Transfer reactions between odd-odd and even-even nuclei by using the interacting boson fermion fermion model and $0\nu\beta\beta$ decay in closure approximation

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Spectroscopic amplitudes (SAs) in the interacting boson fermion fermion model (IBFFM) are necessary for the computation of $0\nu\beta\beta$ decay but also for cross sections of heavy-ion reactions, in particular, double charge exchange reactions for the NUMEN Collaboration, if one does not want to use the closure limit. We present for the first time (i) the formalism and operators to compute in a general case the spectroscopic amplitudes in the IBFFM scheme from even-even to odd-odd nuclei, in a way suited to be used in reaction code, i.e., extracting the contribution of each orbital; (2) the odd-odd nuclei as described by the old IBFFM are obtained for the first time with the new implementation of machine learning (ML) techniques for fitting the parameters, getting a more realistic description. The one-body transition densities for ^{116}Cd $\rightarrow$ ^{116}In and ^{116}In $\rightarrow$ ^{116}Sn are part of the experimental program of the NUMEN experiment, which aims to find constraints on $0\nu\beta\beta$ decay matrix elements. As a first application of the spectroscopic amplitudes we present the calculation of the $0\nu\beta\beta$ nuclear matrix elements in the IBFFM formalism in closure approximation.

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I. INTRODUCTION

The first articles on odd-odd nuclei with algebraic models date back to the 1980s, when some odd-odd nuclei were studied by a dynamical symmetry scheme and a supersymmetry scheme [1–3]. Unfortunately, that approach is suitable only for a few nuclei that correspond to dynamical supersymmetry. It must be emphasized that nuclear supersymmetry places very severe constraints on the form of the boson-fermion Hamiltonian used to describe nuclei belonging to a supermultiplet. Consequently, applications are restricted to only a few regions of the nuclear chart. The first possible example of a supermultiplet involving an odd-odd nucleus was proposed in the Pt-Au region [1,4] and subsequent investigations [2,3] focused on the odd-odd member of the quartet, ^{198}Au in [5], ^{196}Au in [6–8], and ^{194}Ir in [9,10].

In a second more general approach, which we will follow in this paper, the interacting boson model (IBM) was extended to describe odd-odd nuclei, in which the two fermions (one neutron and one proton) are coupled to the even-even core described either by IBM-1 [11–13], or by IBM-2 [14,15]. These extensions to odd-odd nuclei are usually referred to as the interacting boson-fermion-fermion model, IBFFM-1 and IBFFM-2, respectively. In this article we use the neutron-proton version, and we will use the shorthand notation IBFFM from here on (strictly speaking it is IBFFM-2). The IBFFM Hamiltonian is derived from semimicroscopic arguments and subsequently diagonalized numerically. It was introduced by Lopac and Bianco [11,12], and it is essentially an IBFM-1 in the proton and an IBFM-1 in the neutron plus a residual interaction as introduced by Brant and Paar [13]. The original approach was extended to include the neutron-proton degree of freedom in the description of the even-even core nucleus in Ref. [14]. In that work Yoshida and Iachello wrote the operator expression for single beta decay and two-neutrino double beta decay, which can be considered as a particular subset of the general one-body transition operator presented in this paper.

The aim of this paper is from one side to investigate the transition from ^{116}Cd to ^{116}In and from ^{116}Sn to ^{116}In, and introduce the formalism of the spectroscopic amplitudes (SAs) between even-even and odd-odd nuclei in the IBFFM that are the key ingredient for planned publications of double charge exchange (DCE) reactions for the NUMEN Collaboration and the study $0\nu\beta\beta$ decay. The spectroscopic amplitudes operator dressed as a charge exchange or as simple transition amplitudes is needed for the theoretical NUMEN project [16,17] to be inserted into a reaction code.

For the NUMEN experiment at INFN-LNS in particular there is interest for ^{116}Cd $\rightarrow$ ^{116}In and ^{116}In $\rightarrow$ ^{116}Sn [18].

In this article, the SA operator in the scheme of IBFFM has been derived in a general way and then calculated for the previous reactions, and the results are given in the form of spectroscopic amplitudes, which will be used to compute the nuclear matrix elements of the $0\nu\beta\beta$ decay in closure approximation.

We introduce a new method to fit the odd-odd nuclei parameters by creating a multidimensional theoretical dataset using machine learning libraries such as SCIKIT-LEARN. The

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operators needed to compute the transitions between odd-odd and even-even nuclei by using IBFFM depend on two factors: the similarity between the states in the initial and final nucleus, which differ by two nucleons, and the transferred pair of nucleons' correlation. We have derived the two different transition operators' cases: the case where the numbers of bosons are conserved between the even-even and odd-odd nuclei and the one where they are not conserved.

The organization of this paper is the following: In Sec. IV we briefly review a simple theory of the one-body transitions density. In Sec. V we present the derivation of the transition operator for the IBFFM. In Sec. II we discuss the theory of the odd-odd nucleus in the implemented IBFFM with machine learning (ML). In Sec. III we compute the nuclear wave functions and spectroscopic amplitudes for the transitions ^{116}Sn to ^{116}In and ^{116}Cd to ^{116}Sn, that will be used to compute the nuclear matrix elements $0\nu\beta\beta$ decay and compare with the IBM-2 formalism. In our last Sec. VI we present our conclusions and discuss our results; also we give the future outlook.

II. DESCRIPTION OF ODD-ODD NUCLEI

In the IBFFM, odd-odd nuclei are described in terms of a system of N_π proton bosons (s_π and d_π) and N_ν neutron bosons (s_ν and d_ν) coupled to a single proton (j_π) and a single neutron (j_ν). The proton and neutron orbitals are those of the active major shell for protons and neutrons, respectively.

