism, albeit in a one-nucleon addition context [22,23] (see also equivalent theories in Refs. [24–26]). We present here an extension of this idea, the Green's function knockout (GFK), which describes one-nucleon removal processes.

In Sec. II, we derive the general formalism for one-nucleon removal reactions, such as knockout and (p, d), and we show in Sec. III its connection to one-neutron addition processes and the GFT formalism. In Sec. IV we discuss different approximations that can be made in the context of the GFK, before concluding in Sec. V with a summary and outlook of future developments.

II. GREEN'S FUNCTION KNOCKOUT FORMALISM

Let us introduce the GFK by considering a reaction involving a projectile nucleus P of mass m_P impinging on a target T of mass m_T. This system is described by the solution Ψ of the many-body Hamiltonian $\hat{H}$,

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{1}$$

where E is the energy of the system in the center of mass frame. We are interested in the reaction channel in which a nucleon N is knocked out from the projectile and only the residual core c is detected with a kinetic energy $E_{f_{cT}}$. Since the final state of the N-T system is often not measured, we focus here on deriving the inclusive cross section in both the core c and N-T systems, i.e., summed over all energy-conserving final states f_{NT} of the N-T system and over all the energetically available states f_c of the core c [19,27,28]. When expressed as a function of the deflection angle of the core Ω and its final kinetic energy $E_{f_{cT}}$, the cross section in the *prior*

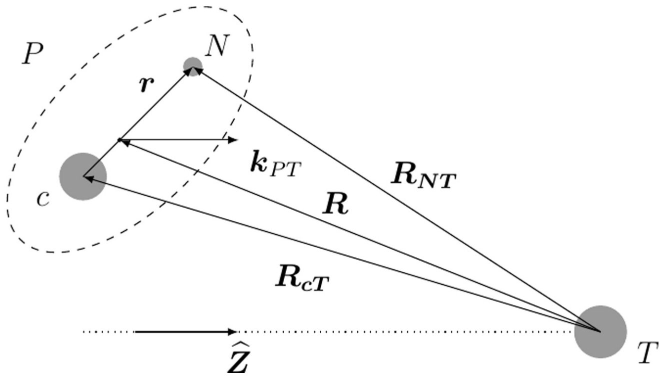

FIG. 1. Set of coordinates used in this article: the c-N, c-T, and N-T relative coordinates $\boldsymbol{r}$, $\boldsymbol{R}_{cT}$, and $\boldsymbol{R}_{NT}$, respectively.

representation reads

$$\begin{aligned}\frac{d\sigma}{dE_{f_{cT}}d\Omega} &= \frac{2\pi\,\mu_{PT}}{\hbar^2 k_{PT}}\,\rho(E_{f_{cT}})\\ &\times \sum_{f_{NT},f_c}\left|\langle\Psi(f_{NT},f_c)\,|\hat{V}_{\rm prior}\big|\phi_T^{(0)}\phi_P^{(0)}\mathcal{F}\rangle\right|^2\\ &\times\delta(E-E_{f_{cT}}-E_{f_c}-E_{f_{NT}}),\end{aligned} \tag{2}$$

where $\Psi(f_{NT}, f_c)$ is the solution of Eq. (1) subject to the boundary condition consisting in having an outgoing wave containing the core in a state f_c moving with kinetic energy $E_{f_{cT}}$ with respect to the target, and the N-T system is in a state f_{NT}. The many-body wave functions $\phi_T^{(0)}$ and $\phi_P^{(0)}$ correspond to the ground state of the target and the projectile, respectively, while $\mathcal{F}$ describes the relative motion between the projectile and the target in the incoming channel. The term $\rho(E_{f_{cT}}) = \mu_{cT}k_{cT}^f/[(2\pi)^3\hbar^2]$ is the asymptotic phase space factor (density of states) of the core fragment, k_{cT}^f and k_{PT} are, respectively, the final c-T and initial P-T wave numbers, μ_{cT} and μ_{PT} being the c-T and P-T reduced masses.

In its exact form, the prior potential $\hat{V}_{\rm prior}$ depends on the c-T and N-T many-body potentials, respectively $\hat{V}_{cT}$ and $\hat{V}_{NT}$, and on the potential $\hat{V}_i$ used to compute the incoming scattering function $\mathcal{F}$,

$$\hat{V}_{\rm prior} = \hat{V}_{cT} + \hat{V}_{NT} - \hat{V}_i. \tag{3}$$

The exact prior potential therefore depends on the N-T and c-T relative coordinates, respectively, $\boldsymbol{R}_{NT}$ and $\boldsymbol{R}_{cT}$ (see Fig. 1), and the intrinsic coordinates of the target and the core, respectively, $\boldsymbol{\xi}_T$ and $\boldsymbol{\xi}_c$. In this formalism, we neglect the dependence of the prior potential on the core intrinsic coordinates $\boldsymbol{\xi}_c$, although the role of reaction channels in exciting the core is accounted for with the inclusion of an imaginary part. In Sec. IV, we will discuss different possible choices of $\mathcal{F}$ and $\hat{V}_i$.

