Spin Boltzmann equation for nonrelativistic spin-1/2 fermions

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We derive the spin Boltzmann equations for spin-1/2 fermions in a nonrelativistic model with four-fermion contact interaction which conserves spin degrees of freedom. A great advantage of the model is that the spin matrix elements in collision terms can be completely worked out and be put into such a compact form that one can clearly see how spins are coupled in particle scatterings. A semiclassical expansion in the Planck constant has been made and the on-shell part of the spin Boltzmann equation up to the next-to-leading order is derived. At the leading order the equilibrium spin distribution can be obtained from the vanishing of the collision term for the spin density. The spin chemical potential emerges as a natural consequence of spin conservation. The off-shell part of the spin Boltzmann equation is also discussed. The work can be extended to more sophisticated interaction such as nuclear force in order to apply to spin polarization phenomena in heavy-ion collisions at low energies.

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I. INTRODUCTION

Very large orbital angular momenta (OAM) are generated in noncentral heavy-ion collisions which can be partially converted into the spin polarization of hadrons along the direction of OAM or with respect to the reaction plane [1–3]. This effect is called the global spin polarization or global polarization for short. The global polarization of Λ hyperons (including $\overline{\Lambda}$) has been measured for the first time by the STAR collaboration in Au + Au collision at 200 GeV and lower energies [4,5]. The data show that the global polarization is about $1.08 \pm 0.15\%$ (Λ) and $1.38 \pm 0.30\%$ ($\overline{\Lambda}$) with a decreasing behavior with the collision energy.

Several theoretical methods have been developed for the global polarization. These theoretical methods can be roughly put into three categories. One category is related to the quantum statistical theory for particle systems with spin degrees of freedom in equilibrium [6–11] [for a recent review, see, e.g., Ref. [12]]. One category is the microscopic transport theory based on kinetic or Boltzmann equations for spin degrees of freedom [13–23] in terms of covariant Wigner functions [24–28] (see, e.g., Refs. [29,30] for recent reviews). Another category is relativistic spin hydrodynamics [10,31–39] (see Ref. [32] for a review), which incorporates spin degrees of freedom into conventional relativistic hydrodynamics applied to the strong interaction matter in heavy-ion collisions [40–43]. There are many phenomenological studies of the global and local polarization using these theoretical methods to describe experimental data [38,44–59] (for recent reviews, see, e.g., Refs. [60–62]).

Recently the HADES collaboration measured the global Λ polarization in Ag + Ag collisions at 2.55 GeV and Au + Au collisions at 2.4 GeV [63], while the STAR collaboration measured the same observable in Au + Au collisions at 3 GeV [64]. Combining all these low energy measurements with the high energy ones, the Λ polarization is observed to continue the increasing trend with decreasing collision energy down to 2.4 GeV. At these collision energies of $\mathcal{O}(m_N)$, where m_N is the nucleon mass, the relativistic effect is small and nonrelativistic theory can be a proper approximation. Experimental data can be described by models such as ultrarelativistic quantum molecular dynamics (UrQMD) and Boltzman-Uehling-Uhlenbeck (BUU) models [65,66]. These models are based on Boltzmann equations for hadrons which do not incorporate spin degrees of freedom.

In this paper, we will derive the spin Boltzmann equation for spin-1/2 fermions in a nonrelativistic model with four-fermion contact interaction similar to the Nambu–Jona-Lasinio (NJL) model in relativistic theory [67,68]. The nonrelativistic model has a feature that the particle's spin is decoupled from its momentum and is conserved in the interaction. This is very different from a relativistic system in which the particle's spin and momentum are entangled. The method is based on a previous work by one of us about the relativistic system of spin-1/2 fermions [19]. A great advantage of the current nonrelativistic model is that the spin matrix elements in collision terms can be completely worked out and be put into such a compact form that one can clearly see how spins are coupled in two-to-two scatterings of particles. This is not the case in the relativistic theory [19]. The current work can be extended to nucleon-nucleon interaction via nuclear force and then can be applied to the global polarization in heavy-ion collisions at low energies.

This paper is organized as follows. In Sec. II, we briefly introduce Green's functions in the closed-time-path (CTP) formalism. In Sec. III, we derive the Kadanoff-Baym (KB) equation from the Schwinger-Dyson equation in a quasiparticle approximation. In Sec. IV, we derive the on-shell part of the spin Boltzmann equation at the leading and

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next-to-leading order in $\hbar$. In Sec. V, the off-shell part of the KB equation is discussed. A summary of the results is given in the final section.

II. GREEN'S FUNCTIONS FOR FERMIONS IN CTP FORMALISM

We consider a nonrelativistic system of spin-1/2 fermions. A general form of the Lagrangian with four-fermion interaction can be written as [69]

$$\begin{aligned}\mathcal{L} = &\int d^3\mathbf{x}\psi_\alpha^\dagger(t,\mathbf{x})\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\right)\psi_\alpha(t,\mathbf{x})\\ &-\frac{1}{2}\int d^3\mathbf{x}d^3\mathbf{x}'\psi_\alpha^\dagger(t,\mathbf{x})\psi_\beta^\dagger(t,\mathbf{x}')\\ &\times V_{\alpha\alpha',\beta\beta'}(\mathbf{x},\mathbf{x}')\psi_{\beta'}(t,\mathbf{x}')\psi_{\alpha'}(t,\mathbf{x}),\end{aligned} \tag{1}$$

where $\alpha,\beta=\pm$ denote spin states, $V_{\alpha\alpha',\beta\beta'}(\mathbf{x},\mathbf{x}')$ is the spin-dependent potential, and repeated indices imply a summation if not explicitly stated. Let us consider the contact interaction of the NJL type [67,68],

$$\begin{aligned}\mathcal{L}_{\rm int} = &-g_0\int d^3\mathbf{x}[\psi_\alpha^\dagger(t,\mathbf{x})\psi_\alpha(t,\mathbf{x})]^2\\ &-g_\sigma\sum_i\int d^3\mathbf{x}\big[\psi_\alpha^\dagger(t,\mathbf{x})\sigma^i_{\alpha\beta}\psi_\beta(t,\mathbf{x})\big]^2,\end{aligned} \tag{2}$$

which corresponds to the potential in the form

$$\begin{aligned}&V_{\alpha\alpha',\beta\beta'}(\mathbf{x},\mathbf{x}')\\ &\quad= 2\delta^{(3)}(\mathbf{x}-\mathbf{x}')\left(g_0\delta_{\alpha\alpha'}\delta_{\beta\beta'}+g_\sigma\sum_i\sigma^i_{\alpha\alpha'}\sigma^i_{\beta\beta'}\right).\end{aligned} \tag{3}$$

In Eqs. (2) and (3), σ^i (or equivalently σ_i) with $i=1,2,3$ are Pauli matrices in spinor space which are also denoted as $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$. We can see that the above nonrelativistic interaction involves two types of vertices: 1 and σ^i. In contrast, the relativistic NJL model [19] includes fives types of vertices that correspond to 16 Clifford algebra: 1, γ^μ, γ^5, $\gamma^5\gamma^\mu$, $\sigma^{\mu\nu}$, for Dirac spinors. The most significant feature of the relativistic theory is that the spin is entangled with momentum, while it is not so in the nonrelativistic theory. How the nonrelativistic and relativistic NJL models are related is a complicated problem which needs a systematic nonrelativistic expansion in small parameters such as the momentum to mass ratio. This is beyond the scope of the current work and deserves a future study.

The Lagrangian (1) with the interaction part (2) is invariant under the global SU(2) transformation defined as

$$\psi'_\alpha = U_{\alpha\beta}\psi_\beta,\ \ \psi'^\dagger_\alpha = \psi^\dagger_\beta U^\dagger_{\beta\alpha}, \tag{4}$$

which means the spin is conserved. The kinetic term of the Lagrangian (1) and the g_0 term of the interaction Lagrangian (2) are obviously invariant under the SU(2) transformation. Let us look at the g_σ term

$$\begin{aligned}\big[\psi'^\dagger_\alpha\sigma^i_{\alpha\beta}\psi'_\beta\big]\big[\psi'^\dagger_{\alpha_1}\sigma^i_{\alpha_1\beta_1}\psi'_{\beta_1}\big] &= \big[\psi^\dagger_\gamma U^\dagger_{\gamma\alpha}\sigma^i_{\alpha\beta}U_{\beta\lambda}\psi_\lambda\big]\\ &\quad\times\big[\psi^\dagger_{\gamma_1}U^\dagger_{\gamma_1\alpha_1}\sigma^i_{\alpha_1\beta_1}U_{\beta_1\lambda_1}\psi_{\lambda_1}\big]\\ &= V_{ij}V_{ik}\big[\psi^\dagger_\gamma\sigma^j_{\gamma\lambda}\psi_\lambda\big]\big[\psi^\dagger_{\gamma_1}\sigma^k_{\gamma_1\lambda_1}\psi_{\lambda_1}\big]\\ &= \big[\psi^\dagger_\gamma\sigma^j_{\gamma\lambda}\psi_\lambda\big]\big[\psi^\dagger_{\gamma_1}\sigma^j_{\gamma_1\lambda_1}\psi_{\lambda_1}\big],\end{aligned} \tag{5}$$

where we have used the property of SU(2) transformation $U^\dagger\sigma_iU = V_{ij}\sigma_j$ with V_{ij} denoting an SO(3) matrix. Corresponding to the SU(2) invariance of the Lagrangian, the Noether charge and current for spin are given by

$$Q_i^{\rm spin} = \frac{\hbar}{2}\psi^\dagger\sigma_i\psi,\ \ J_{ij}^{\rm spin} = \frac{i\hbar^2}{4m}(\nabla_j\psi^\dagger\sigma_i\psi-\psi^\dagger\sigma_i\nabla_j\psi), \tag{6}$$

which satisfy the conservation equation

$$\frac{\partial}{\partial t}Q_i^{\rm spin}+\nabla_jJ_{ij}^{\rm spin}=0. \tag{7}$$

Note that the Noether charge is a spin vector while the Noether current is a tensor.

