Two-dimensional extrapolation procedure for an *ab initio* study of nuclear size parameters and the properties of the halo nucleus ^{6}He

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A new two-dimensional procedure for extrapolation of the values of matter, neutron, and proton radii obtained in no-core shell model (NCSM) calculations to infinite size of its basis is proposed. A relationship between the radii is used as an additional test. Together with the JISP16 potential, which is frequently used in NCSM calculations of the radii, the Daejeon16 potential is applied for these purposes. Halo nucleus ^{6}He is the object of studies. The small spread of radii values and reasonable agreement between the obtained results and experimental data as well as the results of other advanced *ab initio* calculations demonstrate the high efficiency of the developed approach and, therefore, good prospects for its application. The performed investigations and analysis of the results of other studies indicate that the halo of ^{6}He has a large size: 0.7–0.8 fm.

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I. INTRODUCTION

Matter and charge radii are one of the most significant observables characterizing atomic nuclei. Due to a variety of experimental techniques, the charge radii of stable and long-lived nuclei have been well studied. A slightly less favorable, but also quite satisfactory situation is the case for matter radii. For nuclei far from the stability band, the study of these quantities causes serious difficulties, which are gradually overcome with the development of methods for obtaining ever more intense beams of short-lived ions. The data obtained during such measurements are of great physical importance, since they provide information about the inhomogeneity of the distribution of protons and neutrons, the presence of "skin" or "halo" in nuclei. However, such experiments are still technically difficult and their results obtained for the same isotope often differ markedly from experiment to experiment. Therefore, the problem of theoretical description the size parameters (including neutron radii not available for direct measurements) of nuclei of the discussed type seems to be very topical.

At the same time, it should be stressed that simplified approaches, such as the shell model with an inert core and, even more so, schemes that consider the core without taking into account its nucleon structure, encounter difficulties in describing long-range nucleon correlations, exchange effects, etc. and, therefore, give results of limited reliability. So, the use of *ab initio* approaches is one of few possible ways to solve this task.

An *ab initio* study of the size parameters of the lightest nucleon-stable neutron-rich ^{6}He nucleus is the subject of this paper. Naturally, the total binding energy of this system is also computed. The choice of this isotope is due to the fact that it is a good testing ground for the studies discussed. Indeed, first, its size characteristics have been fairly well studied experimentally. Second, the relatively small number of its constituent nucleons makes it possible to test a wide variety of high-precision theoretical methods on it. Third, this "canonical" example of the lightest halo nucleus turns out to be difficult to describe even within the framework of many *ab initio* schemes due to its large size and the great difference in neutron and charge radii (halo size $r_n - r_c$), unique for nuclei with A $\leqslant$ 10. The main motivation behind this choice is that the ^{6}He nucleus is a very physically interesting (so-called Borromean) three-body ($\alpha + 2n$) system characterized by a large distribution radius of weakly bound neutrons.

The list of works devoted to the same subject is quite large. The most important of these are presented in Refs. [1–14] which are based on wide diversity of *ab initio* schemes. Various Monte Carlo approaches [1,2,6,12], the hyperspherical harmonics method [5], no-core configuration interaction (NCCI) calculations [10,11], the coupled-cluster method [14], etc. are used in these works. Various versions of the no-core shell model (NCSM) play the most prominent role among these schemes. The NCSM and similar in structure NCCI calculations of matter and charge radii together with the binding energy ones are performed using realistic *NN* [7] and *NN* + 3*N* [3,9,12] interactions. The bases of the usual NCCI and NCSM are exploited, as well as symmetry-adapted SU(3)-based NCSM [8]. NCSM with ($\alpha + 2n$) continuum (NCSMC) [13] also used to solve the above-mentioned problem.

It should be emphasized that the problem of high-precision *ab initio* computation of the size parameters of nuclei is qualitatively different from the problem of calculation their binding energy. The main reason for this is that the sizes of nuclei are largely determined by long-range nucleon correlations. The consequence of that is, first, that it is hard task to achieve convergence of the results of calculating the radii of nuclei on a basis that is available to modern computers. This, in turn, forces one to use procedures for extrapolating the results to large dimensions of a basis. Approaches of such a kind in the calculations of ^{6}He nucleus are presented in Refs. [15,16]. Also, there is a second problem along this way. In contrast to the regular tendency for the binding energy to change monotonically with increasing basis size, the convergence of nuclear radii is not monotonic. Because of this, approaches that include extrapolation procedures to an infinite basis need to be more sophisticated.

The results obtained in the above works describe the experimental data with varying degrees of success (see the analysis below). However, the approaches developed in these works are often insufficient to answer the question of whether the source of the discrepancy between the theoretical results and experiment is the model of internucleon interaction or the inaccuracy of the calculation method itself. This is true even for the canonical ^{6}He nucleus example. Therefore, new *ab initio* calculations of the radii of the six-nucleon nuclei are needed and developing new methods for extrapolation of computed data obtained with an achievable basis cutoff parameter to infinity seem to be a vital issue.

