Neutron-proton interaction in odd-odd nuclei from statistical analysis

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The statistical distribution and correlation relationship of the empirical neutron-proton (np) interaction are analyzed, whereby the interaction strengths are extracted from the binding energies using a known four-point formula. By comparing the correlations of the data and those from numerical simulations of the random number method, it is shown that an additional attractive np interaction persists between the last proton and last neutron in odd-odd nuclei. It provides evidence of the residual np interaction from statistical analysis. The adopted new analytical method might be a useful way to clarify the inherent correlation.

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I. INTRODUCTION

The atomic nucleus is a unique quantum many-body system comprising two types of fermions, protons and neutrons. It has been well established that there exists a strong pairing correlation between like particles in nuclei, which leads to an additional binding for systems with even Z and/or even N [1,2]. From a general perspective, systematic studies on nuclear binding energy have revealed rich information on nuclear structure and the underlying effective interaction. However, our knowledge of nuclear forces is still limited. In particular, the possible onset of pairing interaction between protons and neutrons is one important open question which has been studied continuously. The proton and neutron can form both isoscalar ($T = 0$) and isovector ($T = 1$) pairs. They are expected to be the strongest in nuclei along the $N = Z$ line see, e.g., Refs. [3–6]). For most other nuclei, the pair interaction between the last neutron and last proton in odd-odd nuclei may not be negligible even though the protons and neutrons near the Fermi surface may occupy different orbitals. Furthermore, strong short-range interactions between correlated neutron-proton (np) pairs might also be expected to be strong at high momentum even in heavy and neutron-rich nuclei [7,8], which is different from the traditional understanding. It will be very interesting to study the possible pair correlation between the last neutron and last proton for $N \neq Z$ nuclei.

It is possible to isolate the np interaction by using appropriate relative mass differences; see, e.g., Refs. [9–32]. Meanwhile, it has been shown that the analysis of statistical laws could shed additional light on the study of different nuclear physics quantities; see, e.g., Refs. [33,34]. Thus, in this paper, considering the fact that there are now more than two thousand data for nuclear mass [35], we would like to research the np interaction from some new viewpoint, i.e., the statistical perspective.

II. SYSTEMATICS OF *np* INTERACTION FROM NUCLEAR BINDING ENERGIES

There exist essentially two different ways to extract the np interaction from the binding energies of neighboring nuclei (see, e.g., Eqs. (1)–(5) in Ref. [19]). In this paper we will focus on the four-point formula which was proposed in Ref. [9], and analyzed recently in detail in Refs. [16,17,36–38]. The formula can be written as

$$\begin{aligned} V_{1p1n}(Z,N) &= B(Z,N) + B(Z-1,N-1) - B(Z-1,N) \\ &\quad - B(Z,N-1) \\ &= S_p(Z,N) - S_p(Z,N-1) \\ &= S_n(Z,N) - S_n(Z-1,N) \\ &= S_{np}(Z,N) - S_p(Z,N-1) \\ &\quad - S_n(Z-1,N), \end{aligned} \tag{1}$$

where B is the binding energy and S is the separation energy of the one-neutron, one-proton, or np pair. The above formula is simple and works for both even-A and odd-A systems. The other family of formulas was proposed in Ref. [10]. It was applied recently and followed up in papers [12,15,19,23]. It is a bit more complex since different expressions were used for even-Z even-N (denoted as EE in this paper), even-Z odd-N (EO), odd-Z even-N (OE), and odd-Z odd-N (OO) nuclei, respectively.

In Fig. 1(a), we plotted the np interaction $V_{1p1n}(Z,N)$ extracted from the latest experimental binding energy data compilation AME2020 [35]. The experimental binding energy data with large error (>100 keV) are not adopted. The results for the nuclei with $|N - Z| > 1$ are shown here while the

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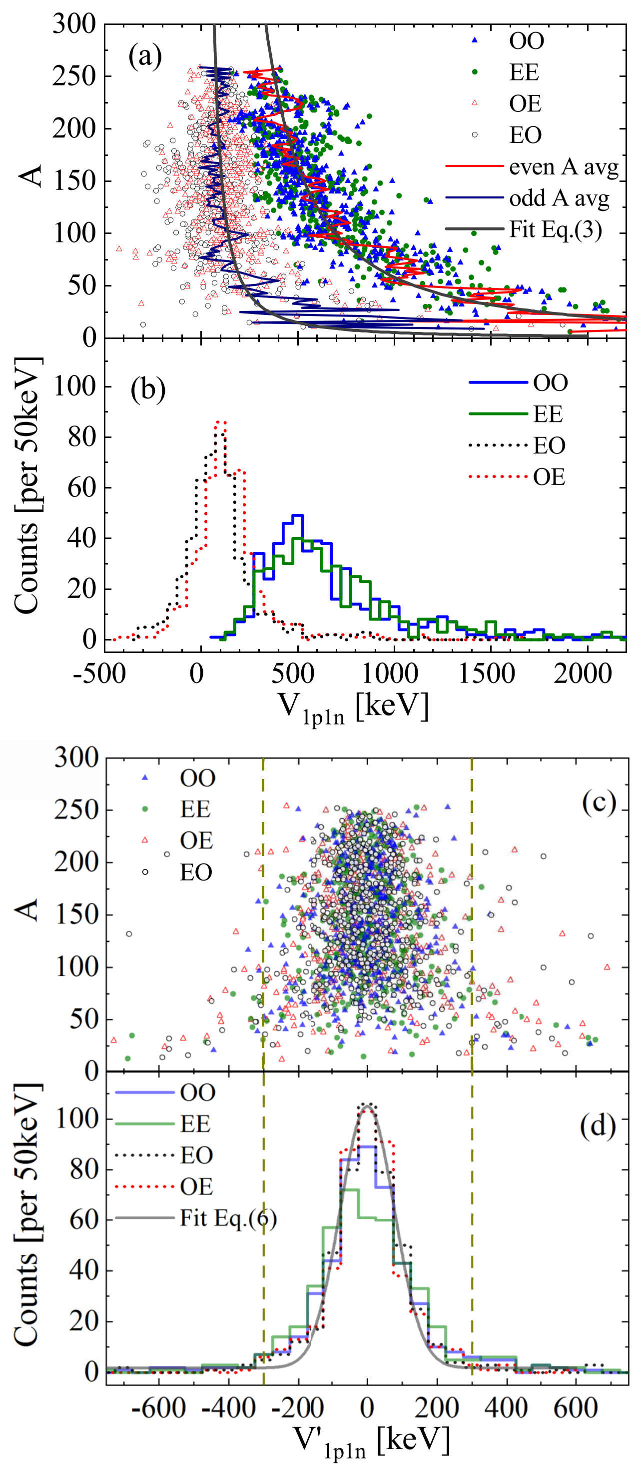

