SS-HORSE extension of the no-core shell model: Application to resonances in ^{7}He

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Theoretical *ab initio* studies of resonances in the unbound ^{7}He nucleus are presented. We perform no-core shell-model calculations with NN interactions Daejeon16 and JISP16 and utilize the SS-HORSE method to calculate the S matrix for two-body channels $n + {}^6$He and $n + {}^6$He* with ^{6}He respectively in the ground and excited 2^+ states as well as for the four-body democratic decay channel ^{4}He $+ n + n + n$. The resonant energies and widths are obtained by numerical location of the S-matrix poles. We describe all experimentally known ^{7}He resonances and suggest an interpretation of an observed wide resonance of unknown spin-parity.

I. INTRODUCTION

A modern trend of nuclear theory is the development of methods for describing nuclear states in the continuum, resonances in particular, and the boundaries of nuclear stability as either neutron number or proton number is increased to the point where the nucleus becomes unbound. The ^{7}He nucleus presents an especially significant challenge since it has no bound states and the experimental information on its resonances is fragmentary. Ideally, an approach with predictive power could help refine the current knowledge of ^{7}He and inform further experimental efforts. For maximal predictive power, *ab initio* ("first-principles") approaches in this field are of primary importance since the only input is the interaction between nucleons.

Currently there are a number of reliable methods for the *ab initio* description of nuclear bound states (see, e.g., the review [1]). Prominent methods include the Green's function's Monte Carlo [2], the no-core shell model (NCSM) [3], the coupled cluster method [4], etc. The NCSM employed here is a modern version of the nuclear shell model which does not introduce an inert core and includes the degrees of freedom

of all nucleons of a given nucleus. The multiparticle wave function is expanded in a series of basis many-body oscillator functions (Slater determinants) which include all many-body oscillator states with total number of excitation quanta above the minimum needed to satisfy the Pauli principle that are less than or equal to some given value $N_{\max}$. This makes it possible to separate the center-of-mass motion. The degree of convergence achieved with NCSM calculations as $N_{\max}$ and/or number of nucleons A increases is governed by the limits of available supercomputers.

The NCSM cannot be directly applied to the description of resonant states. Energies of resonant states are positive with respect to some breakup threshold so that one needs to consider decay modes. Special methods taking into account the continuum are therefore needed for the description of resonances.

There are well-developed methods for *ab initio* descriptions of continuum spectrum states based on Faddeev and Faddeev–Yakubovsky equations that are successfully applied to systems with $A \leqslant 5$ nucleons (see, e.g., the review [1] and Ref. [5]). A very important breakthrough in developing *ab initio* theory of low-energy reactions in heavier systems was achieved by combining the NCSM and the resonating group method to build the NCSM with continuum (NCSMC) approach [6] which has been applied to nuclear systems up to $A = 12$ [7,8]. Nuclear resonances can also be obtained in

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the no-core Gamow shell model (GSM) [9]. However, these methods provide significant numerical challenges for no-core systems [9–11]. At higher energies, above the resonance region, alternative *ab initio* methods are developed and applied (see, e.g., Ref. [12]).

Recently we proposed the SS-HORSE method [13–18], which generalizes the NCSM to the continuum states. The SS-HORSE allows one to calculate the single-channel S matrix and resonances by a simple analysis of NCSM eigenenergy behavior as a function of parameters of the many-body oscillator basis. The SS-HORSE extension of the NCSM was successfully applied to the calculation of the neutron-α and proton-α scattering and resonant states in the ^{5}He and ^{5}Li nuclei in Refs. [13,17]; a generalization of this approach to the case of the democratic decay provided a prediction of a resonance in the system of four neutrons (tetraneutron) [19,20] whose first low-statistics observation [21] has been followed by its discovery and characterization in a high-statistics experiment [22].

The unbound ^{7}He nucleus presents a new challenge for *ab initio* theory but is especially interesting since its experimental information is fragmentary and conflicting. A few resonances have been observed in ^{7}He but all have weak spin-parity assignments if any [23]. In particular, the lowest resonance with a width of 0.18 MeV at the energy of 0.43 MeV above the $n + {}^6$He threshold [24] has a tentative spin-parity of $3/2^-$. There is also a resonance at 3.36 MeV with the width of 1.99 MeV which is tentatively assigned $J^\pi = 5/2^-$ and another resonance at 6.2 ± 0.3 MeV with the width of 4 ± 1 MeV of unknown spin-parity [23]. The most complicated situation is with the $1/2^-$ resonance which was observed in Refs. [25–27]: According to these works, its energy ranges from 1 [27] to 3.5 MeV [26] and the width from 0.75 [27] to 10 MeV [26]. Thus, ^{7}He represents a very good candidate for invoking the predictive power of *ab initio* scattering theory. Therefore, we predict the resonances of ^{7}He within the SS-HORSE–NCSM approach. We find additional broad resonances that suggest a new interpretation of ^{7}He resonant structure.

Recent many-body calculations of the ^{7}He nucleus, explicitly accounting for the continuum spectrum effects, include a GSM study of Ref. [28], Gamow-density-matrix renormalization-group (G-DMRG) calculations of Ref. [29], an investigation within the complex-scaled cluster-orbital shell model (CS-COSM) in Ref. [30], a NCSMC study of Refs. [31,32], and recent calculations of Ref. [33] which we shall refer to as NCSMch where the NCSM wave functions of ^{7}He are matched with the wave functions in a particular decay channel. The G-DMRG approach is based on the GSM but utilizes the many-body technique of the density matrix renormalization group [34,35] to speed up the convergence. The GSM [28], G-DMRG [29], and CS-COSM [30] calculations are performed with the ^{4}He core and nucleons in the *psdf* (GSM) or *spd* (G-DMRG and CS-COSM) valence spaces interacting by phenomenological effective potentials. The NCSMC calculations of Refs. [31,32] use a similarity renormalization group (SRG) evolved chiral next-to-next-to-next-to-leading order (N3LO) nucleon-nucleon (*NN*) potential of Refs. [36,37] while the

NCSMch calculations of Ref. [33] utilize the Daejeon16 *NN* interaction [38] originating from the same chiral N3LO interaction and adjusted with unitary transformations that preserve the *NN* phase shifts to describe accurately binding energies and spectra of *p*-shell nuclei without three-nucleon (*NNN*) forces.

