Relation to a property of the angular-momentum-zero space of states of four fermions in an angular momentum $j = 9/2$ shell unexpectedly found to be stationary for any rotationally invariant two-body interaction

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(Received 7 June 2022; accepted 22 July 2022; published 4 August 2022)

The existence of states with angular momenta $I = 4$ and 6 of four fermions in an angular momentum $j = 9/2$ shell that are stationary for any rotationally invariant two-body interaction despite the presence of other states with the same angular momentum, the *Escuderos-Zamick* states, is shown to be equivalent to the invariance to any such interaction of the span of states generated from $I = 0$ states by one-body operators. This invariance is verified by exact calculation independently of previous verifications of the equivalent statement. It explains the occurrence of the Escuderos-Zamick states for just $I = 4$ and 6. The action of an arbitrary interaction on the invariant space and its orthogonal complement is analyzed, leading to a relation of the Escuderos-Zamick energy levels to levels with $I = 10$ and 12. Aspects of the observed spectra of ^{94}Ru, ^{96}Pd, and ^{74}Ni are discussed in the light of this relation.

DOI: 10.1103/PhysRevC.106.024308

I. INTRODUCTION

Escuderos and Zamick found in a numeric study of the system of four nucleons in an angular momentum $j = 9/2$ shell of a semimagic nucleus that, for each angular momentum $I = 4$ and 6, the states in one irreducible module of the angular momentum algebra, briefly a *multiplet*, are stationary for any rotationally invariant two-body interaction [1], that is, *solvable* in Talmi's terminology [2], despite the presence of other multiplets with the same I. It follows that they have definite seniority v [2], which gives rise, in certain nuclei, to particular patterns of transition rates in $E2$ decay and single-nucleon transfer [3–5]. The solvability of the Escuderos-Zamick states was subsequently confirmed in exact calculations by Van Isacker and Heinze [3,4] and Qi, Xu, and Liotta [6]. These calculation are case-by-case examinations of the individual instances of two-body and total angular momentum, which led the authors of [4] to conclude that "a simple, intuitive reason for [the solvability] is still lacking." I show below that the existence of the Escuderos-Zamick states is equivalent to a property of the space of $I = 0$ states of the system. The verification of this property again leads to an examination of several cases one by one. The equivalence explains, however, that the solvable multiplets occur for exactly $I = 4$ and 6.

Throughout this paper, $j = 9/2$. Let Φ_0 denote the space of $I = 0$ states of the four-fermion system, and let a_m be the annihilator of a fermion in the state $|jm\rangle$ in the conventional notation [7]. One can then define a space

$$\Phi_4 = \operatorname{span}_{m,m'} a_m^\dagger a_{m'} \Phi_0. \tag{1}$$

The property to be verified below and shown there to be equivalent to the existence of the Escuderos-Zamick states is the following. Φ_4 *is invariant to any rotationally invariant two-body interaction*. To see how this explains the appearance of solvable multiplets for just $I = 4$ and 6, note that the tensor operators $T_{IM_I} = \sum_{mm'} (-)^{j-m'} \langle jmj-m'|IM_I\rangle a_m^\dagger a_{m'}$, where $\langle j_1 m_1 j_2 m_2 | jm\rangle$ is the vector coupling coefficient [7], form a basis for the span of operators $a_m^\dagger a_{m'}$. The subspace of Φ_4 carrying quantum numbers I, M_I is $T_{IM_I}\Phi_0$. Now consider Table I, obtained by a straightforward count of m-combinations. Since Φ_0 is two-dimensional, $T_{IM_I}\Phi_0$ has dimension 2, at most. Angular momenta $I = 4$ and 6 are the only ones allowing more than two linearly independent multiplets in the four-fermion system, exactly three in both cases. It may be verified by direct calculation, and also follows from a general result in Sec. III, that in each case $T_{IM_I}\Phi_0$ is exactly two-dimensional. If Φ_4 is invariant to a Hermitian and rotationally invariant operator V, then so is also $T_{IM_I}\Phi_0$, and so is then also its one-dimensional orthogonal complement within the space of states with quantum numbers I, M_I. This means that the states in the orthogonal complement are eigenstates of V.

The proof of equivalence is completed in Sec. II, and the verification of the invariance of Φ_4 in Sec. IV. Analyzing the actions of an arbitrary V on Φ_4 and its orthogonal complement $\Phi_4^\perp$ reveals remarkable regularities, one of which leads to a rule for relative level spacings that is accessible to experimental verification and so far lacks fundamental explanation. This analysis is the topic of Secs. V and VI, followed by my conclusion in Sec. VII. A detail of my formalism is discussed and one other observed regularity explained in two appendices.

II. ANALYSIS

Below, $I_\text{E} = 4$ or 6. Important for the following is also the space Φ_3 of states with $I = j$ of three $j = 9/2$ fermions, which is spanned by single multiplets Φ_{3v} with $v = 1$ and 3.

TABLE I. Multiplicities of multiplets of four angular momentum $j = 9/2$ fermions per angular momentum I and seniority v.

v	I	0	2	3	4	5	6	7	8	9	10	12
0		1	0	0	0	0	0	0	0	0	0	0
2		0	1	0	1	0	1	0	1	0	0	0
4		1	1	1	2	1	2	1	1	1	1	1

For each v, at most one multiplet with a given I can be formed by adding a $j = 9/2$ fermion to the states in Φ_{3v}. This multiplet can be written $P_I \operatorname{span}_m a_m^\dagger \Phi_{3v}$, where P_I is the projection onto angular momentum I. It may be verified by direct calculation, and also follows from the general result in Sec III, that for each I_E these two multiplets are independent. The space $\Phi_{I_\text{E}\gamma} = P_{I_\text{E}} \operatorname{span}_m a_m^\dagger \Phi_{31}$ necessarily has $v = 2$. By Table I, its orthogonal complement $\Phi_{I_\text{E}\beta}$ within $P_{I_\text{E}} \operatorname{span}_m a_m^\dagger \Phi_3$ then has $v = 4$. The Escuderos-Zamick multiplet $\Phi_{I_\text{E}\alpha}$ is, by definition, the orthogonal complement of $\Phi_{I_\text{E}\beta}$ within the space of states with $I = I_\text{E}$ and $v = 4$ of the four fermions. It may be characterized also among such states by vanishing parentage by Φ_{33} [1]. Evidently, it is also the orthogonal complement of $P_{I_\text{E}} \operatorname{span}_m a_m^\dagger \Phi_3$ within the space of states with $I = I_\text{E}$.

In the remainder of this paper, V denotes any rotationally invariant two-body interaction. Because V acts as a scalar on the irreducible module $\Phi_{I_\text{E}\alpha}$, the states in $\Phi_{I_\text{E}\alpha}$ being eigenstates of V is equivalent to $\Phi_{I_\text{E}\alpha}$ being invariant to V. By Hermiticity of V and conservation of angular momentum, this is, in turn, equivalent to $P_{I_\text{E}} \operatorname{span}_m a_m^\dagger \Phi_3$ being invariant to V. The space $\operatorname{span}_m a_m^\dagger \Phi_3$ cannot contain states with $I > 2j$. For every $I \leqslant 2j$ except $I = 4$ and 6, it may be verified by direct calculation, and also follows from the general result in Sec III, that $P_I \operatorname{span}_m a_m^\dagger \Phi_3$ exhausts the space of states of the four fermions with angular momentum I and thus is invariant to any rotationally invariant operator. Invariance of both spaces $P_{I_\text{E}} \operatorname{span}_m a_m^\dagger \Phi_3$ to V is then equivalent to $\Phi_4 = \operatorname{span}_m a_m^\dagger \Phi_3$ being invariant to V. In summary, the existence of the Escuderos-Zamick states is equivalent to Φ_4 being invariant to any V.