The Hamiltonian of the IBFFM for the odd-odd nuclei can be written as

$$H = H^B + H^F_\pi + V^{BF}_\pi + H^F_\nu + V^{BF}_\nu + V_{\rm res}. \tag{1}$$

H^B is the IBM-2 Hamiltonian. H^F_π and H^F_ν are the proton and neutron single-particle terms

$$H^F_\rho = \sum_{j_\rho} \epsilon_{j_\rho} \hat{n}_{j_\rho}, \tag{2}$$

where ϵ_{j_ρ} is the quasiparticle energy of the extra nucleon and $\hat{n}$ is the number operator. The quasiparticle energies ϵ_{j_ρ} are obtained within the BCS approximation with a gap $\Delta = 12/\sqrt{A}$ MeV, where A is the mass number of the even-even core nucleus. In the BCS approximation, the quasi-particle energies are related to the single-particle level E_j, the occupation probabilities v_j, and the Fermi level λ as follows:

$$\begin{aligned} \epsilon_j &= \sqrt{(E_j - \lambda)^2 + \Delta^2}, \\ v_j^2 &= \frac{1}{2}\left(1 - \frac{E_j - \lambda}{\epsilon_j}\right), \\ u_j^2 &= 1 - v_j^2. \end{aligned} \tag{3}$$

V^{BF}_π and V^{BF}_ν describe the core-particle coupling of the odd proton and odd neutron in the IBFM-2 model [19–21] as the sum of a quadrupole term (Γ_ρ), an exchange term (Λ_ρ), and a monopole term (A_ρ):

$$V^{BF}_\rho = \Gamma_\rho Q^{(2)}_{\rho'} \cdot q^{(2)}_\rho + \Lambda_\rho F_{\rho'\rho} + A_\rho \hat{n}_{d_{\rho'}} \cdot \hat{n}_\rho \tag{4}$$

where $\rho' \neq \rho$ and $\rho, \rho' = \nu, \pi$. The first term in Eq. (4) is a quadrupole-quadrupole interaction with

$$\begin{aligned} q^{(2)}_\rho &= \sum_{j_\rho, j'_\rho} (u_{j_\rho} u_{j'_\rho} - v_{j_\rho} v_{j'_\rho}) Q_{j_\rho j'_\rho} (a^\dagger_{j_\rho} \times \tilde{a}_{j'_\rho})^{(2)}, \\ Q^{(2)}_\rho &= (s^\dagger_\rho \times \tilde{d}_\rho + d^\dagger_\rho \times \tilde{s}_\rho)^{(2)} + \chi_\rho (d^\dagger_\rho \times \tilde{d}_\rho)^{(2)}. \end{aligned} \tag{5}$$

The second term is the exchange interaction

$$\begin{aligned} F_{\rho,\rho'} = &- \sum_{j_\rho j'_\rho j''_\rho} \beta_{j_\rho j'_\rho} \beta_{j''_\rho j_\rho} \sqrt{\frac{10}{N_\rho (2j_\rho + 1)}} \\ &\times Q^{(2)}_{\rho'} \cdot : [(d_\rho \times \tilde{a}_{j''_\rho})^{(j_\rho)} \times (a^\dagger_{j'_\rho} \times \tilde{s}_\rho)^{(j'_\rho)}]^{(2)} : + \text{H.c.} \end{aligned} \tag{6}$$

The coefficients $\beta_{j_\rho j'_\rho}$ are related to the single-particle matrix elements of the quadrupole operator $Q_{j_\rho j'_\rho}$ by

$$\begin{aligned} \beta_{j_\rho j'_\rho} &= (u_{j_\rho} v_{j'_\rho} + v_{j_\rho} u_{j'_\rho}) Q_{j_\rho j'_\rho}, \\ Q_{j_\rho j'_\rho} &= \langle j_\rho \| Y^{(2)} \| j'_\rho \rangle. \end{aligned} \tag{7}$$

The last term is the monopole-monopole interaction with

$$\begin{aligned} n_{d_\rho} &= \sum_m d^\dagger_{\rho,m} d_{\rho,m}, \\ \hat{n}_\rho &= \sum_{j_\rho} \hat{n}_{j_\rho} = \sum_{j_\rho, m} a^\dagger_{j_\rho, m} a_{j_\rho, m}. \end{aligned} \tag{8}$$

The residual interaction between the odd-proton and odd-neutron is defined as [13,14]

$$V_{\rm res} = H_\delta + H_{\sigma\sigma\delta} + H_{\sigma\sigma} + H_T, \tag{9}$$

with

$$\begin{aligned} H_\delta &= 4\pi V_\delta \delta(\vec{r}_\pi - \vec{r}_\nu) \delta(r_\pi - R_0) \delta(r_\nu - R_0), \\ H_{\sigma\sigma\delta} &= 4\pi V_{\sigma\sigma\delta} \delta(\vec{r}_\pi - \vec{r}_\nu)(\vec{\sigma}_\pi \cdot \vec{\sigma}_\nu) \\ &\quad \times \delta(r_\pi - R_0) \delta(r_\nu - R_0), \\ H_{\sigma\sigma} &= -\sqrt{3} V_{\sigma\sigma} \vec{\sigma}_\pi \cdot \vec{\sigma}_\nu, \\ H_T &= V_T \left[\frac{3(\vec{\sigma}_\pi \cdot \vec{r}_{\pi\nu})(\vec{\sigma}_\nu \cdot \vec{r}_{\pi\nu})_{\pi\nu}}{r^2} - \vec{\sigma}_\pi \cdot \vec{\sigma}_\nu \right], \end{aligned} \tag{10}$$

where $\vec{r}_{\pi\nu} = \vec{r}_\pi - \vec{r}_\nu$ and $R_0 = 1.2A^{1/3}$fm. The matrix elements of the residual interaction are calculated in the quasiparticle basis, which is related to the particle basis by [14]

$$\begin{aligned} \langle j'_\nu j'_\pi; J | V_{\rm res} | j_\nu j_\pi; J \rangle_{qp} = &(u_{j'_\nu} u_{j'_\pi} u_{j_\nu} u_{j_\pi} + v_{j'_\nu} v_{j'_\pi} v_{j_\nu} v_{j_\pi}) \\ &\times \langle j'_\nu j'_\pi; J | V_{\rm res} | j_\nu j_\pi; J \rangle \\ &- (u_{j'_\nu} v_{j'_\pi} u_{j_\nu} v_{j_\pi} + v_{j'_\nu} u_{j'_\pi} v_{j_\nu} u_{j_\pi}) \\ &\times \sum_{J'} (2J' + 1) \begin{Bmatrix} j'_\nu & j_\pi & J' \\ j_\nu & j'_\pi & J \end{Bmatrix} \\ &\times \langle j'_\nu j_\pi; J' | V_{RES} | j_\nu j'_\pi; J' \rangle, \end{aligned} \tag{11}$$

where $v_j^2 = 1 - u_j^2$ denotes the occupation probability.

