Charge-symmetry-breaking effects on neutron β decay in nonrelativistic quark models

Jacob W. Crawford and Gerald A. Miller
Department of Physics, University of Washington, Seattle, Washington 98195-1560, USA

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A formalism for the study of charge-symmetry-breaking (CSB) effects is discussed and used to analyze the effects of charge-symmetry breaking on neutron β decay. The effect of including CSB reduces the β-decay matrix element by an amount on the order of 10^{-4}, a value much larger than the previous estimate. The earlier calculation used the neutron-proton mass difference as the CSB operator instead of the matrix element of the sum of the individual terms between the ground and excited states. The electromagnetic and dynamic effects of the up-down quark mass difference oppose the up-down quark mass difference leading to a small $n - p$ mass difference, but add coherently in computing the excitation matrix elements causing large enhancements. The current uncertainty in the value of V_{ud} is also on the order of 10^{-4}. An improvement of that uncertainty by an order of magnitude would require that charge-symmetry-breaking effects should be included in future analyses.

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I. INTRODUCTION

There is a great modern interest in precisely determining the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. This is particularly true of V_{ud}, since it provides the greatest contribution to the unitary condition $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$, which is a testing ground for searches for physics beyond the standard model. The uncertainty of this condition comes in comparable parts from V_{ud} and V_{us} [1], meaning stronger statements about the possibility of a nonunitary CKM matrix can be made by reducing the uncertainty of V_{ud}. Of particular interest to this work are the measurements of V_{ud} via neutron β decay. A benefit of these experiments is a lack of nuclear-structure-dependent corrections that add theoretical uncertainty to a measurement. So the uncertainty in these experiments is largely experimental in nature. This means of measuring V_{ud} is expected to reach levels of precision in competition with superallowed decay experiments within the next decade [1].

The need for greater precision impels us to re-examine the connection between the standard model Lagrangian expressed in terms of quarks and neutron β decay. The implicit assumption is that the quark-level isospin operator is the same as the nucleon isospin operator. This is only true if the nucleon wave function is invariant under the isospin rotation known as the charge-symmetry rotation. The accuracy of the implicit assumption was examined by Behrends and Sirlin [2] who found, using an order-of-magnitude estimate, very small corrections on the order of 10^{-6}. The present paper is aimed at providing a more detailed estimate.

Neglecting the mass difference and electromagnetic effects of the up and down quarks leads to an invariance in the QCD Lagrangian under the interchange of up and down quarks. This invariance is called charge symmetry, which is a more restrictive symmetry than isospin. The fundamental reason why the neutron is more massive than the proton is the fact that the down quark is more massive than the up quark. This positive contribution to $M_n - M_p$ is tempered by electromagnetic effects and the influence of quark masses on the one-gluon-exchange potential (see the reviews [3–5]). The influence of charge-symmetry-breaking operators on the proton wave function and the resulting electromagnetic form factors is discussed in Ref. [6].

Here is an outline. Section II introduces the necessary definitions and the perturbation theory that are the basis for our understanding of the weak operator and charge symmetry. The nonrelativistic quark model and the charge-symmetry-breaking interactions are discussed in Sec. III. This section includes the explicit definitions of our Hamiltonian, charge-symmetry-breaking (CSB) operators, and all nucleon states used in this work. The CSB effects are the mass difference between up and down quarks, its influence in the kinetic energy and one-gluon-exchange operators, and electromagnetic effects. In Sec. IV, the parameters of the models are determined by the need to reproduce the measured mass difference between the neutron (n) and the proton (p). Section V displays the relevant perturbation theory. Evaluations are performed in Sec. VI, and the results are interpreted in Sec. VII.

II. FORMALISM

The weak operator that dictates the decay $n \rightarrow p + e + \overline{\nu}_e$ can be written in terms of operators acting on quarks using first-quantized notation as

$$H_w := V_{ud} \sum_{i=1}^{3} \tau_+(i) \equiv V_{ud}\tau_+, \tag{1}$$

where $\tau_+|d\rangle = |u\rangle$ and $\tau_+|u\rangle = 0$. The property is a statement that the u, d system is a fundamental isospin doublet. If the

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same is true of the neutron-proton system we may state that $\tau_+|n\rangle = |p\rangle$, so that $\langle p|H_w|n\rangle = V_{ud}$. This expression is used in all analyses that aim to extract the value of V_{ud}.