The fermion fields can be quantized as

$$\begin{aligned}\psi(x) &= \int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}e^{-ip\cdot x/\hbar}\sum_{s=\pm}a(s,\mathbf{p})\chi(s),\\ \psi^\dagger(x) &= \int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}e^{ip\cdot x/\hbar}\sum_{s=\pm}a^\dagger(s,\mathbf{p})\chi^\dagger(s),\end{aligned} \tag{8}$$

where $x\equiv(x_0,\mathbf{x})\equiv(t,\mathbf{x})$, $p\equiv(p_0,\mathbf{p})\equiv(\omega_p,\mathbf{p})$ with $\omega_p=\mathbf{p}^2/(2m)$, $p\cdot x\equiv\omega_pt-\mathbf{p}\cdot\mathbf{x}$, $a(s,\mathbf{p})$ and $a^\dagger(s,\mathbf{p})$ are annihilation and creation operators associated with $\mathbf{p}$ and the spin state s, respectively, and $\chi(s)$ is the spin state (Pauli spinor) which satisfies $(\mathbf{n}\cdot\boldsymbol{\sigma})\chi(s)=s\chi(s)$ with $\mathbf{n}$ being the spin quantum direction $\mathbf{n}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. The anticommutators of $a(s,\mathbf{p})$ and $a^\dagger(s,\mathbf{p})$ are given by

$$\begin{aligned}\{a(s_1,\mathbf{p}_1),a^\dagger(s_2,\mathbf{p}_2)\} &= (2\pi\hbar)^3\delta_{s_1s_2}\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2),\\ \{a(s_1,\mathbf{p}_1),a(s_2,\mathbf{p}_2)\} &= \{a^\dagger(s_1,\mathbf{p}_1),a^\dagger(s_2,\mathbf{p}_2)\}=0,\end{aligned} \tag{9}$$

which lead to equal-time anticommutators for fermion fields

$$\begin{aligned}&\{\psi_\alpha(t,\mathbf{x}),\psi^\dagger_\beta(t,\mathbf{x}')\}=\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{x}'),\\ &\{\psi_\alpha(t,\mathbf{x}),\psi_\beta(t,\mathbf{x}')\}=\{\psi^\dagger_\alpha(t,\mathbf{x}),\psi^\dagger_\beta(t,\mathbf{x}')\}=0.\end{aligned} \tag{10}$$

Now we define the two-point Green's function in the CTP formalism [70,71] (see, e.g., Refs. [72–75] for reviews) as

$$G^C_{\alpha\beta}(x_1,x_2)=\langle T_C[\psi_\alpha(x_1)\psi^\dagger_\beta(x_2)]\rangle, \tag{11}$$

where T_C denotes the time-ordered product on the CTP, and the angular brackets denote averages weighted by the density operator at the initial time $\rho(t_0)$. Note that our definition for the Green's function is different from Ref. [69] without the additional factor $i=\sqrt{-1}$. Depending on whether the two space-time points are on the positive or negative time branch, there are four types of two-point functions

$$\begin{aligned}G^F_{\alpha\beta}(x_1,x_2) &= G^{++}_{\alpha\beta}(x_1,x_2)=\langle T\psi_\alpha(x_1)\psi^\dagger_\beta(x_2)\rangle,\\ G^{\overline{F}}_{\alpha\beta}(x_1,x_2) &= G^{--}_{\alpha\beta}(x_1,x_2)=\langle T_A\psi_\alpha(x_1)\psi^\dagger_\beta(x_2)\rangle,\end{aligned}$$

$$G^{<}_{\alpha\beta}(x_1,x_2) = G^{+-}_{\alpha\beta}(x_1,x_2) = -\langle\psi^{\dagger}_{\beta}(x_2)\psi_{\alpha}(x_1)\rangle,$$
$$G^{>}_{\alpha\beta}(x_1,x_2) = G^{-+}_{\alpha\beta}(x_1,x_2) = \langle\psi_{\alpha}(x_1)\psi^{\dagger}_{\beta}(x_2)\rangle, \quad (12)$$

where $+/-$ stands for the positive/negative time branch, respectively, and T and T_A denote the time-ordered and reverse-time-ordered product, respectively. Note that only three out of four types of two-point functions in Eq. (12) are independent due to the identity

$$G^F + G^{\overline{F}} = G^< + G^>. \quad (13)$$

We can choose $(G^F, G^<, G^>)$ as three independent two-point functions. Equivalently we can also choose (G^R, G^A, G^S) as independent ones

$$G^R = G^F - G^< = \theta(t_1 - t_2)(G^> - G^<),$$
$$G^A = G^F - G^> = \theta(t_2 - t_1)(G^< - G^>),$$
$$G^S = G^< + G^>, \quad (14)$$

where G^R and G^A are retarded and advanced two-point functions.

The Wigner function is the building-block of the quantum transport theory since it is the quantum analog of the particle distribution in phase space. The Wigner function can be obtained by taking the Fourier transformation with respect to the distance of two space-time points in a two-point function. Then the Wigner functions for $G^{<}_{\alpha\beta}(x_1,x_2)$ and $G^{>}_{\alpha\beta}(x_1,x_2)$ are defined by

$$G^{<}_{\alpha\beta}(x,p) = \int d^4y e^{ip\cdot y/\hbar} G^{<}_{\alpha\beta}(x_1,x_2)$$
$$= -\int d^4y e^{ip\cdot y/\hbar}\left\langle\psi^{\dagger}_{\beta}\left(x-\frac{y}{2}\right)\psi_{\alpha}\left(x+\frac{y}{2}\right)\right\rangle,$$
$$G^{>}_{\alpha\beta}(x,p) = \int d^4y e^{ip\cdot y/\hbar} G^{>}_{\alpha\beta}(x_1,x_2)$$
$$= \int d^4y e^{ip\cdot y/\hbar}\left\langle\psi_{\alpha}\left(x+\frac{y}{2}\right)\psi^{\dagger}_{\beta}\left(x-\frac{y}{2}\right)\right\rangle, \quad (15)$$

where $p\cdot y \equiv \omega_p y_0 - \mathbf{p}\cdot\mathbf{y}$, $x_1 = x + y/2$, and $x_2 = x - y/2$.

In nonequilibrium, we do not know $G^{<}_{\alpha\beta}(x,p)$ and $G^{>}_{\alpha\beta}(x,p)$ exactly due to unknown ensemble averages. However, using Eq. (8), we can make an ansatz for their forms in a power expansion of $\hbar$. Inserting Eq. (8) to Eq. (15), the leading order contributions to the Wigner functions have the form

$$G^{<(0)}_{\alpha\beta}(x,p)$$
$$= -(2\pi\hbar)\delta\left(p_0 - \frac{\mathbf{p}^2}{2m}\right)\sum_{s_1,s_2=\pm}\chi_{\alpha}(s_1)\chi^{\dagger}_{\beta}(s_2)f^{(0)}_{s_1s_2}(x,\mathbf{p}),$$
$$G^{>(0)}_{\alpha\beta}(x,p)$$
$$= (2\pi\hbar)\delta\left(p_0 - \frac{\mathbf{p}^2}{2m}\right)\sum_{s_1,s_2=\pm}\chi_{\alpha}(s_1)\chi^{\dagger}_{\beta}(s_2)$$
$$\times\left[\delta_{s_1s_2} - f^{(0)}_{s_1s_2}(x,\mathbf{p})\right], \quad (16)$$

where $p \equiv (p_0,\mathbf{p}) \equiv (\omega,\mathbf{p})$ with p_0 or ω being an independent variable, and the matrix valued spin-dependent distribution (MVSD) at $\mathcal{O}(\hbar^0)$ is defined as [19]

$$f^{(0)}_{s_1s_2}(x,\mathbf{p}) = \int\frac{d^3\mathbf{q}}{(2\pi\hbar)^3}\exp\left[i\frac{1}{\hbar}\left(-\frac{\mathbf{p}\cdot\mathbf{q}}{m}t + \mathbf{q}\cdot\mathbf{x}\right)\right]$$
$$\times\left\langle a^{\dagger}\left(s_2,\mathbf{p}-\frac{\mathbf{q}}{2}\right)a\left(s_1,\mathbf{p}+\frac{\mathbf{q}}{2}\right)\right\rangle. \quad (17)$$