In this paper we start from one of the most reliable and justified *ab initio* approaches, the M-scheme of the no-core shell model. Two different versions of the *NN* interaction together with Coulomb proton-proton interaction have been incorporated in the model Hamiltonian. The first of these two—the JISP16-potential—has been built from nucleon scattering data using the *J*-matrix inverse scattering method [17]. It has already been used to calculate the size parameters of the ^{6}He nucleus in Refs. [6–8,10], so in this paper it is exploited, in particular, to test other new elements of our approach.

An important part of the current investigation is that the Daejeon16 potential [18] is also applied for the NCSM computations of the ^{6}He nuclear radius as a model of the *NN* interaction. It is built using the N3LO limitation of chiral effective field theory [19] softened via a similarity renormalization group (SRG) transformation [20]. Both JISP16 and Daejeon16 versions of internucleon are "global," i.e., designed to calculate all kinds of characteristics of nuclei with mass $A \leqslant 16$. They were tested in the framework of large-scale computations of the total binding energies, nucleon and cluster binding energies, excitation energies, radii, moments of nuclear states, and the reduced probabilities of electromagnetic transitions. These tests demonstrated that such characteristics are, in general, reproduced well. In some cases, the quality of describing these characteristics using the Daejeon16 interaction is somewhat better than that for the JISP16 one.

The NCSM calculations were performed by using the BIGSTICK code [21]. In our computations of the ^{6}He system the oscillator bases are limited by the values of the cutoff parameter of the number of excitation quanta $\mathcal{N}_{\max} \leqslant 14$.

As mentioned above, new extrapolation methods are needed. The central point of this work is a new two-dimensional procedure for extrapolating the calculated values of radii to an infinite basis. The object of extrapolation is the shape of the surface of values of one of the radii over the plane $(\mathcal{N}_{\max}, \hbar\omega)$ and both of these coordinates are contained in the extrapolation formula. The proposed procedure was verified in the framework of calculating the size parameters of the ^{4}He nucleus, for which the convergence of these parameters was achieved in direct NCSM calculations.

In this way the matter, neutron, and proton radii as well as the size of the neutron halo of ^{6}He isotopes are obtained. A comparative analysis of the obtained results, experimental data, and results of other theoretical works is given.

Let us start the description of the experimental situation and the developed approach with the terminology. As in most modern works, point-nucleon rms radii—parameters that characterize the distributions of all nucleons (nuclear matter) $r_m \equiv (\bar{r}_m^2)^{1/2}$, as well as neutrons r_n and protons r_p, are considered. The last parameter is obtained from the measured charge radius r_c using the expression presented in Ref. [22]:

$$r_p^2 = r_c^2 - R_p^2 - (N/Z)R_n^2 - 3\hbar^2c^2 \big/ 4(M_pc^2) - r_{so}^2. \quad (1)$$

Here R_p^2 and R_n^2 are the proton and neutron mean-square charge radii, $3\hbar^2c^2/4(M_pc^2)$ is a relativistic Darwin-Foldy correction, and r_{so}^2 is a spin-orbit nuclear charge-density correction. The following values are usually chosen: $R_n^2 = -0.1161$ fm^2, $3\hbar^2c^2/4(M_pc^2) = 0.033$ fm^2, $r_{so}^2 = 0.08$ fm^2. The last estimate is valid for ^{6}He only. For R_p, the Particle Data Group [23] value is 0.877(7) fm. It is this quantity that is used when processing the measurement results to obtain point-proton nuclear radii. However, this value has also been precisely determined from the spectroscopy of muonic hydrogen [24]. It turned out to be 0.84184(67) fm. Therefore, by using these two values for R_p, two different experimental values of r_p can be obtained. The matter radius r_m is deduced directly from differential cross sections of elastic proton scattering at high momentum transfer using Glauber multiple-scattering theory. Radius r_n is unavailable for measurements and therefore is calculated by using the expression

$$Ar_m^2 = Zr_p^2 + Nr_n^2. \quad (2)$$

As in other papers when studying the properties of systems with a neutron halo or skin, a measure of their thickness—the difference of point-nucleon sizes $r_h = r_n - r_p$ is used by us.

II. OUTLINE OF METHODOLOGY OF THE RESEARCH

A. Formalism of calculating nuclear radii

In NCSM computations we use the universal approach, which makes it possible to calculate all three size parameters. After calculating the binding energy and wave function, we proceed to calculate matter, neutron, and proton radii for point nucleons. The basic expressions of the formalism intended to describe these quantities are the following: In the shell model, the squared radius of a corresponding system is defined as

$$r_{m(n,p)}^2 = (1/N_{A(N,Z)}) \sum_i (\vec{r}_{m(n,p),i} - \vec{r}_{\text{c.m.}})^2,$$

TABLE I. The total binding energies (MeV), matter, neutron, and point-proton radii of ^{6}He nucleus (fm) obtained in direct calculations with the use of Daejeon16 NN-interaction.