To establish the equivalence stated in the Introduction, it remains to show that Φ_4 can be written in the form (1). To this end, notice $\operatorname{span}_m a_m \Phi_0 \subset \Phi_3$. It may be verified by direct calculation, and also follows from the general result in Sec III, that the left-hand side exhausts Φ_3 so that $\operatorname{span}_m a_m \Phi_0 = \Phi_3$. This evidently leads to the expression (1). The remainder of this paper is dedicated to a proof (independent of the proofs in [3,4,6] of the equivalent statement) that Φ_4 as given by (1) is actually invariant to any V, and analyses of the actions of an arbitrary V on Φ_4 and its orthogonal complement.

III. SPACES Φ_0 AND Φ_4

The structure of multifermion states in the $j = 9/2$ shell is conveniently described in terms of creation operators

$$\alpha_m^\dagger = \sqrt{\frac{(j+m)!}{(j-m)!}}\, a_m^\dagger, \tag{2}$$

corresponding to unnormalized single-fermion states. In terms of the usual complex coordinates $(I_0, I_\pm)$ of the total angular momentum $\boldsymbol{I}$ [7], these operators obey

$$\begin{aligned}
[I_0, \alpha_m^\dagger] &= m\alpha_m^\dagger,\\
[I_+, \alpha_m^\dagger] &= \begin{cases} \alpha_{m+1}^\dagger, & m < j,\\ 0, & m = j,\end{cases}\\
[I_-, \alpha_m^\dagger] &= \begin{cases} (j+m)(j-m+1)\alpha_{m-1}^\dagger, & m > -j,\\ 0, & m = -j.\end{cases}
\end{aligned} \tag{3}$$

A state of four $j = 9/2$ fermions can be expanded on the states

$$|m_1 m_2 m_3 m_4\rangle = \left(\prod_{i=1}^{4} \alpha_{m_i}^\dagger\right)|\rangle \tag{4}$$

with $j \geqslant m_1 > m_2 > m_3 > m_4 \geqslant -j$, where $|\rangle$ is the vacuum. The eigenspaces of I_0 with eigenvalues M_I are spanned by the states with $\sum_i m_i = M_I$. The space Φ_0 is the subspace of the $M_I = 0$ space characterized by $J_+\Phi_0 = 0$. Since there are 18 states $|m_1 m_2 m_3 m_4\rangle$ with $M_I = 0$ and 16 with $M_I = 1$ (in accordance with the total multiplicities for $I \geqslant 0$ and 1 in Table I), this constraint can be expressed by a homogeneous system of 16 linear equations in 18 expansion coefficients. The equations turn out independent in accordance with the dimension 2 of Φ_0. Two linearly independent solution are

$$\begin{aligned}
\phi_0 = {}& \big|\tfrac{9}{2}\tfrac{7}{2}\tfrac{-7}{2}\tfrac{-9}{2}\big\rangle - \big|\tfrac{9}{2}\tfrac{5}{2}\tfrac{-5}{2}\tfrac{-9}{2}\big\rangle + \big|\tfrac{9}{2}\tfrac{3}{2}\tfrac{-3}{2}\tfrac{-9}{2}\big\rangle - \big|\tfrac{9}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-9}{2}\big\rangle\\
&+ \big|\tfrac{7}{2}\tfrac{5}{2}\tfrac{-5}{2}\tfrac{-7}{2}\big\rangle - \big|\tfrac{7}{2}\tfrac{3}{2}\tfrac{-3}{2}\tfrac{-7}{2}\big\rangle + \big|\tfrac{7}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-7}{2}\big\rangle + \big|\tfrac{5}{2}\tfrac{3}{2}\tfrac{-3}{2}\tfrac{-5}{2}\big\rangle\\
&- \big|\tfrac{5}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-5}{2}\big\rangle + \big|\tfrac{3}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-3}{2}\big\rangle,\\
\phi_1 = {}& -5\big|\tfrac{9}{2}\tfrac{7}{2}\tfrac{-7}{2}\tfrac{-9}{2}\big\rangle + 5\big|\tfrac{9}{2}\tfrac{5}{2}\tfrac{-5}{2}\tfrac{-9}{2}\big\rangle + \big|\tfrac{9}{2}\tfrac{3}{2}\tfrac{-3}{2}\tfrac{-9}{2}\big\rangle - 7\big|\tfrac{9}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-9}{2}\big\rangle\\
&+ 9\big|\tfrac{7}{2}\tfrac{5}{2}\tfrac{-5}{2}\tfrac{-7}{2}\big\rangle - 3\big|\tfrac{7}{2}\tfrac{3}{2}\tfrac{-3}{2}\tfrac{-7}{2}\big\rangle - 9\big|\tfrac{7}{2}\tfrac{1}{2}\tfrac{-1}{2}\tfrac{-7}{2}\big\rangle\\
&- 6\big|\tfrac{5}{2}\tfrac{3}{2}\tfrac{-3}{2}\tfrac{-5}{2}\big\rangle - 14\big|\tfrac{9}{2}\tfrac{3}{2}\tfrac{-5}{2}\tfrac{-7}{2}\big\rangle - 6\big|\tfrac{7}{2}\tfrac{5}{2}\tfrac{-3}{2}\tfrac{-9}{2}\big\rangle\\
&+ 16\big|\tfrac{9}{2}\tfrac{1}{2}\tfrac{-3}{2}\tfrac{-7}{2}\big\rangle + 6\big|\tfrac{7}{2}\tfrac{3}{2}\tfrac{-1}{2}\tfrac{-9}{2}\big\rangle - 25\big|\tfrac{9}{2}\tfrac{-1}{2}\tfrac{-3}{2}\tfrac{-5}{2}\big\rangle\\
&- 6\big|\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{-9}{2}\big\rangle + 9\big|\tfrac{7}{2}\tfrac{1}{2}\tfrac{-3}{2}\tfrac{-5}{2}\big\rangle + 6\big|\tfrac{5}{2}\tfrac{3}{2}\tfrac{-1}{2}\tfrac{-7}{2}\big\rangle.
\end{aligned} \tag{5}$$

Here, ϕ_0 evidently has $v = 0$. The state $\phi_0 + 2\phi_1$ is orthogonal to ϕ_0 and thus has $v = 4$. In the expansion of ϕ_1, the coefficients of $|\tfrac{7}{2}\tfrac{5}{2}\tfrac{-3}{2}\tfrac{-9}{2}\rangle$ and $|\tfrac{7}{2}\tfrac{3}{2}\tfrac{-1}{2}\tfrac{-9}{2}\rangle$ are equal except for opposite signs. This is because in the expansion of $I_+\phi_1$, the coefficient of $|\tfrac{7}{2}\tfrac{5}{2}\tfrac{-1}{2}\tfrac{-9}{2}\rangle$ gets contributions only from these two coefficients. Similar comparisons explain that all the four states $|\tfrac{7}{2}\tfrac{5}{2}\tfrac{-3}{2}\tfrac{-9}{2}\rangle$, $|\tfrac{7}{2}\tfrac{3}{2}\tfrac{-1}{2}\tfrac{-9}{2}\rangle$, $|\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{-9}{2}\rangle$, and $|\tfrac{5}{2}\tfrac{3}{2}\tfrac{-1}{2}\tfrac{-7}{2}\rangle$ have equal coefficients except for a sign. The ratios of coefficients of $|m_1 m_2 m_3 m_4\rangle$ and $|-m_4, -m_3, -m_2, -m_1\rangle$ are $\prod_{i=1}^4 (j - m_i)!/(j + m_i)!$ so that the corresponding ratios in the basis of states $(\prod_{i=1}^4 a_{m_i}^\dagger)|\rangle$ equal 1, as required by the symmetry under half-turn rotations about axes perpendicular to the quantization axis.