One can check that $G^{\lessgtr(0)}(x,p)$ are Hermitian matrices because $f^{(0)*}_{s_1s_2} = f^{(0)}_{s_2s_1}$, i.e., $f^{(0)}$ is a Hermitian matrix in spin space. The first order contributions at $\mathcal{O}(\hbar^1)$ are assumed to have the form

$$\hbar G^{<(1)}_{\alpha\beta}(x,p) = \hbar G^{>(1)}_{\alpha\beta}(x,p) = -\hbar(2\pi\hbar)\delta\left(p_0 - \frac{\mathbf{p}^2}{2m}\right)$$
$$\times\sum_{s_1,s_2=\pm}\chi_{\alpha}(s_1)\chi^{\dagger}_{\beta}(s_2)f^{(1)}_{s_1s_2}(x,\mathbf{p}), \quad (18)$$

where $f^{(1)}_{s_1s_2}(x,\mathbf{p})$ is unknown and can be determined by solving the evolution equations. We assume $f^{(1)*}_{s_1s_2} = f^{(1)}_{s_2s_1}$, which means $f^{(1)}$ and then $G^{\lessgtr(1)}(x,p)$ are Hermitian matrices in spin space. Note that in Eq. (18) we only include on-shell contributions to $G^{\lessgtr(1)}(x,p)$ which are proportional to $\delta[p_0 - \mathbf{p}^2/(2m)]$. In principle there are off-shell contributions which are proportional to $\delta'[p_0 - \mathbf{p}^2/(2m)]$. We will address off-shell contributions in Sec. V separately.

Combining Eqs. (16) and (18) we obtain $G^{\lessgtr}(x,p)$ up to $\mathcal{O}(\hbar)$,

$$G^<(x,p) = G^{<(0)}(x,p) + \hbar G^{<(1)}(x,p) + \mathcal{O}(\hbar^2)$$
$$= -(2\pi\hbar)\delta\left(p_0 - \frac{\mathbf{p}^2}{2m}\right)$$
$$\times\sum_{s_1,s_2=\pm}\chi_{\alpha}(s_1)\chi^{\dagger}_{\beta}(s_2)f_{s_1s_2}(x,\mathbf{p}),$$
$$G^>(x,p) = G^{>(0)}(x,p) + \hbar G^{>(1)}(x,p) + \mathcal{O}(\hbar^2)$$
$$= (2\pi\hbar)\delta\left(p_0 - \frac{\mathbf{p}^2}{2m}\right)$$
$$\times\sum_{s_1,s_2=\pm}\chi_{\alpha}(s_1)\chi^{\dagger}_{\beta}(s_2)[\delta_{s_1s_2} - f_{s_1s_2}(x,\mathbf{p})]. \quad (19)$$

Here, the MVSD $f_{s_1s_2}(x,\mathbf{p})$ is given by

$$f_{s_1s_2}(x,\mathbf{p}) = f^{(0)}_{s_1s_2}(x,\mathbf{p}) + \hbar f^{(1)}_{s_1s_2}(x,\mathbf{p}) + \mathcal{O}(\hbar^2)$$
$$= \delta_{s_1s_2}\overline{f}(x,\mathbf{p}) + \boldsymbol{\tau}_{s_1s_2}\cdot\mathbf{t}(x,\mathbf{p}) + \mathcal{O}(\hbar^2), \quad (20)$$

where $\boldsymbol{\tau} = (\tau_1,\tau_2,\tau_3)$ and $\mathbf{t}(x,\mathbf{p}) = [t_1(x,\mathbf{p}), t_2(x,\mathbf{p}), t_3(x,\mathbf{p})]$ denote Pauli matrices and the polarization vector in (s_1,s_2) space, respectively. It is important to note about the difference between $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, the former is defined in (s_1,s_2) space or spin space while the latter is defined in spinor space. It can be easily checked that $G^{\lessgtr}(x,p)$ are Hermitian. For convenience we can write $G^{\lessgtr}(x,p)$ as

$$G^{\lessgtr}(x,p) = -(2\pi\hbar)\delta\left(p_0 - \frac{\mathbf{p}^2}{2m}\right)g^{\lessgtr}(x,p), \quad (21)$$

where $g^{\lessgtr}(x,p)$ denotes the part of $G^{\lessgtr}(x,p)$ without the delta function and it can be decomposed in terms of 1 and Pauli

matrices $\boldsymbol{\sigma}$ in spinor space,

$$\begin{aligned} g^<(x,p) &\equiv \sum_{s_1,s_2=\pm} \chi(s_1)\chi^\dagger(s_2) f_{s_1s_2}(x,\mathbf{p}) \\ &= \overline{f}(x,\mathbf{p}) + \boldsymbol{\sigma}\cdot\mathbf{S}(x,\mathbf{p}), \\ g^>(x,p) &\equiv -\sum_{s_1,s_2=\pm} \chi(s_1)\chi^\dagger(s_2)[\delta_{s_1s_2} - f_{s_1s_2}(x,\mathbf{p})] \\ &= \overline{f}(x,\mathbf{p}) + \boldsymbol{\sigma}\cdot\mathbf{S}(x,\mathbf{p}) - 1. \end{aligned} \tag{22}$$

These components can be extracted by taking traces

$$\overline{f}(x,\mathbf{p}) = \tfrac{1}{2}\mathrm{Tr}[g^<(x,p)], \quad \mathbf{S}(x,\mathbf{p}) = \tfrac{1}{2}\mathrm{Tr}[\boldsymbol{\sigma} g^<(x,p)]. \tag{23}$$

So $\mathbf{S}(x,\mathbf{p})$ is given by

$$\begin{aligned} \mathbf{S}(x,\mathbf{p}) &= \frac{1}{2}\sum_{s_1,s_2=\pm} \chi^\dagger(s_2)\boldsymbol{\sigma}\chi(s_1) f_{s_1s_2}(x,\mathbf{p}) \\ &= \frac{1}{2}\sum_{s_1,s_2=\pm} (\mathbf{n}_1\tau_1 + \mathbf{n}_2\tau_2 + \mathbf{n}_3\tau_3)_{s_2s_1} f_{s_1s_2}(x,\mathbf{p}) \\ &= \mathbf{n}_i t_i(x,\mathbf{p}), \end{aligned} \tag{24}$$

where we have used

$$\begin{aligned} \chi^\dagger(s_2)\boldsymbol{\sigma}\chi(s_1) &= (\mathbf{n}_1\tau_1 + \mathbf{n}_2\tau_2 + \mathbf{n}_3\tau_3)_{s_2s_1} \\ &= \begin{pmatrix} \mathbf{n}_3 & \mathbf{n}_1 - i\mathbf{n}_2 \\ \mathbf{n}_1 + i\mathbf{n}_2 & -\mathbf{n}_3 \end{pmatrix}_{s_2s_1}. \end{aligned} \tag{25}$$

Here, $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ are three basis vectors given by

$$\begin{aligned} \mathbf{n}_1 &= (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta), \\ \mathbf{n}_2 &= (-\sin\phi, \cos\phi, 0), \\ \mathbf{n}_3 &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \end{aligned} \tag{26}$$

Note that $\mathbf{n}_3 = \mathbf{n}$ is the direction of the spin quantization.

III. KADANOFF-BAYM EQUATION

The time evolution of a many-body quantum system in nonequilibrium is described by the KB equation [76] (see, e.g., Ref. [77] for a review). The KB equations can be derived from the Dyson-Schwinger equations for two-point Green functions on the CTP,

$$\begin{aligned} -i\left(i\hbar\frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m}\nabla_{x1}^2\right)G_C(x_1,x_2) &= \hbar\delta_C^{(4)}(x_1-x_2) + \hbar\int_C d^4x'\Sigma_C(x_1,x')G_C(x',x_2), \\ -iG_C(x_1,x_2)\left(-i\hbar\frac{\overleftarrow{\partial}}{\partial t_2} + \frac{\hbar^2}{2m}\overleftarrow{\nabla}_{x2}^2\right) &= \hbar\delta_C^{(4)}(x_1-x_2) + \hbar\int_C d^4x' G_C(x_1,x')\Sigma_C(x',x_2), \end{aligned} \tag{27}$$

where the index C stands for the CTP, $\delta_C^{(4)}(x_1-x_2)$ and $\Sigma_C(x_1,x')$ are the δ function and self-energy on the CTP, respectively. Note that G_C and Σ_C are 2×2 matrices in the spinor space. In the case that (t_1,t_2) are on $(+,-)$ time branch, Eq. (27) can be put into the conventional coordinate form

$$-i\left(i\hbar\frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m}\nabla_{x1}^2\right)G^<(x_1,x_2) = \hbar\int d^4x'[\Sigma^R(x_1,x')G^<(x',x_2) + \Sigma^<(x_1,x')G^A(x',x_2)], \tag{28}$$

$$-i\left(-i\hbar\frac{\partial}{\partial t_2} + \frac{\hbar^2}{2m}\nabla_{x2}^2\right)G^<(x_1,x_2) = \hbar\int d^4x'[G^R(x_1,x')\Sigma^<(x',x_2) + G^<(x_1,x')\Sigma^A(x',x_2)], \tag{29}$$

where Σ^R and Σ^A are the retarded and advanced self-energy, respectively. Performing the Wigner transform for above equations, we obtain