TABLE I. IBM-2 parameters in MeV taken from Ref. [22]. χ_ν and χ_π are dimensionless.

Nucleus	N_ν	N_π	ϵ_d	κ	χ_ν	χ_π	$\xi_1 = \xi_3$	ξ_2	$c_\nu^{(0)}$	$c_\nu^{(2)}$	$c_\nu^{(4)}$	v_0^ν	v_2^ν
^{116}Sn	8	0	1.32						−0.50	−0.22	−0.07	−0.06	0.04
^{118}Sn	7	0	1.31						−0.50	−0.23	−0.08	−0.11	0.09
^{116}Cd	7	1	0.85	−0.27	−0.58	0.00	−0.18	0.24	−0.15	−0.06			

III. NUMERICAL RESULTS

In the context of the IBFFM, the odd-odd nucleus is described by coupling a proton and a neutron to its core (even-even nucleus). Thus the first step is to construct the core nucleus. It is described by the IBM-2 Hamiltonian (H^B) [23], which only depends on the neutron- and proton-boson degrees of freedom, the expression of the boson Hamiltonian H^B is taken as in Ref. [14].

In the IBM and its extensions the boson and fermion degrees of freedom are counted with respect to the nearest closed shell. The initial even-even nucleus $^{116}_{48}Cd_{68}$ has $N_\pi = 1$ proton bosons and $N_\nu = 7$ neutron bosons, and the final nucleus $^{116}_{50}Sn_{66}$ has $N_\pi = 0$ and $N_\nu = 8$. The intermediate odd-odd nucleus $^{116}_{49}In_{67}$ is described in the IBM as a proton hole and a neutron hole coupled to the core nucleus $^{118}_{50}Sn_{68}$ with $N_\pi = 0$ and $N_\nu = 7$. The charge-exchange reaction from ^{116}Cd to ^{116}In involves a change in the number of proton bosons, and from ^{116}In to ^{116}Sn a change in the number of neutron bosons. The parameters in the boson-fermion Hamiltonians are determined in a study of the corresponding odd-even nucleus: V_ν^{BF} from the odd-neutron nucleus ^{117}Sn and V_π^{BF} from the odd-proton nucleus ^{117}In.

The parameters for the even-even nucleus are taken from the literature [22]. They are shown in Table I. In Figs. 1 and 2 we compare the calculated spectra of ^{116}Cd and ^{116}Sn with the experimental data [24]. We can see that the agreement is quite good [25].

The second step is to construct the two odd-even associated nuclei, i.e., the core nucleus plus an extra neutron and the core nucleus plus an extra proton. It allows us to get a reliable set of parameters that we use to construct the odd-odd wave function. In this case, the odd-even nuclei are described in the context of the IBFM-2 [19,20,26], where the degrees of freedom of the extra nucleon are taken into account. In the IBFM-2, the Hamiltonian is given by

$$H = H^B + H_\rho^F + V_\rho^{BF}, \tag{12}$$

where H^B is the boson Hamiltonian that describes the core nucleus, and the label ρ refers to the π (extra proton) or ν (extra neutron) is added in the even-even core to form the odd-even nucleus.

The quasiparticle energies and occupation probabilities were obtained solving the BCS equations with the single-particle energies calculated and reported in Table II. The value of λ is constrained to the conservation of the number of particles as follows:

$$2N_\rho = \sum_{j_\rho} v_{j_\rho}^2 (2j_\rho + 1). \tag{13}$$

The odd-odd ^{116}In nucleus uses the quasiparticle energies obtained from the single-particle energies for the odd-neutron nucleus ^{117}Sn and the odd-proton nucleus ^{117}In. Both odd-

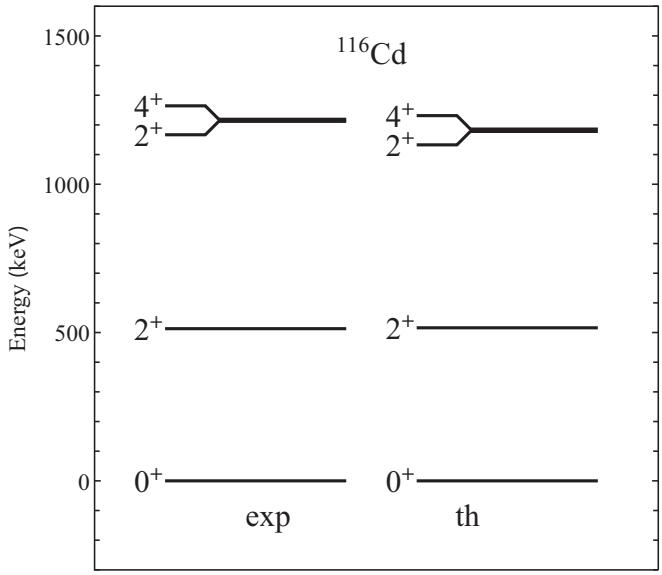

FIG. 1. The energy levels obtained in the calculation in comparison with the available experimental data for even-even nucleus ^{116}Cd [24].

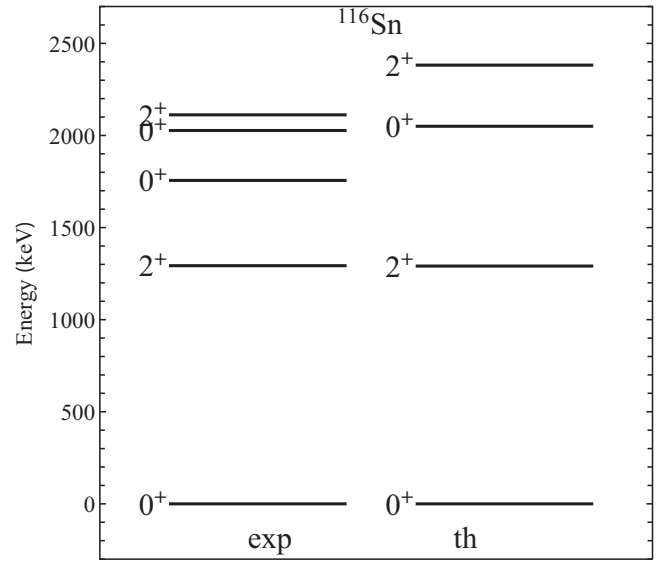

FIG. 2. The spectrum of ^{116}Sn in comparison with the experimental data [24].