	E_b			Matter			Neutron			Proton		
$\hbar\omega/N_{\max}$	10	12	14	10	12	14	10	12	14	10	12	14
6.0	23.677	25.454	26.752	2.721	2.661	2.612	2.948	2.889	2.842	2.201	2.132	2.076
7.0	25.941	27.274	28.141	2.548	2.508	2.482	2.760	2.724	2.701	2.058	2.007	1.971
7.5	26.757	27.871	28.550	2.485	2.457	2.443	2.693	2.669	2.659	2.006	1.966	1.939
8.0	27.398	28.309	28.830	2.436	2.419	2.416	2.640	2.628	2.630	1.964	1.934	1.917
9.0	28.258	28.839	29.134	2.368	2.372	2.386	2.567	2.578	2.599	1.907	1.894	1.892
10.0	28.725	29.086	29.258	2.329	2.348	2.372	2.525	2.552	2.583	1.874	1.874	1.880
11.0	28.957	29.194	29.302	2.306	2.334	2.362	2.501	2.537	2.572	1.856	1.864	1.873
12.5	29.093	29.239	29.313	2.284	2.318	2.348	2.476	2.518	2.555	1.841	1.853	1.866
15.0	29.093	29.210	29.279	2.252	2.288	2.318	2.438	2.482	2.519	1.822	1.838	1.851
17.5	28.988	29.127	29.214	2.215	2.252	2.284	2.394	2.440	2.478	1.804	1.824	1.834

where $\vec{r}_{\rm c.m.} = (1/N_A)\sum_i \vec{r}_{m,i}$ Mean square radius takes the form

$$\bar{r}^2_{m(n,p)} = -\frac{4}{N_A N_{A(N,Z)}}\langle\Psi_A|\sum_{i<j}\vec{r}_{m(n,p,i)}\vec{r}_{m,j}|\Psi_A\rangle + \langle\Psi_A|r^2_{\rm c.m.}|\Psi_A\rangle + \frac{N_A-2}{N_A N_{A(N,Z)}}\langle\Psi_A|\sum_i r^2_{m(n,p),i}|\Psi_A\rangle. \tag{3}$$

Here $N_{A(N,Z)}$ denotes the number of nucleons, A (or neutrons N or protons Z), in the system,

$$\langle\Psi_A|r^2_{\rm c.m.}|\Psi_A\rangle = \frac{3(\hbar c)^2}{2mc^2\hbar\omega N_A}, \tag{4}$$

$$\langle\Psi_A|\sum_i \vec{r}_i^{\,2}|\Psi_A\rangle = \frac{1}{\sqrt{2J+1}} \times \sum_{k_a,k_b} OBTD(k_a,k_b,\lambda=0)\langle k_a||r^2||k_b\rangle, \tag{5}$$

and

$$\langle\Psi_A|\sum_{i<j}\vec{r}_i\vec{r}_j|\Psi_A\rangle = \frac{1}{\sqrt{2J+1}}\sum_{k_a\leqslant k_b,k_c\leqslant k_d,J_0}\langle k_a k_b J_0||\vec{r}_1\vec{r}_2||k_c k_d J_0\rangle TBTD \times (k_a,k_b,k_c,k_d,J_0). \tag{6}$$

The one-body and two-body transition densities (OBTD and TBTD) included in these formulas are expressed in terms of the matrix elements of the products of fermion second quantization operators:

$$OBTD(k_a,k_b,\lambda=0) = \langle\Psi_A||[a^+_{k_a}\otimes\tilde{a}_{k_b}]^{\lambda=0}||\Psi_A\rangle. \tag{7}$$

$$TBTD(k_a,k_b,k_c,k_d,J_0) = \langle\Psi_A||[[a^+_{k_a}\otimes a^+_{k_b}]_{J_0}\otimes[\tilde{a}_{k_c}\otimes\tilde{a}_{k_d}]_{J_0}]^{\lambda=0}||\Psi_A\rangle. \tag{8}$$

where $\otimes$ is the sign of a tensor product of rank J_0 or λ.

The results of the performed NCSM calculations of the matter, neutron, and point-proton radii of the ^{6}He nucleus for different values of $\hbar\omega$ and $\mathcal{N}_{\max}$ in which the Daejeon16 potential was used, as well as the corresponding values of total binding energies, are shown in Table I.

B. Extrapolation procedure

A new two-dimensional extrapolation procedure proposed by us is based on NCSM calculations of each of these radii for different values of $\hbar\omega$ and $\mathcal{N}_{\max}$. To understand the reasoning behind this idea consider the surface formed by the values of any of the investigated radii over the plane $(\mathcal{N}_{\max},\hbar\omega)$. The surfaces formed by the values of the matter, neutron, and proton radii of the ^{4}He nucleus, calculated within the framework of the NCSM in a wide range of H values, can serve as good examples. They are presented in Table II.

