Double-β transition $0^+ \to 2^+$ within a fully renormalized proton-neutron quasiparticle random-phase approximation with the gauge symmetry restored

A. A. Raduta,[1,2] C. M. Raduta,[1] and R. Poenaru[1,3]

[1]*Department of Theoretical Physics, Institute of Physics and Nuclear Engineering, P.O. Box MG6, Bucharest 077125, Romania*
[2]*Academy of Romanian Scientists, 54 Splaiul Independentei, Bucharest 050094, Romania*
[3]*Doctoral School of Physics, University of Bucharest, Bucharest 050663, Romania*

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The $2\nu\beta\beta$ decay from the ground to the first excited 2^+ state is considered for eight nuclei where experimental data are available. The transition amplitude is calculated with a projected spherical single-particle basis, a fully renormalized proton-neutron quasiparticle random-phase approximation (pnQRPA) for the Gamow-Teller (GT) dipole transition operator and a renormalized QRPA for charge conserving quadrupole operators. Also for the transition operator the first-order boson expansion expression is employed. Using the phase space integral corresponding to the transition $0^+ \to 2^+$ and the GT transition amplitude, the process half-life is readily obtained. The single-$\beta^{\mp}$ transition strengths are studied as function of the energies of the fully renormalized pnQRPA with the gauge symmetry restored. The single-β transition operators are used to calculate the $\log_{10} ft$ values for the electron capture of the intermediate odd-odd nucleus to the mother nucleus as well as for the β^- transition to the daughter nucleus. For the final state in the daughter nucleus the $B(E2)$ values for the transition $2^+ \to 0^+$ and the half-life of the electromagnetic decaying state are calculated. The Ikeda sum rule for the mother nucleus is satisfied. The mentioned results are compared with the corresponding available data and a reasonable agreement is shown. The gauge projection quenches the half-life in the case of ^{150}Nd, ^{116}Cd, and ^{100}Mo and enhances it for the remaining considered nuclei. Keeping the same parameters for the model Hamiltonian the ground to ground double-β transition is also treated and a good agreement with the existing data is obtained.

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I. INTRODUCTION

One of the hot subjects of nuclear physics concerns double-β decay. The process may take place through two modes, neutrinoless double β ($0\nu\beta\beta$) and double β with two neutrinos in the final state ($2\nu\beta\beta$). The first mode is most interesting since its existence decides whether a neutrino is a Dirac or a Majorana particle. However, there is no direct reliable test for the matrix elements which are used for the $0\nu\beta\beta$ process.

The $2\nu\beta\beta$ decay is interesting on its own but is also very attractive since it provides a test for some matrix elements which are used for the process of $0\nu\beta\beta$. The history of the subject was outlined in several review papers [1–11]. The formalism which yields closest results to the experimental data is the proton-neutron quasiparticle random-phase approximation (pnQRPA) involving particle-hole (ph) and particle-particle (pp) as independent interactions. This is caused by the fact that the second leg of the decay matrix element is very sensitive to changing the relative strength of the pp interaction. Since the pp interaction is attractive, increasing its strength results in the transition amplitude decreasing and consequently one reaches a critical value where the first root of the pnQRPA becomes imaginary. In the decreasing interval one also meets the value which corresponds to the experimental data. Unfortunately this value lies close to the critical value where the ground state is unstable. To stabilize the ground state we have to go beyond the pnQRPA approach, which was achieved either by the boson expansion method [12,13] applied to the Gamow-Teller (GT) transition operator or by renormalizing the pnQRPA equation [14]. Later on, the renormalization procedure was improved by adding the scattering terms [15]. The drawback of the higher pnQRPA formalisms is that the Ikeda sum rule is violated. Restoring the sum rule is a challenge for any description of the process. It is worth mentioning that at the pnQRPA level the transition from the ground to the excited 2^+ state is forbidden. However, going beyond the pnQRPA, such a process is allowed. Moreover, there are experimental data concerning the low bounds of the half-life of the transition. A large volume of work has been devoted to the description of the relevant properties for the double-β transition $0^+ \to 2^+$ [11,16–31].

It is commonly accepted that double-β decay takes place via two successive virtual β^- decays. It is a natural question whether by replacing the β^- with β^+ the resulting process exists or not. A possible answer was attempted in Ref. [32]. This subject was *in extenso* treated in a review paper [33] about neutrinoless double electron capture. It was shown that the process exhibits a resonance when the initial and final states are degenerate.

In the present paper we study the double-β decay on the first excited 2^+ state within a formalism of fully renormalized pnQRPA with the gauge symmetry restored (GRFRpnQRPA).

It is well established that any symmetry is associated with the conservation of a certain physical quantity. In the present context the gauge symmetry determines the conservation of the total number of nucleons. The mentioned symmetry reflects the system invariance to rotations around the z axis in the space of quasispin [34]. This is different from the gauge symmetry for electromagnetic interaction where the charge is conserved whenever a U(1) transformation (a phase factor) is performed. Note that although both the third isospin component, T_3, and the nuclear charge, Q, are not preserved in the double-β process, the total number of nucleons, $N + Z = 2(-T_3 + Q/e)$, is conserved.

One important ingredient of our approach is the use of the projected spherical single-particle basis whose main properties are briefly described in Sec. II. The many-body Hamiltonian which describes the ground to 2^+ transitions is introduced in Sec. III. In Sec. IV one describes the fully renormalized pnQRPA, while the gauge symmetry is projected out in Sec. V. The amplitude of the GT transition is considered in Sec. VI and the numerical calculations are presented in Sec. VII. The summary and final conclusions are given in Sec. VIII.

II. PROJECTED SINGLE-PARTICLE BASIS

The angular momentum projected single-particle basis, defined in Ref. [35], seems to be suitable for the description of the single-particle motion in a deformed mean field generated by the particle-core interaction. Such a projected spherical single-particle basis has been used to study the collective $M1$ states in deformed nuclei [35–37] as well as the rate of the double-β process [38–41].

Other groups also used various deformed single-particle bases corresponding to specific deformed mean field potentials, like the SU(3), Nilsson, Skyrme interaction, or Woods-Saxon potential, to evaluate the double-β decay rate [42–51].

To fix the necessary notations and moreover for the sake of a self-contained presentation, we describe briefly the main ideas underlying the construction of the projected single-particle basis.

The single-particle mean field is determined by a particle-core Hamiltonian:

$$\tilde{H} = H_{sm} + H_{\rm core} - M\omega_0^2 r^2 \sum_{\lambda=0,2}\sum_{-\lambda\leqslant\mu\leqslant\lambda} \alpha^*_{\lambda\mu} Y_{\lambda\mu}, \tag{2.1}$$

where H_{sm} denotes the spherical shell model Hamiltonian, while $H_{\rm core}$ is a harmonic quadrupole boson (b^+_μ) Hamiltonian associated to a phenomenological core. The interaction of the two subsystems is accounted for by the third term of the above equation, written in terms of the shape coordinates $\alpha_{00}, \alpha_{2\mu}$. The quadrupole coordinates are related to the quadrupole boson operators by the canonical transformation:

$$\alpha_{2\mu} = \frac{1}{k\sqrt{2}}(b^\dagger_{2\mu} + (-)^\mu b_{2,-\mu}), \tag{2.2}$$

where k is an arbitrary C number. The monopole shape coordinate is to be determined from the volume conservation condition.

Averaging $\tilde{H}$ on a given eigenstate of H_{sm}, denoted as usual by $|nl\, jm\rangle$, one obtains a deformed quadrupole boson Hamiltonian which admits the axially symmetric coherent state

$$\Psi_g = \exp[d(b^+_{20} - b_{20})]|0\rangle_b \tag{2.3}$$

as eigenstate. $|0\rangle_b$ stands for the vacuum state of the boson operators while d is a real parameter which simulates the nuclear deformation. However, averaging $\tilde{H}$ on Ψ_g, one obtains a single-particle mean field operator for the single-particle motion, similar to the Nilsson Hamiltonian. Concluding, by averaging the particle-core Hamiltonian with a factor state the rotational symmetry is broken and the mean field mentioned above may generate, by diagonalization, a deformed basis for treating the many-body interacting systems. However, this standard procedure is tedious since the final many-body states should be projected over the angular momentum.

Our procedure defines first a spherical basis for the particle-core system, by projecting out the angular momentum from the deformed state:

$$\Psi^{pc}_{nlj} = |nl\, jm\rangle\Psi_g. \tag{2.4}$$

The projected states are obtained, in the usual manner, by acting on these deformed states with the projection operator

$$P^I_{MK} = \frac{2I+1}{8\pi^2}\int {D^I_{MK}}^*(\Omega)\hat{R}(\Omega)d\Omega. \tag{2.5}$$

We consider the subset of projected states

$$\Phi^{IM}_{nlj}(d) = \mathcal{N}^I_{nlj} P^I_{MI}[|nl\, jI\rangle\Psi_g] \equiv \mathcal{N}^I_{nlj}\Psi^{IM}_{nlj}(d), \tag{2.6}$$

which are orthonormalized and form a basis for the particle-core system. This basis exhibits useful properties which have been presented in some of our previous publications.

To the projected spherical states, one associates the "deformed" single-particle energies defined as the average values of the particle-core Hamiltonian $H' = \tilde{H} - H_{\rm core}$:

$$\epsilon^I_{nlj} = \langle \Phi^{IM}_{nlj}(d)\big|H'\big|\Phi^{IM}_{nlj}(d)\rangle. \tag{2.7}$$

Since the core contribution to this average value does not depend on the quantum numbers of the single-particle energy levels, it produces a constant shift for all energies. For this reason such a term is omitted in Eq. (2.7). The deformation dependence of the new single-particle energies is similar to that shown by the Nilsson model [52]. Therefore, the average values ϵ^I_{nlj} may be viewed as approximate single-particle energies in deformed Nilsson orbits [52]. We may account for the deviations from the exact eigenvalues by considering, at a later stage when a specific treatment of the many-body system is performed, the exact matrix elements of the two-body interaction. The dependence of single-particle energies on deformation parameter d is shown in Fig. 1 for protons and neutrons, respectively, in the major shell with $N = 5$ and $N = 6$.

Although the energy levels are similar to those of the Nilsson model, the quantum numbers in the two schemes are different. Indeed, here we generate from each j a multiplet of $(2j + 1)$ states distinguished by the quantum number I, which plays the role of the Nilsson quantum number Ω and runs from $1/2$ to j. Moreover, the energies corresponding to the quantum

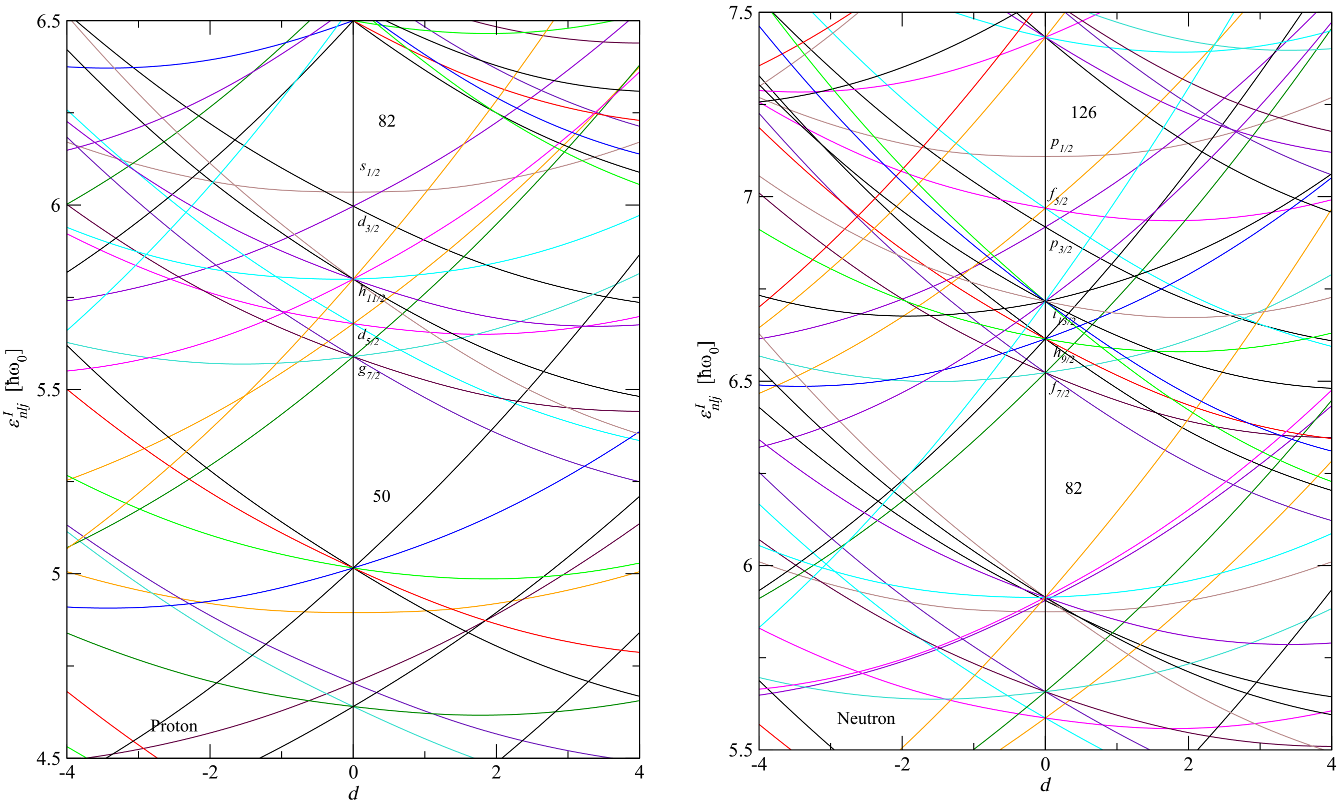

FIG. 1. Proton and neutron single-particle energies in the region of $N = 5$ and $N = 6$ shells, respectively, given by Eq. (2.7) where the shell model parameters $\kappa = 0.0637$ and $\mu = 0.60$ for protons and $\mu = 0.42$ for neutrons were used. The canonical transformation constant is fixed to $k = 10$.

numbers K and $-K$ are equal to each other. However, for a given I there are $2I + 1$ degenerate substates, while the Nilsson states are only double degenerate. As explained in Ref. [15], the redundancy problem can be solved by changing the normalization of the model functions:

$$\langle \Phi^{IM}_\alpha | \Phi^{IM}_\alpha \rangle = 1 \Longrightarrow \sum_M \langle \Phi^{IM}_\alpha | \Phi^{IM}_\alpha \rangle = 2. \qquad (2.8)$$

Due to this weighting factor the particle density function is providing the consistency result that the number of particles which can be distributed on the $(2I + 1)$ substates is at most 2, which agrees with the Nilsson model. Here α stands for the set of shell model quantum numbers nlj. Due to this normalization, the states Φ^{IM}_α used to calculate the matrix elements of a given operator should be multiplied with the weighting factor $\sqrt{2/(2I+1)}$.