We present here *ab initio* SS-HORSE–NCSM calculations performed using the code MFDn [39,40] with realistic Daejeon16 [38] and JISP16 [41] *NN* interactions. The difference with the approach presented in Ref. [33] is that we obtain the resonant energies and widths by locating the S-matrix poles. The S-matrix elements in all channels have the poles at the same location in the complex energy plane. Therefore our resonance widths are the total widths of resonances associated with decay in all possible channels; they may be very different from the partial widths which are obtained within NCSMch [33] or within the NCSMC [31,32] that characterize the probability of the decay in a particular channel.

Our SS-HORSE-NCSM approach is sketched in Sec. II. Results of our calculations of ^{7}He resonances are presented in Sec. III. Section IV includes a summary and conclusions.

II. SS-HORSE–NCSM APPROACH

Our calculations are based on the formalism of the harmonic-oscillator representation of scattering equations (HORSE) [42], which is a particular version of the J-matrix formalism in quantum-scattering theory. The J-matrix formalism was initially proposed [43] for atomic problems with the use of the so-called Laguerre basis convenient in atomic physics; however, a possibility of the utilization of the harmonic-oscillator basis within this formalism was also described in Ref. [43]. In nuclear physics, the version of the J-matrix approach with the oscillator basis has been later rediscovered independently [44–47]. The harmonic-oscillator basis suggests various convenient features and simplifications of the general J-matrix formalism (see, e.g., Refs. [42,48,49]); therefore we prefer to use the term HORSE when using the harmonic-oscillator version of the J-matrix formalism to distinguish it from the general J-matrix.

The utilization of the general HORSE formalism in combination with the large-scale NCSM calculations is very challenging. However, for calculations of nuclear resonance energies and widths based on the *ab initio* NCSM, one can employ its simplified version, the so-called single state HORSE (SS-HORSE) method [13–18].

Our SS-HORSE–NCSM approach to obtaining resonance parameters is to locate S-matrix poles. The S-matrix in the channel with orbital momentum ℓ, $S_\ell = e^{2i\delta_\ell}$, can be expressed through the effective range function,

$$K_\ell(E) = k^{2\ell+1} \cot \delta_\ell(E), \tag{1}$$

where $\delta_\ell(E)$ is the phase shift, E is the energy of relative motion in a given channel and $k = \sqrt{2\mu E}/\hbar$ is the relative momentum while μ is the reduced mass. The effective range function (1) has good analytical properties and may be expanded in a power series of the energy E (the so-called effective range expansion). The function $K_\ell(E)$ within the

SS-HORSE method is calculated at the eigenenergies of relative motion E_i of a decaying resonant state obtained in the NCSM as [13–15]

$$K_\ell(E_i) = -k_i^{2\ell+1} \frac{C_{\mathbb{N}^i+2,\ell}(E_i)}{S_{\mathbb{N}^i+2,\ell}(E_i)}. \quad (2)$$

Here $S_{n,\ell}(E)$ and $C_{n,\ell}(E)$ are the regular and irregular solutions of the free Hamiltonian in the oscillator representation for which analytical expressions in the case of two-body channels can be found in Refs. [42,43] and in the case of democratic four-body decay channels in Refs. [48,49]; $\mathbb{N}^i$ is the maximal oscillator quanta of the relative motion in the decaying channel allowed in the NCSM calculation for ^{7}He. Note that the functions $S_{\mathbb{N}^i+2,\ell}(E_i)$ and $C_{\mathbb{N}^i+2,\ell}(E_i)$ depend on the oscillator quantum $\hbar\Omega^i$ used in the respective NCSM calculations with the maximal number of excitation quanta N_{max}^i as well as the energies $E_i = E_i(N_{max}^i, \hbar\Omega^i)$.

All resonant states in ^{7}He can decay via the $n + {}^6$He channel with ^{6}He in the ground state. Additionally, all examined resonances with the exception of the low-lying $3/2^-$ resonance, can decay also via a two-body channel $n + {}^6$He* with ^{6}He in the excited 2^+ state or via a four-body channel $n + n + n + {}^4$He.

The channel wave functions of the four-body $n + n + n + {}^4$He democratic decay, i.e., the decay where none of the two- or three-body subsystems has a bound state [50,51], cannot be orthogonalized to the two-body $n + {}^6$He and $n + {}^6$He* channel wave functions; thus the democratic channel should be examined separately from the two-body channels. Most of the ^{7}He resonances under consideration have two or more open two-body channels. The SS-HORSE method employed in our study is designed for treating single-channel problems. A very appealing feature of the SS-HORSE method is that it suggests a simple calculation of nuclear resonance energies and widths provided that the results of standard NCSM calculations are known. This is very different from the NCSMC [6] or no-core GSM [9] approaches which complexify the NCSM calculations. The SS-HORSE extension to the multichannel case is possible but it requires extracting some information from the NCSM eigenfunctions which significantly complicates this approach. We study the two-body channels $n + {}^6$He and $n + {}^6$He* separately. That means that we treat one of the two-body channels $n + {}^6$He and $n + {}^6$He* as an open channel in our SS-HORSE–NCSM approach; we note, however, that the alternative channel as well as a huge number of other channels including democratic ones are coupled to the specified open channel as closed channels in the ^{7}He NCSM calculation.

It is a common approximation in nuclear physics to treat a nuclear system with few open channels as a system with a single open channel and neglecting the contributions in the continuum from other channels when one of the channels is supposed to dominate or when calculating partial widths in different channels like in the NCSMch ^{7}He study of Ref. [33]. Even in the case of multichannel calculations, usually not all possible channels are included as open channels; for example, in the coupled-channel NCSMC ^{7}He calculations of Refs. [31,32] the channels with $\ell \geqslant 3$ were neglected in calculations of negative-parity states. Open three- and four-body

channels are conventionally neglected in theoretical studies of resonances with open two-body channels.