$$\left(i\hbar\frac{1}{2}\partial_t + i\frac{\hbar}{2m}\mathbf{p}\cdot\nabla_x + p_0 - \frac{\mathbf{p}^2}{2m} + \frac{\hbar^2}{8m}\nabla_x^2\right)G^<(x,p) = I_{\mathrm{coll}} \tag{30}$$

and

$$\left(-i\hbar\frac{1}{2}\partial_t - i\frac{\hbar}{2m}\mathbf{p}\cdot\nabla_x + p_0 - \frac{\mathbf{p}^2}{2m} + \frac{\hbar^2}{8m}\nabla_x^2\right)G^<(x,p) = I_{\mathrm{coll}}^\dagger, \tag{31}$$

where I_{coll} and $I_{\mathrm{coll}}^\dagger$ are given by

$$I_{\mathrm{coll}} = i\hbar[\Sigma^R(x,p)G^<(x,p) + \Sigma^<(x,p)G^A(x,p)] + \tfrac{1}{2}\hbar^2[\{\Sigma^R(x,p), G^<(x,p)\}_{P.B.} + \{\Sigma^<(x,p), G^A(x,p)\}_{P.B.}], \tag{32}$$

$$I_{\mathrm{coll}}^\dagger = i\hbar[G^R(x,p)\Sigma^<(x,p) + G^<(x,p)\Sigma^A(x,p)] + \tfrac{1}{2}\hbar^2[\{G^R(x,p), \Sigma^<(x,p)\}_{P.B.} + \{G^<(x,p), \Sigma^A(x,p)\}_{P.B.}]. \tag{33}$$

Here, the Poisson bracket of two matrices is defined as

$$\{A,B\}_{\mathrm{PB}} \equiv \partial_t A\partial_{p_0}B - \partial_{p_0}A\partial_t B - (\nabla_x A\cdot\nabla_p B - \nabla_p A\cdot\nabla_x B). \tag{34}$$

We see that $I_{\mathrm{coll}}^\dagger$ can be obtained by interchange of Σ and G from I_{coll}, and vice versa. With the relations for $O = G, \Sigma$,

$$[O^R]^\dagger = -O^A, \quad [O^{\lessgtr}]^\dagger = [O^{\lessgtr}], \tag{35}$$

one can check that $I^{\dagger}_{\rm coll}$ is really the Hermitian conjugate of $I_{\rm coll}$. Taking the sum and difference of Eqs. (32) and (33), we obtain an equation for the dispersion relation or the on-shell equation

$$\left(p_0 - \frac{\mathbf{p}^2}{2m} + \frac{\hbar^2}{8m}\nabla_x^2\right)G^<(x,p) = \frac{1}{2}(I_{\rm coll} + I^{\dagger}_{\rm coll}), \tag{36}$$

and the evolution equation

$$\hbar\left(\partial_t + \frac{\mathbf{p}}{m}\cdot\nabla_x\right)G^<(x,p) = -i(I_{\rm coll} - I^{\dagger}_{\rm coll}). \tag{37}$$

Equations (36),(37) are some of the main results in this paper and the starting point for the derivation of Boltzmann equations.

In the quasiparticle picture, the retarded and advanced self-energies and two-point functions can be approximated as

$$O^{R/A}(x,p) = \frac{1}{2\pi i}\int dk_0 \frac{1}{k_0 - p_0 \mp i\epsilon}[O^>(x,k_0,\mathbf{p}) - O^<(x,k_0,\mathbf{p})] = \pm\frac{1}{2}[O^>(x,p) - O^<(x,p)] + O^{R/A}_{\rm pr}, \tag{38}$$

where $O \equiv G, \Sigma$, and the principal part $O^{R/A}_{\rm pr}$ is related to off-shell effects that modify the dispersion relation of the quasiparticle which we will not consider in this paper. Using Eq. (38), the collision terms (32) and (33) can be simplified as

$$I_{\rm coll} = \tfrac{1}{2}i\hbar[\Sigma^>(x,p)G^<(x,p) - \Sigma^<(x,p)G^>(x,p)] + \tfrac{1}{4}\hbar^2[\{\Sigma^>(x,p), G^<(x,p)\}_{P.B.} - \{\Sigma^<(x,p), G^>(x,p)\}_{P.B.}] \tag{39}$$

and

$$I^{\dagger}_{\rm coll} = \tfrac{1}{2}i\hbar[G^>(x,p)\Sigma^<(x,p) - G^<(x,p)\Sigma^>(x,p)] + \tfrac{1}{4}\hbar^2[\{G^>(x,p), \Sigma^<(x,p)\}_{P.B.} - \{G^<(x,p), \Sigma^>(x,p)\}_{P.B.}]. \tag{40}$$

With collision terms (39) and (40), Eq. (37) is our starting point for the derivation of Boltzmann equations.

The Boltzmann equations for $\overline{f}(x,\mathbf{p})$ and $\mathbf{S}(x,p)$ can be obtained by taking a trace of Eq. (37) and a trace of Eq. (37) multiplied by $\boldsymbol{\sigma}$ as

$$\hbar\left(\partial_t + \frac{1}{m}\mathbf{p}\cdot\nabla_x\right){\rm Tr}(G^<) = 2{\rm Im}{\rm Tr}(I_{\rm coll}), \tag{41}$$

$$\hbar\left(\partial_t + \frac{1}{m}\mathbf{p}\cdot\nabla_x\right){\rm Tr}(\boldsymbol{\sigma}G^<) = 2{\rm Im}{\rm Tr}(\boldsymbol{\sigma}I_{\rm coll}). \tag{42}$$

From Eq. (19) we know that there is an on-shell δ function $\delta[p_0 - \mathbf{p}^2/(2m)]$ in both sides of the above equations, therefore one can drop these δ functions and derive equations for $\overline{f}(x,\mathbf{p})$ and $\mathbf{S}(x,p)$ without δ functions.

IV. SPIN BOLTZMANN EQUATIONS: ON-SHELL PARTS

In this section we consider the contact interaction of the NJL type, one of the simplest cases for interaction. The interaction Lagrangian is given in Eq. (2). The collision term $I_{\rm coll}$ in Eq. (39) and $I^{\dagger}_{\rm coll}$ in Eq. (40) depend on the self-energy $\Sigma^>$ and $\Sigma^<$ whose Feynman diagrams are shown in Fig. 1. The self-energies can be written as

$$\begin{aligned}\Sigma^>(x,p) = {}& 4\sum_{c1,c2} g_{c1}g_{c2}\int \frac{d^4p_1}{(2\pi\hbar)^4}\frac{d^4p_2}{(2\pi\hbar)^4}\frac{d^4p_3}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(p+p_3-p_1-p_2){\rm Tr}[\Gamma^{(c2)}G^>(p_1)\Gamma^{(c1)}G^<(p_3)]\Gamma^{(c2)}G^>(p_2)\Gamma^{(c1)}\\ &- 4\sum_{c1,c2} g_{c1}g_{c2}\int \frac{d^4p_1}{(2\pi\hbar)^4}\frac{d^4p_2}{(2\pi\hbar)^4}\frac{d^4p_3}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(p+p_3-p_1-p_2)\\ &\times\Gamma^{(c2)}G^>(p_1)\Gamma^{(c1)}G^<(p_3)\Gamma^{(c2)}G^>(p_2)\Gamma^{(c1)},\quad \Sigma^<(x,p) = \Sigma^>[G^> \leftrightarrow G^<],\end{aligned} \tag{43}$$

where g_{c1} and g_{c2} can be g_0 or g_σ, and we have suppressed the x dependence of $G^<$ and $G^>$. If g_c is g_σ, a summation over i for $\Gamma^{(\sigma)} = \sigma_i$ is implied.

A. Leading order

Using Eqs. (19), (43), and (39) in Eqs. (41) and (42) and performing an integration of p_0 over 0 to $+\infty$, we obtain the Boltzmann equations for the scalar and polarization part of the distribution at the leading order

$$\begin{aligned}\hbar\left(\partial_t + \frac{1}{m}\mathbf{p}\cdot\nabla_x\right)\overline{f}^{(0)}_p = {}& 4\hbar(g_0 - 3g_\sigma)^2\int \frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_2}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_3}{(2\pi\hbar)^3}(2\pi\hbar)^4\delta^{(4)}(p+p_3-p_2-p_1)\\ &\times\big[\overline{f}^{(0)}_1\overline{f}^{(0)}_2(1-\overline{f}^{(0)}_3)(1-\overline{f}^{(0)}_p) - (1-\overline{f}^{(0)}_1)(1-\overline{f}^{(0)}_2)\overline{f}^{(0)}_3\overline{f}^{(0)}_p\\ &- (1-\overline{f}^{(0)}_3-\overline{f}^{(0)}_p)\mathbf{S}^{(0)}_1\cdot\mathbf{S}^{(0)}_2 + (1-\overline{f}^{(0)}_1-\overline{f}^{(0)}_2)\mathbf{S}^{(0)}_3\cdot\mathbf{S}^{(0)}_p\big],\end{aligned} \tag{44}$$