The projected states might be thought of as eigenstates of an effective rotational invariant fermionic one-body Hamiltonian $H_{\rm eff}$, with the corresponding energies given by Eq. (2.7):

$$H_{\rm eff}\Phi^{IM}_\alpha = \epsilon^I_\alpha(d)\Phi^{IM}_\alpha. \qquad (2.9)$$

As shown in Ref. [35] in the vibrational limit, $d \rightarrow 0$, the projected spherical basis goes to the spherical shell model basis and ϵ^I_{nlj} to the eigenvalues of H_{sm}.

A fundamental result obtained in Ref. [53] for the product of two single-particle states, which comprises a product of two core components, deserves to be mentioned. Therein, we have proved that the matrix elements of a two-body interaction corresponding to the present scheme are very close to the matrix elements corresponding to spherical states projected from a deformed product state with one factor being a product of two spherical single-particle states, and a second factor consisting of a unique collective core wave function. The small discrepancies of the two types of matrix elements could be washed out by using slightly different strengths for the two-body interaction in the two methods. Due to this property the basis (2.6) might be used for studying any two-body interaction.

As for the matrix elements of a one-body operator T^k_μ, the result is

$$\langle \Phi^I_{nlj} || T^k || \Phi^{I'}_{n'l'j'} \rangle = f^{n'l'j'I'}_{nljI}(d)\langle nlj||T^k||n'l'j'\rangle, \quad \text{with}$$

$$f^{n'l'j'I'}_{nljI}(d) = \mathcal{N}^I_{nlj}(d)\mathcal{N}^{I'}_{n'l'j'}(d)\hat{j}\hat{I}' \times \sum_J C^{j\,J\,I}_{I\,0\,I} C^{j'\,J\,I'}_{I'\,0\,I'} W(jkJI';j'I)\left(N^g_J\right)^{-2}. \qquad (2.10)$$

This expression is used to calculate the reduced matrix elements of the Gamow-Teller interaction as well as of the quadrupole interaction. $\mathcal{N}^I_{nlj}(d)$ denotes the norm of the projected spherical single-particle state, while N^g_J is the norm of the core projected state. Also, the Rose convention is used for the reduced matrix elements [54].

III. THE MODEL HAMILTONIAN

We suppose that the states describing the nuclei involved in a $2\nu\beta\beta$ process are described by a many-body Hamiltonian which may be written in the projected spherical basis as

$$\begin{aligned} H = & \sum_{\tau,\alpha,I,M} \frac{2}{2I+1}(\epsilon_{\tau\alpha I} - \lambda_{\tau\alpha}) c^\dagger_{\tau\alpha IM} c_{\tau\alpha IM} \\ & - \sum_{\tau,\alpha,I,I'} \frac{G_\tau}{4} P^\dagger_{\tau\alpha I} P_{\tau\alpha I'} \\ & + 2\chi \sum_{pn;p'n';\mu} \beta^-_\mu(pn)\beta^+_{-\mu}(p'n')(-)^\mu - 2\chi_1 \\ & \times \sum_{pn;p'n';\mu} P^-_\mu(pn) P^+_{-\mu}(p'n')(-)^\mu \\ & - \sum_{\mu,\tau} X_{\tau,\tau'} Q^\tau_\mu Q^{\tau'}_{-\mu'}(-)^\mu, \end{aligned} \tag{3.1}$$

where $c^\dagger_{\tau\alpha IM}$ ($c_{\tau\alpha IM}$) denotes the creation (annihilation) operator of one nucleon of the type $\tau(=p,n)$ in the state Φ^{IM}_α, with α being an abbreviation for the set of quantum numbers nlj. The Hamiltonian H contains the mean field term, the pairing interaction for alike nucleons and the Gamow-Teller dipole-dipole interaction in the ph and pp channels, characterized by the strengths χ and χ_1, respectively, and the quadrupole-quadrupole interaction.

To simplify the notations, hereafter the set of quantum numbers $\alpha(=nlj)$ will be omitted. All the two-body interactions are separable with the factors defined by the following expressions:

$$\begin{aligned} P^\dagger_{\tau I} &= \sum_M \frac{2}{2I+1} c^\dagger_{\tau IM} c^\dagger_{\widetilde{\tau IM}}, \\ \beta^-_\mu(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_\mu|nI'M'\rangle \frac{\sqrt{2}}{\hat{I}'} c^\dagger_{pIM} c_{nI'M'}, \\ P^-_{1\mu}(pn) &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle pIM|\sigma_\mu|nI'M'\rangle \frac{\sqrt{2}}{\hat{I}'} c^\dagger_{pIM} c^\dagger_{\widetilde{nI'M'}}, \\ Q^\tau_\mu &= \sum_{M,M'} \frac{\sqrt{2}}{\hat{I}} \langle \tau IM|\sqrt{\frac{16\pi}{5}} r^2 Y_{2_\mu}|\tau I'M'\rangle \frac{\sqrt{2}}{\hat{I}'}. \end{aligned} \tag{3.2}$$

The remaining operators from Eq. (3.1) can be obtained from the above-defined operators, by Hermitian conjugation.

Passing to the quasiparticle representation through the Bogoliubov-Valatin transformation,

$$\begin{aligned} a^\dagger_{\tau IM} &= U_{\tau I} c^\dagger_{\tau IM} - s_{IM} V_{\tau I} c_{\tau I -M}, \quad s_{IM} = (-)^{I-M}, \\ \tau &= p, n, \quad U^2_{\tau I} + V^2_{\tau I} = 1, \end{aligned} \tag{3.3}$$

the first two terms of H are replaced by the independent quasiparticle term, $\sum E_{\tau I} a^\dagger_{\tau IM} a_{\tau IM}$, while the ph and pp interactions are expressed in terms of the dipole two quasiparticle (qp) and the qp density operators:

$$\begin{aligned} A^\dagger_{1\mu}(pn) &= \sum C^{I_p\ I_n\ 1}_{m_p\ m_n\ \mu} a^\dagger_{pI_pm_p} a^\dagger_{nI_nm_n}, \\ A_{1\mu}(pn) &= (A^\dagger_{1\mu}(pn))^\dagger, \\ B^\dagger_{1\mu}(pn) &= \sum C^{I_p\ I_n\ 1}_{m_p\ -m_n\ \mu} a^\dagger_{pj_pm_p} a_{nI_nm_n}(-)^{I_n-m_n}, \\ B_{1\mu}(pn) &= (B^\dagger_{1\mu}(pn))^\dagger, \\ A^\dagger_{2\mu}(\tau\tau') &= \sum C^{I_\tau\ I_{\tau'}\ 2}_{m_\tau\ m_{\tau'}\ \mu} a^\dagger_{\tau I_\tau m_{\tau'}} a^\dagger_{\tau' I_{\tau'} m_{\tau'}}, \\ A_{2\mu}(\tau\tau') &= (A^\dagger_{2\mu}(\tau\tau'))^\dagger, \\ B^\dagger_{2\mu}(\tau\tau') &= \sum C^{I_\tau\ I_{\tau'}\ 2}_{m_\tau\ -m_{\tau'}\ \mu} a^\dagger_{\tau j_\tau m_\tau} a_{\tau' I_{\tau'} m_{\tau'}}(-)^{I_{\tau'}-m_{\tau'}}, \\ B_{2\mu}(\tau\tau') &= (B^\dagger_{2\mu}(\tau\tau'))^\dagger. \end{aligned} \tag{3.4}$$

IV. THE FULLY RENORMALIZED pnQRPA

A. The case of the proton-neutron interaction

In Ref. [15], we showed that all these operators can be renormalized as suggested by the commutation equations:

$$\begin{aligned} [A_{1\mu}(k), A^\dagger_{1\mu'}(k')] &\approx \delta_{k,k'}\delta_{\mu,\mu'}\left[1 - \frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right], \\ [B^\dagger_{1\mu}(k), A^\dagger_{1\mu'}(k')] &\approx [B^\dagger_{1\mu}(k), A_{1\mu'}(k')] \approx 0, \\ [B_{1\mu}(k), B^\dagger_{1\mu'}(k')] &\approx \delta_{k,k'}\delta_{\mu,\mu'}\left[\frac{\hat{N}_n}{\hat{I}^2_n} - \frac{\hat{N}_p}{\hat{I}^2_p}\right],\ k = (I_p, I_n). \end{aligned} \tag{4.1}$$

Indeed, denoting by $C^{(1)}_{I_p,I_n}$ and $C^{(2)}_{I_p,I_n}$ the averages of the right-hand sides of Eqs. (4.1) with the renormalized random-phase approximation (RPA) vacuum state, the renormalized operators defined as

$$\bar{A}_{1\mu}(k) = \frac{1}{\sqrt{C^{(1)}_k}} A_{1\mu}, \quad \bar{B}_{1\mu}(k) = \frac{1}{\sqrt{|C^{(2)}_k|}} B_{1\mu}, \tag{4.2}$$

obey bosonlike commutation relations:

$$\begin{aligned} [\bar{A}_{1\mu}(k), \bar{A}^\dagger_{1\mu'}(k')] &= \delta_{k,k'}\delta_{\mu,\mu'}, \\ [\bar{B}_{1\mu}(k), \bar{B}^\dagger_{1\mu'}(k')] &= \delta_{k,k'}\delta_{\mu,\mu'} f_k, \quad f_k = \mathrm{sign}\big(C^{(2)}_k\big). \end{aligned} \tag{4.3}$$

Further, these operators are used to define the phonon operator:

$$\begin{aligned} C^\dagger_{1\mu} = \sum_k & [X_1(k)\bar{A}^\dagger_{1\mu}(k) + Z_1(k)\bar{D}^\dagger_{1\mu}(k) - Y_1(k) \\ & \times \bar{A}_{1-\mu}(k)(-)^{1-\mu} - W_1(k)\bar{D}_{1-\mu}(k)(-)^{1-\mu}], \end{aligned} \tag{4.4}$$

where $\bar{D}^\dagger_{1\mu}(k)$ is equal to $\bar{B}^\dagger_{1\mu'}(k')$ or $\bar{B}_{1\mu}(k)$ depending on whether f_k is $+$ or $-$. The phonon amplitudes are determined by the equations

$$[H, C^\dagger_{1\mu}] = \omega C^\dagger_{1\mu}, \quad [C_{1\mu}, C^\dagger_{1\mu'}] = \delta_{\mu\mu'}. \tag{4.5}$$

Interesting properties for these equations and their solutions are discussed in our previous publications [15]. The renormalized pnQRPA (pnRQRPA) vacuum describes the ground state of the parent nucleus, while the excited one phonon states describe the dipole states of the intermedite odd-odd nucleus.