FIG. 1. (a) Proton-neutron interactions $V_{1p1n}(Z,N)$ extracted from experimental binding energy data by using Eq. (1). The results for the nuclei with $|N-Z|>1$ are shown here. The average values of $V_{1p1n}(A)$ for the same mass number A for even-A and odd-A nuclei are plotted as red and purple lines. (b) Histogram for $V_{1p1n}(Z,N)$, where the bin values correspond to the number of nuclei within for $V_{1p1n}\pm 25$ keV. V_{1p1n} takes values of $\ldots,-50,0,50,100,\ldots$ keV. Panels (c) and (d) are similar to (a) and (b) but for the results of $V'_{1p1n}(Z,N)$ in Eq. (4). The gray line in (d) gives an approximate Gaussian distribution $T(\mu,\sigma)$ in Eq. (6) with $\mu=1,\sigma=77$.

$|N-Z|=0$ or 1 are discussed later. One can safely assume that all those nuclei in Fig. 1 have ground state isospin values $T=|N-Z|/2$.

One may easily picture that $V_{1p1n}(Z,N)$ for OO nuclei can measure the energy gain (i.e., difference between the *np* pair separation energy and the sum of the single-proton and -neutron separation energies) as induced by the interaction between the last odd proton and neutron. That may be compared to the energy gain by the pairing interaction among like particles, which can be extracted from the binding energy as [2,39,40]

$$\begin{aligned}2\Delta_C^{(3)}(N) &= B(N,Z)+B(N-2,Z)-2B(N-1,Z)\\ &= S_{2n}(N,Z)-2S_n(N-1,Z),\end{aligned} \tag{2}$$

which measures the energy gain by the neutron pairing in an even-N system relative to the neighboring odd-N system.

A striking feature one notices immediately in Fig. 1(a) is that, as also pointed out in Refs. [11,18], $V_{1p1n}(Z,N)$ for even-A nuclei (EE and OO) are systematically larger than those of neighboring odd-A nuclei (EO and OE). That kind of odd-even staggering can also be seen in Fig. 1(b) where we plotted the probability distributions for $V_{1p1n}(Z,N)$. It is found that probability distributions for V_{1p1n} of EE nuclei are almost same as those of OO nuclei, while the distributions of EO nuclei are similar to those of OE nuclei. The average of V_{1p1n} was empirically expressed with constant term and $1/A$ term [16,36,38] for the large A region. We fit a new formula for all mass regions as

$$\overline{V_{1p1n}}=[9+(-1)^A\times\ 6]\times A^{-2/3}[\mathrm{MeV}], \tag{3}$$

and show it in Fig. 1(a).

One thing one has to bear in mind is that the values for $V_{1p1n}(Z,N)$ extracted from Eq. (1) may still contain a smooth contribution from the change in nuclear mean field when one goes from system A to system $A-1$. One may expect that those mean field effects can be averaged out by subtracting a smooth energy as mass number A. We therefore introduce

$$V'_{1p1n}(Z,N)=V_{1p1n}(Z,N)-V^{\mathrm{smooth}}_{1p1n}(Z,N), \tag{4}$$

with the smooth energy taken as [12]

$$\begin{aligned}V^{\mathrm{smooth}}_{1p1n} &= [V_{1p1n}(Z-2,N)+V_{1p1n}(Z+2,N)\\ &\quad +V_{1p1n}(Z,N-2)+V_{1p1n}(Z,N+2)]/4.\end{aligned} \tag{5}$$

For the nuclei in the boundary of nuclear chart which lack complete data of fourneighboring nuclei in Eq. (5), $V^{\mathrm{smooth}}_{1p1n}$ takes the average value of two or three neighboring nuclei. After the smooth energy is extracted, all data points look like random values being scattered around zero. It is interesting to find that all V'_{1p1n} for EE, OO, OE, EO nuclei satisfy approximately the same Gaussian distribution. The Gaussian probability distribution is expressed as

$$T(\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \tag{6}$$

where μ and σ are the mean and standard deviation, respectively.