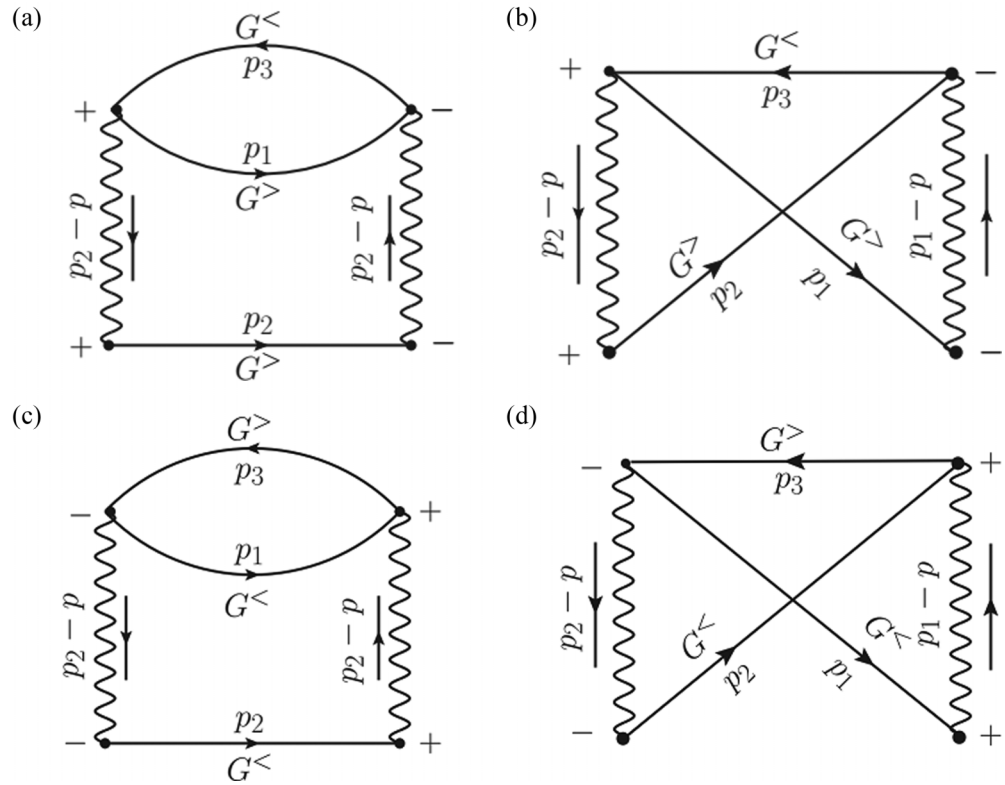

FIG. 1. Feynman diagrams for $\Sigma^>(x,p)$ (upper panel) and $\Sigma^<(x,p)$ (lower panel). In the NJL model the wavy lines attached with two solid circles represent vertices $2g_{c1}\Gamma^{(c1)}$ (left) and $2g_{c2}\Gamma^{(c2)}$ (right).

$$\hbar\left(\partial_t + \frac{1}{m}\mathbf{p}\cdot\nabla_x\right)\mathbf{S}_p^{a,(0)} = 4\hbar(g_0 - 3g_\sigma)^2 \int \frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_2}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_3}{(2\pi\hbar)^3}(2\pi\hbar)^4\delta^{(4)}(p+p_3-p_2-p_1)$$
$$\times\{[(1-\overline{f}_p^{(0)})\overline{f}_1^{(0)}\overline{f}_2^{(0)} + \overline{f}_p^{(0)}(1-\overline{f}_1^{(0)})(1-\overline{f}_2^{(0)})]\mathbf{S}_3^{a,(0)}$$
$$-[(1-\overline{f}_3^{(0)})\overline{f}_1^{(0)}\overline{f}_2^{(0)} + \overline{f}_3^{(0)}(1-\overline{f}_1^{(0)})(1-\overline{f}_2^{(0)})]\mathbf{S}_p^{a,(0)} + \mathbf{S}_1^{(0)}\cdot\mathbf{S}_2^{(0)}(\mathbf{S}_p^{a,(0)} - \mathbf{S}_3^{a,(0)})\}, \tag{45}$$

where $\overline{f}_i \equiv \overline{f}(x,\mathbf{p}_i)$, $\overline{f}_p \equiv \overline{f}(x,\mathbf{p})$, $\mathbf{S}_i \equiv \mathbf{S}(x,\mathbf{p}_i)$, $\mathbf{S}_p \equiv \mathbf{S}(x,\mathbf{p})$, and the index '(0)' denotes the leading order. If the system has no polarization at the leading order, i.e., $\mathbf{S}_i^{(0)} = 0$ for $i = 1,2,3,p$, Eq. (44) is reduced to the conventional Boltzmann equation for $\overline{f}$, with Eq. (45) for $\mathbf{S}^a$ being trivially satisfied (both sides are vanishing). If we assume polarized distributions are much smaller in magnitude than unpolarized ones, i.e., $|\mathbf{S}_i^{(0)}| \ll \overline{f}_j$ for $i,j = 1,2,3,p$, then we can neglect quadratic terms of polarized distributions in Eq. (44) relative to terms with only unpolarized distributions and neglect cubic terms of polarized distributions in Eq. (45) relative to linear terms. In this case the vanishing of the collision term in Eq. (44) gives the equilibrium condition for unpolarized distributions

$$\overline{f}_1^{(0)}\overline{f}_2^{(0)}(1-\overline{f}_3^{(0)})(1-\overline{f}_p^{(0)}) = (1-\overline{f}_1^{(0)})(1-\overline{f}_2^{(0)})\overline{f}_3^{(0)}\overline{f}_p^{(0)}. \tag{46}$$

Similarly we can also obtain the equilibrium condition for unpolarized distributions from the vanishing of the collision term in Eq. (45)

$$[(1-\overline{f}_p^{(0)})\overline{f}_1^{(0)}\overline{f}_2^{(0)} + \overline{f}_p^{(0)}(1-\overline{f}_1^{(0)})(1-\overline{f}_2^{(0)})]\mathbf{S}_3^{a,(0)} = [(1-\overline{f}_3^{(0)})\overline{f}_1^{(0)}\overline{f}_2^{(0)} + \overline{f}_3^{(0)}(1-\overline{f}_1^{(0)})(1-\overline{f}_2^{(0)})]\mathbf{S}_p^{a,(0)}, \tag{47}$$

which leads to

$$\frac{\mathbf{S}_3^{(0)}}{(1-\overline{f}_3^{(0)})\overline{f}_3^{(0)}} = \frac{\mathbf{S}_p^{(0)}}{(1-\overline{f}_p^{(0)})\overline{f}_p^{(0)}}, \tag{48}$$

using the equilibrium condition (46). The equilibrium condition (46) implies that $\overline{f}_p^{(0)}$ follows the Fermi-Dirac distribution

$$\overline{f}_p^{(0)} = \frac{1}{\exp[\beta(\omega_p - \mu)] + 1}. \tag{49}$$

If we assume the ratio in Eq. (48) is a constant vector $\mathbf{c}$ which is related to the spin potential $\boldsymbol{\mu}_{\rm spin}$, then at the leading order the MVSD in Eq. (20) has the form

$$f^{(0)}_{s_1s_2}(x,\mathbf{p}) = \delta_{s_1s_2}\overline{f}^{(0)}_p + \overline{f}^{(0)}_p\big(1-\overline{f}^{(0)}_p\big)\boldsymbol{\tau}_{s_1s_2}\cdot\boldsymbol{\mu}_{\rm spin} + \mathcal{O}(\hbar), \tag{50}$$

where the components of $\boldsymbol{\mu}_{\rm spin}$ are $\mu^i_{\rm spin} = \mathbf{n}_i\cdot\mathbf{c}$ $(i=1,2,3)$ with three directions $(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)$ being given by Eq. (26). Since we have assumed $|\mathbf{S}^{(0)}_i| \ll \overline{f}_j$, i.e., $|\boldsymbol{\mu}_{\rm spin}| \ll 1$, $f^{(0)}_{s_1s_2}$ can be put into an approximated matrix form

$$f^{(0)}(x,\mathbf{p}) \approx \frac{1}{\exp[\beta(\omega_p-\mu)-\boldsymbol{\tau}\cdot\boldsymbol{\mu}_{\rm spin}]+1}. \tag{51}$$

Here, $\boldsymbol{\tau} = (\tau_1,\tau_2,\tau_3)$ are Pauli matrices in spin space. The MVSD in Eq. (51) is the equilibrium distribution for fermions with spin degrees of freedom.