B. The case of the quadrupole-quadrupole (QQ) interaction

For the charge preserving operators we can apply similar considerations. Indeed, the commutators of the quadrupole two-quasiparticle and quadrupole quasiparticle density operators may be approximated as

$$\begin{aligned}
&[A_{2\mu}(\tau,a,b),A^{\dagger}_{2\mu'}(\tau',a',b')]\\
&\quad\approx \delta_{\tau,\tau'}\delta_{a,a'}\delta b,b'\delta_{\mu,\mu'}\left[1-\frac{\hat{N}_a}{\hat{I}_a^2}-\frac{\hat{N}_b}{\hat{I}_b^2}\right],\\
&[B^{\dagger}_{2\mu}(\tau,a,b),A^{\dagger}_{2\mu'}(\tau',a',b')]\\
&\quad\approx [B^{\dagger}_{1\mu}(\tau,a,b),A_{2\mu'}(\tau',a',b')]\\
&\quad\approx [B_{2\mu}(\tau,a,b),B^{\dagger}_{2\mu'}(\tau',a',b')]\approx 0.
\end{aligned}\tag{4.6}$$

Note that in contradistinction to the case of the *pn* interaction the quadrupole scattering operators are not normalizable to unity. As for the two quasiparticle operators, denoting by F_2 the average of the approximated commutators with the quadrupole boson vacuum,

$$F_2 = {}_2\langle 0|\left[1-\frac{\hat{N}_a}{\hat{I}_a^2}-\frac{\hat{N}_b}{\hat{I}_b^2}\right]|0\rangle_2,\tag{4.7}$$

one obtains the normalized to unity boson operator:

$$\bar{A}_{2\mu}(\tau,a,b)=\frac{1}{\sqrt{F_2}}A_{2\mu}(\tau,a,b).\tag{4.8}$$

Indeed it is conspicuous that

$$[\bar{A}_{2\mu}(\tau,a,b),\bar{A}^{\dagger}_{2\mu}(\tau',a',b')]=\delta_{\tau,\tau'}\delta_{a,a'}\delta_{b,b'}.\tag{4.9}$$

Therefore, we can define the quadrupole phonon operator:

$$\begin{aligned}
C^{\dagger}_{2\mu}=\sum&[X_2(\tau,a,b)\bar{A}^{\dagger}_{2\mu}(\tau,a,b)\\
&-Y_2(\tau,a,b)\bar{A}_{2,-\mu}(\tau,a,b)(-)^{\mu}],
\end{aligned}\tag{4.10}$$

such that the following two equations are fulfilled:

$$[H,C^{\dagger}_{2\mu}]=\omega_2C^{\dagger}_{2\mu},\quad [C_{2\mu},C^{\dagger}_{2\mu'}]=\delta_{\mu\mu'}.\tag{4.11}$$

The vacuum state $|0\rangle_2$ describes the ground state of the daughter nucleus, while the quadrupole phonon operator excites the daughter nucleus to a 2^+ state.

V. GAUGE PROJECTION OF THE FULLY RENORMALIZED pnQRPA EQUATIONS

The beautiful feature of the quasiparticle random-phase approximation (QRPA) formalism is that the so-called Ikeda sum rule (ISR) is exactly satisfied. The sum rule is considered a measure of how realistic is the approach which is used. Therefore, going beyond the QRPA we have to take care of the sum rule in order to get a consistent description. Unfortunately, the higher QRPA approaches violate the Ikeda sum rule. Indeed the sum rule is satisfied neither by the boson expansion procedure nor by the renormalization approach. However, in order to describe the double-β decay ground state to the first quadrupole phonon state we have to go beyond the QRPA level. It seems that renormalizing the QRPA equations underestimates the ISR, while the boson expansion overestimates it. This feature suggested use of the boson expansion on top of the renormalized pnQRPA state, i.e., that the Gamow-Teller transition operators be expanded in terms of the renormalized phonon operators. In this way the calculated ISR was brought close to the $N-Z$ value. We recall the fact that the pnQRPA which includes also the quasiparticle scattering terms is called the fully renormalized pnQRPA (FRpnQRPA) equation. The renormalized ground state, i.e., the vacuum state for the phonon operator defined by the FRpnQRPA approach, is a superposition of components describing the neighboring nuclei $(N-1,Z+1)$, $(N+1,Z-1)$, $(N+1,Z+1)$, $(N-1,Z-1)$. The first two components conserve the total number of nucleons $(N+Z)$ but violate the third component of isospin, T_3. By contrast, the last two components violate the total number of nucleons but preserve T_3. Actually, the last two components contribute to the violation of the ISR. One can construct linear combinations of the basic operators $A^{\dagger}, A, B^{\dagger}, B$ which excite the nucleus (N,Z) to the nuclei $(N-1,Z+1)$, $(N+1,Z-1)$, $(N+1,Z+1)$, $(N-1,Z-1)$, respectively. These operators are

$$\begin{aligned}
\mathcal{A}^{\dagger}_{1\mu}(pn)&=U_pV_nA^{\dagger}_{1\mu}(pn)+U_nV_pA_{1,-\mu}(pn)(-)^{1-\mu}\\
&\quad+U_pU_nB^{\dagger}_{1\mu}(pn)-V_pV_nB_{1,-\mu}(pn)(-)^{1-\mu}\\
&=-[c^{\dagger}_pc_{\tilde{n}}]_{1\mu},\\
\mathcal{A}_{1\mu}(pn)&=U_pV_nA_{1\mu}(pn)+U_nV_pA^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}\\
&\quad+U_pU_nB_{1\mu}(pn)-V_pV_nB^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}\\
&=-[c^{\dagger}_pc_{\tilde{n}}]^{\dagger}_{1\mu},\\
\mathbf{A}^{\dagger}_{1\mu}(pn)&=U_pU_nA^{\dagger}_{1\mu}(pn)-V_pV_nA_{1,-\mu}(pn)(-)^{1-\mu}\\
&\quad-U_pV_nB^{\dagger}_{1\mu}(pn)-V_pU_nB_{1,-\mu}(pn)(-)^{1-\mu}\\
&=[c^{\dagger}_pc^{\dagger}_n]_{1\mu},\\
\mathbf{A}_{1\mu}(pn)&=U_pU_nA_{1\mu}(pn)-V_pV_nA^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}\\
&\quad-U_pV_nB_{1\mu}(pn)-V_pU_nB^{\dagger}_{1,-\mu}(pn)(-)^{1-\mu}\\
&=[c^{\dagger}_pc^{\dagger}_n]^{\dagger}_{1\mu}.
\end{aligned}$$

Thus, the operators from the first two rows excite the nucleus (N,Z) to the nuclei $(N-1,Z+1)$ and $(N+1,Z-1)$, respectively, while the operators $\mathbf{A}^{\dagger}_{1\mu}(pn)$ and $\mathbf{A}_{1\mu}(pn)$ bring (N,Z) to $(N+1,Z+1)$ and $(N-1,Z-1)$, respectively. In the quasiparticle Hamiltonian we keep only the terms which preserve the total number of nucleons. Similarly, the quadrupole two-quasiparticle operators which conserve the total number of nucleons are

$$\begin{aligned}
\mathcal{A}^{\dagger}_{2\mu}(\tau,a,b)&=U_aU_bB^{\dagger}_{2\mu}(\tau,a,b)+U_aV_bA^{\dagger}_{2\mu}(\tau,a,b)\\
&\quad+V_aU_b(-)^{\mu}A_{2,-\mu}(\tau,a,b)\\
&\quad-V_aV_b(-)^{\mu}B_{2,-\mu}(\tau,a,b)\\
&=[c^{\dagger}_ac_{\tilde{b}}]_{2\mu}.
\end{aligned}\tag{5.1}$$

Note that the particle-particle interaction violates the gauge and therefore is neglected. However, one knows that an attractive interaction is necessary in order to bring the transition amplitude close to the experimental value. For that reason we introduced an attractive two-body interaction which preserves the total number of nucleons:

$$\Delta H = -X_{dp}\sum_{\substack{pn;p'\\ n';\mu}}(\beta^-_\mu(pn)\beta^-_{-\mu}(p'n') + \beta^+_{-\mu}(p'n')\beta^+_\mu(pn))(-1)^{1-\mu}. \tag{5.2}$$

The picture for the quadrupole interaction is opposite. Indeed, the interaction is attractive and the minimal two quasiparticle energies in the gauge invariance picture become close to zero and therefore a repulsive interaction is needed. The simplest form for such an interaction is a diagonal one. With these details the final Hamiltonian to be used looks like

$$\begin{aligned} H &= \sum_{\tau jm}E_{\tau j}a^\dagger_{\tau jm}a_{\tau jm} \\ &\quad + 2\chi\sum_{pn,p'n';\mu}\sigma_{pn;p'n'}\mathcal{A}^\dagger_{1\mu}(pn)\mathcal{A}_{1\mu}(p'n') \\ &\quad - X_{dp}\sum_{\substack{pn;p'\\ n';\mu}}\sigma_{pn;p'n'}(\mathcal{A}^\dagger_{1\mu}(pn)\mathcal{A}^\dagger_{1,-\mu}(p'n') \\ &\quad + \mathcal{A}_{1,-\mu}(p'n')\mathcal{A}_{1\mu}(pn))(-)^{1-\mu} \\ &\quad - \sum X_{\tau,\tau'}q^{\tau,\tau'}(ik;i'k')\mathcal{A}^\dagger_{2\mu}(\tau;I_iI_k)\mathcal{A}_{2\mu}(\tau';I_{i'}I_{k'}) \\ &\quad + X_2\sum\mathcal{A}^\dagger_{2\mu}(\tau;I_iI_k)\mathcal{A}_{2\mu}(\tau;I_iI_k), \end{aligned} \tag{5.3}$$

where the following abbreviations have been used:

$$\begin{aligned} \sigma_{pn;p'n'} &= \frac{2}{\sqrt{3}\hat{I}_n}\langle I_p||\sigma||I_n\rangle\frac{2}{\sqrt{3}\hat{I}_{n'}}\langle I_{p'}||\sigma||I_{n'}\rangle, \\ q^{\tau\tau'}(ab;a'b') &= \frac{2}{\sqrt{5}\hat{I}_b}\langle\tau a||r^2Y_2||\tau b\rangle\frac{2}{\sqrt{5}\hat{I}_{b'}}\langle\tau'a'||r^2Y_2||\tau'b'\rangle. \end{aligned} \tag{5.4}$$

The dipole and quadrupole operators commute with each other and therefore the pnQRPA equations will be decoupled. Since the above-defined Hamiltonian preserves the gauge, the resulting harmonic approximated equation will be conventionally called the GRFRpnQRPA equation. For this reason they will be separately treated. Adding the first-order corrections to the quasiboson approximation, we have

$$\begin{aligned} &[\mathcal{A}_{1\mu}(pn), \mathcal{A}^\dagger_{1\mu'}(p'n')] \\ &\approx \delta_{\mu,\mu'}\delta_{j_p,j_{p'}}\delta_{j_n,j_{n'}}\left[U_p^2 - U_n^2 + \frac{U_n^2 - V_n^2}{\hat{I}_n^2}\hat{N}_n - \frac{U_p^2 - V_p^2}{\hat{I}_p^2}\hat{N}_p\right]. \end{aligned} \tag{5.5}$$

The average of the rhs of this equation with the GRFRpnQRPA vacuum state is denoted by:

$$\begin{aligned} D_1(pn) &= U_p^2 - U_n^2 + \frac{1}{2I_n+1}s\big(U_n^2 - V_n^2\big)\langle\hat{N}_n\rangle \\ &\quad - \frac{1}{2I_p+1}\big(U_p^2 - V_p^2\big)\langle\hat{N}_p\rangle. \end{aligned} \tag{5.6}$$

The equations of motion show that the two qp energies are renormalized too:

$$E^{\rm ren}(pn) = E_p\big(U_p^2 - V_p^2\big) + E_n\big(V_n^2 - U_n^2\big). \tag{5.7}$$

The space of pn dipole states, $\mathcal{S}$, is written as a sum of three subspaces defined as:

$$\begin{aligned} &\mathcal{S}_+ = \{(p,n)|D_1(pn) > 0, \quad E^{\rm ren}(pn) > 0, \}, \\ &\mathcal{S}_- = \{(p,n)|D_1(pn) < 0, \quad E^{\rm ren}(pn) < 0, \}, \\ &\mathcal{S}_{sp} = \mathcal{S} - (\mathcal{S}_+ + \mathcal{S}_-), \quad \mathcal{N}_\pm = \dim(\mathcal{S}_\pm), \quad \mathcal{N}_{sp} = \dim(\mathcal{S}_{sp}), \\ &\mathcal{N} = \mathcal{N}_+ + \mathcal{N}_- + \mathcal{N}_{sp}. \end{aligned} \tag{5.8}$$

The third line of the above equations specify the dimensions of these subspaces. In $\mathcal{S}_+$ one defines the renormalized operators:

$$\begin{aligned} \bar{\mathcal{A}}^\dagger_{1\mu}(pn) &= \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}^\dagger_{1\mu}(pn), \\ \bar{\mathcal{A}}_{1\mu}(pn) &= \frac{1}{\sqrt{D_1(pn)}}\mathcal{A}_{1\mu}(pn), \end{aligned} \tag{5.9}$$

while in $\mathcal{S}_-$ the renormalized operators are

$$\begin{aligned} \bar{\mathcal{F}}^\dagger_{1\mu}(pn) &= \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}_{1\mu}(pn), \\ \bar{\mathcal{F}}_{1\mu}(pn) &= \frac{1}{\sqrt{|D_1(pn)|}}\mathcal{A}^\dagger_{1\mu}(pn). \end{aligned} \tag{5.10}$$

Indeed, the operator pairs $\mathcal{A}_{1\mu}, \mathcal{A}^\dagger_{1\mu}$ and $\mathcal{F}_{1\mu}, \mathcal{F}^\dagger_{1\mu}$ satisfy commutation relations of boson type. An RPA treatment within $\mathcal{S}_{sp}$ would yield either vanishing or negative energies. The corresponding states are therefore spurious. FRpnQRPA with the gauge symmetry projected defines the phonon operator as

$$\begin{aligned} \Gamma^\dagger_{1\mu} &= \sum_k[X(k)\bar{\mathcal{A}}^\dagger_{1\mu}(k) + Z(k)\bar{\mathcal{F}}^\dagger_{1\mu}(k) - Y(k)\bar{\mathcal{A}}_{1-\mu}(k)(-)^{1-\mu} \\ &\quad - W(k)\bar{\mathcal{F}}_{1-\mu}(k)(-)^{1-\mu}], \end{aligned} \tag{5.11}$$

with the amplitudes determined by the equations

$$[H, \Gamma^\dagger_{1\mu}] = \omega\Gamma^\dagger_{1\mu}, \quad [\Gamma_{1\mu}, \Gamma^\dagger_{1\mu'}] = \delta_{\mu,\mu'}. \tag{5.12}$$