However, the neutron and the proton are composite particles. The u and d quarks within have different masses and undergo electromagnetic interactions. Thus the expression $\tau_+|n\rangle = |p\rangle$ must be modified. It is necessary to introduce the isospin formalism [3] to understand the modifications.

The isospin rotation known as the charge-symmetry rotation operator is used to obtain the nucleon matrix element. The logic is as follows. The charge-symmetry operator is the 180° isospin rotation operator defined as

$$P_{\rm cs} = e^{i\pi T_2}, \tag{2}$$

with

$$P_{\rm cs}^\dagger u P_{\rm cs} = d \tag{3}$$

and

$$T_2 = \tfrac{1}{2} q^\dagger \tau_2 q, \tag{4}$$

where $q = u$ or d is the quark-field operator.

If charge symmetry holds, the neutron is obtained from the proton by the $P_{\rm cs}$ isospin rotation. However, the Hamiltonian H can be expressed in terms of a charge-symmetry-conserving term, H_0, and a breaking term, H_1, with $H = H_0 + H_1$ and $[H_0, P_{\rm cs}] = 0$. The eigenstates of H_0 are denoted in a round bracket notation, $|\cdots)$ states, and $|\cdots\rangle$ is used to denote the physical eigenstates of H. Then

$$H_0|p, m_s) = \sqrt{\overline{M}^2 + \vec{p}^2}|p, m_s), \tag{5}$$

with the label p, m_s representing a proton of momentum $\vec{p}$ of spin m_s and $\overline{M}$ is the average of the neutron and proton masses. We treat the physical wave function using first-order perturbation theory in H_1:

$$|p, m_s\rangle \approx \sqrt{Z}|p, m_s) + \frac{1}{\overline{M} - H_0}\Lambda H_1|p, m_s), \tag{6}$$

where the projection operator Λ defined by

$$\Lambda := 1 - |p, m_s)(p, m_s| - |n, m_s)(n, m_s| \tag{7}$$

projects out the ground-state degrees of freedom. The normalization factor Z is defined so that

$$1 = Z + (p, m_s|H_1 \frac{\Lambda}{(\overline{M} - H_0)^2} H_1|p, m_s). \tag{8}$$

The expression Eq. (6) is sufficient to account for terms of order H_1^2 in the β-decay matrix element. Second-order terms in the wave function lead to higher-order contributions to the matrix element.

Using charge symmetry, the neutron and proton states obey the relation

$$|n, m_s) = P_{\rm cs}|p, m_s). \tag{9}$$

The charge-symmetry-breaking piece of our Hamiltonian is H_1 and in first-quantized notation contains operators $\tau_3(i)$, where i labels a quark. The use of the identity

$$P_{\rm cs}^\dagger \tau_3(i) P_{\rm cs} = -\tau_3(i), \tag{10}$$

along with Eqs. (8) and (9), informs us that Z of the neutron is the same as Z of the proton.

It is also useful to define the quantity

$$\Delta H := P_{\rm cs}^\dagger H P_{\rm cs} - H \tag{11}$$

$$= P_{\rm cs}^\dagger H_1 P_{\rm cs} - H_1 = -2H_1. \tag{12}$$

The relation

$$\begin{aligned}(p|\Delta H|p) &= (p|P_{\rm cs}^\dagger H P_{\rm cs}|p) - (p|H|p)\\ &= (n|H|n) - (p|H|p) = M_n - M_p\end{aligned} \tag{13}$$

will be used to fix the model parameters in Sec. IV.

III. NONRELATIVISTIC QUARK MODEL

In nonrelativistic quark models the spin and the momentum of the proton are unrelated, so we can write our proton state as $|p, i) \to |p, \uparrow)$ for a spin up proton. The spin index will be treated implicitly so that $|p, \uparrow) \to |p)$.