Contrary to most of the other theoretical studies, we calculate parameters of nuclear resonances by locating numerically the respective S-matrix poles. In a complete multichannel investigation, all S-matrix elements should be obtained with the poles at the same total complex energy which includes excitation energy of the fragments. Our NCSM calculations with large bases include all possible channels though treated as closed. This is an approximation, however all closed channels contribute to the global features of the S matrix in a particular single channel open by the SS-HORSE. Thus it is interesting to compare the locations of the S-matrix poles obtained with different open channels: These poles should be close to each other if our approximation is accurate. We shall see that our resonance parameters obtained with the $n + {}^6$He and $n + {}^6$He* open channels are really close supporting the validity of our approach. More, we also obtain close resonance energies in calculations with open four-body ^{4}He $+ n + n + n$ democratic channels. However the democratic decays are suppressed by the strong hyperspherical centrifugal barrier which decreases their widths and makes two-body decays dynamically preferable. We evaluate the accuracy of resulting resonant energies and widths by studying the spread of results obtained in different channels excluding the underestimated widths obtained in democratic channels. We note that the calculation of the S-matrix poles results in total widths of resonances associated with decay in all possible channels.

Within our SS-HORSE–NCSM approach, we start from the NCSM calculations of the ^{7}He eigenenergies E_i^7 corresponding to a set of pairs of the NCSM basis parameters N_{max}^i and $\hbar\Omega^i$, as well as, depending on the channel of interest, the energies E_i^6 of the ground state ($n + {}^6$He channel) or the lowest 2^+ state of ^{6}He ($n + {}^6$He* channel) or the ground-state energy E_i^4 of ^{4}He ($n + n + n + {}^4$He channel) obtained by NCSM with the same $\hbar\Omega^i$ and the maximal excitation quanta N_{max}^i or $N_{max}^i - 1$ depending on the parity of the states of interest in ^{7}He. The number of oscillator quanta of the relative motion $\mathbb{N}^i$ entering Eq. (2) is defined as

$$\mathbb{N}^i = N_{max}^i + N_{min}^7 - N_{min}^A, \quad (3)$$

where N_{min}^7 and N_{min}^A are the minimal total oscillator quanta consistent with the Pauli principle in ^{7}He and ^{A}He, $A = 6$ or 4 in the current work. The eigenenergies of relative motion $E_i = E_i^7 - E_i^A$.

In the case of the four-body decay channel, we use the democratic decay approximation (also known as true four-body scattering or $4 \rightarrow 4$ scattering) suggested in Refs. [50,51]. Democratic decay implies a description of the continuum using a hyperspherical harmonics (HH) basis. We use here the minimal approximation for the four-body decay mode; i.e., only HH with hyperspherical momentum $K = K_{min} = 0$ or 1 for positive- or negative-parity resonances, respectively, are retained in the SS-HORSE extension of the NCSM. This approximation relies on the fact that the decay in the hyperspherical states with $K > K_{min}$ is strongly suppressed by a large hyperspherical centrifugal barrier $\mathscr{L}(\mathscr{L} + 1)/\rho^2$, where the effective momentum $\mathscr{L} = K + 3$ and the

hyperradius $\rho^2 = \sum_{i=1}^{4}(\mathbf{r}_i - \mathbf{R})^2$, $\mathbf{R}$ is the center-of-mass coordinate, and $\mathbf{r}_i$ are the coordinates of decaying neutrons and ^{4}He. Note that all possible HH are retained in the NCSM basis; thus the hyperspherical states with $K = K_{\min} = 0$ are treated as open channels while all the remaining hyperspherical states with $K > K_{\min}$ are treated as closed channels. The accuracy of this approximation was confirmed in studies of democratic decays in cluster models [52–55]; it was also utilized in our successful study of the tetraneutron [19,20]. In this case we should set $\ell = \mathscr{L}_{\min} = 3$ in Eqs. (1) and (2), the relation between the HH momentum k and energy E can be found in Refs. [48,49].

We perform the NCSM calculations with various choices of the basis parameters $N_{\max}^i$ and $\hbar\Omega^i$ and obtain a set of values of the effective range function $K_\ell(E_i)$ using Eq. (2) in some energy interval since $E_i = E_i(N_{\max}^i, \hbar\Omega^i)$. Next we perform a parametrization of the function $K_\ell(E)$ which makes it possible to calculate the S matrix and its poles associated with the resonant states in ^{7}He. The effective range function $K_\ell(E)$ has good analytical properties and may be expanded with a Taylor series in E [56] except for energies in the vicinity where the phase shift takes the values of $0, \pm\pi, \pm 2\pi, ...$(this can happen in the resonance region) and the effective range function $K_\ell(E)$, according to Eq. (1), tends to infinity. Therefore, we use Padé approximants to parametrize $K_\ell(E)$; the number of fit parameters in the numerator and denominator of the Padé approximant taken individually in each case to obtain a reasonable description of selected NCSM eigenenergies.

With any set of the Padé approximant parameters we obtain $K_\ell(E)$ as a function of energy E and solve Eq. (2) to obtain the eigenenergies E_i^{th} which should be obtained in the NCSM calculations with any given combination of $N_{\max}^i$ and $\hbar\Omega^i$ to describe exactly this function. These energies E_i^{th} are compared with the NCSM eigenenergies E_i; the optimal values of the fit parameters are found by minimizing the sum of squares of deviations of E_i^{th} and E_i with weights enhancing the contribution of energies obtained with larger $N_{\max}$ values,

$$\Xi = \sqrt{\frac{1}{p}\sum_{i=1}^{p}\left[\left(E_i^{\text{th}} - E_i\right)^2 \left(\frac{N_{\max}^i}{N_M}\right)^2\right]}. \quad (4)$$

Here p is the number of basis parameter pairs and N_M is the largest value of $N_{\max}^i$ used in the fit.