The first equation may be written as

$$\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ -\mathcal{B} & -\mathcal{A}\end{pmatrix}\begin{pmatrix}X(pn)\\ Z(pn)\\ Y(pn)\\ W(pn)\end{pmatrix} = \omega_1\begin{pmatrix}X(pn)\\ Z(pn)\\ Y(pn)\\ W(pn)\end{pmatrix}, \tag{5.13}$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ have the dimension $(\mathcal{N}_+ + \mathcal{N}_-) \times (\mathcal{N}_+ + \mathcal{N}_-)$ and the analytical expressions

$$(\mathcal{A}) = \begin{pmatrix} \left(E^{\rm ren}(pn)\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_+\\(p_1,n_1)\in\mathcal{S}_+}} & -2X_{dp}\left(\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_+\\(p_1,n_1)\in\mathcal{S}_-}} \\ -2X_{dp}\left(\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_-\\(p_1,n_1)\in\mathcal{S}_+}} & \left(|E^{\rm ren}(pn)|\delta_{pn;p_1n_1} + 2\chi\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_-\\(p_1,n_1)\in\mathcal{S}_-}} \end{pmatrix},$$

$$(\mathcal{B}) = \begin{pmatrix} -2X_{dp}\left(\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_+\\(p_1,n_1)\in\mathcal{S}_+}} & 2X_{dp}\left(\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_+\\(p_1,n_1)\in\mathcal{S}_-}} \\ 2X_{dp}\left(\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_+\\(p_1,n_1)\in\mathcal{S}_+}} & -2X_{dp}\left(\sigma^{(1)T}_{p_1n_1;pn}\right)_{\substack{(p,n)\in\mathcal{S}_+\\(p_1,n_1)\in\mathcal{S}_+}} \end{pmatrix}, \tag{5.14}$$

where the matrix $\sigma^{(1)}_{pn;p'n'}$ has the expression

$$\sigma^{(1)}_{pn;p'n'} = |D_1(pn)|^{1/2}\sigma_{pn;p'n'}|D_1(p'n')|^{1/2}, \tag{5.15}$$

while the index T suggests that the accompanying matrix is to be transposed.

To solve the equation we need the renormalization factors D_1 which in turn depend on the averages $\langle\hat{N}_p\rangle$ and $\langle\hat{N}_n\rangle$. Using the boson expansion principle the quasiparticle number operators are expressed as a linear combination of $\mathcal{A}^\dagger\mathcal{A}$ and $\mathcal{F}^\dagger\mathcal{F}$ determined such that their commutators with $\mathcal{A}^\dagger$, $\mathcal{A}$ and $\mathcal{F}^\dagger$, $\mathcal{F}$ are preserved. The results are

$$\langle\hat{N}_p\rangle = V_p^2(2I_p+1) + 3\left(U_p^2 - V_p^2\right)\left(\sum_{\substack{n',k\\(p,n')\in\mathcal{S}_+}}(Y_k(p,n'))^2 - \sum_{\substack{n',k\\(p,n')\in\mathcal{S}_-}}(W_k(p,n'))^2\right),$$

$$\langle\hat{N}_n\rangle = V_n^2(2I_n+1) + 3\left(U_n^2 - V_n^2\right)\left(\sum_{\substack{p',k\\(p',n)\in\mathcal{S}_+}}(Y_k(p',n))^2 - \sum_{\substack{p',k\\(p',n)\in\mathcal{S}_-}}(W_k(p',n))^2\right). \tag{5.16}$$

Equations (5.13) and (5.16) and the norm restriction,

$$\sum_{\substack{n',k\\(p,n')\in\mathcal{S}_+}}(X(pn)^2 - Y(pn)^2) + \sum_{\substack{n',k\\(p,n')\in\mathcal{S}_-}}(Z(pn)^2 - W(pn)^2) = 1, \tag{5.17}$$

are to be simultaneously considered and solved iteratively. It is worth mentioning that using the quasiparticle representation for the basic operators $\mathcal{A}^\dagger_{1\mu}$, $\mathcal{F}^\dagger_{1\mu}$, $\mathcal{A}_{1,-\mu}(-1)^{1-\mu}$, and $\mathcal{F}_{1,-\mu}(-)^{1-\mu}$, one obtains for $\Gamma^\dagger_{1\mu}$ an expression which involves the scattering *pn* operators. Thus, the present approach is, indeed, the GRFRpnQRPA.

The case of quadrupole interaction is much simpler since there is no term involved in the model Hamiltonian which might generate back-going phonon amplitude. We denote

$$\begin{aligned} Q^{(\tau,\tau')}(mn;ik) &= D_2^{1/2}(\tau;mn)q^{(\tau,\tau')}(mn;ik)D_2^{1/2}(\tau';ik), \\ E^{\rm ren}(\tau;ab) &= E_{\tau a}\left(U^2_{\tau a} - V^2_{\tau a}\right) + E_{\tau b}(U^2_{\tau b} - V^2_{\tau b}) \\ &= \epsilon_{\tau a} - \epsilon_{\tau b}, \\ D_2(\tau;ab) &= U^2_{\tau a} - U^2_{\tau b}. \end{aligned} \tag{5.18}$$

The quadrupole phonon operator is defined as

$$\Gamma^\dagger_{a,2\mu} = \sum_{tau;ik} X_a(\tau,ik)\bar{\mathcal{A}}^\dagger_{2\mu}(\tau;ik), \tag{5.19}$$

with the renormalized boson operator

$$\bar{\mathcal{A}}^\dagger_{2\mu}(\tau;ik) = \frac{1}{\sqrt{D_2(\tau,ab)}}\mathcal{A}^\dagger_{2\mu}(\tau,ab). \tag{5.20}$$

The phonon amplitudes satisfy the equation

$$\begin{aligned} \mathbf{A}_{\tau ik;\tau'mn}X_a(\tau,ik) &= \omega_2 X_a(\tau',mn), \quad \text{with} \\ \mathbf{A}_{\tau ik;\tau'mn} &= (E^{\rm ren}(\tau,ik) + X_2)\delta_{\tau,\tau'}\delta_{ik,mn} \\ &\quad - X_{\tau\tau'}Q^{(\tau'\tau)}(mn;ik) \end{aligned} \tag{5.21}$$

and the normalization condition

$$\sum_{\tau,mn} X_a(\tau,mn)^2 = 1. \tag{5.22}$$

VI. THE GAMOW-TELLER TRANSITION AMPLITUDE

The GRFRpnQRPA states defined in the previous section are used to describe the amplitude of the double-β transition $0^+ \to 2^+$:

$$M^{(02)}_{GT} = \sqrt{3}\sum_{k,m}\frac{{}_i\langle 0||\beta_i^+||k,m\rangle_{ii}\langle k,m|k'm'\rangle_{ff}\langle k'm'||\beta_f^+||2_1^+\rangle_f}{(E_{km} + \Delta E + E_{1^+})^3}. \tag{6.1}$$

In the above equation, the denominator consists of three terms: (a) ΔE, the energy carried by leptons in the intermediate state approximated by the sum of the rest energy of the emitted electron and the half of the Q value of the $\beta\beta$ decay process

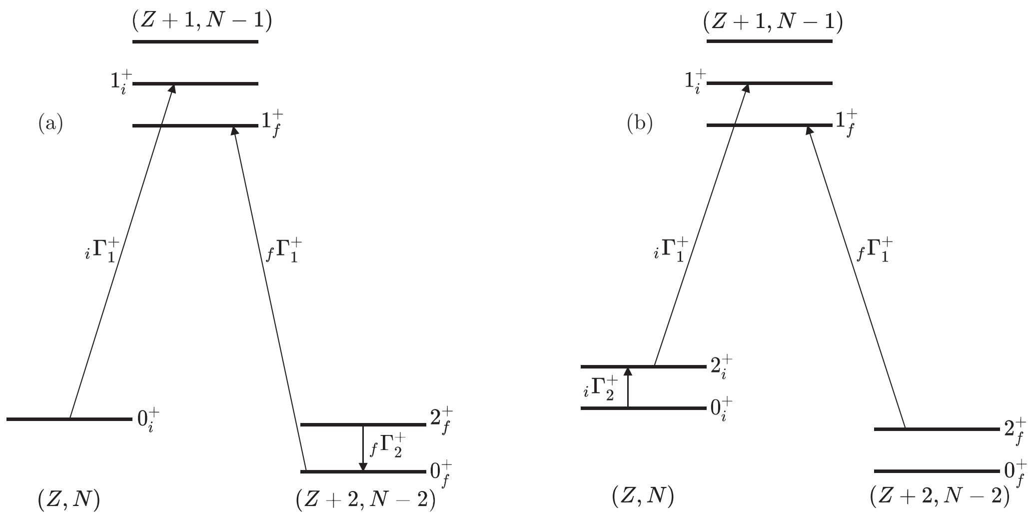

FIG. 2. Illustration of the GT transition $0^+ \rightarrow 2^+$ via (a) one- and (b) two-phonon state.

from the ground state of the mother nucleus to the first excited 2^+ state of the daughter nucleus,

$$\Delta E = m_e c^2 + \tfrac{1}{2} Q_{\beta\beta}^{(0\rightarrow 2)}; \tag{6.2}$$

(b) the average value of the mth GRFRpnQRPA energy for the k-boson state normalized to the particular value corresponding to $m = 1$; and (c) the experimental energy for the lowest 1^+ state. The indices carried by the β^+ operators indicate that they act in the space spanned by the GRFRpnQRPA states associated to the initial (i) or final (f) nucleus. The overlap matrix elements (m.e.) of the k phonon states in the initial and final nuclei, respectively, are calculated within GRFRpnQRPA. In Eq. (6.1), the Rose convention for the reduced m.e. is used [54]. The ground state of the mother nucleus, $|0\rangle_i$, and the first excited 2^+ state of the daughter nucleus may be excited to the kth phonon state built up with the mth root of the GRFRpnQRPA equations by means of the β^- and β^+ transition operators, respectively. The connection between the states excited from the mother nucleus and those excited from the daughter nucleus is achieved by the overlap matrix ${}_i\langle k, m|k'm'\rangle_f$. In Eq. (6.1), the index k takes the values 1 and 2 while m runs over the complete set of the GRFRpnQRPA equations for the dipole phonons and $m = 1$ for the quadrupole phonon. The mechanisms which contribute to the double-β process are illustrated in Fig. 2. Note that Fig. 2(a) suggests that the first leg of the transition is determined by the one dipole phonon operator, while the second transition is caused by the dipole-quadrupole double phonon operator. The scenario of Fig. 2(b) is different; namely, the first β^- transition has a double phonon character, while the second one is a single dipole phonon transition.

Once the transition amplitude is calculated, the half-life of the process is readily obtained:

$$T_{1/2}^{2\nu}(0_i^+ \rightarrow 2_f^+)^{-1} = G_{02}\big|M_{GT}^{(02)}\big|^2, \tag{6.3}$$

where G_{02} is an integral on the phase space, independent of the nuclear structure.

VII. NUMERICAL RESULTS

The formalism presented in the previous sections was applied to eight double-β emitters with the one quadrupole phonon state 2^+ as the final state. The spherical shell model single-particle basis is defined using the parameters given in Ref. [52]:

$$\hbar\omega_0 = 41A^{-1/3},\ C = -2\hbar\omega_0\kappa,\ D = -\hbar\Omega_0\mu. \tag{7.1}$$

The parameters $(\kappa;\mu)$ for the proton system are (0.08; 0) for ^{76}Ge, ^{76}Se, ^{82}Se, ^{82}Kr, and ^{116}Cd, and (0.0637; 0.6) for ^{116}Sn, ^{128}Te, ^{128}Xe, ^{130}Te, ^{130}Xe, ^{150}Nd, and ^{130}Xe, while for the neutron systems of the two groups of nuclei mentioned above, the values are (0.08; 0) and (0.0637; 0.42), respectively. The proton and neutron pairing strengths are listed in Table I and illustrated in Fig. 3.

The pairing calculation corresponds to Z protons and N neutrons, respectively, while the quasiparticle correlations are accounted for by means of the states from outside the core mentioned in Table I. For the strengths from Fig. 3 we solved the pairing equations and obtained the gaps plotted in Fig. 4 as a function of A and compared them with the experimental data approximated by $12/\sqrt{A}$ and $13/\sqrt{A}$, respectively [55]. The calculated gaps are spread around the mentioned experimental data. What is generating the discrepancies? The sum rule is sensitive to changing the dimension of the single-particle basis as well as to the pairing strength. Therefore, we slightly modified the pairing strengths towards improving the agreement with the sum rule.