We now approximate the exact wave function as

$$\Psi(f_{NT}, f_c) \approx \chi_{cT}^{(f_{cT})}(\boldsymbol{R}_{cT})\psi_{NT}^{(f_{NT})}(\boldsymbol{R}_{NT},\boldsymbol{\xi}_T)\psi_c^{(f_c)}(\boldsymbol{\xi}_c), \tag{4}$$

where $\chi_{cT}^{(f_{cT})}$ is the wave function describing the final c-T relative motion, and the final wave functions of the core and the

N-T systems, respectively, $\psi_c^{(f_c)}$ and $\psi_{NT}^{(f_{NT})}$, are many-body objects. These functions satisfy the Schrödinger equations

$$(E_{f_c} - \hat{h}_c)\,\psi_c^{(f_c)}(\boldsymbol{\xi}_c) = 0, \tag{5}$$

$$(E_{f_{NT}} - \hat{T}_{NT} - \hat{V}_{NT} - \hat{h}_T)\,\psi_{NT}^{(f_{NT})}(\boldsymbol{R}_{NT}, \boldsymbol{\xi}_T) = 0, \tag{6}$$

where we define the many-body core ($\hat{h}_c$) and target ($\hat{h}_T$) Hamiltonian operators, and the kinetic energy operator $\hat{T}_{NT}$. Then,

$$\frac{d\sigma}{dE_{f_{cT}}d\Omega} = \frac{2\pi\mu_{PT}}{\hbar^2 k_{PT}}\,\rho(E_{f_{cT}})\sum_{f_{NT},f_c} \times \left|\left\langle \chi_{cT}^{(f_{cT})}\,\psi_{NT}^{(f_{NT})}\psi_c^{(f_c)}\,\middle|\hat{V}_{\rm prior}\middle|\phi_T^{(0)}\phi_P^{(0)}\mathcal{F}\right\rangle\right|^2 \times \delta(E - E_{f_{cT}} - E_{f_c} - E_{f_{NT}}). \tag{7}$$

We now write the sum over δ functions in Eq. (7), which enforces energy conservation, in terms of the imaginary part of the Green's function [29]

$$\sum_{f_{NT},f_c}\left|\psi_{NT}^{(f_{NT})}\psi_c^{(f_c)}\right\rangle\left\langle\psi_c^{(f_c)}\psi_{NT}^{(f_{NT})}\right|\delta(E - E_{f_{cT}} - E_{f_c} - E_{f_{NT}}) = -\frac{1}{\pi}{\rm Im}\,\hat{G}(E - E_{f_{cT}}), \tag{8}$$

which expresses the relationship between the spectral function and the many-body Green's function

$$\hat{G}(\mathcal{E}) = \lim_{\eta\to 0}\frac{1}{\mathcal{E} - \hat{h}_c - \hat{T}_{NT} - \hat{V}_{NT} - \hat{h}_T + i\eta}. \tag{9}$$

Making use of the complete set of product states $\psi_{NT}^{(f_{NT})}$ and $\psi_c^{(f_c)}$, the Green's function can be written in the Lehmann representation,

$$\hat{G}(\mathcal{E}) = \lim_{\eta\to 0}\sum_{f_{NT},f_c}\frac{\left|\psi_{NT}^{(f_{NT})}\psi_c^{(f_c)}\right\rangle\left\langle\psi_c^{(f_c)}\psi_{NT}^{(f_{NT})}\right|}{\mathcal{E} - E_{f_c} - E_{f_{NT}} + i\eta}, \tag{10}$$

or in two other equivalent representations,

$$\hat{G}_h(\mathcal{E}) = \lim_{\eta\to 0}\sum_{f_{NT}}\frac{\left|\psi_{NT}^{(f_{NT})}\right\rangle\left\langle\psi_{NT}^{(f_{NT})}\right|}{\mathcal{E} - E_{f_{NT}} - \hat{h}_c + i\eta}, \tag{11}$$

$$\hat{G}_{NT}(\mathcal{E}) = \lim_{\eta\to 0}\sum_{f_c}\frac{\left|\psi_c^{(f_c)}\right\rangle\left\langle\psi_c^{(f_c)}\right|}{\mathcal{E} - E_{f_c} - \hat{T}_{NT} - \hat{V}_{NT} - \hat{h}_T + i\eta}. \tag{12}$$

Using Eq. (10) we can rewrite Eq. (7) as

$$\frac{d\sigma}{dE_{f_{cT}}d\Omega} = -\frac{2\mu_{PT}}{\hbar^2 k_{PT}}\,\rho(E_{f_{cT}})\left\langle\phi_T^{(0)}\phi_P^{(0)}\mathcal{F}\,\middle|\hat{V}_{\rm prior}\middle|\,\chi_{cT}^{(f_{cT})}\right\rangle \times {\rm Im}\,\hat{G}(E - E_{f_{cT}})\left\langle\chi_{cT}^{(f_{cT})}\,\middle|\hat{V}_{\rm prior}\middle|\phi_T^{(0)}\phi_P^{(0)}\mathcal{F}\right\rangle, \tag{13}$$

or, equivalently,

$$\frac{d\sigma}{dE_{f_{cT}}d\Omega} = -\frac{2\mu_{PT}}{\hbar^2 k_{PT}}\,\rho(E_{f_{cT}})\left\langle\phi_T^{(0)}\phi_P^{(0)}\mathcal{F}\,\middle|\hat{V}_{\rm prior}\middle|\,\chi_{cT}^{(f_{cT})}\right\rangle \times {\rm Im}\,\hat{G}_h(E - E_{f_{cT}})\left\langle\chi_{cT}^{(f_{cT})}\,\middle|\hat{V}_{\rm prior}\middle|\phi_T^{(0)}\phi_P^{(0)}\mathcal{F}\right\rangle. \tag{14}$$

This expression is still difficult to handle as it contains the many-body operator $\hat{G}_h$.