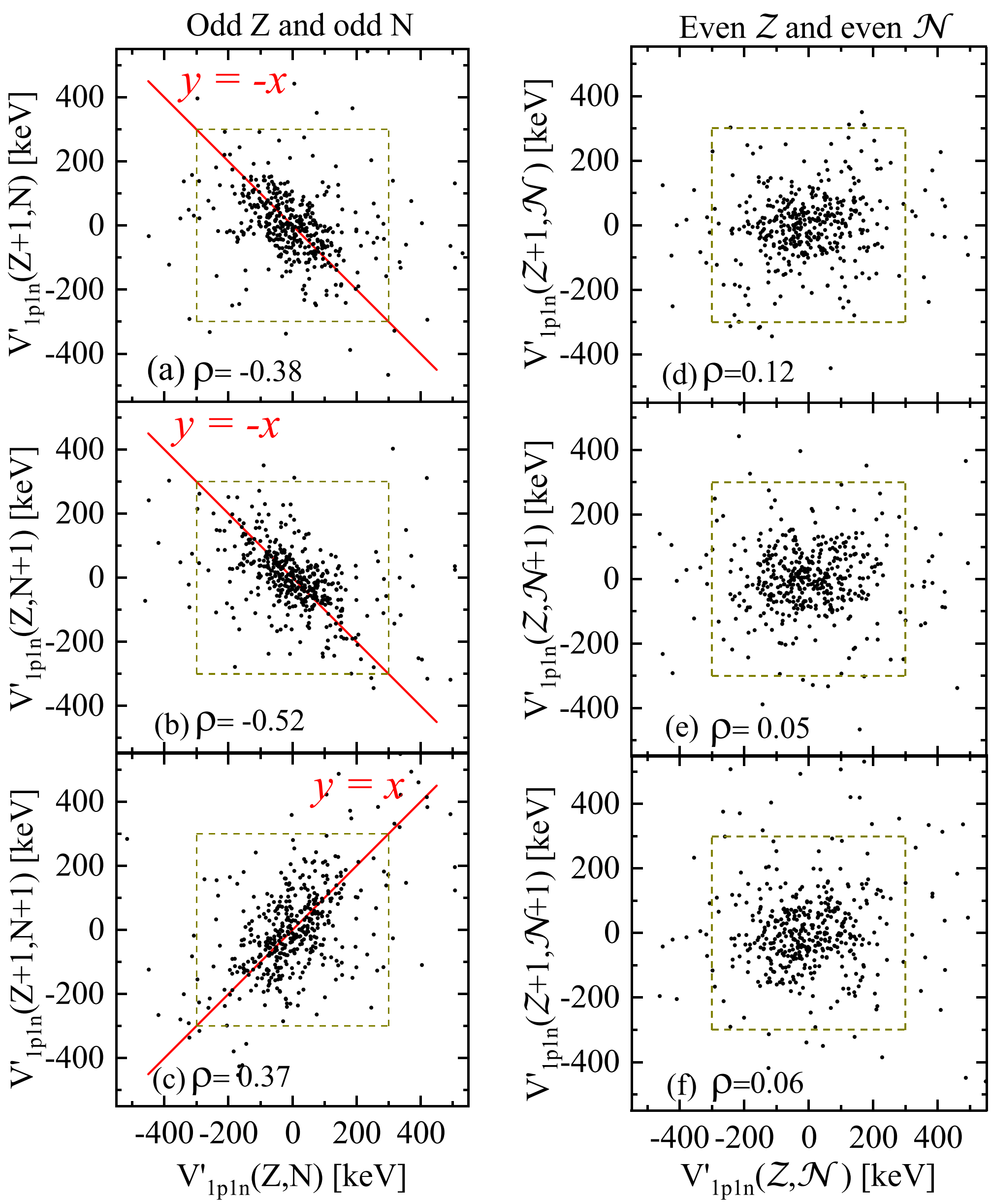

FIG. 2. Two-dimensional distributions of the extracted interaction V'_{1p1n}. Left panels the data distributions between the OO nuclei (Z, N) and the neighboring EO, OE, and EE nuclei which are plotted in (a), (b), and (c), respectively. Right panels the data distributions between EE nuclei $(\mathcal{Z}, \mathcal{N})$ and the neighboring OE, EO, and OO nuclei which are plotted in (d), (e), and (f), respectively. The calculated coefficient of correlation ρ is also shown.

Now the question is whether we can learn anything new on the nature of the *np* interaction from those randomly distributed $V'_{1p1n}(Z, N)$ values.

III. STATISTICAL ANALYSIS OF THE STAGGERING BETWEEN EVEN-*A* AND ODD-*A* NUCLEI

The systematic staggering of *np* interactions between even-*A* and odd-*A* nuclei had been tentatively explained in terms of the additional *np* interaction in OO nuclei relative to the neighboring EE and odd-*A* systems [19]. Before studying the physics meaning of those $V_{1p1n}(Z, N)$ values, we would like to explore first their statistical properties.

In Fig. 2, the correlation properties between different $V'_{1p1n}(Z, N)$ values are given. In the left panel, we plot V'_{1p1n} for neighboring EE, EO, and OE nuclei (with one or two particles above the corresponding OO nuclei) as a function of V'_{1p1n} for OO (Z, N) nuclei. The right panel shows instead the correlation between EE $(\mathcal{Z}, \mathcal{N})$ nuclei and neighboring nuclei with one or two particles above such EE nuclei.

A very striking feature one notices immediately is that there exists a quite strong and positive correlation in Fig. 2(c) between V'_{1p1n} for OO nuclei and those for EE nuclei with one more *np* pair on top. However, there is almost no correlation between V'_{1p1n} for OO nuclei and those for the corresponding EE cores with one less *np* pair, as shown in Fig. 2(f).