The projected spherical single-particle basis depends on two parameters, namely, the deformation d and the parameter k relating the quadrupole boson operator and the quadrupole collective coordinate. These parameters were taken as in Ref. [56]. Their connection with the deformation β was *in extenso* studied in Refs. [39,56]. From Table I we see that we meet the situation when the mother and daughter nuclei have similar deformation ((^{82}Se; ^{82}Kr), (^{128}Te; ^{128}Xe), (^{150}Nd; ^{150}Sm)) and the case where the initial and final nuclei are

TABLE I. The parameters of the projected spherical single-particle basis (the deformation d and the k defining the canonical transformation relating the quadrupole bosons and collective coordinates), the strengths of the pairing interactions, G_p and G_n, the strengths of the Gamow-Teller interactions, $\chi(1)$, $\chi(2)$, and χ_{dp}, and the strengths of the QQ interaction are listed. We also give the dimension of the inert core, the number of states for protons and neutrons, respectively, (p, n), the number of iterations needed to find the solution of the pnQRPA equations, and the Ikeda sum rule. All the mentioned parameters correspond to the parent and daughter nuclei listed in the first column.

	d	k	G_p (keV)	G_n (keV)	$\chi(i)$ (MeV)	X_{dp} (MeV)	b^4X_{pp} (keV)	X_2 (MeV)	Core (Z, N)	No. states (p, n)	No. iterations	ISR/3
^{76}Ge	1.6	10	360	380	0.25	0.20	31.5	3.260	(20,20)	(18,18)	4	12.09
^{76}Se	1.9	10	240	325	0.25	0.20	19.7	4.017	(20,20)	(18,18)	4	
^{82}Se	0.2	9	340	360	0.01	0.05	19.1	2.031	(26,26)	(20,20)	5	14.05
^{82}Kr	0.2	9	340	360	0.01	0.05	19.7	2.173	(26,26)	(20,20)	5	
^{96}Zr	1.5	10	180	433	0.22	0.11	25.8	3.635	(20,20)	(20,20)	6	16.1
^{96}Mo	1.2	12	220	338	0.22	0.11	25.8	2.636	(20,20)	(20,20)	6	
^{100}Mo	−1.4	10	380	360	0.06	1.65	31.5	1.857	(28,28)	(19,19)	4	16.03
^{100}Ru	−0.6	3.6	385	365	0.06	1.65	28.0	1.872	(28,28)	(19,19)	4	
^{116}Cd	−1.8	3.0	200	245	0.98	1.60	30.0	2.187	(26,26)	(27,27)	4	19.96
^{116}Sn	−1.2	3.0	135	275	0.98	1.60	7.00	1.148	(26,26)	(27,27)	4	
^{128}Te	1.7	7.17	270	220	0.80	1.56	12.0	1.852	(38,38)	(30,30)	5	24.09
^{128}Xe	1.7	8.0	270	220	0.80	1.56	12.0	1.240	(38,38)	(30,30)	5	
^{130}Te	1.0	8.0	270	240	0.30	0.33	12.1	1.753	(40,40)	(29,29)	6	26.00
^{130}Xe	1.4	8.0	260	220	0.30	0.33	17.3	2.130	(40,40)	(29,29)	6	
^{150}Nd	1.952	3.0	240	260	0.64	1.45	27.3	2.187	(50,50)	(25,25)	5	29.77
^{150}Sm	1.952	2.0	220	240	0.64	1.45	27.3	2.148	(50,50)	(25,25)	5	

characterized by different deformations. Our calculations show that deformation enhances the half-life of the process. The same effect of deformation on the GT matrix elements was pointed out also by Zamick and Auerbach [42]. In the quoted reference the mentioned authors calculated the GT transition matrix elements for the neutrino capture ν_μ +

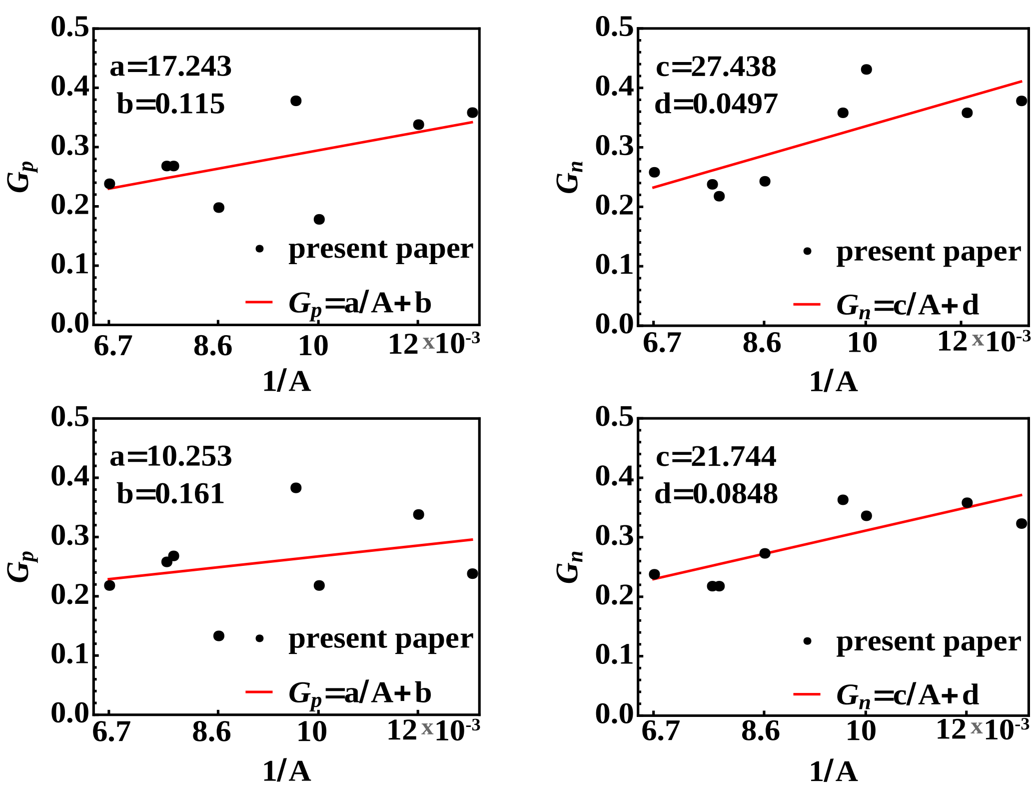

FIG. 3. The proton and neutron pairing strengths for mother (first row) and daughter nuclei (second row), respectively. The results were interpolated by a linear function of $1/A$.

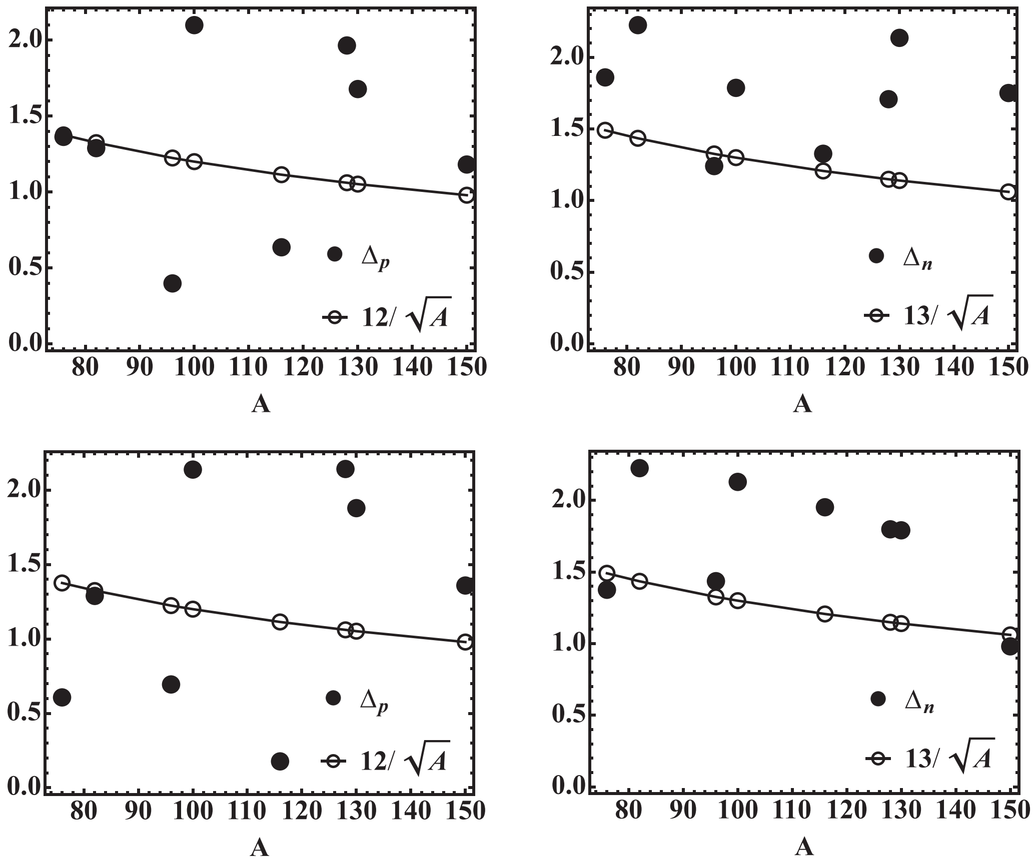

FIG. 4. Calculated proton (Δ_p) and neutron (Δ_n) gap parameters for mother (first row) and daughter (second row) nuclei, compared with experimental data, approximated by $12/\sqrt{A}$ and $13/\sqrt{A}$, respectively.

$^{12}\mathrm{C} \rightarrow {}^{12}\mathrm{N} + \mu^-$ using different structures for the ground state: (a) a spherical ground state, (b) asymptotic limits of the wave functions, and (c) deformed states with the deformation $\delta = -0.3$. The results for the transition rate were 5.333, 0, and 0.987, respectively. The ratio between the transition rates obtained with spherical and deformed bases explains the factor of 5 overestimate in the calculations of Ref. [57].

The strength of the repulsive *pn* interaction was fixed such that the position of the giant Gamow-Teller resonance was reproduced, while the attractive interaction strength is chosen so that the result for the $\log_{10} ft$ value associated with the single-β^- decay of the intermediate odd-odd nucleus to the ground state of the daughter nucleus was close to the corresponding experimental data. If the experimental data are missing the restriction refers to the exiting data in the neighboring region.

The QQ interactions are fixed so that the properties of the first excited 2^+ state are properly described. In Table I we also give the value of the Ikeda sum rule. The number of proton and neutron single-particle states used in our calculations is also mentioned in Table I.

To calculate the half-lives for ground to 2^+ and ground to ground transitions, one needs the phase space factors G_{02} and G_{00}, respectively. These were determined following the procedure described in Ref. [4]. Having the parameters involved in the model Hamiltonian fixed, the amplitude of the transition $0^+ \rightarrow 2^+$ is obtained from Eq. (6.1). Results are listed in Table II together with the half-life of the process. These are compared with the predictions of different approaches. For the transitions of the parent nuclei ^{76}Ge, ^{76}Ge, ^{82}Se, ^{96}Zr, ^{128}Te, and ^{139}Te, projecting the gauge symmetry enhances the half-lives, while for ^{100}Mo, ^{116}Cd, and ^{150}Nd the effect is opposite. As we showed before, breaking the spherical symmetry makes the process less probable. Also, in Ref. [51] the SU(4) symmetry, broken by the mean field approximation, was restored by the Pyatov method [65] and thus the effect of restoring the symmetry on the transition rates of four double-β processes was investigated. For these transitions a quenching of the matrix elements was pointed out. From the three examples discussed above we cannot draw a definite conclusion about quenching or enhancing the transition rates when a symmetry is restored. Note that the results for the half-lives are in the range of the measured data.

Unfortunately, only the low limits of the half-lives are experimentally known. Therefore, in order to point out the virtues of the proposed model, the extension to the ground to ground transition for the same parameters as those used for the ground to 2^+ transition is necessary. Thus the amplitude describing the transition $0^+ \rightarrow 0^+$ is given

TABLE II. The double-β transition amplitudes the half lives obtained with our formalism are compared with the corresponding experimental data as well as with those provided by other formalism. On the first column, the double-β emitters are listed. For ^{150}Nd the first data corresponds to the nuclear deformation $\beta = 0.28$, while the second one to $\beta = 0.19$. The last three columns concern the data for the ground to ground double-β transition.