For large values of $N_{\max}$ this surface is a horizontal plane in a fairly wide range of values of $\hbar\omega$. As $\mathcal{N}_{\max}$ decreases, the values of the radius increase for small $\hbar\omega$ and decrease for large $\hbar\omega$, which was noted in several previous works, in particular in Refs. [7,10], devoted, in particular, to calculations of the radii of light nuclei, and stressed by illustrations of the latter. The wide horizontal part of the surface narrows and, in the range of maximal $\mathcal{N}_{\max}$ values achievable for computer calculations, degenerates into an (approximately) horizontal line in some domain of values $\mathcal{N}_{\max}$. The values of the radius on this line, it was proposed in Refs. [7,10] to consider as a value of the corresponding size parameter. It is the so-called "crossover prescription." The value of $\hbar\omega$ at which the value of this parameter approximately stabilizes is called crossover point. A detailed description of this concept can be found in Ref. [10]. This behavior is well illustrated in Table II. Indeed, the horizontal plane, which is wide at large values of $\mathcal{N}_{\max}=16$, 18, degenerates into a narrow horizontal strip at $\mathcal{N}_{\max}=10$–14 and $\hbar\omega=15.0$–17.5 MeV for any for each of the radii. The same property of size parameters is even more pronounced for the ^{6}He nucleus (see Table I). Note, for the sake of completeness, that despite the complete convergence of the results, they do not reproduce the experimental value of the point-proton radius $r_p^{\rm expt}=1.455(1)$. This, obviously, is not related to the quality of the computer calculation, but is determined by the properties of the potential Daejeon16.

TABLE II. The matter, neutron, and point-proton radii of ^{4}He nucleus (fm) obtained from direct calculations with the use of the Daejeon16 NN-interaction.

	Matter					Neutron					Proton				
$\hbar\omega/N_{max}$	10	12	14	16	18	10	12	14	16	18	10	12	14	16	18
10.0	1.5304	1.5181	1.5135	1.5118	1.5112	1.5278	1.5154	1.5107	1.5091	1.5085	1.5330	1.5208	1.5162	1.5145	1.5139
12.5	1.5133	1.5116	1.5111	1.5109	1.5109	1.5106	1.5089	1.5084	1.5082	1.5082	1.5160	1.5143	1.5138	1.5136	1.5136
15.0	1.5113	1.5108	1.5109	1.5109	1.5109	1.5086	1.5081	1.5082	1.5081	1.5081	1.5140	1.5135	1.5136	1.5136	1.5136
17.5	1.5105	1.5107	1.5108	1.5108	1.5109	1.5078	1.5080	1.5081	1.5081	1.5081	1.5132	1.5134	1.5135	1.5135	1.5136
20.0	1.5097	1.5104	1.5107	1.5108	1.5108	1.5070	1.5077	1.5079	1.5081	1.5081	1.5124	1.5131	1.5134	1.5135	1.5135
22.5	1.5086	1.5098	1.5104	1.5106	1.5107	1.5059	1.5071	1.5077	1.5079	1.5080	1.5112	1.5125	1.5131	1.5133	1.5134
25.0	1.5068	1.5088	1.5098	1.5103	1.5106	1.5042	1.5061	1.5071	1.5076	1.5079	1.5095	1.5115	1.5125	1.5130	1.5133

The geometric image of the surface as a whole in this case is a tape twisted around this line as an axis. So, the idea is to determine trends in surface properties with increasing $\mathcal{N}_{\max}$ using an extrapolation procedure basing on two-dimensional data with limited values of this parameter. We called this procedure twisted tape extrapolation (TTE). Within the framework of the TTE procedure, the following formula is proposed:

$$r^2_{m(n,p)}(\mathcal{N}_{\max}, \hbar\omega) = r^2_{\infty,m(n,p)} + P_k(\hbar\omega)\exp(-\alpha\sqrt{\mathcal{N}_{\max}}), \tag{9}$$

where $P_k(x)$—a polynomial of degree k whose coefficients are fitting parameters, $r^2_{\infty,m(n,p)}$ is the extrapolation result—is the squared radius for infinite oscillator basis and $r^2_{m(n,p)}(\mathcal{N}_{\max}, \hbar\omega)$ are theoretically obtained results for squared radii.

It is of value to highlight the main features of the discussed procedure:

1. First of all, the procedure is phenomenological one. To simultaneously involve the values of radii computed in the bases with different $\mathcal{N}_{\max}$ and $\hbar\omega$, we had to abandon the idea of extrapolating the binding energy of the nucleus and its size parameters using logically related formulas, which is implemented in most modern works. Bearing in mind the successful description of the radii of the ^{6}He nucleus obtained in this work, as well as the prospects for the description of other observables of various nuclei (γ-transition probabilities, etc.), we consider this justified.
2. At the same time the proposed procedure is based to a certain extent on the currently popular parametrization of nuclear radii given by the expression (see, for example, Ref. [16])
$$r^2_{m(n,p)}(L, \Lambda) = r^2_{\infty,m(n,p)}(\Lambda) - a(\Lambda)[k_\infty(\Lambda)L]^3\exp[-k_\infty(\Lambda)L], \tag{10}$$
where $a(\Lambda)$ and $k_\infty(\Lambda)$ are adjustable parameters having dimensions fm^2 and fm^{-1}, respectively, Λ—the ultraviolet linear momentum cutoff parameter, and L—a size parameter that characterizes the distance at which the basis used correctly reproduces the wave function of a nucleus. The last value is proportional to $\sqrt{2N+3/2}$ when $N \gg 1$. Index N is not well defined for a multiparticle problem, however, for a sufficiently light nucleus, it is assumed to be proportional to $\mathcal{N}_{\max}$. Under conditions of such uncertainty, the term $3/2$ can be neglected, and the value $\sqrt{\mathcal{N}_{\max}}$ with the corresponding adjustable parameter may be introduced into the exponent.
3. If one does not try to describe the radii and the binding energy of the nucleus in logically connected schemes, the introduction of the momentum cutoff parameter Λ loses its meaning. With the fixed form of internucleon interaction, the real linear momentum cutoff parameter is determined by the size of the basis, i.e., by $\mathcal{N}_{\max}$ (thus, in practice, the power of the computer used) and the chosen value of $\hbar\omega$.
4. Finally, the pre-exponential factor of the formula (10) could, in principle, be introduced into the formula (9). However, this introduction together with the polynomial $P_k(x)$ would complicate the approach too much. Therefore, it is not included in our extrapolation formula.