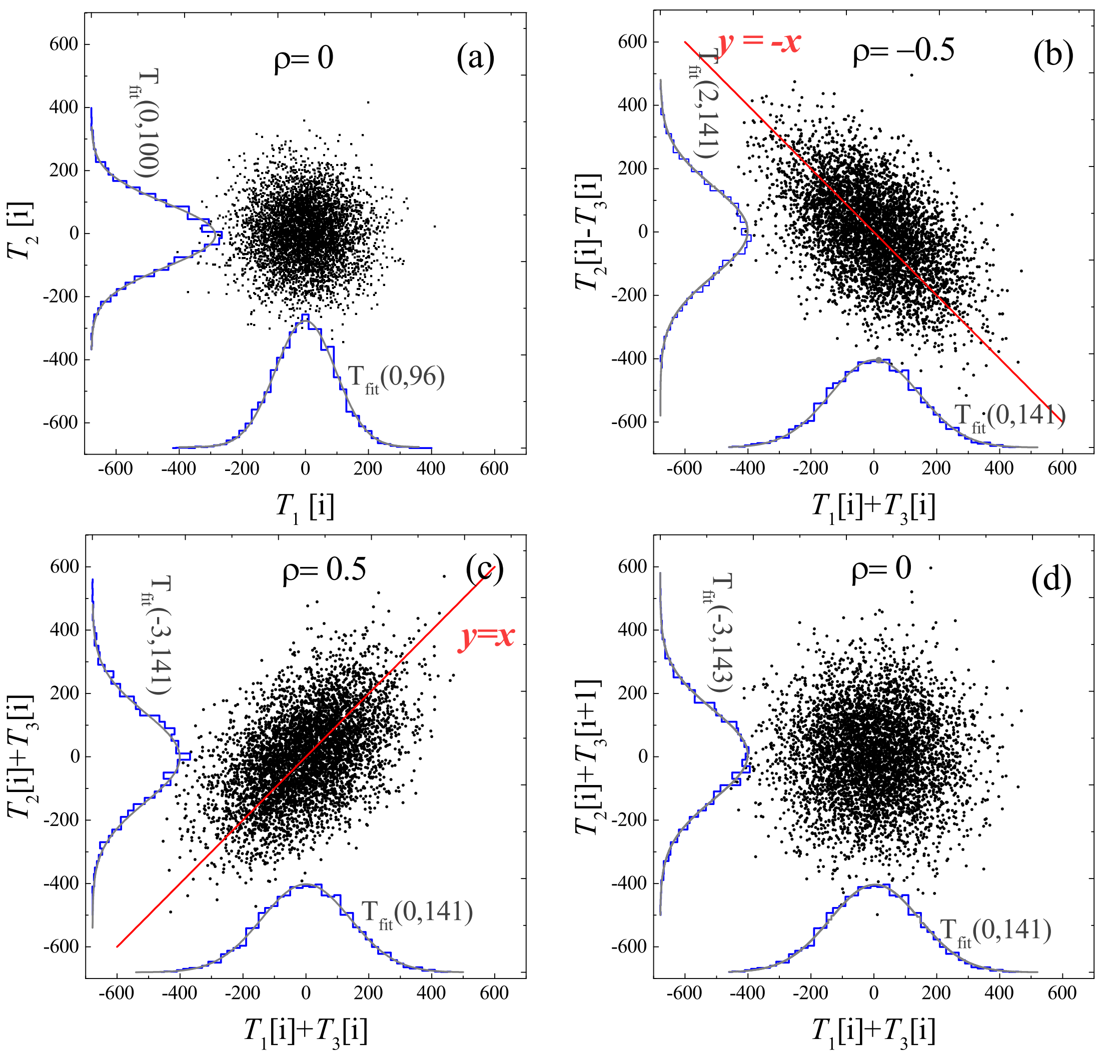

FIG. 3. Distribution of numerically simulated samples of no correlation [(a) and (d)], negative correlation (b) and positive correlation (c), constructed by the random number method. T_1, T_2, T_3 are three independent groups of random numbers. Each black dot represent a data point $(T_1[i], T_2[i])$ in panel (a), $(T_1[i] + T_3[i], T_2[i] \mp T_3[i])$ in panels (b) and (c), and $(T_1[i] + T_3[i], T_2[i] + T_3[i+1])$ in panel (d), with $i = 1, 2, \dots, 5000$. The blue line denotes the distribution of T_1, T_2, $T_1 \pm T_3$, or $T_2 \pm T_3$, and gray lines are Gaussian fits $T_{\rm fit}(\mu, \sigma)$ for the blue lines.

Meanwhile, the EO and OE nuclei has strong negative correlation to the corresponding OO nuclei with one less particle, shown in Figs. 2(a) and 2(b), while there is almost no correlation to the corresponding EE nuclei with one less particle, shown in Figs. 2(d) and 2(e).

In statistics, the coefficient of correlation ρ for two random variables X and Y is defined as [41]

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_x\sigma_y} = \frac{E\{[X - E(X)][Y - E(Y)]\}}{\sigma_x\sigma_y}, \quad (7)$$

where σ_x and σ_y are the standard deviations of X and Y, $\mathrm{Cov}(X,Y)$ is the covariance of two random variables X and Y, and $E(X)$ is the mean of a random variable X. Values of $\rho = \pm 1$ imply perfect straight-line relationships between X and Y, and $\rho = 0$ implies no linear relationship between X and Y.

The coefficient of correlation ρ is calculated based on the V'_{1p1n} data and shown in Fig. 2, in which the data satisfying $V'_{1p1n} < 300$ keV ($\approx 4\sigma$) are adopted. In the following, we perform some numerically simulations to help to understand the correlation properties between the V'_{1p1n} in different nuclei.

With the Monte Carlo random number method, several groups of random numbers $T(\mu, \sigma)$ were produced which satisfy a Gaussian distribution. In Fig. 3, T_1, T_2, and T_3 are three independent groups of random numbers with expected value 0 and standard deviations 100. $T_n[i]$ denotes the ith random number in the nth group, where $i = 1, 2 \dots, M$ and $n = 1, 2, 3$. The results with $M = 5000$ (about ten times the data in Fig. 2) are shown in Fig. 3.

In Fig. 3(a), we show a sample of 5000 pairs of values $(T_1[i], T_2[i])$, in which each black dot represents a data point. T_1 and T_2 are produced independently, thus they have no correlation, and the coefficient of correlation ρ should be 0. In Fig. 3(b), we construct two new variables $T_2[i] - T_3[i]$ and $T_1[i] + T_3[i]$. They are both still groups of random numbers