After obtaining an accurate parametrization, we express the S matrix through $K_\ell(E)$ and search numerically for the S-matrix poles in the complex energy plane, as described in Ref. [17]. These poles produce the energies E_r and widths Γ of the ^{7}He resonances.

III. RESONANCES IN ^{7}He

We perform the NCSM calculations of ^{7}He with $N_{\max}^i$ up to 16 for negative-parity states and up to 17 for positive-parity states with $\hbar\Omega^i$ ranging from 10 to 50 MeV and of ^{6}He and ^{4}He with the same $\hbar\Omega^i$ values and respective $N_{\max}$ to get the set of relative motion eigenenergies E_i.

As stated in Refs. [13–19], we cannot use all energies E_i for the SS-HORSE analysis. In particular, the SS-HORSE equa-
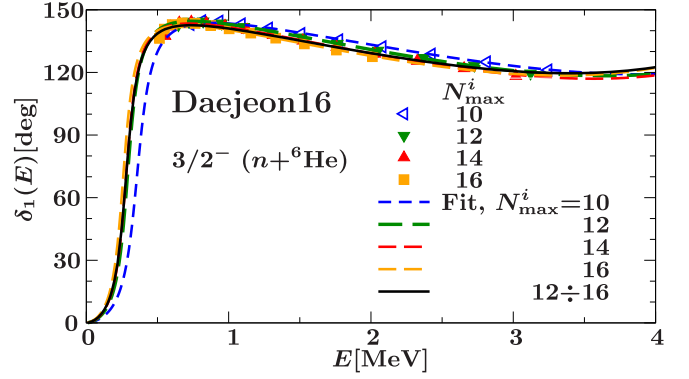

FIG. 1. Convergence of phase shifts of the $n + {}^6$He scattering in the $3/2^-$ state in the vicinity of the low-lying resonance in calculations with Daejeon16 NN interaction. Symbols are phase shifts $\delta_1(E_i)$ at selected energies E_i; curves are SS-HORSE fits to the NCSM results from different model spaces. Energies are given relative to the $n + {}^6$He threshold.

tions are consistent only with those E_i obtained at any given $N_{\max}$ which increase with $\hbar\Omega$. Therefore, from the E_i obtained by NCSM with any $N_{\max}$ we should select only those which are obtained with $\hbar\Omega > \hbar\Omega_{\min}$, where $\hbar\Omega_{\min}$ corresponds to the minimum of the E_i.

Next, for the $K_\ell(E)$ parametrization, we should select only the results obtained with large enough $N_{\max}$ and in the ranges of $\hbar\Omega$ values for each $N_{\max}$ where the continuum state calculations converge, at least, approximately. The convergence means that the $K_\ell(E_i)$ values [as well as the respective phase shifts $\delta_\ell(E_i)$] obtained with different pairs of $N_{\max}^i$ and $\hbar\Omega^i$ values form a single smooth curve as a function of energy. Our method for the selection of the NCSM results is described in detail with a number of illustrations in Refs. [13–19].

We illustrate in Fig. 1 the convergence in calculations with Daejeon16 NN interaction of the $n + {}^6$He elastic-scattering phase shifts for the $3/2^-$ state in the vicinity of the low-lying resonance. The energies E_i selected from the results of NCSM calculations with $N_{\max}^i = 12$, 14 and 16 generate a set of the phase shifts $\delta_1(E_i)$ shown by closed symbols which approximately form a single smooth curve. The SS-HORSE parametrization of these three sets of $3/2^-$ phase shifts $\delta_3(E_i)$ (solid curve labeled $12 \div 16$) accurately describes them.

We present in Fig. 1 also the parametrizations fit to the NCSM eigenenergies from the selection obtained individually with each of these three $N_{\max}^i$ values. These three parametrizations nearly coincide up to 3.1 MeV which is the largest of the NCSM eigenenergies corresponding to the ^{7}He ground state included in the fit. In particular, these parametrizations are nearly indistinguishable in the resonance region. As a result, we obtain very similar resonance energies E_r and widths Γ with these three parametrizations (see Table I).

To further elucidate the convergence trends, we present in Fig. 1 also the phase shifts obtained from the NCSM results with $N_{\max}^i = 10$ and $\hbar\Omega^i$ ranging from 15 to 40 MeV together with the respective parametrization. We do not include these $N_{\max}^i = 10$ results in our selection of the NCSM eigenenergies used in Eq. (4) since the respective phase shifts $\delta_1(E_i)$ show

TABLE I. Convergence with increasing N_{max}^i of energy E_r (relative to the $n + {}^6$He threshold) and width Γ of the low-lying resonance $3/2_1^-$ in ^{7}He in the $n + {}^6$He channel in calculations with Daejeon16 NN interaction.

N_{max}^i	10	12	14	16	$12 \div 16$
E_r (MeV)	0.356	0.289	0.279	0.259	0.279
Γ (MeV)	0.155	0.127	0.127	0.123	0.131

more significant deviations from the common curve formed by the NCSM results in the three larger model spaces at the energies above the resonance region. However, the deviation of these parametrized $N_{max}^i = 10$ phase shifts from those obtained in larger model spaces is not large in the resonance region, which is of our primary interest. As a result, the $N_{max}^i = 10$ resonance parameters (see Table I) are within 30% of those obtained in larger model spaces.