It should be noted that the interpolation procedure to determine the ranges of possible values of the radii r_p and r_m using two-dimensional $(\mathcal{N}_{\max}, \hbar\omega)$ input data was presented in Ref. [10]. Namely, the radii as functions of $\hbar\omega$ at fixed $\mathcal{N}_{\max}$ values from the domain of approximate stability are computed by cubic one-dimensional interpolation of the calculated data points at different $\hbar\omega$. A more complicated two-dimensional-basing extrapolation procedure which includes the chi-squared fit to the number of points at which the values of the point-proton radius were calculated for various $\mathcal{N}_{\max}$ and $\hbar\omega$ is presented in Ref. [25]. The object of the study was the ^{6}Li nucleus. Some part of the calculations was carried out with the JISP16 potential. The weight of each point was determined by the difference in the values of the radius in neighboring $\mathcal{N}_{\max}$ points.

The main novelty of our procedure is that the interdependence of r^2_∞ and $\hbar\omega$ is directly included in the extrapolation formula (9). Another difference of our extrapolation procedure is a choice of the range of $\hbar\omega$ values which is fundamentally different from the one that is preferred in Ref. [25]. It seems to us preferable to search in the vicinity to crossover point as it is done in work [10]. At the same time, in our work, we used the same method for determining the weight of each point, as proposed in Ref. [25]. For implement-

TABLE III. Approximate positions of the crossover points (MeV) and the values of: the total binding energy (MeV), the matter, neutron, and point-proton radii of ^{6}He nucleus (fm) at this point at $\mathcal{N}_{\max} = 14$ for the JISP16 interaction.

	Mater	Neutron	Proton
$\hbar\omega_{co}$	10	10	12.5
$E_{\rm tot}$	26.41	26.41	27.86
r_∞	2.334	2.550	1.797

ing the extrapolation procedure we have chosen chi-square fit method realized in TMinuit minimization package which is included in open-source ROOT CERN data analysis framework.

III. RESULTS AND DISCUSSION

In this paper we present the results of the first test of TTE which is a rather general and, in our opinion, promising approach for studying observables that reflect long-range internuclear correlations in various nuclei. Therefore, here we limit ourselves to the simplest, taking into account the main property of the surface, namely, the "twisting," linearized version of the $\hbar\omega$ dependence of the surface shape, in which $\hbar\omega$ dependence is given by a first-order polynomial $P_1(\hbar\omega) = A + B\hbar\omega$, and a relatively narrow mesh of calculated data. We have included in it only three values of $\mathcal{N}_{\max}$ in which the crossover conditions take place: 10, 12, and 14. It is clear from our results and known from Refs. [7,10] that, as $\mathcal{N}_{\max}$ decreases, the values of the radii increase for small $\hbar\omega$ significantly faster than decrease for large $\hbar\omega$. To take that into account we have used nonuniform mesh with the points at $\hbar\omega = 6$, 7.5, 8, 9, 10, 11, 12.5, 15, and 17.5 MeV—for the Daejeon16 potential, and $\hbar\omega = 7.5$, 8, 9, 10, 11, 12.5, 15, and 17.5 MeV—for the JISP16 potential. This is partly justified by the results obtained. The prospects for TTE lie, obviously, in the field of using polynomials of a higher degree $\hbar\omega$, a denser and wider mesh. This will require a large amount of computer time.

Let us begin discussion from the results of NCSM computations in which the JISP16 potential has been used since, as indicated above, this version of interaction has already been used to calculate the radii of light nuclei in previous works. These results are the following: Positions of the crossover points, values of the total binding energy and the radii in these points are presented in Table III. It can be seen that these positions for the matter (as well as neutron) and point-proton radii differ noticeably. In view of this fact, the measure of violation of the relation (2) is important to estimate the reliability of these results. This measure is chosen to be $\Delta = 1 - [(Zr_p^2 + Nr_n^2)/Ar_m^2]^{1/2}$. We call it the violation factor. In the case discussed it is equal to 0.67%.

Total binding energies $E_{\rm tot}$ of ^{6}He nucleus in these points are strongly underestimated compare with the experimental one. From a formalistic point of view, the system appears to be unbound with respect to the emission of two neutrons. When computing total binding energy, the optimal value of $\hbar\omega$ is 17.5 MeV with $E_{\rm tot} = 28.47$ MeV at this point, the system is bound, but the binding energy of neutrons is very small.