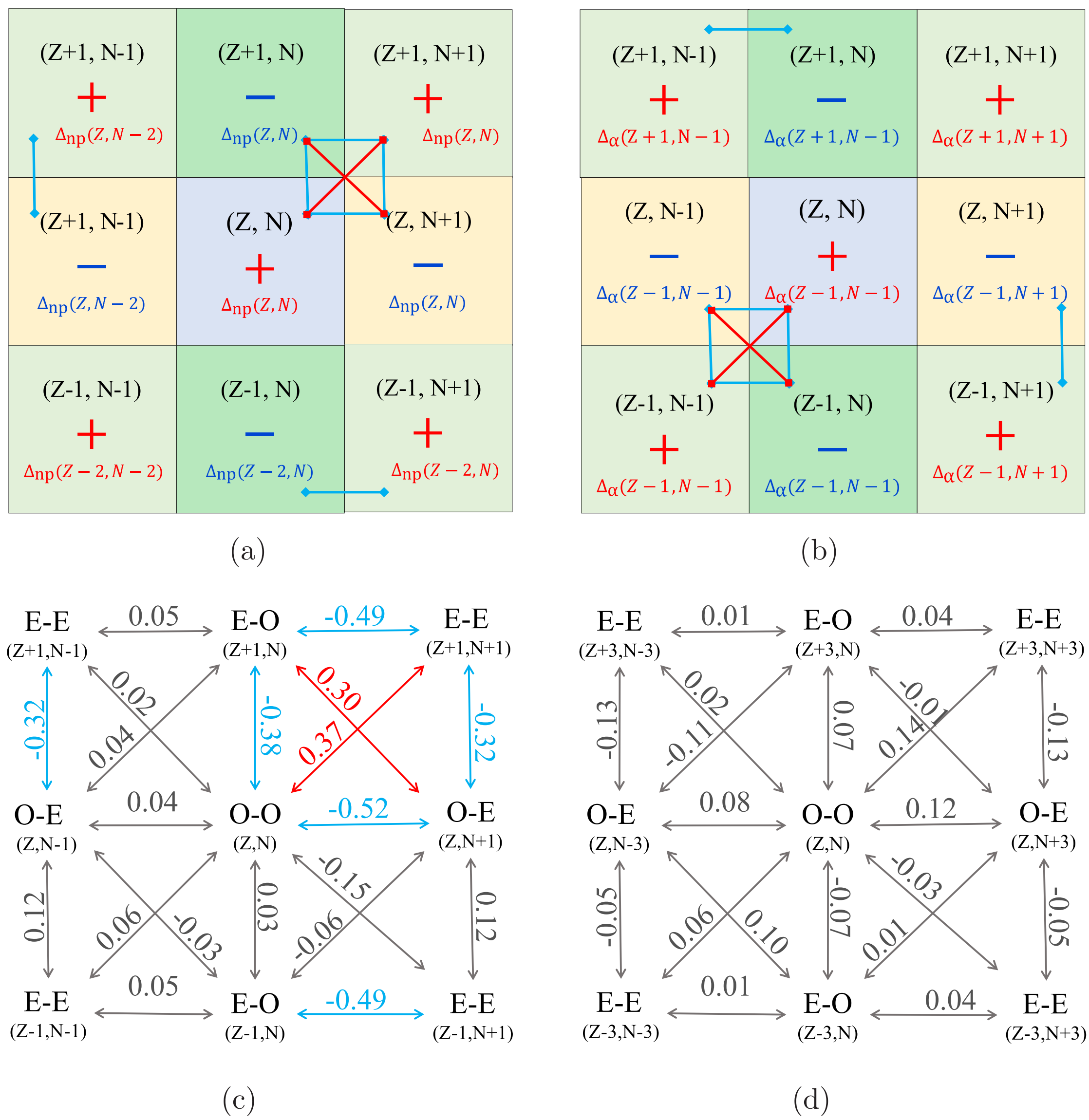

FIG. 4. The expected correlation relationship between neighboring nuclei (EE, EO, OE, OO nuclei) for scenario I (a) and scenario II (b). Z and N are odd numbers in the figure. The red (blue) lines represent positive (negative) correlation relationship. (c) The coefficient of correlation ρ of $Z \pm 1$ or/and $N \pm 1$ nuclei calculated based on the V'_{1p1n} data. (d) Similar to (c) but for $Z \pm 3$ or/and $N \pm 3$ nuclei.

which satisfy the Gaussian distribution with expected value 0 and standard deviations $100 \times \sqrt{2}$. Due to both groups involving the same component of T_3, the coefficient of correlation $\rho(T_2 - T_3, T_1 + T_3)$ could be proved as -0.5 [41]. More black dots (data points) are in the upper left or lower right quadrant of Fig. 3(b). The distribution is similar to an oblique ellipse, and the major axis is in the direction of $y = -x$. Similarly to Fig. 3(b), we construct new variables to show the positive correlation in Fig. 3(c). As shown in Fig. 3(d), we construct two new variables $T_1[i] + T_3[i]$ and $T_2[i] + T_3[i+1]$. As $T_3[i]$ and $T_3[i+1]$ are random, these two variables have no correlation.

Figure 3 provides a visualized picture of two physical quantity with no correlation, positive correlation, or negative correlation. By comparing the two-dimensional distributions of the data shown in Fig. 2 with the ideal cases in Fig. 3, the correlation relationship between different V'_{1p1n} can be clearly analyzed.

IV. PHYSICAL ORIGINS

From a binding energy perspective $V_{1p1n}(Z,N) = B(Z,N) + B(Z-1,N-1) - B(Z-1,N) - B(Z,N-1)$ as shown in Eq. (1), it is easy to infer that, if an extra interaction $\Delta_x(Z,N)$ exists in binding energy of nucleus (Z,N), it will only contribute to the V_{1p1n} of the four neighboring nuclei. The contribution will be

$$\begin{aligned} V_{1p1n}(Z,N) &\ \text{involve}\ [+\Delta_x(Z,N)],\\ V_{1p1n}(Z,N+1) &\ \text{involve}\ [-\Delta_x(Z,N)],\\ V_{1p1n}(Z+1,N) &\ \text{involve}\ [-\Delta_x(Z,N)],\\ V_{1p1n}(Z+1,N+1) &\ \text{involve}\ [+\Delta_x(Z,N)]. \end{aligned} \tag{8}$$