The Hamiltonian is specified by the terms

$$H = K + V_{\rm con} + V_{\rm em} + V_g, \tag{14}$$

which are the kinetic energy K, the confining potential $V_{\rm con}$ (which respects charge symmetry), the electromagnetic interaction $V_{\rm em}$, and the gluon-exchange interaction V_g. The charge-symmetry-breaking part of the Hamiltonian is then

$$\Delta H = \Delta K + \Delta V_{\rm em} + \Delta V_g, \tag{15}$$

with each of the terms defined as in Eqs. (13) and (14).

We proceed to determine the individual contributions to ΔH. The first step is to define the quark masses: $m_i = \overline{m} + \frac{\Delta m}{2}\tau_3(i)$ and $\Delta m = m_u - m_d$. Then the nonrelativistic kinetic energy [7] is given as

$$K = \sum_i \left(m_i + \frac{p_i^2}{2m_i}\right), \tag{16}$$

and

$$\Delta K = \Delta m \sum_i \tau_3(i) + \frac{\Delta m}{\overline{m}} \sum_i \frac{p_i^2}{2\overline{m}} \tau_3(i). \tag{17}$$

The first term of ΔK does not contribute to any excitations and may be neglected in the calculation of the β-decay amplitude. The electromagnetic interaction is given by [8]

$$V_{\rm em} = \alpha \sum_{i<j} q_i q_j \left[\frac{1}{r_{ij}} - \frac{\pi}{2\overline{m}^2}\delta(\vec{r}_{ij})\left(\frac{2}{\overline{m}^2} + \frac{4}{3}\frac{\vec{\sigma}(i)\cdot\vec{\sigma}(j)}{\overline{m}^2}\right)\right], \tag{18}$$

where $q_i = \frac{1}{6} + \frac{1}{2}\tau_3(i)$ and $r_{ij} = |\vec{r}_i - \vec{r}_j|$. The charge-asymmetric contribution from this operator is [6]

$$\begin{aligned}\Delta V_{\rm em} = &-\frac{\alpha}{6}\sum_{i<j}[\tau_3(i) + \tau_3(j)]\\ &\times \left[\frac{1}{r_{ij}} - \frac{\pi}{2\overline{m}^2}\delta(\vec{r}_{ij})\left(1 + \frac{2}{3}\vec{\sigma}(i)\cdot\vec{\sigma}(j)\right)\right].\end{aligned} \tag{19}$$

The gluon-exchange operator is taken to be

$$V_g = -\alpha_s \sum_{i<j} \lambda_i \cdot \lambda_j \left[\frac{\pi}{2}\delta(\vec{r}_{ij}) \left(\frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{4}{3}\frac{\vec{\sigma}(i)\cdot\vec{\sigma}(j)}{m_i m_j} \right) \right], \tag{20}$$

where for three-quark baryons $\lambda_i \cdot \lambda_j = -\frac{2}{3}$ [7,8]. The long-range $1/r_{ij}$ term of V_g respects charge symmetry and so is not included. The charge-symmetry-breaking piece of this interaction is given by [6]

$$\Delta V_g = \alpha_s \frac{2\pi}{3}\frac{\Delta m}{\overline{m}^3} \sum_{i<j} [\tau_3(i) + \tau_3(j)]\delta(\vec{r}_{ij}) \left[1 + \frac{2}{3}\vec{\sigma}(i)\cdot\vec{\sigma}(j) \right]. \tag{21}$$

We use an SU(6) space-spin wave function along with oscillator confinement to represent the charge-symmetric wave functions. Then we write

$$|p) = |\psi_0\rangle \frac{1}{\sqrt{2}}(|\phi_s\rangle|\chi_s\rangle + |\phi_a\rangle|\chi_a\rangle), \tag{22}$$

where

$$\langle \vec{r}_i|\psi_0\rangle = \psi_0(\rho,\lambda) = N e^{-\frac{\rho^2+\lambda^2}{2\beta^2}}. \tag{23}$$

Here $\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$, $\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)$, and the center-of-mass dependence is not made explicit. The standard mixed-symmetry flavor ($\phi_{s,a}$) and spin ($\chi_{s,a}$) wave functions are used [9].