TABLE II. Single-particle energies E_j (MeV), quasiparticle energies ϵ_j (MeV), and occupation probabilities v_j^2.

	E_j	ϵ_j	v_j^2
$2\nu d_{5/2}$	0.20	2.17	0.93
$1\nu g_{7/2}$	0.60	1.84	0.90
$3\nu s_{1/2}$	2.10	1.11	0.48
$1\nu h_{11/2}$	3.00	1.45	0.18
$2\nu d_{3/2}$	2.60	1.23	0.28
$2\pi p_{1/2}$	1.05	2.69	0.96
$2\pi p_{3/2}$	0.20	3.48	0.97
$1\pi f_{5/2}$	0.45	3.25	0.97
$1\pi g_{9/2}$	1.50	2.29	0.94

even nuclei have the same even-even core nucleus ^{118}Sn. The core-particle coupling parameters reported in Table III were obtained by fitting the experimental data for ^{117}Sn and ^{117}In.

For the odd-odd nucleus ^{116}In, the tensor and surface delta interaction (SDI) play an essential role. In this work, the parameters are obtained by minimization of the mean square error (MSE) on the test set given by $\text{MSE} = \frac{1}{N}\sum_n^N (f_n(x) - c_n)^2$, where $f_n(x)$ are our theoretical calculations and c_n are the experimental values [24]. We are interested in studying the low-lying spectra of the odd-odd nucleus. We consider the first three levels of ^{116}In, and this procedure may be applied to as many levels as are needed. We computed the mean square error for different values of the parameters tensor and SDI. We see that the MSE is very sensitive to the parameters, as shown in Fig. 3. The theoretical energy surface of the 1^+ ground state of ^{116}In is depicted in Fig. 4.

The minimization procedure consists of the creation of an uninterpolated contour surface with a given set of parameters over a grid of dimensions at least of dimension 11×11. After that, we generate an interpolated function of a grid dimension 30×30, which is used to determine the best parameters to fit the experimental data. The fitting method is given by the Nelder-Mead simplex algorithm to find the minimum of a function of one or more variables. The library used is the that uses NUMPY, the machine learning Python library used for scientific and technical computing. We obtain the minimum of the surface for $V_\delta = -0.40$ MeV and $V_T = 0.80$ MeV with min MSE $= 0.011$ MeV. The remaining parameters are zero (see Table IV). The result is depicted in Fig 5.

The spectrum of ^{116}In is presented in Fig. 6.

This method also can be used for the fitting parameters for the odd-even nuclei and even-even nuclei. In this work, the parameters are fitted to reproduce the experimental level energies and are depicted in Fig. 6 and reported in Table V. In the following section, we compute the spectroscopic amplitudes by using the previous nuclear wave functions.

TABLE III. Parameters in boson-fermion interaction in MeV.

ρ	Γ_ρ	Λ_ρ	A_ρ
π	0.2	0.0	−0.8
ν	0.0	0.0	0.0

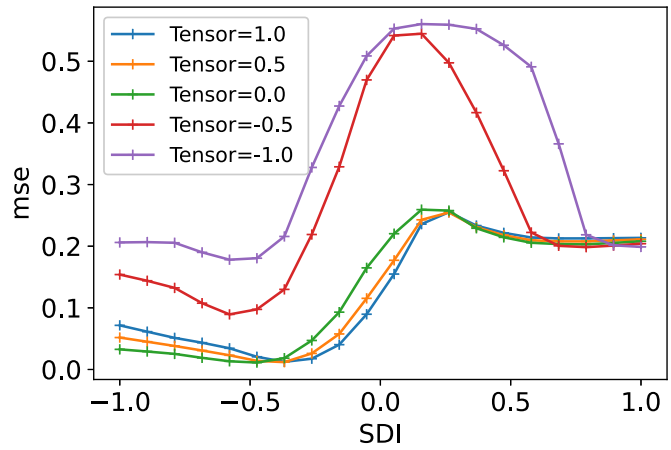

FIG. 3. Values of the values of the mean square error (MSE) for different values of the parameters SDI V_δ and tensor V_T.

IV. SPECTROSCOPIC AMPLITUDES

The microscopic theory for direct charge exchange process by heavy ions following the approach by Greiner [27] and Etchogoyen [28] requires the one-body transition densities (OBTDs) that are needed in the form factors of the direct charge exchange reactions. Nuclear initial and final states can be presented in the proton-neutron formalism; the one-body transition densities without the isospin indices can be represented by

$$\text{OBTD}(AB;\lambda) = \frac{\langle J_B \| [c^\dagger_{j_\pi} \times \tilde{c}_{j_\nu}]^{(\lambda)} \| J_A \rangle}{\sqrt{2\lambda+1}}, \tag{14}$$

where $|J_A\rangle$ represents the vector of the state of the initial nucleus and $|J_B\rangle$ the final nucleus, and the tilde denotes a time-conjugated state $\tilde{c}_j = (-1)^{j-m} c_{j,-m}$. We define the spectroscopic amplitudes (SAs) as the the reduced matrix element $\langle J_B \| [c^\dagger_{j_\pi} \times \tilde{c}_{j_\nu}]^{(\lambda)} \| J_A \rangle$.

In the isospin formalism, the nuclear states have a certain isospin, which results in a nontrivial isospin factor. The IBFFM can be mapped into the proton-neutron formalism

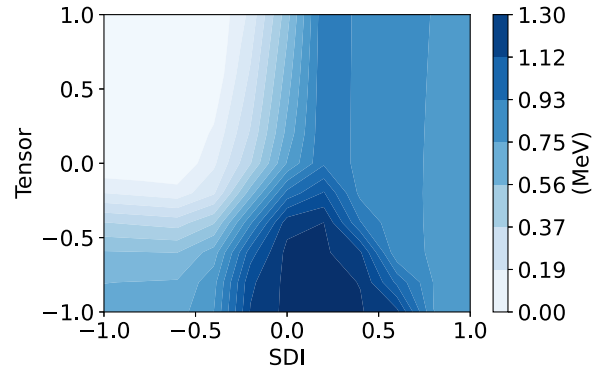

FIG. 4. Energy surface for the 1^+ ground state of ^{116}In.