Since Φ_4 is rotationally invariant, its invariance to V is equivalent to invariance of its $M_I = 0$ subspace

$$\Phi_{40} = \operatorname{span}_m n_m \Phi_0, \tag{6}$$

where $n_m = a_m^\dagger a_m$. This space is spanned by the 20 states $n_m\phi_i$ with $m = j, j-1, \dots, -j$ and $i \in \{0, 1\}$. Each of these states is obtained by selecting in the expansion (5) of ϕ_i the terms where the orbit $|jm\rangle$ is occupied. Not all of them are linearly independent. Thus evidently $n_m\phi_0 = n_{-m}\phi_0$. Further, $I_0\phi_1 = 0$ is a linear relation among the 10 states $n_m\phi_1$. There remain 14 states, which turn out linearly independent. This number coincides with the total multiplicity for $I \leqslant 2j$, excepting the Escuderos-Zamick multiplets. The $M_I = 0$ state in every remaining multiplet thus belongs to Φ_{40}. Consequently, every such multiplet is contained in Φ_4. This requires, in turn, that the multiplets $P_{I_{\rm E}}\,\mathrm{span}_m\, a_m^\dagger \Phi_{3v}$ with $v = 1$ and 3 be independent, that for $I \leqslant 2j$ and $I \neq 4, 6$ the space $P_I\,\mathrm{span}_m\, a_m^\dagger \Phi_3$ exhausts the space of states with angular momentum I, and that equality hold in the inclusion $\mathrm{span}_m\, a_m \Phi_0 \subset \Phi_3$, all of which was used in Secs. I and II.

I choose in Φ_{40} a basis $(\psi_i | i = 1, \ldots, 14)$, where $(\psi_i | i = 1, \ldots, 5)$ are the states $n_m\phi_0$ with $m = j, j-1, \ldots, 1/2$ in this order, and $(\psi_i | i = 6, \ldots, 14)$ are the states $n_m\phi_1$ with $m = j, j-1, \ldots, -j+1$ in this order. By Hermiticity and angular momentum conservation, Φ_{40} is invariant to V if and only if its orthogonal complement $\Phi_{40}^\perp$ within the $M_I = 0$ space is so. This space is spanned by the states

$$\begin{aligned}
\chi_1 &= 14\big|\tfrac92\,\tfrac32\,\tfrac{-5}2\,\tfrac{-7}2\big\rangle + 6\big|\tfrac72\,\tfrac52\,\tfrac{-3}2\,\tfrac{-9}2\big\rangle + \tfrac{2}{153}\big(-768\big|\tfrac92\,\tfrac72\,\tfrac{-7}2\,\tfrac{-9}2\big\rangle + 231\big|\tfrac92\,\tfrac52\,\tfrac{-5}2\,\tfrac{-9}2\big\rangle + 927\big|\tfrac92\,\tfrac32\,\tfrac{-3}2\,\tfrac{-9}2\big\rangle - 72\big|\tfrac92\,\tfrac12\,\tfrac{-1}2\,\tfrac{-9}2\big\rangle \\
&\quad + 159\big|\tfrac72\,\tfrac52\,\tfrac{-5}2\,\tfrac{-7}2\big\rangle - 553\big|\tfrac72\,\tfrac32\,\tfrac{-3}2\,\tfrac{-7}2\big\rangle + 56\big|\tfrac72\,\tfrac12\,\tfrac{-1}2\,\tfrac{-7}2\big\rangle - 640\big|\tfrac52\,\tfrac32\,\tfrac{-3}2\,\tfrac{-5}2\big\rangle - 712\big|\tfrac52\,\tfrac12\,\tfrac{-1}2\,\tfrac{-5}2\big\rangle - 840\big|\tfrac32\,\tfrac12\,\tfrac{-1}2\,\tfrac{-3}2\big\rangle\big), \\
\chi_2 &= 16\big|\tfrac92\,\tfrac12\,\tfrac{-3}2\,\tfrac{-7}2\big\rangle + 6\big|\tfrac72\,\tfrac32\,\tfrac{-1}2\,\tfrac{-9}2\big\rangle + \tfrac{16}{1071}\big(516\big|\tfrac92\,\tfrac72\,\tfrac{-7}2\,\tfrac{-9}2\big\rangle - 105\big|\tfrac92\,\tfrac52\,\tfrac{-5}2\,\tfrac{-9}2\big\rangle - 171\big|\tfrac92\,\tfrac32\,\tfrac{-3}2\,\tfrac{-9}2\big\rangle + 450\big|\tfrac92\,\tfrac12\,\tfrac{-1}2\,\tfrac{-9}2\big\rangle \\
&\quad + 345\big|\tfrac72\,\tfrac52\,\tfrac{-5}2\,\tfrac{-7}2\big\rangle + 1225\big|\tfrac72\,\tfrac32\,\tfrac{-3}2\,\tfrac{-7}2\big\rangle + 364\big|\tfrac72\,\tfrac12\,\tfrac{-1}2\,\tfrac{-7}2\big\rangle + 430\big|\tfrac52\,\tfrac32\,\tfrac{-3}2\,\tfrac{-5}2\big\rangle + 880\big|\tfrac52\,\tfrac12\,\tfrac{-1}2\,\tfrac{-5}2\big\rangle + 966\big|\tfrac32\,\tfrac12\,\tfrac{-1}2\,\tfrac{-3}2\big\rangle\big), \\
\chi_3 &= 25\big|\tfrac92\,\tfrac{-1}2\,\tfrac{-3}2\,\tfrac{-5}2\big\rangle + 6\big|\tfrac52\,\tfrac32\,\tfrac12\,\tfrac{-9}2\big\rangle + \tfrac{25}{1071}\big(834\big|\tfrac92\,\tfrac72\,\tfrac{-7}2\,\tfrac{-9}2\big\rangle + 2184\big|\tfrac92\,\tfrac52\,\tfrac{-5}2\,\tfrac{-9}2\big\rangle + 1629\big|\tfrac92\,\tfrac32\,\tfrac{-3}2\,\tfrac{-9}2\big\rangle + 279\big|\tfrac92\,\tfrac12\,\tfrac{-1}2\,\tfrac{-9}2\big\rangle \\
&\quad - 750\big|\tfrac72\,\tfrac52\,\tfrac{-5}2\,\tfrac{-7}2\big\rangle - 847\big|\tfrac72\,\tfrac32\,\tfrac{-3}2\,\tfrac{-7}2\big\rangle - 931\big|\tfrac72\,\tfrac12\,\tfrac{-1}2\,\tfrac{-7}2\big\rangle - 376\big|\tfrac52\,\tfrac32\,\tfrac{-3}2\,\tfrac{-5}2\big\rangle - 3310\big|\tfrac52\,\tfrac12\,\tfrac{-1}2\,\tfrac{-5}2\big\rangle - 2100\big|\tfrac32\,\tfrac12\,\tfrac{-1}2\,\tfrac{-3}2\big\rangle\big), \\
\chi_4 &= 9\big|\tfrac72\,\tfrac12\,\tfrac{-3}2\,\tfrac{-5}2\big\rangle + 6\big|\tfrac52\,\tfrac32\,\tfrac{-1}2\,\tfrac{-7}2\big\rangle + \tfrac{3}{119}\big(-150\big|\tfrac92\,\tfrac72\,\tfrac{-7}2\,\tfrac{-9}2\big\rangle - 231\big|\tfrac92\,\tfrac52\,\tfrac{-5}2\,\tfrac{-9}2\big\rangle - 162\big|\tfrac92\,\tfrac32\,\tfrac{-3}2\,\tfrac{-9}2\big\rangle - 81\big|\tfrac92\,\tfrac12\,\tfrac{-1}2\,\tfrac{-9}2\big\rangle \\
&\quad + 45\big|\tfrac72\,\tfrac52\,\tfrac{-5}2\,\tfrac{-7}2\big\rangle + 196\big|\tfrac72\,\tfrac32\,\tfrac{-3}2\,\tfrac{-7}2\big\rangle + 301\big|\tfrac72\,\tfrac12\,\tfrac{-1}2\,\tfrac{-7}2\big\rangle + 232\big|\tfrac52\,\tfrac32\,\tfrac{-3}2\,\tfrac{-5}2\big\rangle + 508\big|\tfrac52\,\tfrac12\,\tfrac{-1}2\,\tfrac{-5}2\big\rangle + 126\big|\tfrac32\,\tfrac12\,\tfrac{-1}2\,\tfrac{-3}2\big\rangle\big).
\end{aligned} \tag{7}$$