B. Next-to-leading order

At next to leading order, the Boltzmann equation for the unpolarized distribution reads

$$\hbar\left(\partial_t + \frac{1}{m}\mathbf{p}\cdot\nabla_x\right)\overline{f}^{(1)}_p = I^{\rm scalar}_{\rm qc}[f^{(0)},f^{(1)}] + I^{\rm scalar}_{\rm PB}[f^{(0)}], \tag{52}$$

where the subscript 'qc' represents the quasiclassical contribution and 'PB' represents the contribution from the Poisson bracket term. The quasiclassical part can be obtained from Eq. (44) by replacing all zeroth order distributions $\overline{f}^{(0)}_i$ and $\mathbf{S}^{(0)}_i$ by $\overline{f}_i = \overline{f}^{(0)}_i + \overline{f}^{(1)}_i$ and $\mathbf{S}_i = \mathbf{S}^{(0)}_i + \mathbf{S}^{(1)}_i$, respectively, and expanding it to the first order, the result is

$$\begin{aligned}
I^{\rm scalar}_{\rm qc}\big[f^{(0)},f^{(1)}\big] = {}& 4\hbar(g_0-3g_\sigma)^2\int\frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_2}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_3}{(2\pi\hbar)^3}(2\pi\hbar)^4\delta^{(4)}(p+p_3-p_2-p_1)\\
&\times\big\{\overline{f}^{(1)}_1\big[\overline{f}^{(0)}_2+\overline{f}^{(0)}_3\overline{f}^{(0)}_p-\overline{f}^{(0)}_2\overline{f}^{(0)}_3-\overline{f}^{(0)}_2\overline{f}^{(0)}_p\big]+\overline{f}^{(1)}_2\big[\overline{f}^{(0)}_1+\overline{f}^{(0)}_3\overline{f}^{(0)}_p-\overline{f}^{(0)}_1\overline{f}^{(0)}_3-\overline{f}^{(0)}_1\overline{f}^{(0)}_p\big]\\
&-\overline{f}^{(1)}_3\big[\overline{f}^{(0)}_p+\overline{f}^{(0)}_1\overline{f}^{(0)}_2-\overline{f}^{(0)}_p\overline{f}^{(0)}_1-\overline{f}^{(0)}_p\overline{f}^{(0)}_2\big]-\overline{f}^{(1)}_p\big[\overline{f}^{(0)}_3+\overline{f}^{(0)}_1\overline{f}^{(0)}_2-\overline{f}^{(0)}_3\overline{f}^{(0)}_1-\overline{f}^{(0)}_3\overline{f}^{(0)}_2\big]\\
&-\big(1-\overline{f}^{(0)}_3-\overline{f}^{(0)}_p\big)\mathbf{S}^{(0)}_1\cdot\mathbf{S}^{(1)}_2-\big(1-\overline{f}^{(0)}_3-\overline{f}^{(0)}_p\big)\mathbf{S}^{(1)}_1\cdot\mathbf{S}^{(0)}_2+\big(1-\overline{f}^{(0)}_1-\overline{f}^{(0)}_2\big)\mathbf{S}^{(1)}_3\cdot\mathbf{S}^{(0)}_p\\
&+\big(1-\overline{f}^{(0)}_1-\overline{f}^{(0)}_2\big)\mathbf{S}^{(0)}_3\cdot\mathbf{S}^{(1)}_p+\mathbf{S}^{(0)}_1\cdot\mathbf{S}^{(0)}_2\big(\overline{f}^{(1)}_3+\overline{f}^{(1)}_p\big)-\big(\overline{f}^{(1)}_1+\overline{f}^{(1)}_2\big)\mathbf{S}^{(0)}_3\cdot\mathbf{S}^{(0)}_p\big\}.
\end{aligned} \tag{53}$$

We see in each term there is only one first order distribution with all other distributions being of zeroth order. One can prove

$$I^{\rm scalar}_{\rm PB}[f^{(0)}] = 0, \tag{54}$$

for the scalar part of the Boltzmann equation.

In order to prove Eq. (54), we use the following property for the trace of Pauli matrices:

$$\mathrm{Tr}[\sigma_i\sigma_j\cdots\sigma_k] = (-1)^n\mathrm{Tr}[\sigma_k\cdots\sigma_j\sigma_i], \tag{55}$$

where the n is number of Pauli matrices. To prove the above relation, we insert $C^2=-1$ between Pauli matrices

$$(-1)^n\mathrm{Tr}[C\sigma_iCC\sigma_jC\cdots C\sigma_kC] = (-1)^n\mathrm{Tr}\big[\sigma_i^T\sigma_j^T\cdots\sigma_k^T\big] = (-1)^n\mathrm{Tr}[\sigma_k\cdots\sigma_j\sigma_i], \tag{56}$$

where $C = i\sigma^2$ and $C\boldsymbol{\sigma}C = \boldsymbol{\sigma}^T$.

With Eq. (56) we consider the quantities in the Poisson bracket collision term $I^{\rm scalar}_{\rm PB}[f^{(0)}]$,

$$I_1 = g_{c1}g_{c2}\mathrm{Im}[\mathrm{Tr}(\Gamma^{(c2)}G_1\Gamma^{(c1)}G_3)\mathrm{Tr}(\Gamma^{(c2)}G_2\Gamma^{(c1)}G_p)],\ \ I_2 = g_{c1}g_{c2}\mathrm{Im}\mathrm{Tr}(\Gamma^{(c2)}G_1\Gamma^{(c1)}G_3\Gamma^{(c2)}G_2\Gamma^{(c1)}G_p). \tag{57}$$

Note that G_i $(i=1,2,3,p)$ can be either $G^{\lessgtr}(x,p_i)$ or derivatives of $G^{\lessgtr}(x,p_i)$. We know that G_i contain the scalar and polarization parts, so we can express $G_i = \sum_{b_i=1,0}G_i(b_i)$, where $G_i(1)$ denotes its scalar part and $G_i(0)$ denotes its polarization part. Let us work on I_1,

$$\begin{aligned}
I_1 &= g_{c1}g_{c2}\mathrm{Im}[\mathrm{Tr}(\Gamma^{(c2)}G_1\Gamma^{(c1)}G_3)\mathrm{Tr}(\Gamma^{(c2)}G_2\Gamma^{(c1)}G_p)]\\
&= g_{c1}g_{c2}\sum_{b_1,b_2,b_3,b}\mathrm{Im}\{\mathrm{Tr}[\Gamma^{(c2)}G_1(b_1)\Gamma^{(c1)}G_3(b_3)]\mathrm{Tr}[\Gamma^{(c2)}G_2(b_2)\Gamma^{(c1)}G_p(b)]\}\\
&= g_{c1}g_{c2}\sum_{n=\rm odd}\mathrm{Im}\{\mathrm{Tr}[\Gamma^{(c2)}G_1(b_1)\Gamma^{(c1)}G_3(b_3)]\mathrm{Tr}[\Gamma^{(c2)}G_2(b_2)\Gamma^{(c1)}G_p(b)]\}\\
&\quad\downarrow \text{insert } C^2
\end{aligned}$$

$$= g_{c1}g_{c2}\sum_{n=\text{odd}}(-1)^n\text{Im}\{\text{Tr}[\Gamma^{(c1)}G_1(b_1)\Gamma^{(c2)}G_3(b_3)]\text{Tr}[\Gamma^{(c1)}G_2(b_2)\Gamma^{(c2)}G_p(b)]\}$$
$$= -g_{c1}g_{c2}\sum_{n=\text{odd}}\text{Im}\{\text{Tr}[\Gamma^{(c2)}G_1(b_1)\Gamma^{(c1)}G_3(b_3)]\text{Tr}[\Gamma^{(c2)}G_2(b_2)\Gamma^{(c1)}G_p(b)]\}, \tag{58}$$

where $n = b_1 + b_2 + b_3 + b$ is the number of scalar parts in two traces, and we have interchanged $c1 \leftrightarrow c2$ in the last equality since a summation over $c1$ and $c2$ is implied. Note that even or odd n also indicates even or odd number of Pauli matrices in two traces, respectively. Only when n is odd does the product of two traces have an imaginary part. By comparing the last equality with the second one of Eq. (58) we arrive at $I_1 = 0$. For I_2, we have

$$I_2 = g_{c1}g_{c2}\text{ImTr}(\Gamma^{(c2)}G_1\Gamma^{(c1)}G_3\Gamma^{(c2)}G_2\Gamma^{(c1)}G_p) = g_{c1}g_{c2}\sum_{b_1,b_2,b_3,b}\text{ImTr}[\Gamma^{(c2)}G_1(b_1)\Gamma^{(c1)}G_3(b_3)\Gamma^{(c2)}G_2(b_2)\Gamma^{(c1)}G_p(b)]$$
$$= g_{c1}g_{c2}\sum_{n=\text{odd}}\text{ImTr}[\Gamma^{(c2)}G_1(b_1)\Gamma^{(c1)}G_3(b_3)\Gamma^{(c2)}G_2(b_2)\Gamma^{(c1)}G_p(b)]$$
$$\downarrow \text{insert } C^2$$
$$= g_{c1}g_{c2}\sum_{n=\text{odd}}(-1)^n\text{ImTr}[\Gamma^{(c1)}G_2(b_2)\Gamma^{(c2)}G_3(b_3)\Gamma^{(c1)}G_1(b_1)\Gamma^{(c2)}G_p(b)]$$
$$= -g_{c1}g_{c2}\sum_{n=\text{odd}}\text{ImTr}[\Gamma^{(c2)}G_1(b_1)\Gamma^{(c1)}G_3(b_3)\Gamma^{(c2)}G_2(b_2)\Gamma^{(c1)}G_p(b)], \tag{59}$$

where $n = b_1 + b_2 + b_3 + b$ is the number of scalar parts in the trace, in the final equality we have interchanged $c1 \leftrightarrow c2$ and $G_1(b_1) \leftrightarrow G_2(b_2)$ since a summation over $c1$ and $c2$ is implied and there is a symmetry in the labels 1 and 2 (in the integration over p_1 and p_2 and the summation over b_1 and b_2). The odd/even n also corresponds to odd/even number of Pauli matrices inside the trace. Only when n is odd does the trace have an imaginary part. By comparing the last equality with the second one of Eq. (59), we obtain $I_2 = 0$.