TABLE IV. Parameters of proton-neutron residual interaction in MeV.

Parameter	Value
V_δ	−0.40
$V_{\sigma\sigma\delta}$	0.00
$V_{\sigma\sigma}$	0.00
V_T	0.80

allowing one to derive the one-body transition densities for the bosonic-fermionic space. In the followings sections, we present the derivation of the OBTD operator for the IBFFM framework, and we study the transitions ^{116}Sn to ^{116}In and ^{116}In to ^{116}Cd, that are of interest for the NUMEN experiment.

V. TRANSITION OPERATOR

The one-body transition density operator from even-even to odd-odd in the IBFFM formalism can be obtained considering the mapping from the fermionic space into the boson-fermion-fermion space. The operator for one-nucleon transfer in which the number of bosons is conserved is given by [21]

$$c^\dagger_{j_\rho} \to P^\dagger_{j_\rho} = \xi_{j_\rho} a^\dagger_{j_\rho} + \sum_{j'_\rho} \xi_{j_\rho j'_\rho} [[s^\dagger_\rho \times \tilde{d}_\rho]^{(2)} \times a^\dagger_{j'_\rho}]^{(j_\rho)}, \quad (15)$$

where $s^\dagger$ is the creation operator and $s = \tilde{s}$ the annihilation operator of the s boson, and $\tilde{d}$ is related to the d-boson annihilation operator by $\tilde{d}_\mu = (-1)^\mu d_{-\mu}$.

In case the number of bosons is changed by one unit,

$$c^\dagger_{j_\rho} \to Q^\dagger_{j_\rho} = \theta_{j_\rho} (s^\dagger_\rho \times \tilde{a}_{j_\rho})^{(j_\rho)} + \sum_{j'_\rho} \theta_{j_\rho j'_\rho} [d^\dagger_\rho \times \tilde{a}_{j'_\rho}]^{(j_\rho)}, \quad (16)$$

The coefficients ξ_{j_ρ}, $\xi_{j_\rho j'_\rho}$ and θ_{j_ρ}, $\theta_{j_\rho j'_\rho}$ are defined in Refs. [21,26].

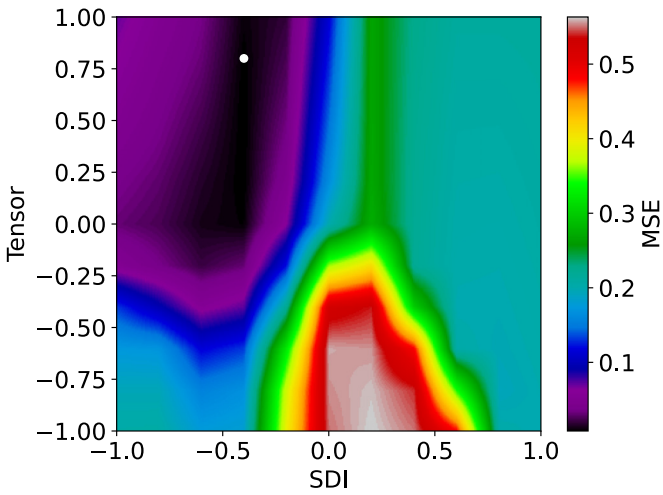

FIG. 5. Interpolated surface of the mean square error. The minimum of the surface is shown with the white dot.

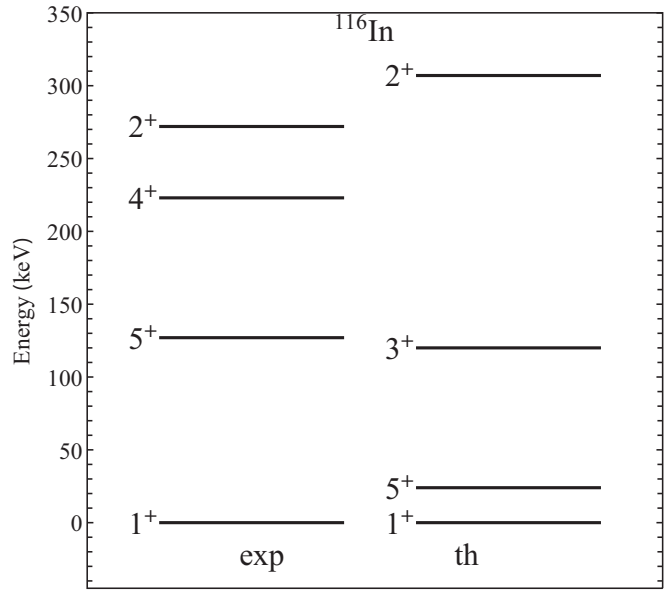

FIG. 6. Energy spectrum of ^{116}In.

The transitions between even-even nuclei and odd-odd nuclei can be computed by considering the tensorial product of the transfer operator of a proton and a neutron coupled to the angular momenta λ, which is the value of the spin of the final state of the odd-odd nucleus. For the transition between ^{116}Cd and ^{116}In, the one-body transition in the IBM and its extensions is given by

$$T^{(\lambda)}_{j_\nu j_\pi} = \left[P^\dagger_{j_\nu} \times \tilde{Q}_{j_\pi}\right]^{(\lambda)} \quad (17)$$

and for the transition between ^{116}In and ^{116}Sn by

$$T^{(\lambda)}_{j_\nu j_\pi} = \left[Q^\dagger_{j_\nu} \times \tilde{P}_{j_\pi}\right]^{(\lambda)} \quad (18)$$

We proceed to compute the matrix elements of the operators of Eqs. (17) and 18.

A. Spectroscopic amplitudes in IBFFM

The model space of transition is given by the model space of the odd-odd nucleus. The model is given by five active neutron orbitals, $2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $1h_{11/2}$, and $2d_{3/2}$, and by four active proton orbitals, $2p_{1/2}$, $2p_{3/2}$, $1f_{5/2}$, and $1g_{9/2}$.