It is straightforward to check $\psi_i^\dagger \chi_k = 0$ for every i, k using $\||m_1m_2m_3m_4\rangle\|^2 = \prod_{i=1}^4 (j+m_i)!/(j-m_i)!$. For example,

$$\begin{aligned}
\psi_6^\dagger\chi_1 = \phi_1^\dagger n_{9/2}\chi_1 &= (-14)\times\tfrac{3}{7}\times 14 \\
&\quad + \tfrac{2}{153}((-5)\times(-786) + 5\times 231 \\
&\quad + 1\times 927 + (-7)\times(-72)) = 0.
\end{aligned} \tag{8}$$

IV. INTERACTION V AND INVARIANCE OF Φ_4

Every V is a linear combination of five basic interactions V_J, where $J = 0, 2, \ldots, 2j-1$. They can be chosen in the form

$$V_J = \frac{1}{2}\sum_{M=-J}^{J} P_{JM}^\dagger P_{JM} \tag{9}$$

with

$$P_{JM} = c_J \sum_{m_1+m_2=M} \langle jm_1 jm_2|JM\rangle a_{m_2} a_{m_1}, \tag{10}$$

where c_J is a positive constant. I set

$$c_J\langle jm_1 jm_2|JM\rangle = \binom{2J}{J+M}^{-1/2}\sqrt{\frac{(j+m_1)!(j+m_2)!}{(j-m_1)!(j-m_2)!}}\,c^J_{m_1m_2}, \tag{11}$$

so that by (2),

$$P_{JM}^\dagger = \binom{2J}{J+M}^{-1/2}\sum_{m_1+m_2=M} c^J_{m_1m_2}\alpha_{m_1}^\dagger\alpha_{m_2}^\dagger. \tag{12}$$

The definition (11) implies $c^J_{m_1m_2} = 0$ for $|m_1+m_2| > J$. It follows from $[I_+, P_{JJ}^\dagger] = 0$, (12), (3), and (11) that c_J can be chosen such that $c^J_{m_1m_2} = (-1)^{j-m_1}$ for $m_1 + m_2 = J$. From $[I_-, P_{JM}^\dagger] = \sqrt{(J+M)(J-M+1)}P^\dagger_{J,M-1}$ for $M > -J$, (12), and (3), one gets the recursion relation

$$\begin{aligned}
(J - m_1 - m_2)c^J_{m_1m_2} &= (j-m_1)(j+m_1+1)c^J_{m_1+1,m_2} \\
&\quad + (j-m_2)(j+m_2+1)c^J_{m_1,m_2+1},
\end{aligned} \tag{13}$$

which then determines $c^J_{m_1m_2}$ for $-J \leqslant m_1 + m_2 < J$. [Continuation of the recursion in fact results in $c^J_{m_1m_2} = 0$ for $m_1 + m_2 < -J$. Terms in (13) with m_1 or m_2 equal to j, which involve undefined values of $c^J_{m_1'm_2'}$, are just omitted.] All $c^J_{m_1m_2}$ turn out integral, which is explained in Appendix A. From the definition (11) and symmetries of the vector coupling coefficients [7], one gets

$$\begin{aligned}
c^J_{m_1m_2} &= -c^J_{m_2m_1}, \\
c^J_{m_1m_2} &= \frac{(j-m_1)!(j-m_2)!}{(j+m_1)!(j+m_2)!}c^J_{-m_2,-m_1},
\end{aligned} \tag{14}$$

whence by (12) and (2) follows

$$\begin{aligned}
P_{JM} &= \binom{2J}{J+M}^{-1/2}\sum_{m_1+m_2=M} c^J_{m_1m_2}\frac{(j+m_1)!(j+m_2)!}{(j-m_1)!(j-m_2)!}\alpha_{m_2}\alpha_{m_1} \\
&= \binom{2J}{J+M}^{-1/2}\sum_{m_1+m_2=M} c^J_{-m_2,-m_1}\alpha_{m_2}\alpha_{m_1}
\end{aligned} \tag{15}$$

in terms of annihilation operators

$$\alpha_m = \sqrt{\frac{(j-m)!}{(j+m)!}}a_m \tag{16}$$

obeying

$$\{\alpha_m, \alpha_{m'}^\dagger\} = \delta_{m,m'}. \tag{17}$$

It follows that the action of V_J on a basic state $|m_1 m_2 m_3 m_4\rangle$ can be described by the following operation u_{Jm}^{pq}. If $m_p + m_q - m$ is outside the range of m's, then $u_{Jm}^{pq}|m_1 m_2 m_3 m_4\rangle = 0$. Otherwise replace m_p and m_q by m and $m_p + m_q - m$. If this results in two m's being equal, $u_{Jm}^{pq}|m_1 m_2 m_3 m_4\rangle = 0$. Otherwise reorder, if necessary, the m's to decreasing order and multiply the state by the sign of the permutation. Finally multiply the state by $\binom{2J}{J+m_p+m_q}^{-1} c^J_{m,m_p+m_q-m} c^J_{-m_q,-m_p}$. Then

$$V_J|m_1 m_2 m_3 m_4\rangle = \sum_{1\leqslant p<q\leqslant 4,m} u_{Jm}^{pq}|m_1 m_2 m_3 m_4\rangle. \tag{18}$$

The state $V_J\chi_i$ is obtained by applying this formula to each term in the expansion (7) of χ_i. I did this calculation for every J, i and found that, in every case, $V_J\chi_i$ is a linear combination of $\{\chi_i|i = 1,\dots,4\}$. This proves that $\Phi_{40}^\perp$, and in turn Φ_{40}, Φ_4, and the orthogonal complement $\Phi_4^\perp$ of the latter, are invariant to every V. For completeness, I also verified directly that every $V_J\psi_i$ is a linear combination of $\{\psi_i|i = 1,\dots,14\}$.