We now derive the Boltzmann equation for the polarization distribution at the next-to-leading order. The contributions can be grouped into the local (quasiclassical) and nonlocal parts of the collision term. The local part contains no space-time derivatives and can be obtained from Eq. (45) by the replacement $\overline{f}_i^{(0)} \to \overline{f}_i = \overline{f}_i^{(0)} + \overline{f}_i^{(1)}$ and $\mathbf{S}_i^{(0)} \to \mathbf{S}_i = \mathbf{S}_i^{(0)} + \mathbf{S}_i^{(1)}$ in the collision term and then by expanding the collision term to the next-to-leading order. The nonlocal part comes from the Poisson bracket term with space-time derivatives. Special care should be taken for the derivatives in ∂_{p_0} and ∇_p which act on the two-point function $G^{\lessgtr}(x, p)$, giving terms with $\delta'(p_0 - \mathbf{p}^2/2m)$. These terms belong to off-shell contributions which we will neglect in this section and leave to the next section for treatment. Substituting Eq. (19) into Eq. (42), we obtain the on-shell Boltzmann equation for the polarization distribution at the next-to-leading order

$$\hbar\left(\partial_t + \frac{1}{m}\mathbf{p}\cdot\nabla_x\right)\mathbf{S}_p^{a,(1)} = I_{\text{qc}}^{\text{pol}}[f^{(0)}, f^{(1)}] + I_{\text{PB}}^{\text{pol}}[f^{(0)}]. \tag{60}$$

The explicit form of the local part (quasiclassical) of the collision term is given by

$$\begin{aligned}
I_{\text{qc}}^{\text{pol}}[f^{(0)}, f^{(1)}] = 4\hbar(g_0 - 3g_\sigma)^2 \int \frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_2}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}_3}{(2\pi\hbar)^3}(2\pi\hbar)^4\delta^{(4)}(p + p_3 - p_2 - p_1)\\
\times\big\{\mathbf{S}_3^{a,(0)}\big[\overline{f}_p^{(1)}\big(1 - \overline{f}_1^{(0)} - \overline{f}_2^{(0)}\big) + \overline{f}_1^{(1)}\big(\overline{f}_2^{(0)} - \overline{f}_p^{(0)}\big) + \overline{f}_2^{(1)}\big(\overline{f}_1^{(0)} - \overline{f}_p^{(0)}\big)\big]\\
- \mathbf{S}_p^{a,(0)}\big[\overline{f}_3^{(1)}\big(1 - \overline{f}_1^{(0)} - \overline{f}_2^{(0)}\big) + \overline{f}_1^{(1)}\big(\overline{f}_2^{(0)} - \overline{f}_3^{(0)}\big) + \overline{f}_2^{(1)}\big(\overline{f}_1^{(0)} - \overline{f}_3^{(0)}\big)\big]\\
+ \mathbf{S}_3^{a,(1)}\big(\overline{f}_p^{(0)} + \overline{f}_1^{(0)}\overline{f}_2^{(0)} - \overline{f}_p^{(0)}\overline{f}_1^{(0)} - \overline{f}_p^{(0)}\overline{f}_2^{(0)}\big)\\
-\mathbf{S}_p^{a,(1)}\big(\overline{f}_3^{(0)} + \overline{f}_1^{(0)}\overline{f}_2^{(0)} - \overline{f}_3^{(0)}\overline{f}_1^{(0)} - \overline{f}_3^{(0)}\overline{f}_2^{(0)}\big)\\
+ \big(\mathbf{S}_1^{(1)}\cdot\mathbf{S}_2^{(0)} + \mathbf{S}_1^{(0)}\cdot\mathbf{S}_2^{(1)}\big)\big(\mathbf{S}_p^{a,(0)} - \mathbf{S}_3^{a,(0)}\big) + \mathbf{S}_1^{(0)}\cdot\mathbf{S}_2^{(0)}\big(\mathbf{S}_p^{a,(1)} - \mathbf{S}_3^{a,(1)}\big)\big\}.
\end{aligned} \tag{61}$$

One can check that $I_{\text{qc}}^{\text{pol}}[f^{(0)}, f^{(1)}]$ is local and on shell.

The nonlocal part of the collision term contains on-shell and off-shell contributions, as we have mentioned, in this section we focus on the on-shell contribution and will treat the off-shell one in Sec. V. The explicit form of the on-shell Poisson bracket

term reads

$$I_{\rm PB}^{\rm pol}[f^{(0)}] \approx 2\hbar^2(g_0-3g_\sigma)^2\epsilon_{aij}\int\frac{d^3p_1}{(2\pi\hbar)^3}\frac{d^3p_2}{(2\pi\hbar)^3}\frac{d^3p_3}{(2\pi\hbar)^3}(2\pi\hbar)^4\delta^{(4)}(p+p_3-p_1-p_2)\big\{\mathbf{S}_3^{(0),i}\big[\nabla_x\big(\overline{f}_1^{(0)}+\overline{f}_2^{(0)}\big)\cdot\nabla_p\mathbf{S}_p^{(0),j} - \big(\nabla_{p_1}\overline{f}_1^{(0)}+\nabla_{p_2}\overline{f}_2^{(0)}\big)\cdot\nabla_x\mathbf{S}_p^{(0),j}\big]+\big(1-\overline{f}_1^{(0)}-\overline{f}_2^{(0)}\big)\big(\nabla_{p_3}\mathbf{S}_3^{(0),i}\cdot\nabla_x\mathbf{S}_p^{(0),j}-\nabla_x\mathbf{S}_3^{(0),i}\cdot\nabla_p\mathbf{S}_p^{(0),j}\big)\big\}. \tag{62}$$

Here, we have neglected a term with

$$\partial_{p_0}\delta^{(4)}(p+p_3-p_1-p_2)\epsilon_{aij}\mathbf{S}_3^{(0),j}\Big(\partial_t+\frac{\mathbf{p}}{m}\cdot\nabla_x\Big)\mathbf{S}_p^{j,(0)}$$

in the integrand which is of order $\mathcal{O}(\hbar^2 g_{\rm couple}^4)$ by Eq. (45), while the terms in Eq. (62) are all of order $\mathcal{O}(\hbar^2 g_{\rm couple}^2)$, where $g_{\rm couple}=g_0$ or g_σ denotes the coupling constant. The collision term in Eq. (62) is nonlocal for it contains derivatives of space-time. From Eq. (62), we see that the collision term vanishes if there is no polarization density at the leading order, i.e., $\mathbf{S}_p^{a,(0)}=0$. This is the result of the non-relativistic coupling of the NJL type. In contrast it has been proved in Ref. [19] that the polarization can be generated from the Poisson bracket term even without polarization density at the leading order in a relativistic NJL model.

V. SPIN BOLTZMANN EQUATIONS: OFF-SHELL PARTS

In this section we will investigate off-shell contributions. From Eqs. (36) and (37) $G^<(x,p)$ actually contains an off-shell part $G^<_{\rm off}(x,p)$ besides the on-shell part in Eq. (19) which we denote as $G^<_{\rm on}(x,p)$, so do $I_{\rm coll}$ and $I^\dagger_{\rm coll}$.

Now we try to derive the connection between $G^<_{\rm on}(x,p)$ and $G^<_{\rm off}(x,p)$. From Eq. (36) we obtain

$$G^<_{\rm off}(x,p)=\frac{1}{2p_0-\mathbf{p}^2/m}(I_{\rm coll}+I^\dagger_{\rm coll})+\mathcal{O}(\hbar^2). \tag{63}$$

Note that the on-shell part $I^{\rm on}_{\rm coll}$ contains $\delta[p_0-\mathbf{p}^2/(2m)]$ which, combining $[p_0-\mathbf{p}^2/(2m)]^{-1}$, gives the derivative of the delta-function, $\delta'[p_0-\mathbf{p}^2/(2m)]$. The explicit form of $G^<_{\rm off}(x,p)$ can be determined by the collision term from Eq. (63) and is at least of the same order as $\hbar G^{\lessgtr(1)}_{\rm on}$ in Eq. (18). We can express $G^<_{\rm off}(x,p)$ as

$$\begin{aligned}G^<_{\rm off}(x,p)=G^>_{\rm off}(x,p)&=-\hbar(2\pi\hbar)\partial_{p_0}\delta\Big(p_0-\frac{\mathbf{p}^2}{2m}\Big)\sum_{s_1,s_2=\pm}\chi(s_1)\chi^\dagger(s_2)f^{\rm off}_{s_1s_2}(x,\mathbf{p})\\&=-\hbar(2\pi\hbar)\partial_{p_0}\delta\Big(p_0-\frac{\mathbf{p}^2}{2m}\Big)[\overline{f}_{\rm off}(x,\mathbf{p})+\boldsymbol{\sigma}\cdot\mathbf{S}_{\rm off}(x,\mathbf{p})]=-\hbar(2\pi\hbar)\partial_{p_0}\delta\Big(p_0-\frac{\mathbf{p}^2}{2m}\Big)\boldsymbol{\sigma}\cdot\mathbf{S}_{\rm off}(x,\mathbf{p}),\end{aligned} \tag{64}$$