TABLE V. Comparison of theoretical and experimental energy levels in keV of the odd-odd nucleus ^{116}In [24].

Expt.		Theor.	
J^P	E	J^P	E
1^+	0.0	1^+	0.0
5^+	127.3	5^+	23.9
4^+	223.3	3^+	120.3
2^+	273.0	2^+	307.9
$4^+, 5^+$	313.5	6^+	359.8
4^+	425.9	4^+	373.9
$4^+, 5^+$	460.0	3^+	401.7
3^+	508.2	5^+	489.8
		4^+	491.1

TABLE VI. Spectroscopic amplitudes (SAs) and one-body transition densities (OBTDs) for transitions from the 0_1^+ ground state of ^{116}Sn and ^{116}Cd to the J_i^P states of ^{116}In.

^{116}In			^{116}Sn(0_1^+)		^{116}Cd(0_1^+)	
J_i^P	j_ν	j_π	SA	OBTD	SA	OBTD
1_1^+	$1g_{7/2}$	$1g_{9/2}$	0.0650	0.0375	−0.0788	−0.0455
5_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0300	−0.0090	0.0055	0.0016
	$1g_{7/2}$	$1g_{9/2}$	−0.0270	−0.0081	0.0187	0.0056
	$3s_{1/2}$	$1g_{9/2}$	−0.0677	−0.0204	0.2289	0.0690
	$2d_{3/2}$	$1g_{9/2}$	−0.0338	−0.0102	0.1576	0.0475
	$1h_{11/2}$	$2p_{1/2}$	0.0134	0.0040	−0.1451	−0.0438
	$1h_{11/2}$	$2p_{3/2}$	−0.0045	−0.0014	0.0666	0.0201
	$1h_{11/2}$	$1f_{5/2}$	0.0042	0.0013	−0.0554	−0.0167
4_1^+	$2d_{5/2}$	$1g_{9/2}$	0.0083	0.0028	−0.0226	−0.0075
	$1g_{7/2}$	$1g_{9/2}$	0.0195	0.0065	0.0015	0.0005
	$3s_{1/2}$	$1g_{9/2}$	0.0271	0.0090	−0.1058	−0.0353
	$2d_{3/2}$	$1g_{9/2}$	0.0323	0.0108	−0.2125	−0.0708
	$1h_{11/2}$	$2p_{3/2}$	0.0006	0.0002	−0.0050	−0.0017
	$1h_{11/2}$	$1f_{5/2}$	0.0032	0.0011	−0.0523	−0.0174
2_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0036	−0.0016	0.0055	0.0025
	$1g_{7/2}$	$1g_{9/2}$	0.0453	0.0203	−0.0667	−0.0298
4_2^+	$2d_{5/2}$	$1g_{9/2}$	0.0014	0.0005	0.0064	0.0021
	$1g_{7/2}$	$1g_{9/2}$	−0.0066	−0.0022	−0.0053	−0.0018
	$3s_{1/2}$	$1g_{9/2}$	0.0205	0.0068	−0.0884	−0.0295
	$2d_{3/2}$	$1g_{9/2}$	0.0138	0.0046	−0.0651	−0.0217
	$1h_{11/2}$	$2p_{3/2}$	−0.0001	−0.0000	0.0035	0.0012
	$1h_{11/2}$	$1f_{5/2}$	0.0007	0.0002	−0.0003	−0.0001
3_1^+	$2d_{5/2}$	$1g_{9/2}$	−0.0196	−0.0074	0.0262	0.0099
	$1g_{7/2}$	$1g_{9/2}$	−0.0359	−0.0136	0.0301	0.0114
	$2d_{3/2}$	$1g_{9/2}$	−0.0437	−0.0165	0.2567	0.0970
	$1h_{11/2}$	$1f_{5/2}$	0.0110	0.0042	−0.1473	−0.0557

The nuclear states of the ^{116}In system are in our model restricted to be $J_P = 0^+$, 1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, and 8^+, within the first 14 excited states. The calculated excitation energies are in MeV. We have decided to use the calculated rather than experimental energies because some of the experimentally known states may not correspond. In the calculations we study the positive parity states in the odd-odd nucleus ^{116}In. In the $0\nu\beta\beta$ nuclear matrix elements we are going to consider contributions from states with positive parity. This affects the selection rules of the available orbitals of the transition operator for neutrons and protons. The neutron and proton model space chosen in the IBFFM depends on the active protons' and neutrons' shells. The neutron's shell chosen for ^{116}In is the 50-82 shell, and for the protons the 28-50 shell. For the numerical calculations of the normalization coefficients of the single nucleon transfer of neutron and proton for particle and hole coupling, we have considered the first 14 excited states. The numerical results of the spectroscopic amplitudes are exhibited in Table VI for various cases of interest. We denote the spectroscopic amplitudes as SA1 for the transitions ^{116}Cd to ^{116}In and SA2 for the transitions ^{116}Sn to ^{116}In. It was found that the squared spectroscopic amplitudes for both transitions (SA1^2 and SA2^2) tends to zero within higher energies of 3 MeV (see Fig. 7). That means that the main contributions of the intermediate states are $\leqslant 3$ MeV. Therefore we do not need an infinite number of intermediate states to compute the SAs.

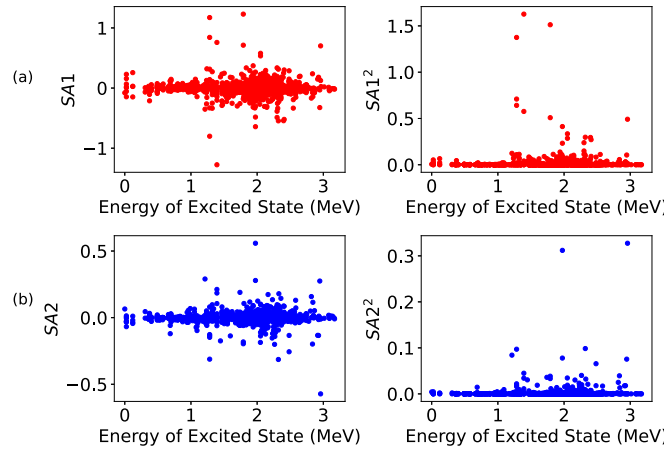

FIG. 7. (a) Spectroscopic amplitudes for the transitions ^{116}Cd → ^{116}In denoted as SA1 and (b) transitions ^{116}Sn → ^{116}In denoted as SA2. The squares of the spectroscopic amplitudes are denoted as SA1^2 and SA2^2 respectively.