In order to reduce the dimensionality of the problem, we average the Green's function over the ground state of the projectile

$$\left\langle\phi_P^{(0)}\middle|\hat{G}_h(E - E_{f_{cT}})\middle|\phi_P^{(0)}\right\rangle = \lim_{\eta\to 0}\sum_{f_{NT}}\left|\psi_{NT}^{(f_{NT})}\right\rangle\left\langle\psi_{NT}^{(f_{NT})}\right| \times \left\langle\phi_P^{(0)}\middle|\frac{1}{E - E_{f_{cT}} - E_{f_{NT}} - \hat{h}_c + i\eta}\middle|\phi_P^{(0)}\right\rangle = \lim_{\eta\to 0}\sum_{f_{NT}}\left|\psi_{NT}^{(f_{NT})}\right\rangle\left\langle\psi_{NT}^{(f_{NT})}\right|\hat{G}_h^{\rm opt}(E - E_{f_{cT}} - E_{f_{NT}}), \tag{15}$$

where we have defined the optical reduction of the Green's function

$$\hat{G}_h^{\rm opt}(E_h) = \lim_{\eta\to 0}\left\langle\phi_P^{(0)}\middle|(E_h - \hat{h}_c + i\eta)^{-1}\middle|\phi_P^{(0)}\right\rangle = \lim_{\eta\to 0}(E_h - \hat{T}_h - \hat{U}_h + i\eta)^{-1}, \tag{16}$$

which is a one-body operator. In this equation, we have defined the one-particle $\hat{T}_h$ kinetic and $\hat{U}_h$ hole potential operators, which in general is nonlocal, complex, and energy-dependent [30]. The eigenstates of the hole Hamiltonian correspond to discrete overlap functions

$$\psi_h^{(f_c)}(\boldsymbol{r}) = \left\langle\phi_P^{(0)}\middle|\psi_c^{(f_c)}\right\rangle, \tag{17}$$

solutions to the Schrödinger equation [29,30]

$$(E_h - \hat{T}_h - \hat{U}_h)\,\psi_h^{(f_c)}(\boldsymbol{r}) = 0. \tag{18}$$

The energy needed to promote a nucleon of the projectile P to a zero-energy state, leaving the core in its ground state, is the nucleon separation energy $S_N^{(P)}$, which we define positive for particle-bound systems, according to the standard convention. Therefore, the core ground state is obtained by creating a hole of energy $E_h = -S_N^{(P)}$ in the projectile. An excited state of the core with energy E_{f_c} is obtained by delivering the corresponding additional energy, thus creating a deeper hole,

$$E_h = -E_{f_c} - S_N^{(P)}. \tag{19}$$

Note that Eq. (18) does not constrain the proper normalization of $\psi_h^{(f_c)}$, namely, its spectroscopic factor S. Since a dispersive optical potential can be identified with the self-energy of the nucleon in the nuclear medium, the spectroscopic factor is connected with the energy dependence of the hole potential,

$$S(E_{f_c}) = \left(1 - \left.\frac{\partial U_h(E)}{\partial E}\right|_{E_h}\right)^{-1}. \tag{20}$$

The above equation is verified only for dispersive potentials, and expresses the relationship between the optical potential and the energy distribution of single-particle strength [29,30]. It highlights the importance of the use of dispersive potentials in the present formalism, where it is essential to have an accurate description of the spectral function, including the verification of sum rules based on the conservation of the number of particles.

By also defining the transition amplitudes

$$T^{(f_{NT})}(\boldsymbol{R}_{NT},\boldsymbol{R}_{cT}) = \int \mathcal{F}(\boldsymbol{R}_{cT},\boldsymbol{R}_{NT},\boldsymbol{\xi}_T)\phi_T^{(0)}(\boldsymbol{\xi}_T) \times \psi_{NT}^{(f_{NT})*}(\boldsymbol{R}_{NT},\boldsymbol{\xi}_T)\hat{V}_{\rm prior}(\boldsymbol{R}_{NT},\boldsymbol{R}_{cT},\boldsymbol{\xi}_T)d\boldsymbol{\xi}_T, \quad (21)$$

and the source term

$$\rho_h^{(f_{NT})}(\boldsymbol{r}) = \int \chi_{cT}^{(f_{cT})*}(\boldsymbol{R}_{cT})\,T^{(f_{NT})}(\boldsymbol{R}_{NT},\boldsymbol{R}_{cT})\,d\boldsymbol{R}_{cT}, \quad (22)$$