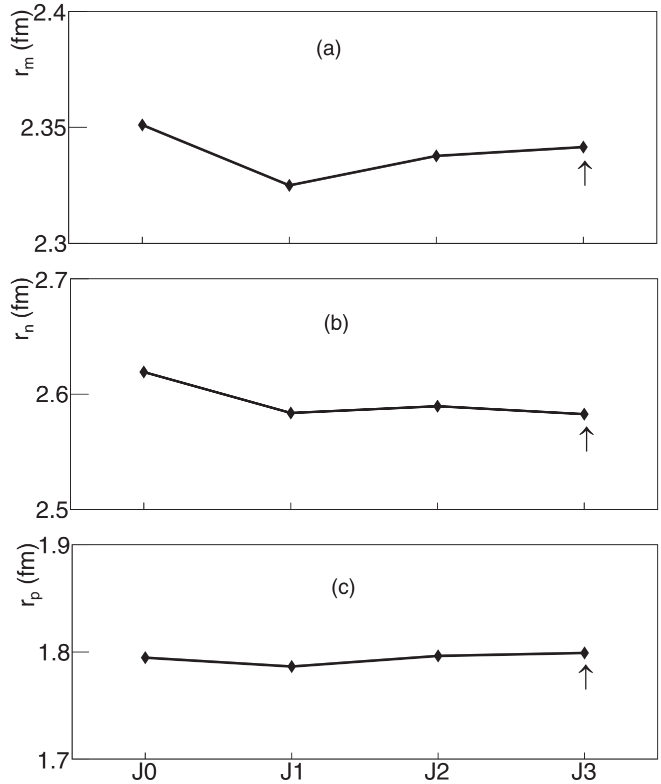

FIG. 1. (a) Matter, (b) neutron, and (c) point-proton radii of ^{6}He nucleus for different $\hbar\omega$ extrapolation domains in case of the use of the JISP16 potential.

To estimate stability of the results of extrapolation of size parameters, the domain of $\hbar\omega$ values has been varied. Namely, the procedure has been performed throughout the above presented range from 7.5 to 17.5 MeV, which is denoted as J0, as well as throughout the narrowed ranges 7.5–15 MeV (J1), 8–15 MeV (J2), and 9–17.5 MeV (J3). The final (optimal) result is the one that corresponds to the smallest value of χ^2.

Extrapolation results for each of these ranges in case of the use of the JISP16 potential are presented in Fig. 1. The optimal values are marked with an arrow. The figure demonstrates the high stability of the obtained radii. The final results for each of them and rms deviations of other ones from the optimal are $r_m = 2.342(7)$ fm, $r_n = 2.582(3)$ fm, and $r_p = 1.799(6)$ fm. The fitted parameters of the procedure (9) at this point are presented in Table IV. The extrapolated quantities sat-

TABLE IV. Values of the fitted parameters of extrapolation formula (9) for the radii of ^{6}He nucleus computed using the JISP16 interaction.

	Matter	Neutron	Proton
A (fm^2)	14.282	15.462	31.796
B (Mev^{-1} fm^2)	−1.4072	−1.6528	−2.3639
α	0.6096	0.5853	0.9556

TABLE V. The same as in Table III for the Daejeon16 interaction.

	Matter	Neutron	Proton
$\hbar\omega_{co}$	8	8	10
$E_{\rm tot}$	28.83	28.83	29.26
r_∞	2.416	2.630	1.880

isfy relation (2) much better than those estimated within the crossover prescription, violation factor Δ is equal to -0.36% for them. This property and, especially, the small values of the standard deviation are, in our opinion, evidence of the high efficiency of the proposed extrapolation method, even in its minimal version.

The proposed procedure somewhat changes the value of r_n and practically does not change the values of other size parameters obtained within the framework of stable crossover points prescription. The results obtained in our calculations, both estimated within the crossover prescription (see Table I) and extrapolated, are in rather good agreement with the results of Ref. [10], also obtained using the JISP16 interaction in different ways, but with up to $\mathcal{N}_{\rm max} = 16$ and based on the crossover prescription. The values $r_p = 1.799$–1.810 fm and $r_m = 2.314$–2.327 fm are presented in this paper.

Analogous computation results in which the Daejeon16 potential has been exploited are presented in Tables V and VI and Fig. 2. As for the JISP16 potential, in this case one can observe a difference in the position of the crossover points. The violation factor is significantly better being equal to -0.26%. For $\hbar\omega = 8$ MeV the binding energy of two neutrons is approximately equal to 0.5 MeV, and for $\hbar\omega = 10$ MeV it is close to that measured experimentally. At $\hbar\omega = 12.5$ MeV, the value of $E_{\rm tot}$ reaches a maximum value of 29.31 MeV.

The extrapolation procedure has been performed throughout the total range 6–17.5 MeV (D0) and the narrowed ranges 6–15 MeV (D1), 7.5–15 MeV (D2), and 7.5–12.5 MeV (D3). The optimal values have been chosen as described above. The obtained results are also stable. These values of the radii and the rms deviations are $r_m = 2.430(6)$ fm, $r_n = 2.663(3)$ fm, and $r_p = 1.871(16)$ fm. In this case the violation factor Δ is very small, namely, equal to 0.09%.