B. Application to $0\nu\beta\beta$ decay in IBFFM in closure approximation

The neutrinoless double-beta decay operators are derived from the Feynman diagram of the $0\nu\beta\beta$ [30]. Considering the situation where the energy of the virtual neutrino ($|\vec{p}| \simeq 100$ MeV) is much larger than the excitation energy of the intermediate states: $|\vec{p}| \gg (E_n - E_i)$, and taking into account this inequality, we can replace the energies of the intermediate states E_n with the average energy $\langle E \rangle$,

$$E_k - (E_i + E_f)/2 \to \langle E \rangle; \tag{19}$$

then neutrino potential $H_\alpha(r)$, may be written in terms of integrals over the neutrino exchange momentum p with parity π:

$$H_\alpha(r) = \frac{2}{\pi} \int_0^\infty \frac{f_\alpha(pr)\tilde{h}_\alpha(p)p dp}{p + \langle E \rangle}, \tag{20}$$

where $f_{GT,F}(pr) = j_0(pr)$ and $f_T(pr) = j_2(pr)$ are spherical Bessel functions. The form factors $\tilde{h}_\alpha(p)$ are defined in Table II of Ref. [22] and they include vector and axial nucleon form factors that take into account nucleon size effects.

The nuclear matrix elements of the $0\nu\beta\beta$ decay can be calculated as

$$M_\alpha^{0\nu} = \sum_\kappa \sum_{j_1 j_2, j_1' j_2'} \langle j_1, j_2; J | \mathcal{O}_\alpha(r) | j_1', j_2'; J \rangle \times \langle f | c_2^\dagger \tilde{c}_1' | \kappa \rangle \langle \kappa | c_1^\dagger \tilde{c}_2' | i \rangle, \tag{21}$$

where $c_{j_\rho}^\dagger$ and $\tilde{c}_{j_{\rho'}}$ are the fermion creation and annihilation operators and $|i\rangle = |0_i^+\rangle$ represents the initial nucleus, $|f\rangle = |0_f^+\rangle$ the final nucleus, and $|\kappa\rangle = |J_\kappa^\pi\rangle$ intermediate nuclear states with certain angular momentum J_κ. The operators $\mathcal{O}_\alpha(r)$, $\alpha = \{GT, F, T\}$, contain neutrino potentials and spin and isospin operators. The standard operator for the $0\nu\beta\beta$ is given by [25]

$$\mathcal{O}_\alpha(r) = \frac{1}{2} \sum_{n,n'} \tau_n^+ \tau_{n'}^+ \left[\Sigma_n^{s_1} \times \Sigma_{n'}^{s_2} \right]^\lambda \cdot V(r) C^\lambda(\Omega_{nn'}), \tag{22}$$

where s_1 and s_2 can be only 0 or 1, $\Sigma^0 = 1$, and $\Sigma^1 = \vec{\sigma}$. There are thee contributions: Fermi ($s_1 = s_2 = \lambda = 0$), Gamow-Teller ($s_1 = s_2 = 1, \lambda = 0$) and tensor ($s_1, s_2 = 1, \lambda = 2$). The Gamow-Teller contribution is multiplied by a factor of $-\sqrt{3}$ and the tensor part $\sqrt{2/3}$ (see Ref. [25]). $V(r)$ is a generic radial form that depends on the mechanism of $0\nu\beta\beta$ and $C^\lambda = \sqrt{4\pi/(2\lambda+1)}Y^\lambda$. In the proton-neutron space, by using the standard second quantized form [32], the nuclear matrix elements of Eq. (21) between the initial nuclear states $\Phi^{(A')}_{J'}$ and the final nuclear states $\Phi^{(A'')}_{J_k}$ with odd-odd intermediate states $\Phi^{(A)}_{J}$ may be written as

$$\begin{aligned} M^{0\nu}_{\alpha} = -\frac{1}{4}\sum_{j_1 j_2}\sum_{j_{1'} j_{2'}}\sum_{J}\sum_{k}(-1)^J \\ \times\sqrt{1+(-1)^J\delta_{j_1 j_2}}\sqrt{1+(-1)^J\delta_{j'_1 j'_2}} \\ \times\sqrt{2J+1}\langle j_1, j_2; J|\mathcal{O}_\alpha(r)\big|j'_1, j'_2; J\rangle \\ \times\big\langle\Phi^{(A')}_{gs}\big|(c^\dagger_{n_2 l_2 j_2}\tilde{c}_{n'_1 l'_1 j'_1})^{(J_k)}\big|\Phi^{(A'')}_{J_k}\big\rangle \\ \times\big\langle\Phi^{(A'')}_{J_k}\big|(c^\dagger_{n_1 l_1 j_1}\tilde{c}_{n'_2 l'_2 j'_2})^{(J_k)}\big|\Phi^{(A)}_{gs}\big\rangle. \end{aligned} \tag{23}$$

For the numerical calculations we consider the active neutrons and protons of identical nucleons coupled to $J = 0, 2$, and the value of the nuclear matrix elements will be obtained from the IBFFM space by

$$\begin{aligned} &\langle\Psi(^{116}\mathrm{Sn})_{\mathrm{gs}}|[Q^\dagger_{j_\nu}\times\tilde{P}_{j'_\pi}]^{(\lambda)}|\Psi(^{116}\mathrm{In})_J\rangle, \\ &\langle\Psi(^{116}\mathrm{In})_J|[P^\dagger_{j_\nu}\times\tilde{Q}_{j'_\pi}]^{(\lambda)}|\Psi(^{116}\mathrm{Cd})_{\mathrm{gs}}\rangle. \end{aligned} \tag{24}$$

The closure energy used for the neutrinoless double beta decay is 12.1 MeV, the same value used in the IBM-2 closure calculation [22].