V. ACTION OF V ON $\Phi_{40}^\perp$

The expansion of $V_J\chi_i$ on states χ_k may be expressed by a matrix $V^{\perp J} = (v_{ik}^{\perp J}|i,k = 1,\dots,4)$ defined by

$$V_J\chi_i = \sum_{k=1}^{4} v_{ki}^{\perp J}\chi_k. \tag{19}$$

These matrices are given by

$$V^{\perp 0} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$\bar V^{\perp 2} = -\frac{1071}{16}V^{\perp 2} = \begin{pmatrix} -318150 & 170280 & 249300 & -18468\\ 90027 & -161100 & 94950 & 59778\\ 0 & 51408 & -420903 & 28917\\ -207088 & 422480 & -1012475 & -50895 \end{pmatrix},$$

$$\bar V^{\perp 4} = -\frac{51}{64}V^{\perp 4} = \begin{pmatrix} -800100 & 170280 & 249300 & -18468\\ 90027 & -643050 & 94950 & 59778\\ 0 & 51408 & -902853 & 28917\\ -207088 & 422480 & -1012475 & -532845 \end{pmatrix},$$

$$\bar V^{\perp 6} = \frac{187}{102400}V^{\perp 6} = \begin{pmatrix} 578277 & 170280 & 249300 & -18468\\ 90027 & 735327 & 94950 & 59778\\ 0 & 51408 & 475524 & 28917\\ -207088 & 422480 & -1012475 & 845532 \end{pmatrix},$$

$$\bar V^{\perp 8} = \frac{2431}{278691840}V^{\perp 8} = \begin{pmatrix} 647892 & 170280 & 249300 & -18468\\ 90027 & 804942 & 94950 & 59778\\ 0 & 51408 & 545139 & 28917\\ -207088 & 422480 & -1012475 & 915147 \end{pmatrix}, \tag{20}$$

where the fractions are determined by the condition that the entries in each $\bar V^J$ be coprime integers. That the pairing force V_0 kills the $v = 4$ space $\Phi_{40}^\perp$ is no surprise. The matrices $\bar V^{\perp J}$ exhibit the remarkable similarity

$$\bar V^{\perp 2} = \bar V^{\perp 4} + 481950 = \bar V^{\perp 6} - 896427 = \bar V^{\perp 8} - 966042 \tag{21}$$

with multiplication of the scalars by the unit matrix understood. Like the invariance of the entire $\Phi_{40}^\perp$, this regularity lacks fundamental explanation. It follows that the interactions $\bar V_J$ represented on $\Phi_{40}^\perp$ by these matrices also act identically on $\Phi_{40}^\perp$ except for these scalar terms. Any linear combination of these interactions, that is, an arbitrary V, then acts on $\Phi_{40}^\perp$ as a linear combination of any one of them and a scalar. This applies, in particular, to the two-body interaction $\boldsymbol{I}^2 - j(j+1)N$, where $N = \sum_m n_m$. Because N acts on the four-body space as the scalar 4, conversely then every V_J acts on $\Phi_{40}^\perp$ as a linear combination of $\boldsymbol{I}^2$ and a scalar. This explains, in particular, the zeros in the third row and first column of every $V^{\perp J}$. For no $|m_1 m_2 m_3 m_4\rangle$ in the expansion (7) of χ_1, the expansion of $\boldsymbol{I}^2|m_1 m_2 m_3 m_4\rangle$ on states $|m_1 m_2 m_3 m_4\rangle$ indeed contains $|\frac{9}{2}\frac{-1}{2}\frac{-3}{2}\frac{-5}{2}\rangle$. Therefore χ_3 cannot appear in the expansion of $\boldsymbol{I}^2\chi_1$ on states χ_i.

One arrives at a prediction that might be tested experimentally. To the extend of validity of the $j = 9/2$ shell model, the spacings of the energy levels with angular momenta I and I' must have the ratio of $I(I+1) - I'(I'+1)$. The nucleus ^{94}Ru has a closed neutron major shell and four protons in the $1g_{9/2}$ subshell. The yrast $I = 4$, 6, 10, and 12 levels (with tentative assignments $I = 10$ and 12) have excitation energies 2186.6, 2498.0, 3991.2, and 4716.6 MeV [8]. The states with $I = 10$ and 12 are expected to have fairly pure $1g_{9/2}$ configurations while, according to Das *et al.* [9], both multiplets with $I = 4$ and 6 could be mixtures of those labeled γ and α in Sec. II due to perturbation by configurations outside the proton $1g_{9/2}$ shell. The pure Escuderos-Zamick energy levels should then be close to the observed yrast levels. Extrapolation by the spacing rule from $I = 10$ and 12 gives excitation energies 2571.9 and 2918.9 MeV, somewhat above the yrast levels. A similar analysis for ^{96}Pd, with four holes in the $1g_{9/2}$ shell (and tentative assignments of the angular momenta concerned), predicts Escuderos-Zamick levels at 2237.9 and 2616.8 MeV, closer to the yrast levels at 2099.01 and 2424.19 MeV. Interpreting the second observed $I = 4$ and 6 levels in ^{74}Ni [10], with a closed proton major shell and four holes in the neutron $1g_{9/2}$ subshell, as Escuderos-Zamick levels leads to the prediction of the $I = 10$ and 12 levels at 4287 and 5577 MeV.

Since the operator $\boldsymbol{I}^2$ acts on the $M_I = 0$ space as I_-I_+, the matrix C representing its action on $\Phi_{40}^\perp$ is easily calculated by (7), (4), and (3). By comparison with (20), one finds in the notation of (21) that

$$\tfrac{1}{3213}\bar{V}^{\perp 2}+156=\tfrac{1}{3213}\bar{V}^{\perp 4}+306=\tfrac{1}{3213}\bar{V}^{\perp 6}-123=\tfrac{1}{3213}\bar{V}^{\perp 8}-\tfrac{434}{3}=C. \quad (22)$$

No simple expression in terms of J seems to reproduce these displacements. For $I=4$ and 6, Van Isacker and Heinze calculated the ratios $r_I^J=\mu_I^J/\nu_J$, where μ_I^J and ν_J are the eigenvalues of V_J in the four-fermion system and a two-fermion state with $I=J$ [3,4]. From (9)–(11), one gets

$$\nu_J=c_J^2=\binom{2J}{J}^{-1}\sum_{m=-j}^{j}\left(c_{m,-m}^J\right)^2. \quad (23)$$

My calculations confirm the values $r_4^J=\frac{68}{33},1,\frac{13}{15},\frac{114}{55}$ and $r_6^J=\frac{19}{11},\frac{12}{13},1,\frac{336}{143}$ for $J=2$–8 reported in [3,4], and further provide $r_{10}^J=\frac{23}{33},\frac{98}{143},\frac{233}{165},\frac{2292}{715}$ and $r_{12}^J=0,\frac{75}{143},\frac{93}{55},\frac{246}{65}$.