The experimental data necessary for comparative analysis are presented in Table VII. The neutron radius given together with the matter radius was calculated by the authors of the experiments using their own values of r_m, the values of point-proton radius from Refs. [22,26,31,32], and relation (2). The experimental value of point-proton radius of the nucleus itself depends on the proton radius R_p - see Ref. [23] and Ref. [24]. With reference to the table, it can be seen that the point-proton radius of ^{6}He nucleus is extracted with a high degree of accuracy and the results of different research groups are in good agreement. It looks a little strange, but against the background of this good agreement between these data, two different versions of the proton radius R_p introduce noticeable duality. The data concerning the matter radius also agree well. The error bars, however, are substantially higher in these measurements. The datum of Ref. [27] somewhat fall out of the systematics, which has a particularly noticeable effect on the size of the halo.

TABLE VI. The same as in Table IV for the Daejeon16 interaction.

	Matter	Neutron	Proton
A	38.813	49.712	100.00
B	−4.9377	−6.6133	−9.5647
α	1.0406	1.0593	1.3689

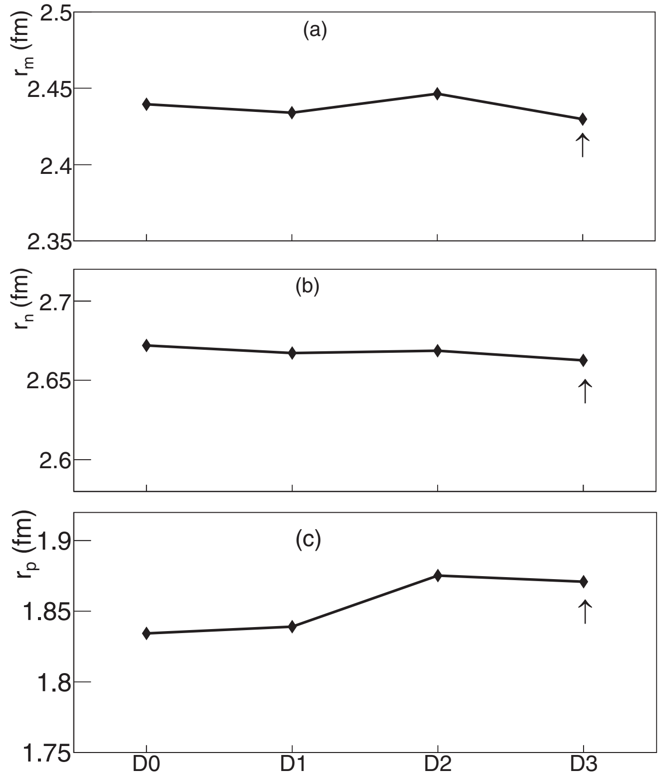

FIG. 2. The same as in Fig. 1 in case of the use of the Daejeon16 potential.

A comparison of the radii obtained for the two versions of the internucleon interaction with each other and with the experimental data leads to the following conclusions: The radii calculated using the Daejeon16 and JISP16 potentials differ significantly and, considering rms deviations, reliably. Daejeon16-based calculations result in larger values of the size parameters. The most likely reason is that the Daejeon16 interaction is softer than JISP16. A very unexpected result is

TABLE VII. Experimental values of matter, neutron, and point-proton radii of ^{6}He nucleus (fm), obtained using the radius of proton R_p from Ref. [23][a] and Ref. [24].[b]

	[28]	[29]	[27]	[30]
r_m	2.33(4)	2.30(7)	2.44(7)	2.29(6)
r_n	2.51(6)	2.47(10)	2.66(10)	2.45(9)
	[31,32]	[22][a]	[22][b]	[26]
r_p	1.925(12)	1.938(23)	1.953(22)	1.934(9)

almost complete coincidence of the size of the neutron halo r_h. Indeed, this value is equal to 0.792 and 0.783 fm in the discussed cases. In our opinion, this is not an artifact, but a real physical result. The values of underestimation of both r_m and r_p are approximating equal in JISP16 calculations. This coincidence also holds for the results obtained via crossover prescription; r_h is equal to 0.750 and 0.733 fm in this approach.

For the Daejeon16 potential, the point-proton radius is slightly, but reliably less than the experimental value. For the JISP16 potential this underestimation is not small. Other theoretical papers also give a significant underestimation of the parameter under discussion: Ref. [9]: 1.82 fm, Ref. [5]: 1.78 fm and 1.82 fm, Ref. [12]: 1.74 fm, 1.81 fm, and 1.84 fm (depending on the choice of interaction version). In the last example the values presented in the original paper for the charge radius r_c have been recalculated by us according to formula (1) with R_p given by Ref. [23]. These works also confirm the trend towards an increase in r_p when using softer interaction options. The exception is part of the data presented in Ref. [13]. As a result of NCSM calculations in the basis limited by $\mathcal{N}_{\max} = 12$ using potential from Ref. [19] softened via the SRG procedure with parameter $\lambda_{SRG} = 1.5$ fm^{-1} (for $\hbar\omega = 14$ MeV basis) and 2.0 fm $^{-1}$ (for $\hbar\omega = 20$ MeV basis) values 1.79 and 1.74 fm were obtained for the point-proton radius, respectively. At the same time in the NCSMC calculations carried out for the same input data, much larger values of the radius 1.85 and 1.87 fm were obtained, and the trend of their change turned out to be opposite. In the most advanced version of NCSMC, i.e., for the biggest model space the value 1.90(2) fm which is close to the experimental one was achieved.