IV. MODEL PARAMETERS

The parameters of the nonrelativistic quark model shall be determined from the neutron-proton mass difference and a consideration of pionic effects. These parameters are β, α_s, and $\bar{m}$, and these are constrained by values of the proton's charge radius, the magnetic moment, and the Δ-nucleon mass splitting. The model does not include the explicit effects of the pion cloud because those are charge symmetric if the pion-nucleon coupling constant is taken (consistent with observations) to be charge symmetric. However, any consideration of the values of parameters must take implicit account of the pion cloud. Here we follow the ideas of the cloudy bag model [10–12] in which a perturbative treatment of pions as quantum fluctuations converges for bag radii (confinement radius) greater than about 0.6 fm. The importance of pionic effects decreases as the confinement radius of the model increases. The effects of the pion cloud contribute to the magnetic moment and to the Δ-nucleon mass splitting.

The quark contribution to the root-mean-square charge radius is β. The measured value is 0.84 fm. The proton magnetic moment is 2.79 nm, and the gluon-exchange contribution (g) to the Δ-nucleon mass splitting of about 303 MeV is given by

$$(M_\Delta - M_N)_g = \frac{2}{3}\sqrt{\frac{2}{\pi}}\frac{\alpha_s}{\bar{m}^2\beta^3}. \tag{24}$$

We use three separate models to evaluate the effects of charge-symmetry breaking on β decay. We start with a large confinement radius of $\beta = 0.837$ fm, with small pionic effects

TABLE I. Model parameters adjusted from Ref. [6].

Model	Model parameters				
	β^2 (fm^2)	$\overline{m}$ (fm^{-1})	α_s	Δm (MeV)	γ
1	0.7	2	6.1	−6.9	0.9
2	0.6	2.1	4.7	−5.3	0.8
3	0.5	2.2	3.5	−4.5	0.7

so that γ, the fraction of the Δ-nucleon mass difference arising from gluon exchange is large, 0.9, and the quark mass is taken to be 2 fm^{-1}, accounting for about 84% of the proton magnetic moment with the pion-cloud accounting for the remainder. The other two models are obtained by increasing the value of $\bar{m}$, decreasing the value of β, and thus decreasing the value of γ. Then the value of Δm is chosen so that according to Eq. (13)

$$(p|\Delta H|p) = M_n - M_p \approx 1.29 \text{ MeV}. \tag{25}$$

Evaluating the individual terms yields

$$(p|\Delta K|p) = \Delta m\left(\frac{1}{2\overline{m}^2\beta^2} - 1\right), \tag{26}$$

$$(p|\Delta V_{\text{em}}|p) = -\frac{\alpha}{3\beta}\sqrt{\frac{2}{\pi}}\left(1 - \frac{5}{12\overline{m}^2\beta^2}\right), \tag{27}$$

$$(p|\Delta V_g|p) = \frac{5\alpha_s\Delta m}{9\overline{m}^3\beta^3}\sqrt{\frac{2}{\pi}}. \tag{28}$$

Then the sum of each term yields approximately the desired mass difference in each nonrelativistic quark model.

The parameter values of each of our models can be found in Table I.

V. β-DECAY MATRIX ELEMENT

We compute the matrix element of τ_+ using the perturbed state of Eq. (6) and the related one for the neutron. The result is

$$\langle p|\tau_+|n\rangle = Z + \left(p|H_1 \tfrac{\Lambda}{\overline{M}-H_0}\tau_+ \tfrac{\Lambda}{\overline{M}-H_0} H_1|n\right), \tag{29}$$

in which the terms of first order in H_1 above vanish because the resolvent $\frac{\Lambda}{\overline{M}-H_0}$ has the states $|p)$ and $|n)$ projected out, and τ_+ can only take nucleons to other nucleons. Moreover, the matrix element of τ_+ between the bare neutron and the state is unity.

The operator H_1 conserves spin angular momentum, so that $(\Delta|H_1|N) = 0$, and there is no contribution due to Δ baryons. There is also no way for H_1 to mix in states containing strange or heavy quarks, so the only contributions are due to spatial excitations of nucleons. Further, recall that $\tau_+|n) = |p)$ and $\tau_+|p) = 0$, and that H_1 introduces no units of angular momentum, so the only excitations can be s waves. This means that the intermediate states appearing in Eqs. (29) and (8) are of the form form $|n^*)(n^*|$ of $|p^*)(p^*|$, in which the * notation refers to radial excitations.