Eq. (14) becomes

$$\frac{d\sigma}{dE_{f_{cT}}d\Omega} = -\frac{2\mu_{PT}}{\hbar^2 k_{PT}}\,\rho(E_{f_{cT}}) \times \sum_{f_{NT}}\big\langle\rho_h^{(f_{NT})}\big|\mathrm{Im}\,\hat{G}_h^{\rm opt}(E_h)\big|\rho_h^{(f_{NT})}\big\rangle, \quad (23)$$

where the argument of the Green's function reflects energy conservation, i.e.,

$$E_{f_c} = E - E_{f_{cT}} - E_{f_{NT}}, \quad (24)$$

complemented with [see Eq. (19)]

$$E_h = E_{f_{cT}} + E_{f_{NT}} - E - S_N^{(P)}. \quad (25)$$

Equation (23) can be interpreted in terms of the source term (22) expressing the probability of the production of a hole in the projectile P at a position $\boldsymbol{r}$, while the Green's function describes the dynamical evolution of the core-hole system. Dissipative effects associated with core excitations are connected with the imaginary part of the optical potential, and are fully accounted for. When these excitations take place at energies above the nucleon emission threshold they can result in particle evaporation, and thus reduce the knockout cross section of deeply-bound nucleons [21].

In order to highlight the effect of the absorption of the core by the hole, we will express Eq. (23) in terms of $\hat{U}_h$ by first defining the free hole propagator $\hat{G}_h^{\rm opt,0}$ as

$$\hat{G}_h^{\rm opt,0}(E_h) = \lim_{\eta\to 0}(E_h - \hat{T}_h + i\eta)^{-1}, \quad (26)$$

so that we can write

$$\big(\hat{G}_h^{\rm opt,0}\big)^{-1} - \big(\hat{G}_h^{\rm opt}\big)^{-1} = \hat{U}_h. \quad (27)$$

By manipulating this expression, we obtain

$$\hat{G}_h^{\rm opt,0}\big[\big(\hat{G}_h^{\rm opt,0}\big)^{-1} - \big(\hat{G}_h^{\rm opt}\big)^{-1}\big]\hat{G}_h^{\rm opt} = \hat{G}_h^{\rm opt} - \hat{G}_h^{\rm opt,0} = \hat{G}_h^{\rm opt,0}\hat{U}_h\hat{G}_h^{\rm opt}, \quad (28)$$

which leads to the Dyson equation

$$\hat{G}_h^{\rm opt} = \hat{G}_h^{\rm opt,0} + \hat{G}_h^{\rm opt,0}\hat{U}_h\hat{G}_h^{\rm opt}. \quad (29)$$

We can now rewrite this propagator as

$$\hat{G}_h^{\rm opt} = \Big(1 + \hat{G}_h^{\rm opt\dagger}\hat{U}_h^{\dagger}\Big)\hat{G}_h^{\rm opt,0}\big(1 + \hat{U}_h\hat{G}_h^{\rm opt}\big) - \hat{G}_h^{\rm opt\dagger}\hat{U}_h^{\dagger}\hat{G}_h^{\rm opt}. \quad (30)$$

The first term of Eq. (30) corresponds to scattering states of the core-hole system. Since the hole Hamiltonian describes the removal of a bound nucleon from the projectile ground state, it does not have any scattering solutions and this first term vanishes. We therefore obtain

$$\mathrm{Im}\,\hat{G}_h^{\rm opt} = \hat{G}_h^{\rm opt\dagger}\,\mathrm{Im}\,\hat{U}_h\hat{G}_h^{\rm opt}. \quad (31)$$

By defining the hole wave function

$$\phi_h^{(f_{NT})}(\boldsymbol{r}) = \hat{G}_h^{\rm opt}(E_h)\,\rho_h^{(f_{NT})}(\boldsymbol{r}), \quad (32)$$

we can then write the cross section (23) as

$$\frac{d\sigma}{dE_{f_{cT}}d\Omega} = -\frac{2\mu_{PT}}{\hbar^2 k_{PT}}\,\rho(E_{f_{cT}})\sum_{f_{NT}}\big\langle\phi_h^{(f_{NT})}\big|\mathrm{Im}\,\hat{U}_h(E_h)\big|\phi_h^{(f_{NT})}\big\rangle. \quad (33)$$

If we assume that for excitation energies above the first particle emission threshold $S_x^{(c)}$ the core will evaporate particles, the experimental cross section for observing the core c is restricted to $0 < E_{f_c} < S_x^{(c)}$ and therefore the hole energy is restricted to $-S_x^{(c)} - S_N^{(P)} < E_h < -S_N^{(P)}$. The cross section therefore reads

$$\frac{d\sigma}{dE_{f_{cT}}d\Omega} = -\frac{2\mu_{PT}}{\hbar^2 k_{PT}}\,\rho(E_{f_{cT}}) \times \sum_{E_h=-S_x^{(c)}-S_N^{(P)}}^{-S_N^{(P)}}\big\langle\phi_h^{(f_{NT})}\big|\mathrm{Im}\,\hat{U}_h(E_h)\big|\phi_h^{(f_{NT})}\big\rangle. \quad (34)$$

Let us stress that the above expression is inclusive in both the N-T and the final states of the core. The sum over all energy-accessible final states of the core is implied in the imaginary part of the hole potential $\mathrm{Im}\,\hat{U}_h$, while it is explicit in the N-T channel, as we sum over the hole states $\phi_h^{(f_{NT})}$, which contain the N-T transition amplitudes in the source term (22). The only approximations made in the derivation of Eq. (34) are the assumption that the prior potential does not depend explicitly on the core internal coordinates, and the factorization of the many-body wave function (4). In particular, no sudden or core spectator approximations has been made concerning the dynamics of the N-T and N-c systems. This is in contrast with the standard eikonal framework [7,19,27], where dynamical, nonsudden effects associated with the nucleon extraction from the projectile (or hole creation) are neglected. It is reasonable to expect that these effects will be particularly important for the knockout of deeply-bound nucleons, creating deeply-bound holes, which could give rise to core excitation above particle emission thresholds and therefore to particle evaporation. The quenching of spectroscopic strength is further enhanced in systems with high neutron-proton asymmetry by the fact that the emission threshold of the deficient species tends to be low, and the sum in Eq. (34) is severely restricted.