Parent nucleus	$M_{GT}^{0^+\rightarrow 2^+}$ [MeV^{-3}]	$T_{1/2}^{2\nu}(0_i^+ \rightarrow 2_f^+)$ (yr) Present	Expt.	Ref. [58]	Ref. [59]	$M_{GT}^{0^+\rightarrow 0^+}$ (MeV^{-1})	$T_{1/2}^{2\nu}(0_i^+ \rightarrow 0_f^+)$ (yr) Present	Expt.
^{76}Ge	0.131×10^{-4}	1.166×10^{34}	$>1.1 \times 10^{21}$ [4] $>1.6 \times 10^{23}$ [17]	5.75×10^{28}	1.0×10^{26}	2.647×10^{-2}	1.16×10^{22}	$(1.5 \pm 0.1) \times 10^{21}$ [9]
^{82}Se	0.677×10^{-5}	2.478×10^{30}	$>1.4 \times 10^{21}$ [4] $>1.0 \times 10^{22}$ [18]	1.70×10^{27}	3.3×10^{26} [60]	2.611×10^{-2}	3.84×10^{20}	$(1.1^{+0.8}_{-0.3}) \times 10^{20}$ [61]
^{96}Zr	0.145×10^{-5}	7.500×10^{30}	$>7.9 \times 10^{19}$ [9]	2.27×10^{25}	4.8×10^{21}	0.816×10^{-2}	3.19×10^{21}	$(2.3 \pm 0.2) \times 10^{19}$ [9]
^{100}Mo	0.426×10^{-2}	1.223×10^{24}	$>2.5 \times 10^{21}$ [20]	1.21×10^{25}	3.9×10^{24}	2.447×10^{-2}	7.22×10^{20}	$(0.115^{+0.03}_{-0.02}) \times 10^{20}$ [61]
^{116}Cd	0.724×10^{-2}	1.671×10^{26}	$>2.3 \times 10^{21}$ [21]	3.4×10^{26}	1.1×10^{24}	0.233	3.33×10^{21}	3.75×10^{19} [62]
^{128}Te	0.606×10^{-3}	6.684×10^{34}	$>4.7 \times 10^{21}$ [4]	4.7×10^{33}	1.6×10^{30}	0.416	0.26×10^{23}	$(1.9 \pm 0.4) \times 10^{24}$ [9]
^{130}Te	0.693×10^{-6}	5.562×10^{32}	$>4.5 \times 10^{21}$ [4] $>2.8 \times 10^{21}$ [22]	6.94×10^{26}	2.7×10^{23}	0.81×10^{-2}	12.00×10^{21}	$(2.7 \pm 0.1) \times 10^{21}$ [63]
^{150}Nd	0.317×10^{-2}	0.461×10^{21}	$>8.0 \times 10^{18}$ [4] $>2.2 \times 10^{20}$ [31]	1.50×10^{23}	7.2×10^{24} [64] 1.2×10^{25} [64]	0.744	0.789×10^{17}	$(8.2 \pm 0.9) \times 10^{19}$ [9]

by

$$M_{GT}^{(00)} = \sqrt{3} \sum_{k,k'} \frac{{}_i\langle 0||\beta_i^+||k\rangle_{ii}\langle k|k'\rangle_{ff}\langle k'||\beta_i^+||0\rangle_f}{E_k + \Delta E_1 + E_{1^+}}, \quad (7.2)$$

where the energy shift from the denominator is the sum between the electron rest mass and half of the ground to ground Q value:

$$\Delta E_1 = m_e c^2 + \tfrac{1}{2} Q_{\beta\beta}^{(0\rightarrow 0)}. \quad (7.3)$$

E_{1^+} denotes the experimental energy of the first state 1^+ in the intermediate odd-odd nucleus, while $|k\rangle_l$ is the kth dipole phonon state obtained by exciting the initial (final) nucleus, for $l = i$ ($l = f$). This transition amplitude determines the half-life of the process by means of

$$T_{1/2}^{2\nu}(0_i^+ \rightarrow 0_f^+)^{-1} = G_{00} \big| M_{GT}^{(00)} \big|^2, \quad (7.4)$$

where G_{00} is the phase space integral specific to the ground to ground transition. The matrix elements involved in Eq. (7.2) have the following expressions:

$$\langle 0||\beta_i^+||k\rangle = \sqrt{3} \sum_{ab} P_1(ab)\sqrt{|D_1(ab)|} X_{1k}(a,b),$$
$$\langle k'||\beta_i^+||0\rangle = \sum_{a',b'} P_1(a'b')\sqrt{|D_1(a'b')|} Y_{1k}(a'b'),$$
$$\langle k|k'\rangle = \sum_{a',b'} (X_{1k}(a'b')X_{1k'}(a'b') - Y_{1k}(a'b')Y_{1k'}(a'b')). \quad (7.5)$$

Results for the transition amplitude and the transition half-life are collected in Table II. By inspection, we notice that the transition amplitude for the transition $0^+ \rightarrow 2^+$ is one to three orders of magnitude smaller than that corresponding to the ground to ground decay. The calculations for the half-lives are in a reasonable agreement with the corresponding experimental data.

We recall that the double-β process is supposed to take place via two consecutive virtual β^- transitions. In other words, the dipole states of the intermediate odd-odd nucleus are fed through either the virtual β^- decay of the parent nucleus or by the β^+ transition of the daughter nucleus. Note that the same matrix elements are involved in the real transitions from the intermediate odd-odd nucleus to the mother nucleus by a β^+/Electron Capture (EC) process or to the daughter via a (p, n) reaction. The two transitions are characterized by the $\log_{10} ft$ value with ft given by the expression

$$ft_{\mp} = \frac{6160}{[[{}_l\langle 1_1||\beta^{\pm}||k\rangle_l g_A]^2}, \quad l = m, d,$$
$$k = 0\delta_{l,m} + (2or0)\delta_{l,d}, \quad (7.6)$$

where $|1_1M\rangle$ denotes the first dipole phonon state in the intermediate odd-odd nucleus, while $|k\rangle_l$ denotes the GRFRpnQRPA ground state if $l = m$ and the state 2^+ if $l = d$. The lower index takes the values m and d depending on whether the end state of the transition is characterizing the mother or the daughter nucleus. f is an integral on the phase space which does not depend on the nuclear structure. This was calculated using the analytical expression from Ref. [4]. We chose $g_A = 1.0$ in order to take account of the distance states responsible for the “missing strength” in the giant GT resonance [4]. Results are compared with the corresponding experimental data in Table III. One notes a reasonable agreement with the existent experimental data. The big discrepancies between the $\log_{10} ft$ value corresponding to the transitions $0^+ \rightarrow 2^+$ and $0^+ \rightarrow 0^+$, respectively, reflect the fact that the transition to the excited state is much less likely than that of ground to ground.

A few comments about the connection between the $\log_{10} ft$ values and ISR are necessary. For small values of the attrac-

TABLE III. The $\log_{10} ft$ values characterizing the β^- transition of the intermediate odd-odd nucleus staying in the state 1^+ to the daughter nucleus in the state 2^+ and the transition β^+/EC from the intermediate odd-odd nucleus in the state 1^+ to the ground state of the mother nucleus, respectively.

Parent nucleus	$\log_{10} ft$	Odd-odd nucleus	$\log_{10} ft$		Daughter nucleus
^{76}Ge	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{76}As	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{76}Se
Theor.	5.59		13.47	7.81	
^{82}Se	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{82}Br	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{82}Kr
Theor.	8.38		12.66	8.6	
^{96}Zr	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{96}Nb	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{96}Mo
Theor.	8.86		14.34	8.2	
^{100}Mo	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{100}Tc	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{100}Ru
Theor.	3.18		9.46	5.16	
Expt.	4.3 [67]		6.4 [67], 6.63 [68]	4.59 [68]	
^{116}Cd	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{116}In	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{116}Sn
Theor.	3.20		10.92	3.44	
Expt.	4.47 [69]		5.85 [69]		
^{128}Te	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{128}I	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{128}Xe
Theor.	4.09		9.71	3.05	
Expt.	6.01 [70]		6.498 [70]	6.061 [71]	
	5.049 [72]				
^{130}Te	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{130}I	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{76}Xe
Theor.	6.49		13.18	9.39	
^{150}Nd	$0^+ \xleftarrow{\beta^+/EC} 1^+$	^{150}Pm	$1^+ \xrightarrow{\beta^-} 2^+$	$1^+ \xrightarrow{\beta^-} 0^+$	^{150}Sm
Theor.	2.84		8.99	3.71	
Expt.			8.62 [73]		

tive interaction strength X_{dp}, the ISR exceeds the value of $N - Z$. However, the β^- matrix element associated to the transition $1^+ \rightarrow 0^+$ from the intermediate odd-odd nucleus to the ground state of the daughter nucleus is mainly determined by the repulsive interaction and consequently is large, which means a small value for the corresponding $\log_{10} ft$. Increasing the strength of X_{dp}, the mentioned matrix element is decreasing and therefore the $\log_{10} ft$ value is increasing. Approaching the critical value where the energy of the first excited dipole state is vanishing, the amplitude of the backgoing graph, $Y_k(ab)$, is increasing and so is the strength of the β^+ transition, which determines a decrease of the ISR. The adopted procedure of fixing the value of X_{dp} consists of fitting the value of $\log_{10} ft$, characterizing the transition from the dipole state 1^+ of the intermediate odd-odd nucleus to the ground state of the daughter nucleus. In some cases the obtained value yields for ISR a value which is different from $N - Z$. In such a case the parameter X_{dp} is modified so that the ISR is brought close to $N - Z$.

Using expression (B2), the single-β transition strengths associated with the decays of the mother nucleus were calculated and the results are represented in Figs. 5 and 6 as functions of the GRFRpnQRPA energies. There we give also the strength of the β^+ transition of the daughter nucleus from the first excited 2^+ state. For ^{76}Ge, ^{128}Te, and ^{150}Nd the β^- strength is accumulated in a narrow peak, while for the remaining emitters the giant resonance exhibits a broad width and a complex structure. For a given nucleus the difference between areas under the curves of figures from the first and second columns, respectively, defines the Ikeda sum rule. We note that the shape of the β^+ strength of the daughter nucleus has a similar shape as that of the β^- strength for the mother decay. Apart from their magnitudes, the maximal strength is reached for the GT resonance energy.

For ^{76}Ge, ^{82}Se, ^{128}Te, and ^{130}Te, the results of our calculations are compared with the corresponding experimental data. One can remark on the quality of the agreement with the experimental data for ^{130}Te. For the other three nuclei the two sets of data agree with each other in the low part of the spectra, while the GT resonance locations are different by 1–2 MeV.

For some of the double-β emitters there are available data concerning the total $B(GT)_-$ strength. Since a limited interval of energies of the odd-odd nucleus is considered, the data are to be compared with 0.6 of the theoretical results. Another feature refers to the fact that the total strength $B(GT)_-$ accounts also for the background contribution. Despite this, the strength for ^{128}Te and ^{130}Te, for example, represents only 72% and 71% of the $N - Z$ value [66]. If the background contribution to the total strength is eliminated, the total measured strength amounts to about 56% and 59%, respectively, of the

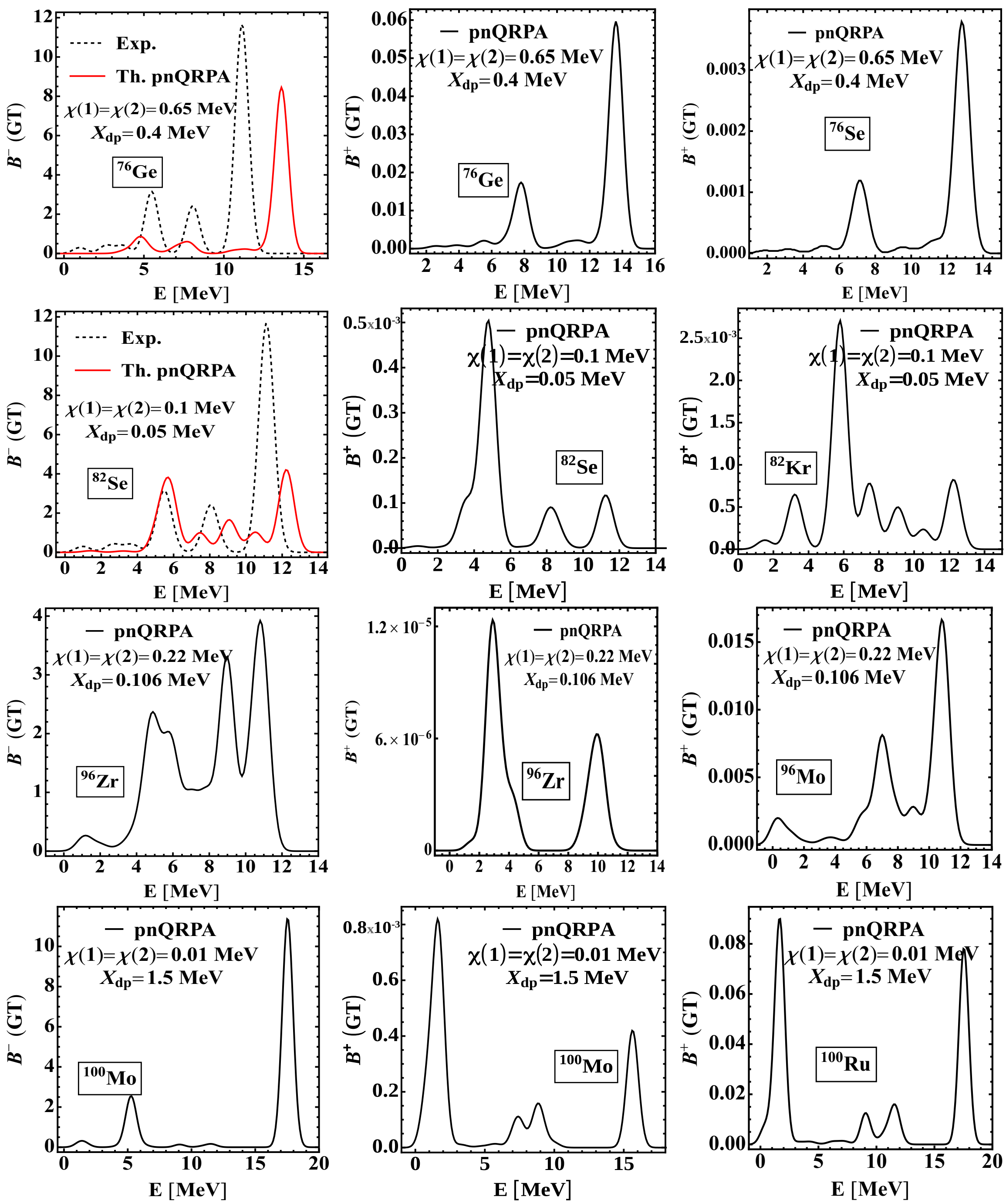

FIG. 5. The strengths for the β^- and β^+ transitions of the double-β emitters are shown in the figures from the first and second columns. Also, the β^+ strengths for the transitions $2^+ \rightarrow 1^+$ in the daughter nuclei are given in the panels of the third column. The strengths were folded with Gaussian functions having a width of 1 MeV. The calculated strength distributions for ^{76}Ge and ^{82}Se are compared with the corresponding experimental data from Ref. [66].