TABLE II. The function $F_1^{(1)}$ is the confluent hypergeometric function of the first kind.

Relevant integrals	
$(\psi_k\|\frac{1}{\rho}\|\psi_0)$	$\frac{\Gamma(k+\frac{1}{2})}{\beta\sqrt{\pi}}\sqrt{\frac{2}{k!\sqrt{\pi}\Gamma(k+\frac{3}{2})}}$
$(\psi_k\|\delta(\vec{\rho})\|\psi_0)$	$\frac{1}{\pi^{3/2}\beta^3}\sqrt{\frac{2\Gamma(k+\frac{3}{2})}{k!\sqrt{\pi}}}$
$(\psi_k\|p_\rho^2\|\psi_0)$	$\frac{3}{4\beta^2}\sqrt{\frac{\pi}{\Gamma(\frac{3}{2})\Gamma(k+\frac{3}{2})}}(\delta_{k0}+\delta_{k1})$
$\frac{\sqrt{6}}{q\beta^3}\int\lambda d\lambda\sin\left(\sqrt{\frac{2}{3}}q\lambda\right)\left(\frac{\lambda^2}{\beta^2}\right)^{i+l}e^{-\frac{\lambda^2}{\beta^2}}$	$\Gamma\left(i+l+\frac{3}{2}\right)F_1^{(1)}\left(i+l+\frac{3}{2},\frac{3}{2},-\frac{Q^2\beta^2}{6}\right)$

With this notation for radial excitations the normalization factor Z of Eq. (8) can be written as

$$Z = 1-\sum_{k\neq 0}\frac{\langle p|H_1|p_k^*\rangle\langle p_k^*|H_1|p\rangle}{(\overline{M}-M_k)^2}, \tag{30}$$

$$= 1+\sum_{k\neq 0}\frac{\langle p|H_1|p_k^*\rangle\langle n_k^*|H_1|n\rangle}{(\overline{M}-M_k)^2}. \tag{31}$$

The second term of this equation is equal to the second term of Eq. (29). The net result is that

$$\langle p|\tau_+|n\rangle = Z+\sum_{k\neq 0}\frac{(p|H_1|p_k^*)(n_k^*|H_1|n)}{(\overline{M}-M_k)^2}. \tag{32}$$

The deviation of Z from unity is the same as that seen in the second term of the above equation. Note that the correction to unity is negative because the neutron and proton matrix elements have opposite signs.

VI. EVALUATION

We next compute the individual contributions to the correction. This will first be done using the assumption that the proton is stationary after the decay, after which we include a nonzero momentum transfer.

We use the notation

$$|p_k^*) = |\psi_k\rangle\frac{1}{\sqrt{2}}(|\phi_s\rangle|\chi_s\rangle+|\phi_a\rangle|\chi_a\rangle) \tag{33}$$

to reference the kth radial excitation, where

$$\langle\rho|\psi_k\rangle = R_{k0}(\rho) := \sqrt{\frac{2(k!)}{\beta^3\Gamma(k+\frac{3}{2})}}\exp\left(\frac{-\rho^2}{2\beta^2}\right)L_k^{1/2}\left(\frac{\rho^2}{\beta^2}\right) \tag{34}$$

is the radial wave function Here, $L_k^{1/2}$ is a generalized Laguerre polynomial [13]. So the quantity to be calculated is the second term of Eq. (32), and the mass denominator can be written

$$M_k-\overline{M} = \frac{2k}{\overline{m}\beta^2}, \tag{35}$$

which is just the energy added due to the harmonic oscillator excitation.