III. APPLICATION TO $A(p,d)B$ AND $B(d,p)A$ TRANSFER REACTIONS

Let us now consider a pickup reaction, in which a neutron is transferred from the projectile $A(\equiv B+n)$ to the proton target p, resulting in the formation of a deuteron d and the core nucleus B. In these reactions, the deuteron is detected with kinetic energy E_{f_d}, and the residual nucleus B is not observed. While this process is still inclusive in the core (B) channel, it is exclusive in the n-p channel, in which the deuteron has been detected in the only available bound state, namely its ground state. According to the notation of the previous section,

$$c \equiv B, \quad T \equiv p, \quad N \equiv n, \quad c + N \equiv A, \quad T + N \equiv d. \tag{35}$$

In this case, the deuteron ground state, with binding energy $\epsilon_d = -2.2246$ MeV, is the only one to be kept in the sum appearing in Eq. (34). The final N-T wave function corresponds to the ground state of the deuteron $\psi_{NT}^{(f_{NT})} \equiv \phi_d$, and the differential cross section becomes

$$\frac{d\sigma}{dE_{f_d} d\Omega} = -\frac{2\mu_{pA}}{\hbar^2 k_{pA}} \rho(E_{f_d}) \langle \phi_h | \mathrm{Im}\, \hat{U}_h(E_h) | \phi_h \rangle, \tag{36}$$

where k_{pA} is the initial p-A wave number, μ_{pA} is the p-A reduced mass, $E_h = E_{f_d} + \epsilon_d - E - S_n^{(A)} = -E_{f_B} - S_n^{(A)}$ is the B-hole energy, E_{f_B} is the energy of the final state of the residual nucleus B, and $S_n^{(A)}$ is the neutron separation energy of the A nucleus. The GFK formalism describes nonsudden, dissipative processes in the core B system, and takes into account the quantum many-body dynamics (encoded here in the imaginary part of the hole optical potential, $\mathrm{Im}\, \hat{U}_h$) to describe its final state. Thanks to the use of dispersive optical potentials, the GFK formalism allows a consistent description of pickup reactions leaving the residual nucleus in a bound and resonant state. Typically, $\mathrm{Im}\, \hat{U}_h$ will be spin- and parity-dependent. Within this context, this framework might be useful in situations in which one is interested in the determination of the energy, spin, and parity of the final state of the residual nucleus, as in, e.g., (p, d) surrogate reactions [31,32].

In reactions where a neutron is transferred from the target to the projectile, such as $B(d, p)A$, the state of the nucleus A is not measured. However, as for pickup reactions, the energy of the state of the nucleus A can be deduced from the energy of the final proton E_{f_p}. In this case, we have

$$\frac{d\sigma}{dE_{f_p} d\Omega} = -\frac{2\mu_d}{\hbar^2 k_d} \rho(E_{f_p}) \langle \phi_n | \mathrm{Im}\, \hat{U}_n(E_n) | \phi_n \rangle, \tag{37}$$

where k_d is the initial wave number of the d-B system, μ_d is the d-B reduced mass, E_n is the excitation energy relative to the ground state of the final nucleus A, and $\hat{U}_n$ is the n-B optical potential. Note that if the incoming scattering wave function $\mathcal{F}$ is obtained within the distorted wave Born approximation (DWBA) described in Sec. IV B, this expression is similar to Eq. (25) of Ref. [22], and the definition of the n-B wave function ϕ_n is analogous to the one in Sec. II. The framework presented here represents therefore a generalization of the GFT [22] to describe inclusive measurements of both one-nucleon addition and removal reactions. Compared to previous models, the GFK is therefore applicable to both transfer and knockout reactions. This general applicability of the GFK is a great advantage, as it provides a theoretical framework to compare their analysis and gives insights on what are the reaction mechanisms at play.

Consequently, the GFK might allow to shed some light on the discrepancy between the analyses of transfer and knockout experiments for projectile with large neutron-to-proton asymmetry [7–9,12,13,17]. Since in transfer reactions, the knocked-out nucleon is measured [in the case of (p, d), it forms a deuteron with the proton target], any dissipative effects associated with the extraction of the neutron from the projectile A, will not impact the cross sections (except if the excited residual nucleus is able to emit deuterons). In knockout reactions, since it is the residual nucleus that is detected, the cross section is therefore directly impacted by these particle emissions of the residual nucleus. Since these effects are expected to be more important for the removal of deeply-bound nucleons, this suggests that that these dissipative effects might contribute to explain the discrepancy between the analyses of transfer and knockout data as discussed in Ref. [21].