$N - Z$ value. The results of such a comparison are given in Table IV.

The final state in the daughter nucleus is the first excited 2^+ state, which decays to the ground state by the γ emission. The half-life of this process is of the order of picoseconds. Therefore, detecting the two electrons emerging from the double-β process in coincidence with the quadrupole γ quanta resulting from the transition of 2^+ to the ground state would be an experimental way of identifying the double-β process to the state 2^+. Using the results of Appendix C, we calculated the $B(E2)$ value of the transition $2^+ \rightarrow 0^+$ and the corresponding half-life. Results were compared with the existent experi-

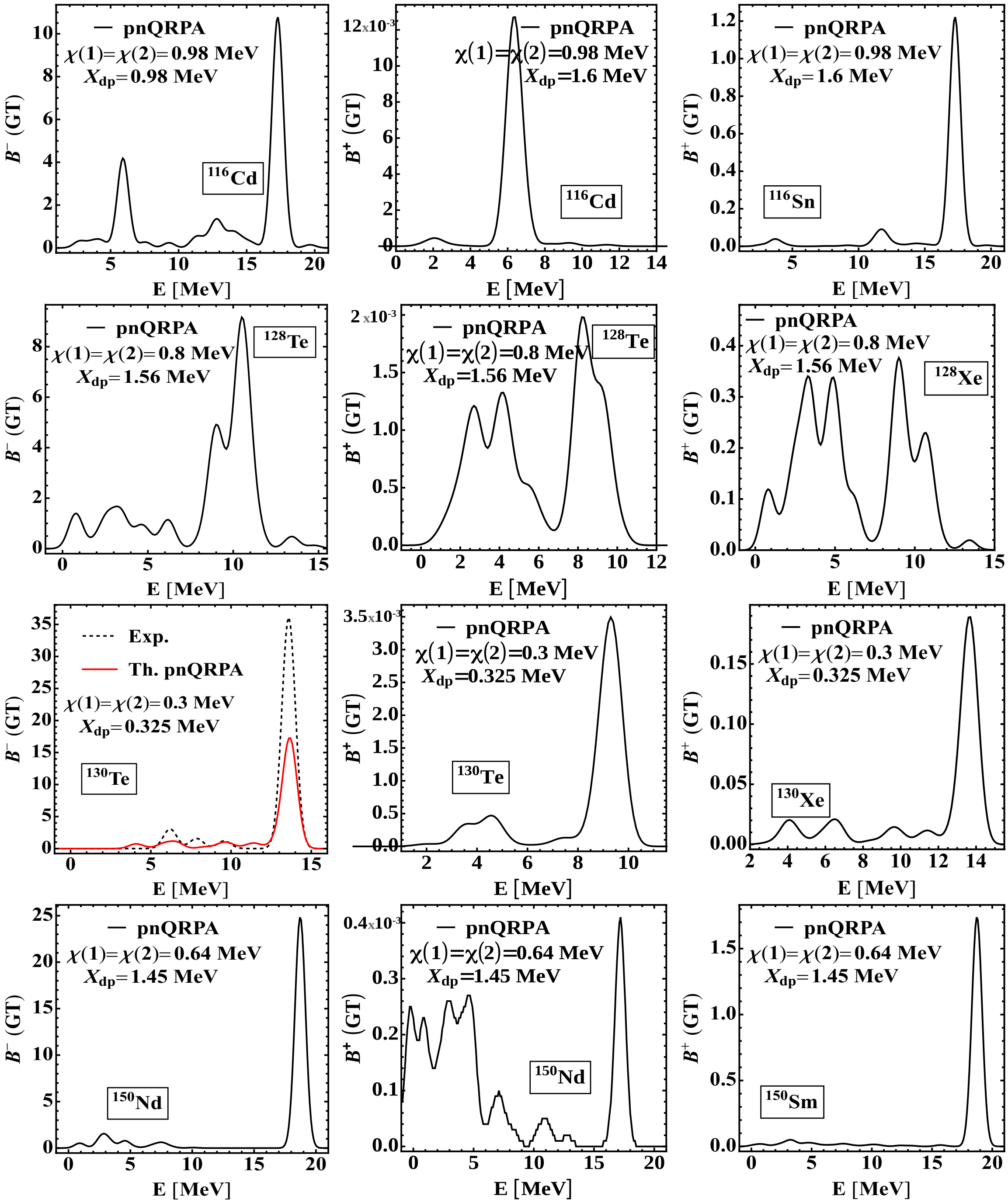

FIG. 6. The same as in Fig. 5 but for other mothers (^{116}Cd, ^{128}Te, ^{130}Te, ^{150}Nd) and daughters (^{116}Sn, ^{128}Xe, ^{130}Xe, ^{150}Sm), respectively. The calculated strength distributions for ^{128}Te and ^{130}Te are compared with the corresponding experimental data from Ref. [66].

mental data, in Table V. The obtained $B(E2)$ value is used to calculate the nuclear deformation. This is compared with the d/k value and the corresponding experimental nuclear deformation. One notes a reasonable agreement between the three sets of data.

VIII. SUMMARY AND CONCLUSIONS

In the previous sections we developed a formalism with gauge invariance restored for the double-β transition $0^+ \rightarrow 2^+$ with two neutrinos in the final state. The aim of this investigation was to bring the Ikeda sum rule close to the $N - Z$ value. Indeed, to describe the transition from the ground to the first excited 2^+ state, one has to go beyond the pnQRPA approach. This is achieved by combining two higher QRPA approaches, namely, the fully renormalized QRPA and the boson expansion approximation. Each of these violates the sum rule. It seems that the renormalization of the QRPA equations underestimates the sum rule, while the boson expansion overestimates it. The idea underlying the present paper is that

TABLE IV. Total strengths for the Gamow-Teller β^- (first column) and β^+ (third column) transitions, quenched by a factor of 0.6, compared with the corresponding available experimental data. Also, the results for the total strength of the β^+ transition from the state 2^+ of the daughter nucleus are given.

Nucleus	$0.6\sum B(GT)_-$	$\sum[B(GT)_-]_{\rm expt}$	Nucleus	$0.6\sum B(GT)_+$	$\sum[B(GT)_+]_{\rm expt}$	$0.6\sum[B(GT)_+]_{2_1^+\to 1_k^+}$
^{76}Ge	21.706	23.3 [66]	^{76}Se	0.0935	1.45 ± 0.07 [74]	0.598×10^{-2}
^{82}Se	25.307	24.6 [66]	^{82}Kr	0.0112		0.324×10^{-6}
^{100}Mo	29.263	26.69 [75]	^{100}Ru	0.4110		0.153×10^{-5}
^{116}Cd	38.675	32.70 [75]	^{116}Sn	2.756		0.6×10^{-7}
^{128}Te	47.113	40.08 [66]	^{128}Xe	3.751		0.111×10^{-4}
^{130}Te	47.372	45.90 [66]	^{130}Xe	0.574		0.111×10^{-6}

breaking the gauge symmetry is responsible for the deviation of the sum rule from the $N - Z$ value. In a previous paper we restored the gauge symmetry for the process of ground to ground double-β decay with two neutrinos in the final state. Here the formalism is extended to the transition from ground to the first excited 2^+ state. In an earlier publication we treated the double-β transition $0^+ \to 2^+$ without projecting the gauge symmetry [58]. Results for the transition $0^+ \to 2^+$ were compared with those of the ground to ground transition as well as with those obtained without the gauge symmetry restored. One can remark on the good agreement between calculated half-lives for ground to ground transition and the corresponding experimental data.

The hypothesis that the gauge symmetry should be conserved is supported, first of all, by the fact that the single β minus transition,

$$n \to p + e + \tilde{\nu}, \tag{8.1}$$

takes place with conserving the gauge.

Several features are addressed in the proposed formalism:

(i) The charge conserved QRPA equations were renormalized. Projecting out the gauge symmetry, the equations attain the Tamm-Dancoff form. Considering the quasiparticle representation for the quadrupole operator $(c_\tau^\dagger c_\tau)_{2\mu}$ results in the specific shape of the quadrupole operators being of renormalized form.

(ii) Using the second order for the perturbation approximation we calculated the amplitude for the transition $0^+ \to 2^+$ and then the half-life of the process. For five transitions, projecting the gauge will enhance the half-life, while for the other three the effect is opposite. Comments on the effect of other symmetries like rotations and SU(4) symmetry are included.

(iii) Although the transition $0^+ \to 2^+$ takes place via two successive single-β virtual transitions, involved matrix elements are the same as for the real transitions β^+/EC and β^- of the intermediate odd-odd nuclei to the mother and daughter nuclei, respectively. For these transitions we calculated the corresponding $\log_{10} ft$ values and compared them with the existent experimental data.

(iv) The single-$\beta^\mp$ transition strengths are presented as a function of the GRFRpnQRPA energies and compared with the existing experimental data.

(v) The final state, i.e., 2^+, is a short lived state decaying by γ emission. For this state we calculated the $B(E2)$ value and the corresponding half-life, $t_{1/2}$. We suggest that by measuring the γ quanta yielded by the decay of 2^+, in coincidence with identifying the two electrons accompanying the double-β transition, we could experimentally point out the $\beta\beta$ transition $0^+ \to 2^+$. Having the $B(E2)$ value calculated, the theoretical nuclear deformation is readily obtained. This is compared with the experimental nuclear deformation as well as with the deformation parameter d.

In conclusion, the present formalism accounts quantitatively for some properties of the double-β transition $0^+ \to 2^+$ and at a time preserves the Ikeda sum rule which is specific to the pnQRPA.

ACKNOWLEDGMENT

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APPENDIX A

Here we give the analytical expressions for the boson expansion associated with the dipole operators defining the single-β^+ transition operator:

$$\begin{aligned}\beta^+_\mu(p,n) &= \sum_M \frac{\sqrt{2}}{\hat{I}}\langle pIM|\sigma_\mu|nI'M'\rangle\frac{\sqrt{2}}{\hat{I'}}c^\dagger_{nIM}c_{pI'M'}\\ &\equiv P_k(pn)(c_n^\dagger c_p)_{1\mu},\\ P_k(ab) &= \frac{2}{\hat{I}_a}{}_k\langle a||\sigma||b\rangle_k\frac{2}{\hat{I}_b},\quad k=i,f.\end{aligned} \tag{A1}$$

Here the indices i and f are for the initial and final nucleus, respectively. In terms of the renormalized pnQRPA phonon operators the dipole operators $(c_n^\dagger c_p)_{1\mu}$ can be expressed as

$$\begin{aligned}(c_n^\dagger c_p)_{1\mu} &= \sum_k (X_1(k;p,n)\Gamma^\dagger_{1\mu}(k) + Y_1(k;p,n)\Gamma_{1\mu}(k))\\ &\quad+ \sum_{i,k} S^1_{i,k}(pn)(\Gamma^\dagger_1(i)\Gamma_2(k))_{1\mu}\\ &\quad+ \sum_{i,k} S^2_{i,k}(pn)(\Gamma_1(i)\Gamma_2(k))_{1\mu},\end{aligned} \tag{A2}$$

where X and Y denote the forward and backward amplitudes, respectively. The coefficients $S^{(1)}$ and $S^{(1)}$ are calculated as follows:

$$S^{(1)}_{i,k}(pn) = [\Gamma_{1\mu_1}(i), [(c^+_n c_p)_{1\mu}, \Gamma^+_{2\mu_2}(k)]]C^{1\ 2\ 1}_{\mu_1\ \mu_2\ \mu},$$
$$S^{(2)}_{i,k}(pn) = [\Gamma^+_{1\mu_1}(i), [(c^+_n c_p)_{1\mu}, \Gamma^+_{2\mu_2}(k)]]C^{1\ 2\ 1}_{\mu_1\ \mu_2\ \mu}. \quad \text{(A3)}$$

The standard notation for the Clebsch-Gordan coefficient has been used. In this way the following expressions for the expansion coefficients are obtained:

$$\begin{aligned} S^{(1)}_{i,k}(pn) &= \sqrt{15}\Bigg[Y_1(i;pn')X_2(k;n'n)\sqrt{\frac{D_1(pn')}{D_2(n'n)}}W(1I_p2I_n;I_{n'}1) \\ &\quad + Y_1(i;p'n)X_2(k;pp')\sqrt{\frac{D_1(p'n)}{D_2(pp')}}W(1I_n2I_p;I_{p'}1)\Bigg], \\ S^{(2)}_{i,k}(pn) &= -\sqrt{15}\Bigg[X_1(i;pn')X_2(k;n'n)\sqrt{\frac{D_1(pn')}{D_2(n'n)}}W(1I_p2I_n;I_{n'}1) \\ &\quad + X_1(i;p'n)X_2(k;pp')\sqrt{\frac{D_1(p'n)}{D_2(pp')}}W(1I_n2I_p;I_{p'}1)\Bigg]. \end{aligned} \quad \text{(A4)}$$

In this expression, $W(abcd;ef)$ denotes the Racah coefficients.