A. Zero recoil

We first turn our attention to the electromagnetic interaction matrix element:

$$(p_k^*|\Delta V_{\rm em}|p) = -\frac{\alpha}{2}(p_k^*|[\tau_3(1)+\tau_3(2)]\left(\frac{1}{\sqrt{2}}\frac{1}{\rho}-\frac{\pi}{\overline{m}^2\sqrt{2}}\delta(\vec{\rho})\left[1+\frac{2}{3}\vec{\sigma}(1)\cdot\vec{\sigma}(2)\right]\right)|p)$$
$$= -\frac{\alpha}{2\sqrt{2}}\frac{2}{3}\left((\psi_k|\frac{1}{\rho}|\psi_0)-\frac{\pi}{\overline{m}^2}\frac{5}{3}(\psi_k|\delta(\vec{\rho})|\psi_0)\right). \tag{36}$$

Then the gluon-exchange term is calculated analogously to the electromagnetic contact term, so we can simply write

$$(p_k^*|\Delta V_g|p) = \frac{20\pi\alpha_s}{9\sqrt{2}}\frac{\Delta m}{\overline{m}^3}(\psi_k|\delta(\rho)|\psi_0). \tag{37}$$

Last, we must calculate the kinetic energy term,

$$(p_k^*|\Delta K|p) = \frac{\Delta m}{3\overline{m}^2}(\psi_k|p_\rho^2|\psi_0). \tag{38}$$

The necessary integrals to complete the above expressions can be found in Table II.

The series of Eq. (32) is evaluated simply by taking the sum of all of the terms, so we explain why the series converges. The large-k values of the contact potential are controlled by the factor

$$\sqrt{\frac{\Gamma(k+3/2)}{k!}}\sim k^{1/4}. \tag{39}$$

This function increases without bound, but the mass difference in the denominator is linear with k, so the series is convergent. The factor $\frac{\Gamma(k+\frac{1}{2})}{\beta\sqrt{\pi}}\sqrt{\frac{2}{k!\sqrt{\pi}\Gamma(k+\frac{3}{2})}}$ associated with the Coulomb correction falls as $1/k^{3/4}$, so it yields a convergent series. The kinetic energy term only enters for $k=1$. The net result is that the perturbation series is convergent.

TABLE III. Results for the three models of Table I are presented. The first column shows the ground-state matrix element of the different contributions to $\Delta H = -2H_1$. The second column shows the connection between the ground state $|0)$ and the first excited state $|1)$. The column labeled "importance" assesses the importance of each term as determined by setting each individually to zero and checking the change in the total correction.

Source	$(0\|\Delta H\|0)$ (MeV)	$(1\|\Delta H\|0)$ (MeV)	Importance
	Model 1		
ΔK	5.667 86	−1.006 04	Second
$\Delta V_{\rm em}$	−0.389 73	−0.103 47	Least
ΔV_g	−3.984 75	−4.880 30	Most
	Model 2		
ΔK	4.298 49	−0.817 73	Second
$\Delta V_{\rm em}$	−0.416 67	−0.106 52	Least
ΔV_g	−2.591 11	−3.173 45	Most
	Model 3		
ΔK	3.570 25	−0.759 14	Second
$\Delta V_{\rm em}$	−0.448 47	−0.106 93	Least
ΔV_g	−1.837 50	−2.250 47	Most

The significance of each contribution obtained with different models is displayed in Table III. The kinetic energy term which contains the effect of the mass difference between up and down quarks controls the sign of the neutron-proton mass difference. The relative importance of each term of ΔH is also displayed and depends on the value of β. It is noteworthy that there are cancellations between the separate terms of ΔH that result in the $n-p$ mass difference of 1.29 MeV for each model. However, the contributions of the separate terms to the dominant matrix element $(1|\Delta H|0)$ all have the same sign. This is mainly because the quark mass difference term does not involve the spatial wave function and so cannot convert the ground state to any excited state. Note especially that the sum of the three terms is much larger than the individual terms and the square that enters in computing the β-decay matrix element.

In particular the ratio R, given by

$$R \equiv \left(\frac{(1|\Delta H|0)}{(0|\Delta H|0)}\right)^2, \tag{40}$$

varies between about 6 and 22 as one changes the models from 3 to 1.