VI. ACTION OF V ON Φ_{40}

Like in (19), the action of V_J on the states ψ_i may be expressed by matrices V^J. They are

$$V^0=2\begin{pmatrix}
4&1&1&1&1&-2&-5&-5&1&7&7&1&-5&-5\\
1&4&1&1&1&-5&-2&9&3&-9&-9&3&9&-2\\
1&1&4&1&1&-5&9&-2&-6&0&0&-6&-2&9\\
1&1&1&4&1&1&3&-6&-2&0&0&-2&-6&3\\
1&1&1&1&4&7&-9&0&0&-2&-2&0&0&-9\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix},$$

$$V^2=\frac{16}{3}\begin{pmatrix}
630&108&-54&-162&-216&594&189&189&351&513&513&351&189&189\\
108&342&-18&-54&-72&189&594&567&405&81&81&405&567&594\\
-54&-18&315&27&36&189&567&594&162&324&324&162&594&567\\
-162&-54&27&387&108&351&405&162&594&324&324&594&162&405\\
-216&-72&36&108&450&513&81&324&324&594&594&324&324&81\\
18&0&0&0&0&522&45&-117&-171&-171&-171&-171&-117&45\\
7&9&0&0&0&79&412&-110&-152&-152&-152&-26&64&-143\\
5&0&9&0&0&32&-13&293&-133&-133&56&-7&-151&83\\
3&0&0&9&0&-15&-42&-96&282&111&-78&-60&12&12\\
1&0&0&0&9&-62&-71&-89&130&301&22&-59&94&-95\\
-1&0&0&0&9&-109&-100&107&-40&41&320&149&-76&-76\\
-3&0&0&9&0&-156&-3&51&-3&-21&168&339&-57&-57\\
-5&0&9&0&0&-149&58&-86&88&151&-38&-38&358&-38\\
-7&9&0&0&0&-34&-178&155&107&-19&-19&-19&-19&377
\end{pmatrix},$$

$$V^4=\frac{1344}{5}\begin{pmatrix}
4518&-1188&-918&162&972&-20574&-6075&-6075&-12393&-18711&-18711&-12393&-6075&-6075\\
-1188&4998&1122&-198&-1188&-6075&-20574&-20817&-14499&-1863&-1863&-14499&-20817&-20574\\
-918&1122&4413&-153&-918&-6075&-20817&-20574&-5022&-11340&-11340&-5022&-20574&-20817\\
162&-198&-153&3573&162&-12393&-14499&-5022&-20574&-11340&-11340&-20574&-5022&-14499\\
972&-1188&-918&162&4518&-18711&-1863&-11340&-11340&-20574&-20574&-11340&-11340&-1863\\
-630&0&0&0&0&8730&1125&1395&585&-495&-495&585&1395&1125\\
-245&-315&0&0&0&655&8500&970&340&1000&1000&-290&-860&1585\\
-175&0&-315&0&0&320&395&8285&2345&245&-700&455&1115&-745\\
-105&0&0&-315&0&-15&30&2370&7590&615&120&390&570&-60\\
-35&0&0&0&-315&-350&1165&445&730&7225&1450&235&-470&1345\\
35&0&0&0&-315&-145&1220&-625&170&1505&7280&665&290&1040\\
105&0&0&-315&0&600&-435&105&195&285&780&7395&1905&-345\\
175&0&-315&0&0&1075&-1370&340&130&-425&520&2020&7510&-230\\
245&-315&0&0&0&470&710&-1945&-745&1385&1385&-115&-115&7625
\end{pmatrix},$$

$$V^6=\frac{1075200}{11}\begin{pmatrix}
4221 & -297 & 270 & 162 & -216 & 21006 & 12339 & 12339 & 9585 & 6831 & 6831 & 9585 & 12339 & 12339\\
-297 & 5229 & -990 & -594 & 792 & 12339 & 21006 & 5913 & 8667 & 14175 & 14175 & 8667 & 5913 & 21006\\
270 & -990 & 5040 & 540 & -720 & 12339 & 5913 & 21006 & 12798 & 10044 & 10044 & 12798 & 21006 & 5913\\
162 & -594 & 540 & 4464 & -432 & 9585 & 8667 & 12798 & 21006 & 10044 & 10044 & 21006 & 12798 & 8667\\
-216 & 792 & -720 & -432 & 4716 & 6831 & 14175 & 10044 & 10044 & 21006 & 21006 & 10044 & 10044 & 14175\\
558 & 0 & 0 & 0 & 0 & 6057 & -954 & -387 & 1179 & 2475 & 2475 & 1179 & -387 & -954\\
217 & 279 & 0 & 0 & 0 & -764 & 5959 & 1630 & 1264 & 472 & 472 & 634 & 328 & 364\\
155 & 0 & 279 & 0 & 0 & -736 & 1253 & 6338 & 728 & 1169 & 224 & -469 & 323 & 476\\
93 & 0 & 0 & 279 & 0 & 183 & 426 & 291 & 6501 & 417 & 714 & 192 & -816 & 237\\
31 & 0 & 0 & 0 & 279 & 1102 & -617 & 445 & 70 & 6334 & 493 & 367 & -470 & -470\\
-31 & 0 & 0 & 0 & 279 & 1373 & -364 & -427 & 236 & 218 & 6059 & -61 & 488 & -511\\
-93 & 0 & 0 & 279 & 0 & 996 & 555 & -687 & -201 & -111 & -408 & 6108 & 420 & 744\\
-155 & 0 & 279 & 0 & 0 & 349 & 1006 & 538 & -1124 & -1151 & -206 & 73 & 6553 & 1783\\
-217 & 279 & 0 & 0 & 0 & -190 & 1106 & 629 & -283 & -1453 & -1453 & 347 & 1931 & 6701
\end{pmatrix},$$

$$V^8=\frac{975421440}{143}\begin{pmatrix}
7143 & -21 & 60 & -84 & 42 & -12342 & -8823 & -8823 & -4845 & -867 & -867 & -4845 & -8823 & -8823\\
-21 & 7287 & -420 & 588 & -294 & -8823 & -12342 & 459 & -3519 & -11475 & -11475 & -3519 & 459 & -12342\\
60 & -420 & 8340 & -1680 & 840 & -8823 & 459 & -12342 & -9486 & -5508 & -5508 & -9486 & -12342 & 459\\
-84 & 588 & -1680 & 9492 & -1176 & -4845 & -3519 & -9486 & -12342 & -5508 & -5508 & -12342 & -9486 & -3519\\
42 & -294 & 840 & -1176 & 7728 & -867 & -11475 & -5508 & -5508 & -12342 & -12342 & -5508 & -5508 & -11475\\
-306 & 0 & 0 & 0 & 0 & 4491 & -72 & 9 & -1053 & -1845 & -1845 & -1053 & 9 & -72\\
-119 & -153 & 0 & 0 & 0 & -122 & 5657 & -2530 & -1068 & -584 & -584 & -438 & -76 & -1078\\
-85 & 0 & -153 & 0 & 0 & 512 & -2111 & 6184 & -2436 & -973 & -28 & 273 & -491 & -542\\
-51 & 0 & 0 & -153 & 0 & -141 & -342 & -2097 & 6723 & -2019 & -588 & -234 & 522 & -369\\
-17 & 0 & 0 & 0 & -153 & -794 & 239 & -445 & -1770 & 7192 & -2321 & -339 & 470 & 20\\
17 & 0 & 0 & 0 & -153 & -1051 & 28 & 469 & -222 & -2116 & 7397 & -1653 & -446 & 247\\
51 & 0 & 0 & -153 & 0 & -912 & -345 & 519 & 117 & 27 & -1404 & 7074 & -2100 & -318\\
85 & 0 & -153 & 0 & 0 & -503 & -502 & -496 & 858 & 997 & 52 & -1851 & 6179 & -2071\\
119 & -153 & 0 & 0 & 0 & 50 & -1022 & -83 & 381 & 851 & 851 & -249 & -2537 & 5713
\end{pmatrix}. \tag{24}$$