where we used $\overline{f}_{\rm off}(x,\mathbf{p})=0$ and

$$\begin{aligned}\mathbf{S}^a_{\rm off}(x,\mathbf{p})&=2(g_0-3g_\sigma)^2\epsilon^{aij}\int\frac{d^4p_1}{(2\pi\hbar)^3}\frac{d^4p_2}{(2\pi\hbar)^3}\frac{d^4p_3}{(2\pi\hbar)^3}(2\pi\hbar)^4\delta(p+p_3-p_2-p_1)\\&\quad\times\delta\Big(p_1^0-\frac{\mathbf{p}_1^2}{2m}\Big)\delta\Big(p_2^0-\frac{\mathbf{p}_2^2}{2m}\Big)\delta\Big(p_3^0-\frac{\mathbf{p}_3^2}{2m}\Big)\big(1-\overline{f}_1^{(0)}-\overline{f}_2^{(0)}\big)\mathbf{S}_3^{i,(0)}\mathbf{S}_p^{j,(0)},\end{aligned} \tag{65}$$

following Eq. (63) at the leading order. Actually $G^<_{\rm off}$ in the form of Eq. (64) would be added to the left-hand side of Eq. (42) at the next-to-leading order [since $\overline{f}_{\rm off}(x,\mathbf{p})=0$, there is no off-shell correction to Eq. (41)], while there is also the off-shell part from the Poisson bracket term in the right-hand side. It can be shown that the off-shell terms in both sides of Eq. (42) are equal at $\mathcal{O}(\hbar^2 g_{\rm couple}^2)$ and thus drop out from the equation to leave Eq. (60) for the on-shell part.

From Eq. (37), we obtain

$$\begin{aligned}\hbar\Big(\partial_t+\frac{1}{m}\mathbf{p}\cdot\nabla_x\Big)G^<_{\rm on}(x,p)&=-i(I_{\rm coll}-I^\dagger_{\rm coll})-\hbar\Big(\partial_t+\frac{\mathbf{p}}{m}\cdot\nabla_x\Big)G^<_{\rm off}(x,p)=-i\big(I^{\rm on}_{\rm coll}-I^{\rm on\dagger}_{\rm coll}\big)-i\big(I^{\rm off}_{\rm coll}-I^{\rm off\dagger}_{\rm coll}\big)\\&\quad-\hbar\frac{1}{2p_0-\mathbf{p}^2/m}\Big(\partial_t+\frac{\mathbf{p}}{m}\cdot\nabla_x\Big)(I_{\rm coll}+I^\dagger_{\rm coll})+\mathcal{O}(\hbar^3).\end{aligned} \tag{66}$$

Now we act the operator in the left-hand side of Eq. (29) to the right-hand side of Eq. (28) and obtain

$$\Big(-i\hbar\frac{\partial}{\partial t_2}+\frac{\hbar^2}{2m}\nabla^2_{x2}\Big)I_{\rm coll}(x_1,x_2)=\mathcal{O}\big(\hbar^2 g_{\rm couple}^4\big). \tag{67}$$

Note that the operator only acts on Green's functions in $I_{\rm coll}$ (32) instead of self-energies since only Green's functions depend on x_2, which gives a contribution of $\mathcal{O}(\hbar^2 g_{\rm couple}^4)$ using Eq. (29). Taking a Wigner transform of Eq. (67) we obtain an equation for

$I_{\rm coll}^{\rm on}$ and $I_{\rm coll}^{\rm off}$

$$I_{\rm coll}^{\rm off} = \frac{1}{2p_0 - \mathbf{p}^2/m} i\hbar\Big(\partial_t + \frac{\mathbf{p}}{m}\cdot\nabla_x\Big) I_{\rm coll} + \mathcal{O}\big(\hbar^2 g_{\rm couple}^4\big) + \mathcal{O}\big(\hbar^3 g_{\rm couple}^2\big), \tag{68}$$

where the first term is at least of $\mathcal{O}(\hbar^2 g_{\rm couple}^2)$. Taking a Hermitian conjugate of the above equation leads to

$$I_{\rm coll}^{\rm off\dagger} = -\frac{1}{2p_0 - \mathbf{p}^2/m} i\hbar\Big(\partial_t + \frac{\mathbf{p}}{m}\cdot\nabla_x\Big) I_{\rm coll}^{\dagger} + \mathcal{O}\big(\hbar^2 g_{\rm couple}^4\big) + \mathcal{O}\big(\hbar^3 g_{\rm couple}^2\big). \tag{69}$$

Substituting Eqs. (68) and (69) into Eq. (66) we arrive at an equation for the on-shell part of the two-point function

$$\hbar\Big(\partial_t + \frac{\mathbf{p}}{m}\cdot\nabla_x\Big) G_{\rm on}^{<}(x,p) = -i\big(I_{\rm coll}^{\rm on} - I_{\rm coll}^{\rm on\dagger}\big) + \mathcal{O}\big(\hbar^2 g_{\rm couple}^4\big) + \mathcal{O}\big(\hbar^3 g_{\rm couple}^2\big). \tag{70}$$

The above equation for on-shell parts is the foundation for discussions in Sec. IV, in which the index "on" has been suppressed for notational simplicity. In this section we have resumed the use of index "on" to denote $G_{\rm on}^{\lessgtr}(x,p)$ and $I_{\rm coll}^{\rm on}$.

Let us analyze $I_{\rm coll}^{\rm off}$ in Eq. (68). From Eq. (39), $I_{\rm coll}$ has two parts: one is the local term or quasiclassical term $I_{\rm qc}$ which is at least of $\mathcal{O}(\hbar)$, and the other is the nonlocal term with Poisson brackets $I_{\rm PB}$ which is at least of $\mathcal{O}(\hbar^2)$. The leading contribution of $I_{\rm coll}^{\rm off}$ is of $\mathcal{O}(\hbar^2 g_{\rm couple}^2)$ from $I_{\rm PB}$ containing derivatives ∂_{p_0} and ∇_p acting on $G^{\lessgtr}(x,p)$, while the contribution from $G_{\rm off}^{<}$ in Eq. (64) to $I_{\rm coll}^{\rm off}$ through the expansion of $I_{\rm qc}$ is of $\mathcal{O}(\hbar^2 g_{\rm couple}^4)$ which is in higher order. Note that the leading contribution in $I_{\rm coll}^{\rm off}$ from $I_{\rm PB}$ is in the same order as the first term in the right-hand side of Eq. (68). So at $\mathcal{O}(\hbar^2 g_{\rm couple}^2)$, Eq. (68) gives a constraint for MVSD $f^{(0)}$ or equivalently $\overline{f}^{(0)}$ and $\mathbf{S}^{(0)}$ in addition to Eqs. (44) and (45).

VI. COMMENTS ON NUCLEAR FORCE THROUGH OBEP

In order to apply our theory to a non-relativistic nucleon system in low energy collisions, one has to go beyond the contact interaction of the NJL type and consider nuclear force as interaction. The main features of nuclear force can be effectively described by one boson exchange potential (OBEP) [78,79]. The OBEPs through scalar, pseudoscalar, and vector meson exchanges have terms with operators $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, $\mathbf{L}\cdot\mathbf{S}$, and S_{12} defined as $\mathbf{L}\equiv -i\mathbf{r}\times\nabla$, $\mathbf{S}\equiv(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$, and

$$S_{12} \equiv \frac{1}{r^2}[3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r}) - r^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2], \tag{71}$$

where $\mathbf{r}\equiv\mathbf{x}-\mathbf{x}'$. So OBEPs are nonlocal and contain couplings between spin and coordinate (equivalently spin and momentum). In this case the spin is not a conserved quantity as in the NJL-like model. It may be converted from local orbital angular momentum or local vorticity [17,20]. The extension to the OBEP is much more complicated and beyond the scope of this work. It will be reserved for a future study.

VII. SUMMARY

We derive spin Boltzmann equations for nonrelativistic spin-1/2 fermions from the KB equation in the CTP formalism. The nonrelativistic model is similar to the NJL model with four-fermion contact interaction which conserves spins in particle scatterings. The great merit of the model is that the spin matrix element in the collision term can be completely worked out and be put into a compact form. One can clearly see how spins are coupled in two-to-two scatterings of particles. In contrast it is hard to envisage the structure of the spin matrix element which is much more complicated in the relativistic theory [19].

Starting from the non-relativistic Lagrangian, the KB equation is derived from the Dyson-Schwinger equation defined on the CTP. The spin Boltzmann equations for the particle number and spin distribution are derived based on Wigner functions and the KB equation. Since the spin polarization is a quantum effect, we make an expansion in the Planck constant $\hbar$ for all quantities in the spin Boltzmann equation. At the leading order, the equilibrium spin distribution can be obtained under the condition of the vanishing collision term for the spin phase space density. A spin chemical potential emerges in the equilibrium spin distribution which is a natural consequence of spin conservation. The off-shell parts of spin Boltzmann equations are also discussed. The work can be extended to a system of nucleons which interact via nuclear forces in low energy heavy-ion collisions.

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