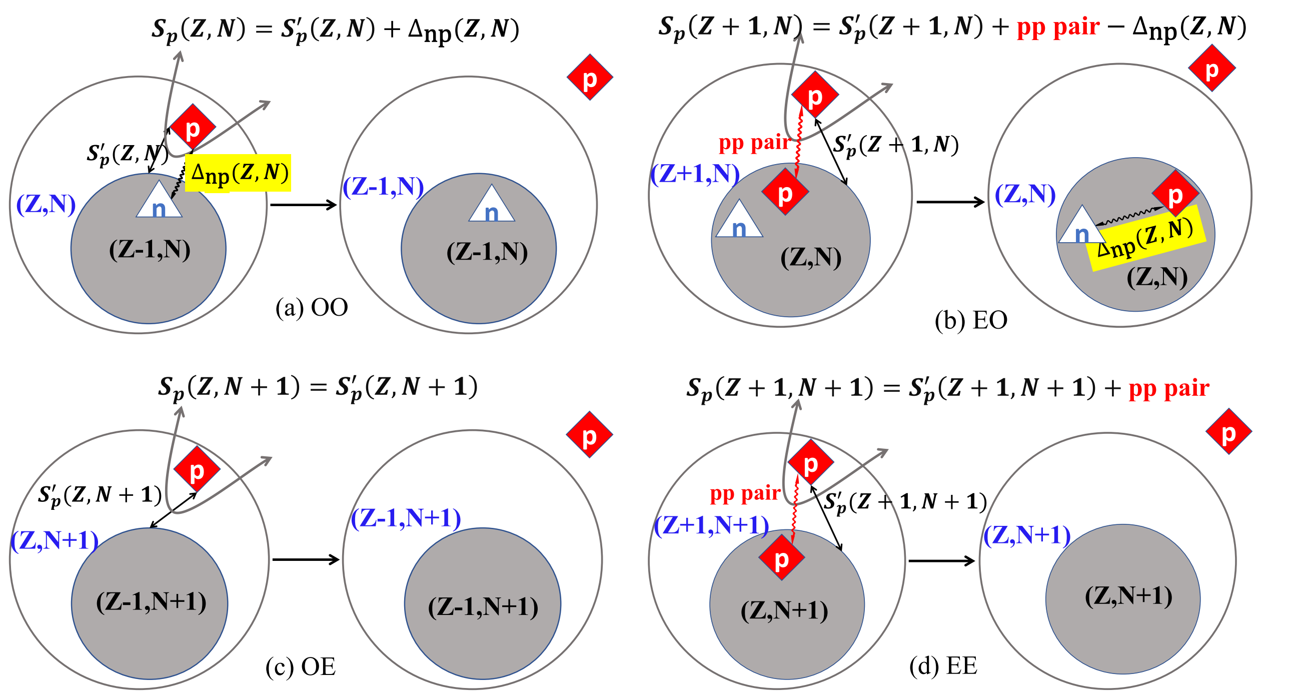

FIG. 5. Schematic diagram of single proton separated energy S_p in OO, EO, OE, EE nuclei shown in (a), (b), (c), and (d), respectively. Z and N are odd numbers in the figure. S'_p represent the mean field part of the interaction between last proton and inner core.

The systematic larger V_{1p1n} values for even-A nuclei can be related to three different scenarios:

I. Extra energy gain by the last np pair in OO nuclei, such as a residual interaction between two independent particles (denoted as Δ_{np}).
II. Extra energy gain by the last two np pairs in EE nuclei, such as an alpha or quartet correlation (denoted as Δ_α).
III. Energy gain in both EE and OO nuclei induced by the pairing correlation among valence np pairs around the Fermi surface.

Scenario III may be safely ruled out by the fact that there is almost no correlation between V_{1p1n} for EE and those of neighboring OO with one more pair. One may expect a rather smooth behavior among neighboring odd-odd and even-even systems if there is a strong np pairing correlation. In addition, the np pairing correlation may not be expected to be strong when one goes away from the $N = Z$ line. There have been significant efforts in recent years studying the possible influence of np pairing correlation and quartetlike np correlation on nuclear mass as well as the low-lying spectrum of even-even nuclei, in particular those around $N = Z$ [3–6,14,17,20,22,42].

In Figs. 4(a) and 4(b), we exhibit the expected correlation relationship between adjacent nuclei (EE, EO, OE, OO nuclei) for scenarios I and II, where Z, N are odd numbers in these panels. If scenario I is reasonable, according to Eq. (8), V_{1p1n} of the neighboring four nuclei (Z, N), $(Z + 1, N)$, $(Z, N + 1)$, $(Z + 1, N + 1)$ involve the same term $\Delta_{np}(Z, N)$, thus these four nuclei have correlations between each other. To make an analogy with the ideal cases in Fig. 3, if $\Delta_{np}(Z, N)$ corresponds to the random group T_3, V'_{1p1n} corresponds to $T_1 \pm T_3$ or $T_2 \pm T_3$. The blue (red) line in Fig. 4 corresponds to correlation shown in Fig. 3(b) [3(c)]. Then, because the V_{1p1n} of OO nuclei (Z, N) and of the EE nuclei $(Z - 1, N - 1)$ involve different terms $\Delta_{np}(Z, N)$ and $\Delta_{np}(Z - 2, N - 2)$, they should have no correlation, similarly to the case in Fig. 3(d).

If scenario II is reasonable, according to Eq. (8), V_{1p1n} of the neighboring four nuclei $(Z - 1, N - 1)$, $(Z, N - 1)$, $(Z - 1, N)$, (Z, N) involve the same term $\Delta_\alpha(Z - 1, N - 1)$, thus they have correlations between each other. Similarly to the above discussion in Fig. 4(a), we should get a correlation relationship like that in Fig. 4(b).

In order to distinguish and clarify these scenarios, we calculate all the coefficients of correlation based on the V'_{1p1n} data and shown them in Fig. 4(c). The negative, positive, and no-correlation relationships in Fig. 4(c) are nicely agreement with the scenario I in Fig. 4(a), and basically in contradiction with scenario II in Fig. 4(b). Thus, we can conclude the systematic staggering of np interactions between even-A and odd-A nuclei should mainly be attributed to the additional np interaction existing only in odd-odd nuclei. This provides evidence of the residual np interaction in odd-odd nuclei from statistical analysis.

Another thing one should bear in mind is that, as deduced from Eq. (8), the correlation relationships of V'_{1p1n} should only appear between neighboring nuclei. In Fig. 4(d), we further give the results for cases of $Z \pm 3$ and/or $N \pm 3$. There are indeed no obvious correlations in this figure, which strongly supports the above idea.

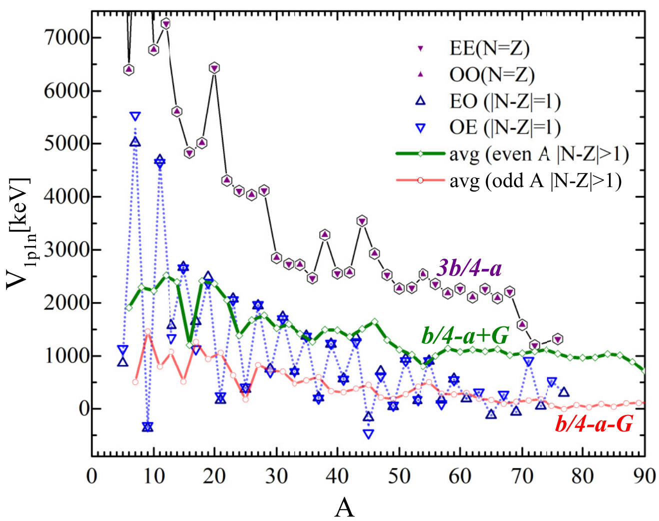

FIG. 6. Experimental V_{1p1n} values of $N = Z$ nuclei and the adjacent odd-A nuclei with $|N - Z| = 1$. For $N = Z$ odd-odd nuclei, we choose the data of the lowest $T = 0$ state. The average of V_{1p1n} for odd-A and even-A nuclei with $|N - Z| > 1$ are also plotted for comparison. The formulas listed in the figure are deduced from Eq. (10).