We use the spread of the results presented in Table I (excluding those obtained with $N_{max}^i = 10$) to evaluate the uncertainties of the obtained resonance and low-energy scattering parameters. To justify these uncertainties, we perform also a few alternative selections of the NCSM energies E_i, e.g., we reduce the set of selected NCSM energies obtained with $N_{max}^i = 12$ and 14 by excluding the eigenstates above the resonant region or extend it by adding the results of calculations with additional $\hbar\Omega^i$ values producing the phase shifts $\delta_1(E_i)$ which deviate more from the common curve $12 \div 16$. Performing the phase shift parametrizations with these energy selections for $N_{max}^i = 12$ and 14 individually as well as parametrizing all these $N_{max}^i = 12$ and 14 results together with previously selected $N_{max}^i = 16$ energies, we obtain the spreads of the ^{7}He resonance parameters within the ranges shown in Table I. We adopt the same approach for the studies of all resonances in each channel. That is we use the results of the NCSM calculations in the three largest model spaces, calculate the resonance energy and width for each of these model spaces individually and for the combination of all selected results from these model spaces, and vary the ranges of energy selections in these model spaces to obtain the spreads of resonance energy and width. These spreads are used as uncertainty estimates while their central values are used as predictions for the energy and width. These predictions together with their uncertainties for various ^{7}He resonances based on our calculations with Daejeon16 and JISP16 NN interactions in two-body decay channels are summarized in Table II.

For comparison, we present in Table II also available resonance parameters from the studies within NCSMch [33], NCSMC [31,32], GSM [28], G-DMRG [29], and CS-COSM [30] (only widths which are given by numbers in Ref. [30]; the energies in this paper are shown only in figures where they are seen to be close to the respective experimental values). Note that all these other theoretical calculations where performed with different interactions with an exception of the NCSMch studies of Ref. [33] where the Daejeon16 was employed.

The low-lying $3/2_1^-$ resonance should be clearly related to the experimental $3/2_1^-$ resonance in ^{7}He. Daejeon16 underestimates while JISP16 overestimates both its energy and

width as compared with experiment; the NCSMC overestimates these resonance parameters while better estimates of this resonance were obtained in the GSM and G-DMRG studies. We note, however, that, contrary to our NCSM and the NCSMC *ab initio* calculations, both of which employ realistic NN interactions, the GSM and G-DMRG approaches utilize phenomenological n-^{4}He and NN interactions fit to spectra of light nuclei.

It is interesting to compare our results with Daejeon16 with those of the NCSMch studies. The NCSMch energy and width of this resonance are respectively nearly twice and three times larger than ours. The NCSMch approach of Ref. [33] is also based on the NCSM calculations (though in smaller model spaces) and utilizes the same Daejeon16 NN interaction. However the resonance parameters are obtained within NCSMch in a very different manner. In particular, the NCSMch resonance energy is obtained using a phenomenological exponential extrapolation A5 [10]. This extrapolation was designed for the bound states and has never been applied to resonances before the investigations of Ref. [33]. There are various phenomenological exponential extrapolations on the market, all of them are known to provide similar results. They are sometimes used to estimate resonance energies, in particular, our group was exploiting exponential extrapolations of Ref. [57] in the studies of resonances, e.g., in Refs. [58,59]. However, the applicability of the exponential extrapolations to estimation of resonant energies is unclear. We have shown in Ref. [60] that the NCSM eigenstates can differ essentially from the resonance energy when the resonance width is comparable to its energy. The resonance width is obtained within the NCSMch by matching the NCSM wave function with the respective channel wave function. The resulting width depends strongly on the resonance energy. For example, the authors of Ref. [33] mention that switching from their extrapolated energy of the $3/2_1^-$ resonance of 547 keV to the experimental energy of 430 keV results in the change of the width $\Gamma = 334$ keV to $\Gamma = 250$ keV. Thus the accuracy of the NCSMch results for this and other resonances is unclear, the authors of Ref. [33] do not report an estimate for it.

We note that our SS-HORSE approach to calculations of resonance energies and widths was carefully tested and justified in Refs. [13,15–17] using model problems where resonant parameters were calculated using other methods. Also, in contrast with the NCSMch, we present the estimates of our uncertainties, and we suggest that our predictions obtained with the same interaction to be reliable within our quoted uncertainties.

The $3/2^-$ phase shifts $\delta_1(E_i)$ discussed in Fig. 1 are given in the vicinity of the lowest $3/2_1^-$ resonance and the symbols in this figure correspond to the NCSM results for the ^{7}He ground state. Combining them with those corresponding to the $3/2^-$ first-excited state, we obtain the phase shifts revealing two resonances. The $3/2^-$ phase shifts in a larger energy scale together with other phase shifts in various two-body channels obtained with both JISP16 and Daejeon16 interactions are shown in Fig. 2. (The phase shift parametrizations shown in Fig. 1 and resonant parameters presented in Table I were

TABLE II. Energies E_r (relative to the $n + {}^6$He threshold) and widths Γ of resonant states in ^{7}He obtained with JISP16 and Daejeon16 in the channels $n + {}^6$He and $n + {}^6\text{He}^*(2^+)$ and our final predictions based on combining the results in these individual channels. Estimates of uncertainties of the quoted results are presented in parentheses. Results from GSM [28], G-DMRG [29], CS-COSM [30] (only widths), NCSMC [31,32], and NCSMch [33] calculations (in the NCSMch case the width in the line "Predictions" is obtained by summing widths in individual channels) together with experimental data are shown for comparison. All values are in MeV unless other units are specified.