We consider the light neutrino potential $V(r)$ as in Ref. [25]. In the calculation of the matrix elements, two corrections are included: the finite nucleon size (FNS) and the short range correlations (SRCs). The short-range correlations are usually taken into account by multiplying the potential $H(r)$ by the Jastrow function squared, $f(r)^2$, with $f(r) = 1 - e^{-ar^2}(1 - br^2)$, where $a = 1.1$ fm^{-2} and $b = 0.68$ fm^{-2}. The finite nucleon size (FNS) is taken into account by taking the coupling constants g_V and g_A as momentum dependent,

$$\begin{aligned} g_V(p^2) &= \frac{g_V}{(1+p^2/M_V^2)^2}, \\ g_A(p^2) &= \frac{g_A}{(1+p^2/M_A^2)^2}. \end{aligned} \tag{25}$$

The value of M_V is well fixed by the electromagnetic form factor of the nucleon, $M_V^2 = 0.71$ (GeV/c^2)^2 [33], and $g_V = 1$ by the hypothesis of conserved vector current (CVC). The value of M_A is estimated to be $M_A = 1.09$ (GeV/c^2)^2 [34] and $g_A = 1.269$.

We have considered the high order corrections (HOCs) [31] of the nuclear matrix elements in neutrinoless double beta decay, denoted by VV, MM, AA, PP, and AP, indicating their origin from the vector, weak-magnetism, axial, and pseudoscalar couplings and the interference of the axial-vector and pseudoscalar couplings, respectively.

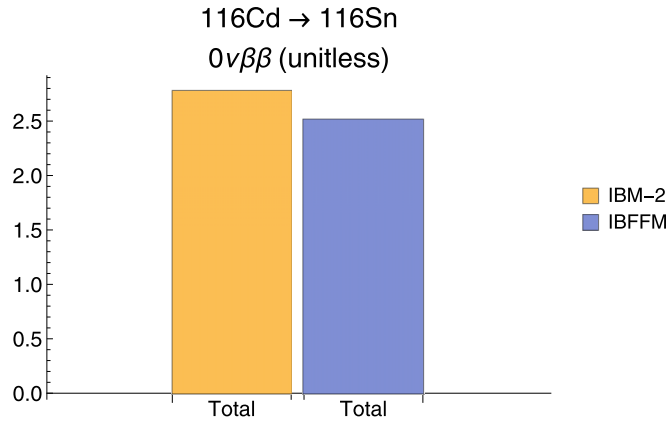

FIG. 8. Numerical results of nuclear matrix elements of the $0\nu\beta\beta$ decay in closure approximation by using IBFFM (yellow) and IBM-2 (blue) The values of the nuclear matrix elements are dimensionless.

We have calculated the nuclear matrix elements in the IBFFM formalism and in IBM-2 [25], and our results, expressed in units of fm^{-1}, are reported in Table VII. The comparison between our results in IBFFM with IBM-2 from Ref. [22] are depicted in Fig. 8. The Fermi, Gamow-Teller, and tensor contributions and the total $M^{0\nu}$ values are reported in the last four columns of Table VII. To make them dimensionless the nuclear matrix elements are multiplied by $2R$, where $R = 1.2 \times A^{1/3}$ is the nuclear radius in fm.

VI. SUMMARY AND CONCLUDING REMARKS

We presented for the first time the explicit operator of the IBFFM for calculating one-body transition densities in terms of active orbitals. We derived the odd-even and odd-odd parameters by implementing a method of fitting using machine learning libraries, which allowed computing more realistic nuclear wave functions, and we presented the spectroscopic amplitudes needed to calculate the double charge exchange without closure. For spins of intermediate states higher than 3 MeV, the SA tends to zero. To summarize, the spectroscopic amplitudes of the ^{116}Cd, ^{116}Sn, and ^{116}In nuclei are investigated within the IBFFM approach. The boson core Hamiltonian and the particle-core couplings were determined in a study of the even-even core nucleus and the neighboring odd-proton and odd-neutron nuclei, were determined.

The only remaining free parameters are a few coupling constants of the boson-fermion Hamiltonians and the residual neutron-proton interaction, and they are determined to reasonably reproduce the low-energy levels of each of the neighboring odd-A and odd-odd nuclei. The IBM and IBFFM wave functions were obtained after diagonalization of the corresponding Hamiltonians for the parent and daughter nuclei. For the first time, the parameters of the odd-odd ^{116}In are presented. This is attributed to the combination of various factors adopted in the theoretical procedure, such as the chosen boson-fermion coupling constants, residual neutron-proton interaction, and underlying microscopic inputs. The spectroscopic amplitudes converge to zero after a certain

TABLE VII. ^{116}Cd $\rightarrow$ ^{116}Sn $0\nu\beta\beta$ matrix elements in IBFFM and IBM-2 in closure approximation with $g_V = 1$ and $g_A = 1.269$ [29]. Here $M^{0\nu} = -(\frac{g_V}{g_A})^2 M_F^{(0\nu)} + M_{GT}^{(0\nu)} + M_T^{(0\nu)}$.

	AA+VV	AP	PP	MM	Sum (fm^{-1})	$M_F^{(0\nu)}$	$M_{GT}^{(0\nu)}$	$M_T^{(0\nu)}$	$M^{0\nu}$
IBM-2	0.251	−0.0304	0.00696	0.00925	0.237	−0.2246	2.481	0.152	2.77
IBFFM	0.228	−0.0236	0.00497	0.00558	0.214	−0.121	2.34	0.0917	2.51

energetic value of the odd-odd nuclei's intermediate states, as shown in Fig 7.

As a first application of the spectroscopic amplitudes obtained in the IBFFM formalism, we presented the evaluation of $0\nu\beta\beta$ nuclear matrix elements in closure approximation and compared them with the $0\nu\beta\beta$ nuclear matrix elements in the framework of the IBM-2 closure approximation. We found that the closure approximation in the IBFFM gives smaller NME, $M^{0\nu}$, by about 9.46%. Increasing the number of states does not increase the value of the matrix elements because the spectroscopic amplitudes converges to zero after 3 MeV. Both in the IBFFM and the IBM-2 calculations the Gamow-Teller is the main contribution. A first application of the spectroscopic amplitudes obtained in the IBFFM formalism is the the evaluation of the $0\nu\beta\beta$ nuclear matrix elements in closure approximation. A complete calculation without closure approximation will be the main subject of future research [35].

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