B. Nonzero proton recoil

The next step is to endow the Hamiltonian H_w to take the momentum transfer $\vec{q}$ to the final proton into account. This is done by making the operator substitution $\tau_+ \rightarrow \tau_+^*(\vec{q})$:

$$\tau_+^*(\vec{q}) = \sum_j \tau_+(j)e^{i\vec{q}\cdot\vec{r}_j}. \tag{41}$$

The second-order quantity to be calculated now is the matrix element $\langle\vec{q}|\tau_+^*|\vec{0}\rangle$ with

$$\langle\vec{q}|\tau_+^*|\vec{0}\rangle \equiv Z(p|e^{-i\sqrt{\frac{2}{3}}\vec{q}\cdot\vec{\lambda}}|n) + \frac{1}{4}\sum_{j,k\neq 0}\frac{(p|\Delta H|p_j^*)(p_j^*|e^{-i\sqrt{\frac{2}{3}}\vec{q}\cdot\vec{\lambda}}|p_k^*)(n_k^*|\Delta H|n)}{(\overline{M}-M_j)(\overline{M}-M_k)}. \tag{42}$$

The correction to the β-decay matrix element is defined to be $\Delta(|\vec{q}|) > 0$ with

$$\langle\vec{q}|\tau_+^*|\vec{0}\rangle = 1 - \Delta(|\vec{q}|). \tag{43}$$

Before calculating the remaining matrix element, we need the average momentum transfer to the proton during β decay. This is accomplished by using the recoil spectrum found on page 14 of the Ph.D. thesis of Konrad [14], originally derived by Nachtmann [15]. The spectrum and related functions are written as follows:

$$w_p(T) \propto g_1(T) + ag_2(T), \tag{44}$$

$$g_1(T) = \left(1-\frac{x^2}{\sigma(T)}\right)^2\sqrt{1-\sigma(T)}\left[4\left(1+\frac{x^2}{\sigma(T)}\right) - \frac{4}{3}\left(1-\frac{x^2}{\sigma(T)}\right)[1-\sigma(T)]\right], \tag{45}$$

$$g_2(T) = \left(1-\frac{x^2}{\sigma(T)}\right)^2\sqrt{1-\sigma(T)}\left[4\left(1+\frac{x^2}{\sigma(T)}-2\sigma(T)\right) - \frac{4}{3}\left(1-\frac{x^2}{\sigma(T)}\right)[1-\sigma(T)]\right], \tag{46}$$

$$\sigma(T) = 1 - \frac{2TM_n}{\Delta^2}, \tag{47}$$

$$x = \frac{m_e}{\Delta}, \quad \Delta = M_n - M_p = 1293.333(33)\ \text{keV}. \tag{48}$$

Note that this spectrum does not take into account Coulomb or radiative corrections, but that level of precision is not necessary for the current application. We use the first moment of a normalized w_p to get the mean kinetic energy of the recoiled proton:

$$\langle T\rangle = \frac{\int T w_p(T)dT}{\int w_p dT} = 357.177\ \text{eV}, \tag{49}$$

where the domain of the given integral is taken as (0, 751 eV). This can then be converted into the average wave number $\langle Q\rangle$ of the recoiled proton:

$$\langle Q\rangle = \frac{\sqrt{2M_p\langle T\rangle}}{\hbar c} = 4.1\times 10^{-3}\ \text{fm}^{-1}. \tag{50}$$

We use $|\vec{q}| = \langle Q\rangle$ in the following calculations.

TABLE IV. Computed changes to the value of the β-decay matrix element caused by charge-symmetry-breaking effects.

	Results	
Model	$\Delta(0)$ $(\times 10^{-4})$	$\Delta(\langle Q\rangle)$ $(\times 10^{-4})$
1	4.0297	4.0101
2	1.4668	1.4500
3	0.6146	0.6005

The effect of a nonzero value of $\langle Q\rangle$ in the first term of Eq. (50) is given by the deviation between the factor $e^{-\frac{Q^2\beta^2}{6}}$ and unity, which is of the order of magnitude 10^{-6}. To compute the analytic expression for the matrix element $\langle\psi_j|e^{-i\sqrt{\frac{2}{3}}\vec{q}\cdot\vec{\lambda}}|\psi_k\rangle$, the closed form for the Laguerre polynomials,

$$L_j^{(\alpha)}(x)=\sum_{i=0}^{j}\frac{(-1)^i}{i!}\binom{j+\alpha}{j-i}x^i, \tag{51}$$

must be used [16]. The final result contains a double sum due to Eq. (51) over the final integrated expression found in Table II. The second term of Eq. (42) contributes only at the level of 10^{-10}.