Resonance			This work		Other theoretical works					Experiment
$J^\pi({}^7\text{He})$	$J^\pi({}^6\text{He})$		JISP16	Daejeon16	NCSMch	NCSMC	GSM	G-DMRG	CS-COSM	
$3/2_1^-$	0^+	E_r	0.665(12)	0.28(4)	0.547	0.71	0.39	0.460(7)		0.430(3) [24]
		Γ	0.57(4)	0.13(2)	0.334	0.30	0.178	0.142	0.048	0.182(5)
$1/2^+$	0^+	E_r			1.696					
		Γ			2.670					
$1/2^-$	0^+	E_r	2.7(8)	2.7(4)	2.318	2.39		1.811(6)		3.0(1) [25]; 3.5 [26]; 1.0(1) [27]
		Γ	5.0(6)	4.3(3)	2.071	2.89		2.150	2.77	2; 10; 0.75(8)
$5/2^-$	0^+	E_r	4.4(4)	3.63(16)	3.437					
		Γ	1.56(4)	1.36(3)	52 eV					
	2^+	E_r	3.85(15)	3.23(25)	3.437	3.13				
		Γ	2.5(2)	2.28(8)	1.941	1.07				
Predictions		E_r	4.1(7)	3.4(4)	3.437	3.13	3.47(2)	3.311(2)		3.36(9) [23]
		Γ	2.0(7)	1.8(5)	1.941	1.07	2.3(3)	1.726	1.80	1.99(17)
$3/2_2^-$	0^+	E_r	5.8(5)	5.0(3)	3.921					
		Γ	4.11(23)	2.84(24)	0.229					
	2^+	E_r	5.3(4)	4.4(4)	3.921					
		Γ	3.9(6)	3.9(3)	1.459					
Predictions		E_r	5.6(7)	4.7(7)	3.921					
		Γ	4.0(7)	3.4(8)	1.796				2.29	
$3/2^+$	0^+	E_r	6.5(1.6)	3.9(6)	3.492					
		Γ	5.9(1.0)	4.2(7)	83.4 keV					
	2^+	E_r			3.492					
		Γ			2.508					
Predictions		E_r	6.5(1.6)	3.9(6)	3.492					
		Γ	5.9(1.0)	4.2(7)	2.591					
$5/2^+$	0^+	E_r	6.7(1.5)	3.7(7)	3.564					
		Γ	5.8(8)	4.4(9)	0.258					
	2^+	E_r			3.564					
		Γ			2.251					
Predictions		E_r	6.7(1.5)	3.7(7)	3.564					
		Γ	5.8(8)	4.4(9)	2.512					

obtained by the fit to both the ground and first-excited states.) The higher lying $3/2_2^-$ resonance can decay not only via the $n + {}^6$He but also via the $n + {}^6\text{He}^*$ channel. The $3/2^-$ phase shifts in this second channel as well as phase shifts in some other states in the $n + {}^6\text{He}^*$ channels are also presented in Fig. 2. The energies and widths of the higher lying $3/2_2^-$ resonance obtained in the $n + {}^6\text{He}^*$ channel are close to those obtained in the $n + {}^6$He channel (see Table II) thus providing *a posteriori* justification of our treatment of coupled-channel resonances in different single channels. Our final prediction for the energy and width of the $3/2_2^-$ resonance and their uncertainties presented in Table II in the line "Predictions" are obtained by combining their spreads in different model spaces in both channels. As was already mentioned, the resonance widths obtained in any open channel as well as those quoted in the line "Predictions" are the total resonance widths associated with the decay in all possible channels.

Within the NCSMch approach, the energy of the $3/2_2^-$ resonance is obtained by extrapolating the first excited $3/2^-$ state obtained in the NCSM calculations which is independent from the decay channel and hence it appears the same in all open channels. However, the width obtained by matching the NCSM wave function with the scattering wave function, depends strongly on the considered channel and has a meaning of the partial width characterizing a probability of the decay in the respective channel. Therefore the NCSMch widths are very different in different channels and should not be compared with ours in each channel. In the line "Predictions" we present the NCSMch result for the total width by summing their widths in individual channels. These total NCSMch widths can be compared with ours.

Our results for the $3/2_2^-$ resonance show that this resonance is one of the candidates for the description of the experimentally observed resonance of unknown spin-parity at the energy
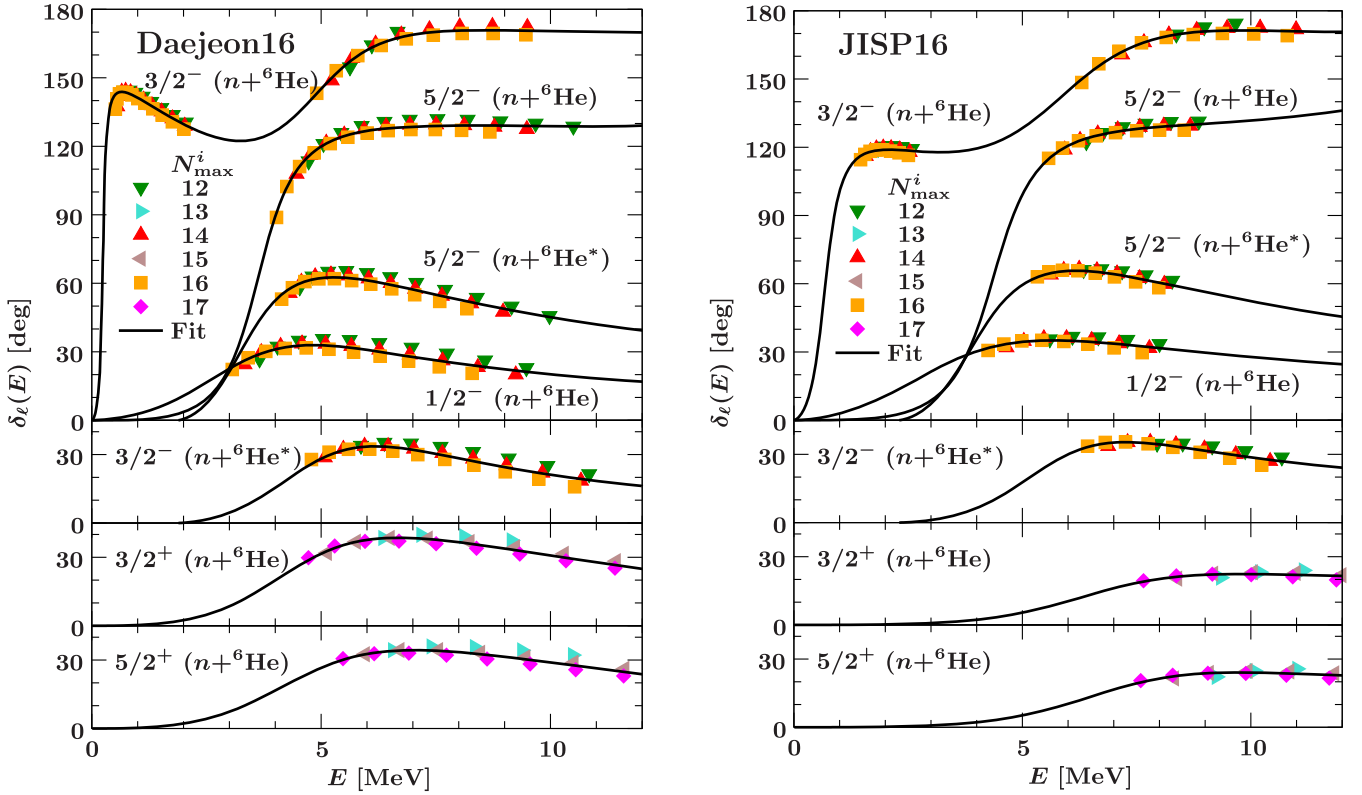
FIG. 2. Phase shifts in the $n + {}^6$He and in some $n + {}^6$He* channels with the Daejeon16 (left) and JISP16 (right) *NN* interactions. See Fig. 1 for details.