Some patterns leap to the eye. The entries in V^0 are easily understood. Thus $V_0|m_1m_2m_3m_4\rangle$ vanishes unless two of the m's form a pair $m, -m$, in which case the remaining two do the same. Further, $V_0|m_1, m_2, -m_2, -m_1\rangle = (-)^{m_1-m_2+1}2(n_{m_1}+n_{m_2})\phi_0$, so the states $n_m\phi_1$ do not contribute to any $V_0\psi_i$. This expression follows from $V_0|m, -m\rangle = 2\sum_{m'}(-)^{m'-m}|m', -m'\rangle$, where $|m_1m_2\rangle = \alpha^\dagger_{m_1}\alpha^\dagger_{m_2}|\rangle$, and the observation that the coefficient of $|m_1, m_2, -m_2, -m_1\rangle$ in the expansion (5) of ϕ_0 is $(-1)^{m_1-m_2+1}$. By using it in combination with the expansions (5), it is, in fact, straightforward to reconstruct every entry in V^0, and in particular, the simple pattern in its upper left 5×5 submatrix. Notice to this end that the last eight terms in the expansion of ϕ_1 do not contribute to $V_0\phi_1$.

For a general J, one notices in the upper right 5×8 submatrix of V^J equal contributions to $V_Jn_{\pm m}\phi_1$ from any $n_{m'}\phi_0$. This is an immediate consequence of $n_{m'}\phi_0 = n_{-m'}\phi_0$ and the symmetry under half-turn rotations about an axis perpendicular to the quantization axis. The same pattern is seen in the parts of the sixth rows just below, which display contributions to $V_Jn_m\phi_1$ from $n_j\phi_1$ for $m\neq\pm j$, and again the reason is the symmetry under half-turn rotations about an axis perpendicular to the quantization axis. Such a rotation thus leads to both the replacement of m by $-m$ and the omission of $n_j\phi_1$ instead of $n_{-j}\phi_1$ in the selection of the states ψ_i. But by $\sum_m mn_m\phi_1 = J_0\phi_1 = 0$, the state $n_{-j}\phi_1$ equals $n_j\phi_1$ plus a linear combination of states that are common to both the original and the new basis. Therefore in the original basis the contribution of $n_j\phi_1$ to $V_Jn_{-m}\phi_1$ equals its contribution to $V_Jn_m\phi_1$. It follows further that when the lower right 9×9 submatrix of V^J is written $(v^{1J}_{mm'}|m, m' > -j)$ with indices referring to the basic states $n_m\phi_1$, then for $m, m' < j$ one should have

$$v^{1J}_{-m',-m} = v^{1J}_{m'm} - \frac{m'}{j}v^{1J}_{jm}. \tag{25}$$

This is verified by inspection. Similar patterns occur when a state $n_m\phi_1$ other than $n_{-j}\phi_1$ is omitted in the selection of the states ψ_i.

It is trivial by $n_m\phi_0 = n_{-m}\phi_0$ that the state $n_{m'}\phi_0$ contributes equally to $V_Jn_{\pm m}\phi_0$. Therefore when $v^{00J}_{mm'}$ denotes the entries in the upper left 5×5 submatrix of V^J with indices referring to the basic states $n_m\phi_0$, and $v^{01J}_{mm'}$ denotes those of its neighboring 5×5 submatrix to the right with indices referring to the basic states $n_m\phi_0$ and $n_{m'}\phi_1$, then for both $i=0$ and 1 one has

$$\frac{1}{2}V_J\bar n_m\phi_i = \sum_{m'>0} v^{0iJ}_{m'm}\bar n_{m'}\phi_0 + \text{linear combination of } n_{m'}\phi_1 \tag{26}$$

with $\bar n_m = n_m + n_{-m}$. The operators $\bar n_m$ can be expressed by the tensor operators T_{I0} defined in the Introduction,

$$\bar n_m = (-)^{j-m}2\sum_{\text{even } I}\langle jmj-m|I0\rangle T_{I0}. \tag{27}$$

Since the states $T_{I0}\phi_i$ span the subspace of Φ_{40} with angular momentum I, which is invariant to V^J, one can write

$$V_JT_{I0}\phi_i = \sum_{i'} w^{IJ}_{i'i}T_{I0}\phi_{i'}. \tag{28}$$

By combining the equations (26)–(28) and the definition of T_{I0} in the Introduction one obtains

$$v_{mm'}^{0iJ} = (-)^{m-m'} \sum_{\text{even } I} \langle jmj-m|I0\rangle\langle jm'j-m'|I0\rangle w_{0i}^{IJ}, \quad (29)$$

which is symmetric in m and m'. This explains that the said submatrices of every V^J are symmetric.

A spectacular pattern emerges in the lower left 9×5 submatrix of every V^J. For $j > m \geqslant 1/2$ the only contribution to $V_J n_m \phi_0$ from states $n_{m'}\phi_1$ is a term $\gamma_J \bar{n}_m \phi_1$, where γ_J is constant. Upon closer inspection, taking into account again $\sum_m m n_m \phi_1 = 0$, this also hold for $m = j$. For every m one thus has

$$\frac{1}{2} V_J \bar{n}_m \phi_0 = \sum_{m'>0} v_{m'm}^{00J} \bar{n}_{m'} \phi_0 + \gamma_J \bar{n}_m \phi_1. \quad (30)$$

(The states $\bar{n}_m\phi_0$ span the $v \leqslant 2$ subspace of Φ_{40}.) This is not dependent on the choice of ϕ_1. Replacing ϕ_1 by any linear combination $\phi_1 + \epsilon\phi_0$ gives a relation of the same structure with the same γ_J. In particular, ϕ_1 could have $v = 4$. The origin of this pattern is explained in Appendix B.

VII. CONCLUSION

The existence of the Escuderos-Zamick states was shown to be equivalent to the invariance to any rotationally invariant two-body interaction of the span Φ_4 of states generated from angular momentum zero states by one-body operators. This equivalence explains the occurrence of the Escuderos-Zamick states for exactly the angular momenta 4 and 6. The said property of the angular momentum zero state space was verified by exact calculation. This verification was facilitated by the observation that it is required only for the subspace Φ_{40} of Φ_4 characterized by magnetic quantum number $M_I = 0$ or its orthogonal complement $\Phi_{40}^{\perp}$ within the $M_I = 0$ space. The actions of five basic rotationally invariant two-body interactions on four basic states in $\Phi_{40}^{\perp}$ and 14 basic states in Φ_{40} were displayed in matrix form, and remarkable regularities in these matrices disclosed. One of them leads to a rule that relates the Escuderos-Zamick energy levels to levels with $I = 10$ and 12. This rule was applied in a discussion of certain aspects of the spectra of ^{94}Ru, ^{96}Pd, and ^{74}Ni. The said regularity so far lacks fundamental explanation. Understanding it could possibly provide a clue towards a more intuitive understanding of the invariance of Φ_4.