IV. COMPUTATION OF THE INCOMING SCATTERING WAVE

To implement the GFK formalism, it is useful to approximate the incoming scattering wave $\mathcal{F}$. We present below two possible approaches, which will also determine the choice of the potential $\hat{V}_{\mathrm{prior}}$ used in the transition amplitudes (21).

A. Eikonal approximation

Reactions measured at energies above 60 MeV/nucleon, such as breakup reactions, are accurately described by the eikonal model [19,33–36]. This approximation [18] assumes that the projectile-target relative motion does not differ much from the initial plane wave χ_i, strongly simplifying the three-body problem (more details can be found in Ref. [3]). The eikonal incoming distorted wave is given by

$$\mathcal{F} \approx S_{NT}(\boldsymbol{R}_{NT})\, S_{cT}(\boldsymbol{R}_{cT})\, \chi_i(\boldsymbol{R}_{NT}, \boldsymbol{R}_{cT}). \tag{38}$$

where S_{NT} and S_{cT} are, respectively, the N-T and c-T eikonal S matrices. The eikonal model exhibit cylindrical symmetry, it is therefore often expressed in terms of the transverse ($b_{(N,c)T}$) and longitudinal ($Z_{(N,c)T}$) coordinates

$$R^2_{(N,c)T} = b^2_{(N,c)T} + Z^2_{(N,c)T}. \tag{39}$$

In these coordinates, the eikonal S matrices can then be written as

$$S_{NT}(Z_{NT}, b_{NT}) = \exp\left[i\delta_{NT}^{\mathrm{eik}}(Z_{NT}, b_{NT})\right], \tag{40}$$

$$S_{cT}(Z_{cT}, b_{cT}) = \exp\left[i\delta_{cT}^{\mathrm{eik}}(Z_{cT}, b_{cT})\right], \tag{41}$$

in terms of the eikonal phases $\delta_{NT}^{\mathrm{eik}}$ and $\delta_{cT}^{\mathrm{eik}}$

$$\delta_{NT}^{\mathrm{eik}}(Z_{NT}, b_{NT}) = -\frac{\mu_{PT}}{\hbar^2 k_{PT}} \int_{-\infty}^{Z_{NT}} U_{NT}(Z'_{NT}, b_{NT})\, dZ'_{NT}, \tag{42}$$

$$\delta_{cT}^{\mathrm{eik}}(Z_{cT}, b_{cT}) = -\frac{\mu_{PT}}{\hbar^2 k_{PT}} \int_{-\infty}^{Z_{cT}} U_{cT}(Z'_{cT}, b_{cT})\, dZ'_{cT} \tag{43}$$

with $\hat{U}_{NT}$ and $\hat{U}_{cT}$ the N-T and c-T optical potentials.

By assuming that the nucleon and the core have the same initial velocity as the projectile, i.e.,

$$k_{NT} = \frac{\mu_{NT}}{\mu_{PT}} k_{PT}; \quad k_{cT} = \frac{\mu_{cT}}{\mu_{PT}} k_{PT}, \tag{44}$$

we have

$$\chi_i(\boldsymbol{R}_{NT}, \boldsymbol{R}_{cT}) = \exp\left[i(k_{NT}Z_{NT} + k_{cT}Z_{cT})\right]. \tag{45}$$

In the eikonal model, the prior potential V_{prior} is chosen as the sum of the c-T and n-T interactions [27,37]

$$\hat{V}_{\text{prior}} = \hat{V}_{cT} + \hat{V}_{NT}. \tag{46}$$

As discussed in Ref. [37], this allows to take into account breakup effects in the entrance channel.

In order to further simplify the calculation, one can also approximate the N-T and c-T final wave functions within the eikonal model

$$\chi_{NT}^{(f_{NT})*}(\boldsymbol{R}_{NT}) = S_{NT}^{f*}(\boldsymbol{R}_{NT}) e^{-i\boldsymbol{k}_{NT}^{f}\boldsymbol{R}_{NT}}, \tag{47}$$

$$\chi_{cT}^{(f_{cT})*}(\boldsymbol{R}_{cT}) = S_{cT}^{f*}(\boldsymbol{R}_{cT}) e^{-i\boldsymbol{k}_{cT}^{f}\boldsymbol{R}_{cT}}. \tag{48}$$

where $S_{(N,c)T}^{f*}$ are the eikonal S-matrices and $\boldsymbol{k}_{(N,c)T}^{f}$ are the (N, c)-T final wave vectors.