of 6.2 MeV with the width of 4 MeV. With Daejeon16 we obtain slightly smaller than experimental values for both its energy and width while JISP16 suggests energy and width closer to the experiment. This resonance has been studied theoretically before within the CS-COSM approach where its width was estimated approximately 30%–40% smaller than in our calculations. The NCSMch predicts the width of this resonance that is approximately two times smaller than ours; the NCSMch energy of this resonance is approximately 1 MeV smaller than ours.

The $1/2^+$ scattering phase shifts are found to decrease monotonically with energy without any signal of a resonant state in calculations with both Daejeon16 and JISP16 interactions. This result is in an agreement with the experimental data and the GSM predictions of Ref. [28] and the NCSMC predictions of Refs. [31,32]. From our parametrization of the effective range function $K_\ell(E)$ we obtain the scattering length $a_0 = 2.2(4)$ fm and the effective radius $r_0 = 2.1(1.1)$ fm for the $n + {}^6$He s-wave scattering.

The NCSMch studies of Ref. [33] propose the $1/2^+$ resonance at the energy $E_r = 1.696$ MeV with the width $\Gamma = 2.670$ MeV. We note here that, as was clearly demonstrated in Ref. [60], not all NCSM eigenstates should be associated with a resonance. Furthermore, the nonresonant scattering requires an appearance of some NCSM eigenstates for compatibility with the respective phase shifts. However, any NCSM eigenstate with positive energy with respect to any threshold can be matched with any open channel thus producing a theoretical prediction for a resonance which may not correspond to a

physical resonance. We suppose this is a drawback of the NCSMch approach which is in particular manifested in the case of the spurious $1/2^+$ resonance. We should note, however, that, according to the NCSMch predictions, the width of the $1/2^+$ resonance is much larger than its energy; thus this resonance will not be pronounced in a scattering experiment though may be detected in other reactions. This latter situation seems to occur in the case of the tetraneutron resonance where theory [19] and some experiments [21] suggest its width is larger than its energy.

The results for the $1/2^-$ resonance presented in Table II, contrary to the $3/2^-_2$ resonance, were obtained only in the channel $n + {}^6$He with the ^{6}He in the ground state. This resonance with the width of approximately 4 MeV or more should have the energy less than 1 MeV in the $n + {}^6$He* channel. Clearly, the $n + {}^6$He* phase shift will be nearly unaffected by the respective S-matrix resonant pole and hence it is not feasible to deduce the pole location from these phase shifts. We obtain the same energy of this resonance with Daejeon16 and JISP16 interactions which are slightly larger the results of other theoretical studies. The widths predicted by Daejeon16 and JISP16 are close to each other and approximately twice as large as those reported in other theoretical papers.

The experimental situation for the $1/2^-$ resonance is not clear. While the resonant energies of Refs. [25,26] are comparable, the widths are very different. The results of our work and other theoretical works for the resonance energy are in fair agreement with the neutron pickup and proton-removal reaction experiments [25]. However for the width of this resonance

we obtain a value that is approximately two times larger than in experiment [25] and approximately two times smaller than in experiment [26]. It is clear that our results do not support the interpretation of experimental data on one-neutron knock-out from ^{8}He of Ref. [27] advocating a low-lying ($E_r \approx 1$ MeV) narrow ($\Gamma < 1$ MeV) $1/2^-$ resonance in ^{7}He.

We obtain very similar results for the $5/2^-$ resonance in the $n + ^6$He and $n + ^6$He* channels as well as in calculations with Daejeon16 and JISP16 interactions. It may look surprising that we got a wide resonance in the $n + ^6$He channel where the orbital momentum $\ell = 3$ produces a high centrifugal barrier. We note again that the respective S-matrix resonant pole appears due to the coupling to other (closed) channels within the NCSM calculations and provides information about the total resonance width associated with all possible channels and may be very different from the partial width associated with the decay probability in one particular channel: The partial width in this channel of 52 eV obtained by the NCSMch is four orders of magnitude smaller. The large width of the $5/2^-$ resonance obtained in the $n + ^6$He channel with the high centrifugal barrier presents an impressive illustration of our proposition that we obtain total resonance width in each channel and justification of our approach. Our results for the energy and total width of the $5/2^-$ resonance are seen to be in good agreement with experiment and with the other available theoretical studies performed with different interactions and using different approaches.

Our results for the positive parity $3/2^+$ and $5/2^+$ resonances presented in Table II were obtained only in the $n + ^6$He channel. Note that these resonances are wide: Their widths obtained with Daejeon16 are more than 4 MeV and are larger than their energies; their widths obtained with JISP16 are close to 6 MeV and their energies are only slightly larger. In the $n + ^6$He* channel their energies become smaller than their widths. Therefore the resonances are not well resolved in this channel and we do not attempt to extract resonance parameters from the $n + ^6$He* phase shifts.