The presented above values of the matter radius obtained in our calculations for both versions of the interaction lie in the range of values presented by different experimental groups. One can see good agreement between the values obtained in the calculations using JISP16 potential and the data from Refs. [28–30], as well as the value obtained in the calculations using the Daejeon16 potential and the datum from Ref. [27]. The trend for a size parameter value to increase when using softer versions of the interaction remains valid for the matter radius. This parameter was also studied in work [14]. The matter radius obtained in the NCSM calculations with the above presented input turned out to be equal to 2.25 and 2.15 fm for $\lambda_{SRG} = 1.5$ and 2.0 fm^{-1}, respectively. At the same time the NCSMC calculations performed in this work gave results of 2.37 and 2.41 fm, respectively, i.e., they sharply increase the values of the matter radius compared with NCSM calculations and reverse their dependence on the hardness of interaction. The computations within the most advanced version of NCSMC resulted in value $r_m = 2.46(2)$ fm.

It is important to ask which of the values of the matter radius obtained in experiments is confirmed in *ab initio* calculations: the larger one, 2.44(7) fm obtained in work [27], or the smaller ones, coinciding with good accuracy [mean value is approximately equal to 2.31(6) fm] presented in Refs. [28–30]. For several reasons, we, evidently, give preference the results obtained with the use of the Daejeon16 interaction. Compared with JISP16 one, this Hamiltonian is newer, provides faster convergence of total binding energies of various nuclei. A lot of other nuclear observables are more reproductive by it. What about ^{6}He example it results in correct binding energies of two neutrons in this isotope and yields better value of the point-proton radius. The extrapolated values of radii obtained with the use of it better satisfy relation (2). Therefore, our study can be considered as a theoretical confirmation of the results of measurements of the matter radius presented in Ref. [27]. The results of NCSMC calculations presented in Ref. [13] are also in good agreement with these two values of the matter radius.

The same question concerning the size of neutron halo is, perhaps, even more intriguing. The measurements presented in Refs. [28–30] resulted in well-consistent small values of r_h: 0.57, 0.53, and 0.51 fm. The result of Ref. [27] is much greater: $r_h = 0.72$ fm. The last value is in reasonable agreement with the results of calculations using the Daejeon16 potential, not only within the framework of the proposed extrapolation procedure, but also within the framework of the crossover prescription. It is noteworthy that this is also true for calculations using the JISP16 potential. An additional confirmation the discussed result may be found in the framework of the analysis of the results presented in Ref. [13]. The value of neutron radius $r_n = 2.70$ fm obtained by us on the basis of the data of this work using formula (2) is, evidently, in good agreement with the data from Ref. [27] and the results of our calculations. So, the computations performed in this work for two versions of the internucleon interaction, the results of advanced NCSMC computations from Ref. [13] and the experimental data presented in Ref. [27] give well-consistent large values of the size of the neutron halo of ^{6}He nucleus r_h: 0.792, 0.783, 0.80, and 0.72 fm, i.e., about one and a half times larger than those presented in Refs. [28–30].

IV. CONCLUSIONS

In conclusion let us list the basic points of the performed studies:

1. A new two-dimensional procedure for extrapolation of the values of matter, neutron, and proton radii obtained in no-core shell-model calculations, using various harmonic-oscillator bases characterized by different parameters of $\mathcal{N}_{\max}$ and $\hbar\omega$, to infinite basis size is proposed.
2. To estimate stability of the results of extrapolation of size parameters, the domain of $\hbar\omega$ values has been varied. A relationship between the values of these three radii is used as an additional test.
3. The JISP16 and Daejeon16 internucleon interactions are used in NCSM computations of halo nucleus ^{6}He. The latter one is involved to the calculations of radii for the first time.
4. The small values of the rms deviations of studied radii together with reasonable agreement between the obtained results and experimental data, as well as successful testing using a relationship between the values of these three radii demonstrate the high efficiency of the developed approach. This merits of it allows one,

probably, in many cases to compare the quality of the description of the size parameters of nuclei by different Hamiltonians.

5. The results of computations of the size of ^{6}He nucleus halo turns out to be the very stable. They are almost the same for the JISP16 and Daejeon16 potentials unlike the results of calculation of neutron and proton radii. The performed investigations and analysis of the results of other *ab initio* studies indicate that the halo of ^{6}He has a large size: 0.7–0.8 fm. These results confirm the material radius measurement datum presented in Ref. [27].
6. Based on the features of the geometry of surfaces of radii, the method as a whole was called twisted tape extrapolation. In our opinion, it looks rather general and promising approach for studying observables that reflect long-range internuclear correlations in various nuclei.

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