Then, by approaching the integral over the target degrees of freedom $\boldsymbol{\xi}_T$ in Eq. (21) in terms of an eikonal N-T final wave function, U_{NT} and U_{cT}, the transition amplitude reads

$$\begin{aligned} T^{(f_{NT})}(\boldsymbol{R}_{NT}, \boldsymbol{R}_{cT}) \approx{}& S_{NT}(\boldsymbol{R}_{NT}) S_{NT}^{f*}(\boldsymbol{R}_{NT}) S_{cT}(\boldsymbol{R}_{cT}) \\ &\times [U_{NT}(\boldsymbol{R}_{NT}) + U_{cT}(\boldsymbol{R}_{cT})] \\ &\times e^{-i\boldsymbol{q}_{NT}(\boldsymbol{R}_{NT} - Z_{NT}\hat{Z})} e^{ik_{cT}Z_{cT}}, \end{aligned} \tag{49}$$

and the source term to be used within this eikonal framework is

$$\begin{aligned} \rho_h^{(f_{NT})}(\boldsymbol{r}) \approx \int & S_{NT}(\boldsymbol{R}_{NT}) S_{NT}^{f*}(\boldsymbol{R}_{NT}) S_{cT}(\boldsymbol{R}_{cT}) S_{cT}^{f*}(\boldsymbol{R}_{cT}) \\ &\times [U_{NT}(\boldsymbol{R}_{NT}) + U_{cT}(\boldsymbol{R}_{cT})] \\ &\times e^{-i\boldsymbol{q}_{NT}(\boldsymbol{R}_{NT} - Z_{NT}\hat{Z})} e^{-i\boldsymbol{q}_{cT}(\boldsymbol{R}_{cT} - Z_{cT}\hat{Z})} d\boldsymbol{R}_{cT}, \end{aligned} \tag{50}$$

where we define the transferred momenta $\boldsymbol{q}_{cT} = \boldsymbol{k}_{cT}^{f} - k_{cT}\hat{Z}$ and $\boldsymbol{q}_{NT} = \boldsymbol{k}_{NT}^{f} - k_{NT}\hat{Z}$. Note that this expression can be further simplifying assuming $k_{(c,N)T}^{f} \approx k_{(c,N)T}$, neglecting the dynamics of the reaction, as done in the usual eikonal model.

Since the wave function (38) takes into account breakup effects [37], the eikonal approximation is able to describe physical process in which the hole or nucleon absorption takes place before as well as after (or simultaneously to) the breakup process. However, it should only be applied when the kinematical conditions are suitable for an eikonal approximation, i.e., when the bombarding energy is large enough. It is important to note that, contrary to the usual eikonal description of knockout reactions [19,27], the cross section (34) accounts for nonsudden effects in the breakup of the projectile, by treating explicitly the dynamics of the core-hole system in terms of the hole optical potential $\hat{U}_h$.

Finally, let us stress that the extension of the eikonal approximation to treat explicitly nonlocal N-T and c-T optical potentials is not straightforward [38]. The issue lies in the fact that nonlocal interactions depend on a integral over the whole space of the wave function and the nonlocal potential, while the eikonal wave function is not accurate at short distances. One way to avoid this issue is to derive the local-equivalent potentials, (i.e., local potentials producing the same elastic phase shifts as the original nonlocal ones), and to use them to compute the eikonal phase shifts (42)–(43).

B. Distorted wave Born approximation

In what we will call the DWBA approximation to the GFK formalism, the distorted wave $\mathcal{F}$ is approximated by

$$\mathcal{F} \approx \chi_{PT}(\boldsymbol{R}), \tag{51}$$

where χ_{PT} is the solution of the Schrödinger equation with the projectile-target optical potential $\hat{U}_{PT}$,

$$(\hat{T}_{PT} + \hat{U}_{PT} - E)\chi_{PT}(\boldsymbol{R}) = 0. \tag{52}$$

The potential $\hat{V}_{\text{prior}}$ will now include the remnant term associated with the standard DWBA,

$$\hat{V}_{\text{prior}} = \hat{V}_{cT} + \hat{V}_{NT} - \hat{U}_{PT}. \tag{53}$$

In a similar spirit as the one we adopted in the eikonal approach, one can also approximate the N-T and c-T final wave functions $\chi_{NT}^{(f_{NT})}$ and $\chi_{cT}^{(f_{cT})}$ by the solutions of the Schrödinger equations with the optical potentials $\hat{U}_{NT}$ and $\hat{U}_{cT}$, respectively. The integral over the target degrees of freedom $\boldsymbol{\xi}_T$ in Eq. (21) can then be evaluated in terms of a N-T final wave function, U_{NT} and U_{cT},

$$\begin{aligned} T^{(f_{NT})}(\boldsymbol{R}_{NT}, \boldsymbol{R}_{cT}) \approx{}& \chi_{PT}(\boldsymbol{R}) \chi_{NT}^{(f_{NT})*}(\boldsymbol{R}_{NT}) \\ &\times [U_{NT}(\boldsymbol{R}_{NT}) + U_{cT}(\boldsymbol{R}_{cT}) - U_{PT}(\boldsymbol{R})], \end{aligned} \tag{54}$$

and the source term becomes

$$\begin{aligned} \rho_h^{(f_{NT})}(\boldsymbol{r}) \approx \int & \chi_{cT}^{(f_{cT})*}(\boldsymbol{R}_{cT}) \chi_{PT}(\boldsymbol{R}) \chi_{NT}^{(f_{NT})*}(\boldsymbol{R}_{NT}) \\ &\times [U_{NT}(\boldsymbol{R}_{NT}) + U_{cT}(\boldsymbol{R}_{cT}) - U_{PT}(\boldsymbol{R})] d\boldsymbol{R}_{cT}. \end{aligned} \tag{55}$$