In Fig. 5, we illustrate schematically the difference between V_{1p1n} for even-A and odd-A nuclei induced from the additional np interaction. Z and N are odd numbers in the panels. The one-proton separation energy S_p for the OO, EE, OE, and EO nuclei are shown in panels (a)–(d), respectively. From the properties of S_p, we can deduce easily the difference of V_{1p1n} according to $V_{1p1n}(Z, N) = S_p(Z, N) - S_p(Z, N - 1)$. S'_p in the figure represents the mean field parts of the interaction between the last proton and inner core. The pairing correlations between like particles have higher pairing energy ($\approx 12A^{-1/2}$ MeV), far more than the odd-even staggerings with np pairing [$\approx 6A^{-2/3}$ MeV, as in Eq. (8)]. In case of even Z, the "pp pair" will break when the last proton is separated, as shown in (b) and (d), and this term will be canceled in the calculation of V_{1p1n}. In the right panel of (b), the remaining unpaired proton and the unpaired neutron which does not participate in the like-particle pairing will contribute an additional binding energy due to the np pair correlation, namely the inner core is dramatically altered by the np pair when the last proton is separated.

V. LIGHT NUCLEI AND $N = Z$ NUCLEI

To understand the np interaction, we consider a system with n_π protons and n_ν neutrons in a single-j shell. The two-body interaction are assumed to obey a simple form [43],

$$\hat{V} = a + b\mathbf{t}_1 \cdot \mathbf{t}_2 - GP_0, \tag{9}$$

where P_0 denotes the monopole pairing interaction. G is the corresponding (positive) coupling strength. The first two terms, which do not depend on the angular momentum J, define the "averaged" monopole interaction. The isovector and isoscalar channels of the monopole interaction are given by $V_{m;T=1} = a + b/4$ and $V_{m;T=0} = a - 3b/4$.

In usual shell-model Hamiltonians the values of $V_{m;T=1}$ are around zero while those of $V_{m;T=0}$ are strongly attractive (see, e.g., Refs. [44,45]), indicating that b should have a positive sign [46]. The $J = 0$ two-body matrix element is given as $\langle j^2|V|j^2\rangle_{J=0,T=1} = a + b/4 - (2j + 1)G$. The total energy of the system can be written analytically as

$$\begin{aligned} E = \varepsilon n + \frac{a}{2}n(n-1) + \frac{b}{2}\left[\mathcal{T}(\mathcal{T}+1) - \frac{3n}{4}\right] \\ -G\left[\frac{n-\nu}{4}(4j+8-n-\upsilon) - \mathcal{T}(\mathcal{T}+1) + s(s+1)\right], \end{aligned} \tag{10}$$

where ε denotes the single-particle energy. The total number of nucleon pairs is $n(n-1)/2$ with $n = n_\pi + n_\nu$ [43]. $\mathcal{T}$ is the total isospin of the system. ν and s denote the seniority and the reduced isospin.

For the $|N - Z| > 1$ nuclei, we have $\mathcal{T} = |n_\pi - n_\nu|/2 = |N - Z|/2$ for the ground state. $s = 0$ and $\upsilon = 0$ for EE nuclei, $s = 1/2$ and $\upsilon = 1$ for EO/OE nuclei, while $s = 1$ and $\upsilon = 2$ for OO nuclei. Thus, based on Eqs. (1) and (10), it is easy to obtain that

$$\begin{aligned} V_{1p1n} &= \frac{b}{4} - a + G \quad \text{for EE and OO nuclei}, \\ V_{1p1n} &= \frac{b}{4} - a - G \quad \text{for EO and OE nuclei}. \end{aligned} \tag{11}$$

The first and second terms in the above expressions could be understood from $\frac{1}{4}V_{2p2n}$ [15]:

$$\frac{b}{4} - a = -\frac{4V_{m;T=1} + 2(V_{m;T=0} - V_{m;T=1})}{4}. \tag{12}$$

On the other hand, in the case of EE nuclei with $n_\pi = n_\nu$, $s = 0$, $\nu = 0$ (i.e., $N = Z$), we have

$$V_{1p1n} = \tfrac{3}{4}b - a = -V_{m;T=0}. \tag{13}$$

The ground state of odd-odd $N = Z$ nuclei may carry isospin quantum numbers $T = 0$ or 1. For the lowest $T = 0$ state, $s = 0$, $\nu = 2$, the same values as in Eq. (13) in the case of $N = Z$ OO nuclei.

For an $I = j, T = 1/2$ system with three particles in a single-j shell, we have $\nu = 1$ and $s = 1/2$. The V_{1p1n} could be obtained from Eq. (10):

$$\begin{aligned} V_{1p1n}(\mathcal{Z}, \mathcal{Z}-1) = V_{1p1n}(\mathcal{Z}-1, \mathcal{Z}) = \frac{b}{4} - a + G, \\ V_{1p1n}(\mathcal{Z}, \mathcal{Z}+1) = V_{1p1n}(\mathcal{Z}+1, \mathcal{Z}) = \frac{b}{4} - a - G, \end{aligned} \tag{14}$$

where $\mathcal{Z}$ takes even values. Thus, the odd-A nuclei with $|N - \mathcal{Z}| = 1$ will separate into two sequences, which is also proved by the data in Fig. 6. All the above analyses are in nice agreement with the bulk properties of the extracted V_{1p1n}.