The numerical results of this and the preceding section can be found in Table IV.

For all of the models the change in the β-decay matrix element is a reduction of the order of 10^{-4}. Table III shows us that this change is about an order of magnitude larger than that caused by any one of the terms of ΔH. The model dependence arises from the different values of the length parameter and the quark masses. These affect the energy denominator. The table shows that the effect of including the nonzero value of the momentum transfer is of the presently negligible order of 10^{-6}.

The current value [17] of V_{ud} is given by

$$|V_{ud}|=0.973\,73\pm0.000\,31, \tag{52}$$

so the size of the charge-symmetry-breaking effect is of the order of the current uncertainty.

VII. DISCUSSION AND ASSESSMENT

A general formalism for including the effects of charge-symmetry breaking (CSB) is discussed and applied to computing neutron β-decay matrix elements. CSB effects are known to enter only at second and higher order [2]. The known CSB effects are the quark mass differences, the effect of quark mass differences on the kinetic energy and gluon-exchange potentials and electromagnetic effects. These are evaluated using three nonrelativistic quark models using oscillator confinement. Our second-order result is that including CSB effects reduces the β-decay matrix element by about 10^{-4}. Thus higher orders need not be included. The calculations involved summing over many intermediate states, but the dominant terms arise from including the first radial excitation. This task was made simpler by the acquisition of analytic results for each matrix element, taking as many terms as necessary for a sufficiently converged result. Three nonrelativistic quark models were compared in the analysis,

Calculations were done with and without including the effects of proton recoil [14]. The latter effect is on the order of 10^{-6} and currently negligible, justifying that the proton can be considered a body at rest in the context of neutron β decay.

It is interesting that our result is about 100 times larger than that of the original work of Behrends and Sirlin [2]. There it was predicted that effects due to charge-symmetry breaking on neutron β decay should be on the order of 10^{-6}. Their schematic calculation correctly used the square of the ratio of a matrix element divided by an energy denominator: $(\frac{\langle p|H_1|p\rangle}{M})^2\approx(\frac{1.3}{940})^2\approx 2\times10^{-6}$. They used the $n-p$ mass difference as a matrix element instead of the matrix element of the sum of the individual terms between the ground and excited states. While the individual terms tend to cancel in computing the $n-p$ difference, they add coherently in computing the excitation matrix elements. This gives rise to enhancements of between about 6 and 21, as seen in Eq. (40). Furthermore, the relevant energy denominator is not the nucleon mass, but the excitation energy which is about half of that. Our lowest and most important energy denominator $\Delta M=\frac{2(\hbar c)^2}{\overline{m}\beta^2}$ varies between 280 and 360 MeV. The values, determined by using the correct approximate size of the nucleon, are lower than the 500 MeV difference between the nucleon mass and its first excited state. This reflects a long-standing problem of the nonrelativistic quark model.

The value of β^2 could be decreased by a factor of about 50–60% to increase the energy difference to about 500 MeV, but the matrix element of, for example, the gluon-exchange term (which is the most important one for each of the models) varies as $1/\beta^3$, so the net result would be an increase the size of the CSB effect by $1/\beta^2$, an increase of 50–60%. Thus we regard the results in Tables III and IV to be reasonable first semirealistic estimates. The general conclusion regarding the order-of-magnitude of the CSB effect of 10^{-4} is independent of the model used. Indeed, this value is corroborated by the earlier bag model calculation of Ref. [18].

We summarize by saying that the size of the CSB effects are to decrease the value of the β-decay matrix element by a factor of about 10^{-4}. This is of the order of the current uncertainty in the measurements. An improvement of that uncertainty by an order of magnitude would require that charge-symmetry-breaking effects be included in future analyses. The present effort is only a first step. An increased experimental precision would require that a more controlled approximation, such as using lattice QCD or effective field theory, be used.

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