Let us emphasize that $\chi_{NT}^{(f_{NT})}$ can be calculated with the help of the optical reduction of the Green's function $\hat{G}_{NT}$ (12),

$$\begin{aligned} &\hat{G}_{NT}^{\text{opt}}(E_{f_{NT}}) \\ &= \lim_{\eta\to 0} \langle \phi_T^{(0)} | (E_{f_{NT}} - \hat{T}_{NT} - \hat{V}_{NT} - \hat{h}_T + i\eta)^{-1} | \phi_T^{(0)} \rangle \\ &= \lim_{\eta\to 0} \left(E_{f_{NT}} - \hat{T}_{NT} - \hat{U}_{NT} - \epsilon_T^{(0)} + i\eta\right)^{-1}, \end{aligned} \tag{56}$$

where $\epsilon_T^{(0)}$ is the target ground-state energy. This single-particle Green's function can be calculated numerically with Lagrange mesh techniques [39]. Although straightforward for local potentials, computing Green's functions for nonlocal potentials is not trivial and will be reported in another contribution [40]. The overlap can then be obtained making use of the relation

$$\chi_{NT}^{(f_{NT})*}(\boldsymbol{r}; E_{f_{NT}}) \chi_{NT}^{(f_{NT})}(\boldsymbol{r}; E_{f_{NT}}) = -\frac{1}{\pi} \operatorname{Im} G_{NT}^{\text{opt}}(\boldsymbol{r}, \boldsymbol{r}; E_{f_{NT}}). \tag{57}$$

As mentioned in Sec. II, this procedure enforces the proper normalization of the overlap as the spectroscopic factor is directly encoded in the energy dependence of the Green's function [see Eq. (20)]. Since the cross section (34) is a functional of $\chi_{NT}^{(f_{NT})*}\chi_{NT}^{(f_{NT})}$, it is unchanged under an arbitrary phase change $\chi_{NT}^{(f_{NT})} \to e^{i\varphi}\chi_{NT}^{(f_{NT})}$, and the expression (57) is enough to provide $\chi_{NT}^{(f_{NT})}$. Once the overlap has been determined, the source term (55) and the hole wave function (32) can be computed by numerical integration.

As for the eikonal model, the DWBA formulation of the GFK accounts for processes in which breakup has been induced. The main difference here is that the approximation is valid for low bombarding energies, for which the eikonal approximation may not be accurate.

V. CONCLUSIONS

One-nucleon knockout and transfer reactions are key probes of the single-particle structure of nuclei away from stability. The standard theoretical approach associated with these observables rely on spectroscopic factors derived within some nuclear structure formalism, and reaction cross sections, and the discrepancy between theory and experiment is often associated with missing correlations in the structure description [17]. A striking feature of the comparison between the theoretical and experimental knockout observables is a marked neutron-to-proton asymmetry dependence which is not observed in the analysis of transfer and quasifree reactions [7–9,12,13,16,17]. In order to understand what causes this discrepancy, it is pressing to describe both of these reaction processes within the same framework, providing a unified description of structure and reaction.

In this work, we introduce a new formalism, the GFK, which describes one-neutron knockout and transfer reactions making use of dispersive optical potentials, hence treating on the same footing bound and scattering states. For one-nucleon addition transfer reactions, which are typically measured at low to medium energies (from few MeVs to 50 MeV/nucleon), the use of a DWBA incoming scattering function leads to the GFT formalism [22]. Moreover, the GFK can also predict one-nucleon removal reactions, such as (p, d) and knockout reactions, thus allowing the description of transfer and knockout reactions within the same framework.

Because the GFK relies on Green's functions, the link between the few-body problem and the underlying nuclear structure of the ground states of the target and the projectile is made explicit. Moreover, no core spectator or sudden approximation is made, which allows to include dynamical effects associated with the nucleon extraction from the projectile, such as excitation of the core above the particle emission threshold during the collision. Our analysis suggests that the discrepancy observed in the analysis of knockout and transfer data might arise from these dynamical effects, that are neglected in the usual eikonal model.

The main approximations made in the GFK are the assumptions that the prior potential does not depend on the intrinsic coordinates of the core and the factorization of the many-body wave function (4), in which the effect of the projectile-target interaction is described by a incoming scattering function $\mathcal{F}$. The choice of this function reflects the approximation of the few-body problem we are willing to make, and can be adapted to a specific energy regime. In particular, we discuss an eikonal and a DWBA approximation. We plan to test the validity of these approximations and verify their applicability for different systems, i.e., with various beam energies and for nuclei ranging from the valley of stability to the proton and neutron driplines.

In a future publication we plan to compare knockout and transfer observables obtained within a standard reaction model, with the GFK calculation along an isotopic chain. For this, we plan to use the dispersive optical model developed in Refs. [30,41–43], which provides a description of structure and reaction properties for nuclei exhibiting different neutron-to-proton asymmetry. This study will provide quantitative estimates of the dynamical effects associated with the extraction of the nucleon, and might help to explain the systematic discrepancy observed by Gade *et al.* [7–9].

ACKNOWLEDGMENTS

The authors thank P. Capel, W. H. Dickhoff, J. E. Escher, T. Frederico, and C. D. Pruitt for insightful discussions and J. E. Escher and C. D. Pruitt for their careful rereading of this manuscript. C.H. acknowledges the support of the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, under the FRIB Theory Alliance award no. DE-SC0013617 and under Work Proposal No. SCW0498. This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract No. DE-AC52-07NA27344.

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