VI. SUMMARY

The empirical np interactions are extracted from the binding energies of nearly 2000 nuclei using a known four-point formula. Statistical analyses are used to explore their properties and physical origin. By comparing the correlations of the data for the neighboring nuclei and those from numerical simulations using the random number method, it is shown that an additional attractive *np* interaction persists between the last proton and last neutron in odd-odd nuclei. It provides evidence of the residual *np* interaction from statistical analysis. Searching for the underlying rules of a complex system is one important goal for both physics and statistics. The present work uses this idea, and the adopted new analytical method might be a useful way to clarify the inherent correlation.

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[1] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer, Berlin, 2004).
[2] A. Bohr and B. R. Mottelson, *Nuclear Structure* (World Scientific, Singapore, 1998), Vol. 1; *ibid.* Vol. 3 (unpublished).
[3] B. Cederwall *et al.*, Nature (London) **469**, 68 (2011).
[4] S. Frauendorf and A. O. Macchiavelli, Prog. Part. Nucl. Phys. **78**, 24 (2014).
[5] B. Cederwall *et al.*, Phys. Rev. Lett. **124**, 062501 (2020).
[6] R. D. O. Llewellyn *et al.*, Phys. Rev. Lett. **124**, 152501 (2020).
[7] O. Hen *et al.*, Science **346**, 614 (2014).
[8] M. Duer *et al.* (The CLAS Collaboration), Nature (London) **560**, 617 (2018).
[9] M. K. Basu and D. Banerjee, Phys. Rev. C **3**, 992 (1971).
[10] J.-Y. Zhang, R. F. Casten, and D. S. Brenner, Phys. Lett. B **227**, 1 (1989).
[11] G. Zaochun and Y. S. Chen, Phys. Rev. C **59**, 735 (1999).
[12] R. R. Chasman, Phys. Rev. Lett. **99**, 082501 (2007).
[13] W. Friedman and G. Bertsch, Eur. Phys. J. A **41**, 109 (2009).
[14] C. Qi, J. Blomqvist, T. Bäck, B. Cederwall, A. Johnson, and R. J. Liotta, R. Wyss, Phys. Rev. C **84**, 021301(R) (2011).
[15] C. Qi, Phys. Lett. B **717**, 436 (2012).
[16] A. Arima and Y. M. Zhao, J. Phys.: Conf. Ser. **403**, 012045 (2012).
[17] G. J. Fu, J. J. Shen, Y. M. Zhao, and A. Arima, Phys. Rev. C **87**, 044309 (2013).
[18] Y. Y. Cheng, Y. M. Zhao, and A. Arima, Phys. Rev. C **91**, 024314 (2015).
[19] Z. Y. Wu, S. A. Changizi, and C. Qi, Phys. Rev. C **93**, 034334 (2016).
[20] C. Qi, and R. Wyss, Phys. Scr. **91**, 013009 (2016).
[21] D. D. Warner, M. A. Bentley, and P. Van Isacker, Nat. Phys. **2**, 311 (2006).
[22] K. Kaneko, Y. Sun, and G. de Angelis, Nucl. Phys. A **957**, 144 (2017).
[23] R. B. Cakirli, D. S. Brenner, R. F. Casten, and E. A. Millman, Phys. Rev. Lett. **94**, 092501 (2005).
[24] R. B. Cakirli and R. F. Casten, Phys. Rev. Lett. **96**, 132501 (2006).
[25] Y. Oktem, R. B. Cakirli, R. F. Casten, R. J. Casperson, and D. S. Brenner, Phys. Rev. C **74**, 027304 (2006).
[26] D. S. Brenner, R. B. Cakirli, and R. F. Casten, Phys. Rev. C **73**, 034315 (2006).
[27] M. Stoitsov, R. B. Cakirli, R. F. Casten, W. Nazarewicz, and W. Satula, Phys. Rev. Lett. **98**, 132502 (2007).
[28] A. O. Macchiavelli *et al.*, Phys. Rev. C **61**, 041303(R) (2000).
[29] W. Satuła and R. Wyss, Phys. Rev. Lett. **86**, 4488 (2001).
[30] W. Satuła and R. Wyss, Phys. Rev. Lett. **87**, 052504 (2001).
[31] W. Satuła, D. J. Dean, J. Gary, S. Mizutori, and W. Nazarewicz, Phys. Lett. B **407**, 103 (1997).
[32] M. Bender and P.-H. Heenen, Phys. Rev. C **83**, 064319 (2011).
[33] G. B. King, A. E. Lovell, L. Neufcourt, and F. M. Nunes, Phys. Rev. Lett. **122**, 232502 (2019).
[34] G. F. Bertsch and D. Bingham, Phys. Rev. Lett. **119**, 252501 (2017).
[35] M. Wang *et al.*, Chin. Phys. C **45**, 030003 (2021).
[36] G. J. Fu, Y. Lei, H. Jiang, Y. M. Zhao, B. Sun, and A. Arima, Phys. Rev. C **84**, 034311 (2011).
[37] Y. Y. Cheng, Y. M. Zhao, and A. Arima, Phys. Rev. C **89**, 061304(R) (2014).
[38] B. B. Jiao, Mod. Phys. Lett. A **33**, 1850156 (2018).
[39] W. Satuła, J. Dobaczewski, and W. Nazarewicz, Phys. Rev. Lett. **81**, 3599 (1998).
[40] S. A. Changizi, C. Qi, and R. Wyss, Nucl. Phys. A **940**, 210 (2015).
[41] W. M. Mendenhall and T. L. Sincich, *Statistics for Engineering and the Sciences* (CRC, Boca Raton, FL, 2016).
[42] D. Negrea, P. Buganu, D. Gambacurta, and N. Sandulescu, Phys. Rev. C **98**, 064319 (2018).
[43] I. Talmi, *Simple Models of Complex Nuclei* (Harwood Academic, Chur, Switzerland, 1993).
[44] B. A. Brown and W. A. Richter, Phys. Rev. C **74**, 034315 (2006).
[45] M. Honma, T. Otsuka, B. A. Brown, and T. Mizusaki, Phys. Rev. C **69**, 034335 (2004).
[46] A. Z. Mekjian and L. Zamick, Phys. Rev. C **85**, 057303 (2012).