The $3/2^+$ and $5/2^+$ phase shifts are seen in Fig. 2 to nearly coincide and behave very similar to the $3/2^-$ phase shifts in the $n + ^6$He* channel that is most noticeable in the case of JISP16. Therefore we obtain the $3/2^+$ and $5/2^+$ resonances at energies close to that of the $3/2_2^-$ resonance but their widths are slightly larger. As a result, we suppose that the wide resonance observed around 6 MeV is formed as a complicated overlap of the $3/2_2^-$, $3/2^+$ and $5/2^+$ resonances. Note that this wide experimental resonance overlaps partially also with the $1/2^-$ and $5/2^-$ resonances.

The $3/2^+$ and $5/2^+$ resonances in the $n + ^6$He channel are characterized by the orbital momentum $\ell = 2$ or higher. Our phase shifts reflect the pole structure of the S matrix. However the orbital momentum $\ell = 2$ suggests a high centrifugal barrier. Therefore, the partial widths of these resonances in the $n + ^6$He channel are suppressed as is manifested in the NCSMch calculations. The total width of the $3/2^+$ and $5/2^+$ resonances in the NCSMch is dominated by contributions from other channels and appears to be much smaller. The energies of these resonances deduced in the NCSMch by exponential extrapolations is also smaller than our predictions with the same interaction and this distinction seems to be a

common feature for all wide resonances reported here. As a result, as seen in Table II, the NCSMch predicts comparable energies and similar total widths for the $5/2^-$, $3/2_2^-$, $3/2^+$, and $5/2^+$ resonances in ^{7}He. In other words, according to the NCSMch, the resonance in ^{7}He at the energy of 3.36 MeV with the width of approximately 2 MeV which spin-parity has a tentative assignment of $5/2^-$, appears as a complicated overlap of $5/2^-$, $3/2_2^-$, $3/2^+$, and $5/2^+$ resonances while, at the same time, there is no NCSMch indication of the wide resonance around 6 MeV.

We also examined the democratic four-body ^{4}He $+ n + n + n$ decay channels of all ^{7}He resonances with the exception of the lowest $3/2^-$ resonance which is below the respective threshold. In all cases we obtain resonances with energies close to those of respective two-body channels but with much smaller widths—at least three times smaller and sometimes more than an order of magnitude smaller. We conclude that the direct democratic four-body decays of ^{7}He resonances are suppressed due to the large hyperspherical centrifugal barrier $\mathscr{L}(\mathscr{L}+1)/\rho^2$ which dynamically pushes the system to form ^{6}He in the ground or excited resonant 2^+ states in the decay process. Therefore we do not present the results for these direct democratic decays of ^{7}He resonances in Table II and in the figures. The democratic decay channels should be treated separately and cannot be included in a multichannel calculation together with two-body decay channels since the democratic and two-body decay channel wave functions cannot be orthogonalized. We note, however, that the four-body ^{4}He $+ n + n + n$ decays of ^{7}He resonances occur as two-step processes in the $n + ^6$He* channels when ^{7}He first emits a neutron leaving the excited ^{6}He* 2^+ state which emits ^{4}He and two neutrons.

IV. SUMMARY AND CONCLUSIONS

Motivated by experimental uncertainties in the properties of the unbound nucleus ^{7}He, we solved for ^{7}He resonances using the SS-HORSE extension of the *ab initio* NCSM with the realistic Daejeon16 and JISP16 NN interactions.

The four-body ^{4}He $+ n + n + n$ direct democratic decays of ^{7}He resonances were found to be suppressed. All examined resonances may decay via the $n + ^6$He channel with ^{6}He in the ground state and, with an exception of the $3/2_1^-$ resonance, via the $n + ^6$He* channel with ^{6}He in the excited 2^+ state. That excited 2^+ state subsequently decays emitting ^{4}He and two neutrons thus resulting in the four-body ^{4}He $+ n + n + n$ final state. The resonance energies and widths quoted in Table II simulate the results of a multichannel calculation; in particular, the widths are a reasonable approximation to the total widths of the resonances as confirmed by similar results obtained with different open channels.

Our predictions for the low-lying narrow $3/2_1^-$ resonance are in reasonable agreement with experiment and with results quoted in the GSM [28], G-DMRG [29], NCSMch [33], and NCSMC [31,32] theoretical studies.

The $1/2^-$ resonance is predicted at the energy in reasonable agreement with the NCSMC [31,32] and NCSMch [33] calculations and results of experiments of Refs. [25,26] and about 1 MeV higher than suggested by the G-DMRG studies

[29]. The width of this resonance is found to be more than 4 MeV which is larger than the width predicted in the NCSMch, NCSMC, G-DMRG, and CS-COSM [30] calculations and larger than the experimental width of Ref. [25]. However, our $1/2^-$ resonance width is less than half the experimental width of Ref. [26]. Our results as well as those of the above-mentioned NCSMch, NCSMC, GSM, G-DMRG, and CS-COSM calculations disagree with the indication of a low-lying narrow resonant $1/2^-$ state suggested in Ref. [27].

Our predictions for the relatively wide $5/2^-$ resonance are in reasonable agreement with experiment and with results quoted in the NCSMC, NCSMch, GSM, G-DMRG, and CS-COSM studies.

We found a wide $3/2_2^-$ resonance around the energy of 5 MeV which was also predicted in the CS-COSM calculations [30] as well as wide $3/2^+$ and $5/2^+$ resonances at nearby energies. Based on our results, it appears reasonable to propose that the observed resonance at the energy of 6.2 MeV with the width of 4 MeV of unknown spin-parity mentioned in the compilation of Ref. [23] is formed as an overlap of the $3/2_2^-$ resonance with $3/2^+$ and $5/2^+$ resonances.

We do not find a resonance in the $1/2^+$ state which is consistent with the findings of the GSM [28], NCSMC [31,32] studies and with the experimental situation.

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