The expansion coefficients $S^{(1)}$ and $S^{(2)}$ are used for calculating the matrix elements characterizing the two legs of the double β. Thus, the product of the single-β transition amplitudes are given analytically by

$$\begin{aligned} &{}_i\langle 0||\beta^+||1_k\rangle_{ii}\langle 1_k|1_j\rangle_{ff}\langle 1_j||\beta^+||2^+_1\rangle_f \\ &\quad = \sum_{ab;a'b'}\sqrt{3}\sqrt{|D_1(ab)|}X_{1k}(ab)(X_{1k}(a'b')X_{1j}(a'b') \\ &\qquad - Y_{1k}Y_{1j}(a'b'))S^{(1)}_{j1}(a'b')P_1(ab)P_2(a'b'). \end{aligned} \quad \text{(A5)}$$

This matrix element corresponds to the graph from Fig. 2(a). The graph in Fig. 2(b) is calculated using the following equation:

$$\begin{aligned} &{}_i\langle 0||\beta^+||1_j2_k\rangle_{ii}\langle 1_j2_k|1_j2_1\rangle_{ff}\langle 1_j2_1||\beta^+||2^+_1\rangle_f \\ &\quad = \sum_{ab;a'b'}\sqrt{3}S^{(2)}_{jk}(ab)P_1(ab) \\ &\qquad \times (X_{1j}(ab)X_{1j'}(ab) - Y_{1j}(ab)Y_{1j'}(ab))X_{2k}(a'b')X_{21}(a'b') \\ &\qquad \times \sqrt{|D_1(a'b')|}Y_{1j'}(a'b')P_2(a'b'). \end{aligned} \quad \text{(A6)}$$

APPENDIX B

The Gamow-Teller interaction generates dipole states whose strengths are governed by the so-called Ikeda sum rule (ISR).This is the nuclear structure counterpart of the famous sum rule from atomic physics, pointed out by Reiche and Kuhn [76–78]. The ISR asserts that for the double-β emitter the difference between the β^- and β^+ strengths equals three times the neutron excess, i.e., $3(N-Z)$. This equality is exactly satisfied within the pnQRPA framework. However, in order to conciliate the agreement with the experimental data and the ground state stability, one has to go beyond the pnQRPA level. This is achieved either by boson expansion of the GT transition operator or by renormalizing the pnQRPA equations by taking care of a piece of the anharmonic interaction. It seems that the two procedures affect the ISM in a different manner. Indeed, while the renormalization formalism underestimates, the boson expansion overestimates the ISR. This fact suggested to one of the authors (A.A.R. in collaboration) to elaborate the boson expansion on the top of a renormalized pnQRPA [53]. Indeed, the GT operator was expressed in terms of the renormalized pnQRPA phonon operators. Thus, the agreement with the ISR was much improved. Although the present formalism restores the gauge symmetry, it remains a higher RPA approach and thereby the ISR is violated. To see what causes such a deviation, a few details about ISR derivation are necessary. Indeed, let us calculate the commutator of the single-β transition operators, written in the second quantization corresponding to the above-defined single-particle basis:

$$\begin{aligned} &[\beta^+_\mu, \beta^-_{-\mu}(-)^\mu] \\ &\quad = \Bigg[\frac{\sqrt{2}}{\hat{I}_n}\langle n|\sigma_\mu|p\rangle\frac{\sqrt{2}}{\hat{I}_p}c^+_{I_nM_n}c_{I_pM_p}, \frac{\sqrt{2}}{\hat{I}_{p'}}\langle p'|\sigma_{-\mu}(-)^\mu|n'\rangle \\ &\qquad \times \frac{\sqrt{2}}{\hat{I}_{n'}}c^+_{I_{p'}M_{p'}}c_{I_{n'}M_{n'}}\Bigg] \\ &\quad = \langle n|\sigma_\mu|p\rangle\langle p|\sigma_{-\mu}|n'\rangle(-)^\mu c^+_{I_nM_n}c_{I'_nM'_n}\frac{2}{2I_n+1} \\ &\qquad - \langle p'|\sigma_\mu|n\rangle\langle n|\sigma_{-\mu}|p\rangle(-)^\mu c^+_{I_{p'}M_{p'}}c_{I_pM_p}\frac{2}{2I_p+1} \\ &\quad = 3(\hat{\mathcal{N}}_n - \hat{\mathcal{N}}_p), \end{aligned} \quad \text{(B1)}$$

where $\hat{\mathcal{N}}_n$ and $\hat{\mathcal{N}}_p$ denote the neutron and proton number operator, respectively. Averaging this equation with the pnQRPA vacuum state and then inserting between the single-β operators the unity operator defined with the phonon dipole state, we obtain

$$\begin{aligned} &\sum_k\langle 0|\beta^+_\mu|1_kM_k\rangle\langle 1_kM_k|\beta^-_{-\mu}(-)^\mu|0\rangle \\ &\qquad - \sum_k\langle 0|\beta^-_{-\mu}(-)^\mu|1_kM_k\rangle\langle 1_kM_k|\beta^+_\mu|0\rangle \\ &\quad = 3(\langle 0|\hat{\mathcal{N}}_n|0\rangle - \langle 0|\hat{\mathcal{N}}_p|0\rangle). \end{aligned} \quad \text{(B2)}$$

Defining the single-*beta* transition strength functions as

$$\beta^{(-)} = \sum_k\langle 0||\beta^+||1_k\rangle^2, \quad \beta^{(+)} = \sum_k\langle 0||\beta^-||1_k\rangle^2, \quad \text{(B3)}$$

and approximating further the pnQRPA vacuum with the BCS vacuum, one obtains

$$\beta^{(-)} - \beta^{(+)} = 3(N-Z). \quad \text{(B4)}$$

TABLE V. Results of our calculations for the $B(E2)$ values and the half-lives of the final state 2^+, in the daughter nucleus, compared with the corresponding experimental data. Energies of the state 2^+ are also listed. The scaling factor Q_0, involved in Eq. (C1) is adimensional.

Daughter		$B(E2; 2^+ \to 0^+)$ (W.u.)		$t_{1/2}$ (ps)		
Nucleus	E_{2^+} (MeV)	Expt.	Theor.	Expt.	Theor.	Q_0
^{76}Se	0.559	44.0	43.60	12.30	12.41	2.82
^{82}Kr	0.776	21.3	12.28	4.45	7.71	1.00
^{96}Mo	0.778	20.7	19.89	3.67	3.81	1.73
^{100}Ru	0.539	0.094	38.45	12.56	11.67	2.82
^{116}Sn	1.293	12.4	11.47	0.374	13.6	1.00
^{128}Xe	0.443	48.0	34.56	18.00	25.04	2.82
^{130}Xe	0.536	38.5	41.04	8.60	7.95	2.82
^{150}Sm	0.331	57.1	30.48	48.40	98.95	2.82

The above-mentioned approximation is not valid within the present formalism. Indeed, writing the particle number operators in terms of the quasiparticle operators, one finds

$$
\begin{aligned}
&(\langle 0|\hat{\mathcal{N}}_n|0\rangle - \langle 0|\hat{\mathcal{N}}_p|0\rangle) \\
&\quad = N - Z + \sum_n \frac{2}{2I_n+1}\left(U_n^2 - V_n^2\right)\sum_{M_n}\langle 0|a^+_{nM_n}a_{nM_n}|0\rangle \\
&\qquad - \sum_n \frac{2}{2I_n+1}U_nV_n \\
&\qquad \times \sum_{M_n}\langle 0|a^+_{nM_n}a^+_{n,-M_n}(-)^{M_n} + a_{n,-M_n}a_{n,M_n}(-)^{M_n}|0\rangle \\
&\qquad - \sum_p \frac{2}{2I_p+1}\left(U_p^2 - V_p^2\right)\sum_{M_p}\langle 0|a^+_{pM_p}a_{pM_p}|0\rangle \\
&\qquad + \sum_p \frac{2}{2I_p+1}U_pV_p \\
&\qquad \times \sum_{M_p}\langle 0|a^+_{pM_p}a^+_{p,-M_p}(-)^{M_p} + a_{p,-M_p}a_{p,M_p}(-)^{M_p}|0\rangle .
\end{aligned}
\tag{B5}
$$

Recalling that a specific feature of our formalism is that the renormalized pnQRPA vacuum state comprises quasiparticles, it becomes conspicuous that the averages involved in the above equation are nonvanishing. However, we found a dimension of the single-particle basis and a set of parameters defining the mean field and the pairing properties which provides a vanishing value for the sum of the mentioned terms in the above equation. Thus, although the present formalism is based on a renormalized pnQRPA and a boson expansion, the ISR is to high accuracy satisfied.

Since satisfying the ISR is an appraisal for the approximation quality of the proposed formalism, one may assert that the present approach is a reliable one.

APPENDIX C

The electric quadrupole transition operator is

$$Q_{2\mu} = Q^{(0)} e_{\rm eff}\sqrt{\frac{16\pi}{5}}\, r^2 Y_{2\mu}. \tag{C1}$$

The scaling factor $Q^{(0)}$ was introduced to account for the contribution of the core nucleons. The reduced matrix element of $Q_{2\mu}$ corresponding to the projected spherical basis is

$$\left\langle \Phi^I_{nlj}\middle|\middle|Q_2\middle|\middle|\Phi^{I'}_{n'l'j'}\right\rangle = f^{j'I'}_{jI;2}(d)\langle nlj||Q_2||n'l'j'\rangle, \tag{C2}$$

with the factor $f^{j'I'}_{jI;2}(d)$ defined as in Ref. [35] and given by Eq. (2.10). Using the second quantization representation, we have

$$
\begin{aligned}
q_\mu &= C^{I_k\,2\,I_i}_{m_k\,\mu\,m_i} Q^{(0)} e_{\rm eff}\langle I_i||\sqrt{\frac{16\pi}{5}}r^2Y_2||I_k\rangle c_i^+ c_k \\
&\equiv q^{(2)}_{ik}(c^+_{I_i}c_{I_k})_{2\mu} \\
&= q^{(2)}_{ik}\sqrt{D_2(\tau, ik)}\bar{\mathcal{A}}^+_{2\mu}(\tau, i, k).
\end{aligned}
\tag{C3}
$$

From here, by simple manipulations one obtains

$$
\begin{aligned}
\langle 2^+||q^{(2)}||0^+\rangle = Q^{(0)}q^{(2)}_{ik}[&e_{\rm eff}(p)\sqrt{D_{2p}(1, ik)}X_{2p}(1, ik) \\
&+ e_{\rm eff}(n)\sqrt{D_{2n}(1, ik)}X_{2n}(1, ik)].
\end{aligned}
\tag{C4}
$$

Furthermore the $B(E2)$ value is obtained from

$$B(E2; 2^+ \to 0^+) = [\langle 2^+||q^{(2)}||0^+\rangle]^2. \tag{C5}$$

TABLE VI. Results for calculated and experimental nuclear deformations compared with the deformation parameter involved in the projected spherical single-particle basis.

Daughter	$\beta_{\rm expt}$	β	d/k
^{76}Se	0.263	0.262	0.190
^{82}Kr	0.168	0.128	0.022
^{96}Mo	0.135	0.132	0.100
^{100}Ru	0.009	0.173	0.167
^{116}Sn	0.082	0.079	0.400
^{128}Xe	0.145	0.123	0.213
^{130}Xe	0.129	0.134	0.175
^{150}Sm	0.131	0.096	0.697

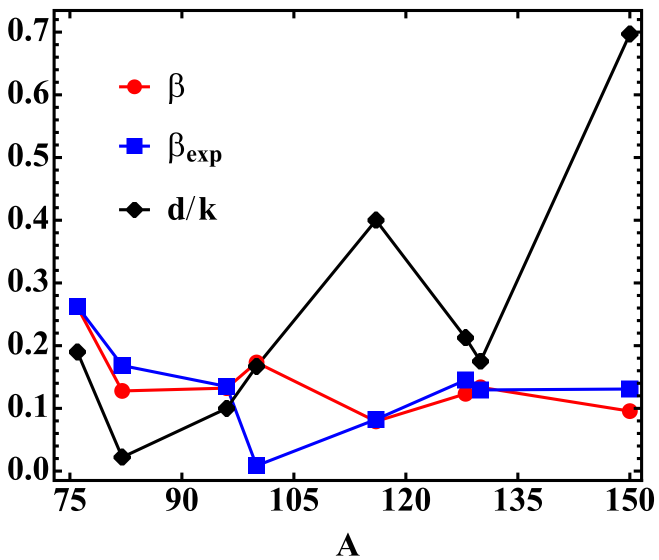

FIG. 7. Calculated and experimental nuclear deformations given by Eq. (C7) are compared with the parameter d defining the projected single-particle basis.

Using this expression the half-life of the collective state 2^+ is

$$t_{1/2} = \frac{15}{5.498} \times 10^{-21} \ln 2 \left[\frac{E_{2^+}[\text{MeV}]}{197.33}\right]^{-5} \times [B(E2; 2^+ \to 0^+)[e^2\,\text{fm}^4]]^{-1}[\text{s}]. \tag{C6}$$

Results for the $B(E2)$ values and half-lives are collected in Table V.

Having the $B(E2)$ values calculated, the nuclear deformation is readily obtained from the equation

$$\sqrt{5B(E2; 2^+ \to 0^+)} = \frac{3}{4\pi} eZR^2\beta, \tag{C7}$$

where R denotes the nuclear radius: $R = r_0 A^1/3$, $r_0 = 1.2$ fm. Similarly, by inserting in the rhs of the above equation the experimental $B(E2)$ value, one obtains the experimental nuclear deformation. The results are collected in Table VI. Note that ß agrees reasonable well with experiment. Except for the spherical nuclei ^{116}Sn and ^{150}Sm, their values are close to the parameter d/k, which defines the projected spherical single-particle basis [79]. This is shown in Fig. 7.

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