APPENDIX A: PROOF THAT $c_{m_1m_2}^J$ ARE INTEGRAL

First notice that the algorithm for $c_{m_1m_2}^J$ described in Sec. IV ensures the proportionality (11) so that $P_{JM}^\dagger$ given by (10) is a tensor operator. Besides (13), $[I_+, P_{JM}^\dagger] = \sqrt{(J-M)(J+M+1)} P_{J,M+1}^\dagger$, (12), and (3) give

$$(J + m_1 + m_2) c_{m_1m_2}^J = c_{m_1-1,m_2}^J + c_{m_1,m_2-1}^J. \quad (A1)$$

With

$$d_{m_1m_2}^J = (J - m_1 - m_2)!\, c_{m_1m_2}^J, \quad (A2)$$

the recursion relations (13) and (A1) take the forms

$$d_{m_1m_2}^J = (j - m_1)(j + m_1 + 1) d_{m_1+1,m_2}^J + (j - m_2)(j + m_2 + 1) d_{m_1,m_2+1}^J, \quad (A3)$$

$$(J + m_1 + m_2)(J - m_1 - m_2 + 1) d_{m_1m_2}^J = d_{m_1-1,m_2}^J + d_{m_1,m_2-1}^J. \quad (A4)$$

(Again, terms with undefined values of $d_{m_1'm_2'}^J$ are omitted.) Setting $m_1 = j$ so that only the second term occurs on the right in (A3), and using also $d_{j,J-j}^J = 1$, one gets for $J - j \geqslant m_2 \geqslant -j$, by repeated application of (A3), an expression for $d_{jm_2}^J$ as a product of two products of $J - j - m_2$ consecutive integers. Hence $d_{jm_2}^J$ is divisible by $(J - j - m_2)!^2$, and, all the more, by $(J - j - m_2)!$. It then follows by induction by means of (A4) that $(J - m_1 - m_2)!$ divides $d_{m_1m_2}^J$ for every m_1, m_2 with $m_1 + m_2 \geqslant 0$. Then by (A2), $c_{m_1m_2}^J$ is integral for $m_1 + m_2 \geqslant 0$. For $m_1 + m_2 < 0$ one can now apply the second equation in (14). In this case, $j + m_1 < j - m_2$ and $j + m_2 < j - m_1$, so the first factor on the right, and hence $c_{m_1m_2}^J$, are integral.

APPENDIX B: EXPLANATION OF (30)

The first equation in (5) can be written

$$\phi_0 = -\tfrac{1}{2} {P^\dagger}^2 |\rangle \quad (B1)$$

with

$$P^\dagger = \sum_{m>0} (-)^{j-m} a_m^\dagger a_{-m}^\dagger. \quad (B2)$$

Hence

$$n_m\phi_0 = a_m^\dagger a_m \phi_0 = -a_m^\dagger P^\dagger [a_m, P^\dagger]|\rangle = -(-)^{j-m} P^\dagger a_m^\dagger a_{-m}^\dagger |\rangle \quad (B3)$$

and

$$V_J n_m \phi_0 = \frac{1}{2}\sum_M P_{JM}^\dagger P_{JM} n_m \phi_0 = -(-)^{j-m} \frac{1}{2} \sum_M P_{JM}^\dagger (P^\dagger P_{JM} + [P_{JM}, P^\dagger]) a_m^\dagger a_{-m}^\dagger |\rangle. \quad (B4)$$

Since $P_{JM} a_m^\dagger a_{-m}^\dagger |\rangle \propto |\rangle$, the first term in the parentheses contributes to $V_J n_m \phi_0$ a term proportional to $P_{JM}^\dagger P^\dagger |\rangle$, which has $v \leqslant 2$. I proceed by calculating the commutator $[P_{JM}, P^\dagger]$.

For convenience, I omit for now the factor c_J in (10). It is reentered at the end of this Appendix. I then have

$$[P_{JM}, P^\dagger] = \frac{1}{2}\sum_{m_1m_2m} (-)^{j-m} \langle jm_1 jm_2|JM\rangle \big[a_{m_2}a_{m_1}, a_m^\dagger a_{-m}^\dagger\big]$$
$$= \sum_{m_1m_2} \langle jm_1 jm_2|JM\rangle \times \big((-)^{j-m_1} a_{m_2} a_{-m_1}^\dagger + (-)^{j-m_2} a_{-m_2}^\dagger a_{m_1}\big)$$
$$= \sum_{m_1m_2} (-)^{j-m_2} \langle jm_1 jm_2|JM\rangle \big[a_{-m_2}^\dagger, a_{m_1}\big]$$
$$= 2\sum_{m_1m_2} (-)^{j-m_2} \langle jm_1 jm_2|JM\rangle a_{-m_2}^\dagger a_{m_1}$$

$$-\sum_m (-)^{j+m}\langle jmj-m|JM\rangle$$
$$=2\sum_{m_1m_2}(-)^{j-m_2}\langle jm_1jm_2|JM\rangle a^\dagger_{-m_2}a_{m_1}$$
$$+\delta_{J0}\sqrt{2j+1}. \quad \text{(B5)}$$

Since $P^\dagger_{J0}\propto P^\dagger$, the last term in this expression is seen by comparison with (B2) to contribute in (B4) a term proportional to $n_m\phi_0$.

Further,

$$a^\dagger_{-m_2}a_{m_1}a^\dagger_m a^\dagger_{-m}|\rangle = \left(\delta_{m_1m}-\delta_{m_1-m}\right)a^\dagger_{-m_2}a^\dagger_{-m_1}|\rangle, \quad \text{(B6)}$$

so the first term in the expression (B5) contributes to the sum in (B4) terms,

$$2a^\dagger_m\sum_{Mm_2}(-)^{j-m_2}\langle j-mjm_2|JM\rangle P^\dagger_{JM}a^\dagger_{-m_2}|\rangle$$
$$-\text{ same with } -m \text{ instead of } m. \quad \text{(B7)}$$

Here [7],

$$\xi=\sum_{Mm_2}(-)^{j-m_2}\langle j-mjm_2|JM\rangle P^\dagger_{JM}a^\dagger_{-m_2}|\rangle$$
$$=-\sqrt{\frac{2J+1}{2j+1}}\sum_{m_2M}\langle j-m_2JM|j-m\rangle a^\dagger_{-m_2}P^\dagger_{JM}|\rangle \quad \text{(B8)}$$

is a member with $M_I=-m$ of the space Φ_3 defined in Sec. II. With standard relative phases within its multiplet, one therefore has

$$\xi=(-)^{j+m}a_m(\zeta\phi_0+\eta\phi_1), \quad \text{(B9)}$$

where ζ and η do not depend on m. Totally, one arrives at

$$V_Jn_m\phi_0=\eta(n_m+n_{-m})\phi_1+v\leqslant 2\text{ state}, \quad \text{(B10)}$$

which is equivalent to (30) with $\gamma_J=\eta c_J^2$.

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