Embedding a critical point in a hadron to quark-gluon crossover equation of state

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Lattice QCD simulations have shown unequivocally that the transition from hadrons to quarks and gluons is a crossover when the baryon chemical potential is zero or small. Many model calculations predict the existence of a critical point at a value of the chemical potential where current lattice simulations are unreliable. We show how to embed a critical point in a smooth background equation of state so as to yield the critical exponents and critical amplitude ratios expected of a transition in the same universality class as the liquid-gas phase transition and the three-dimensional Ising model. There are only two independent critical exponents; the relations $\alpha + 2\beta + \gamma = 2$ and $\beta(\delta - 1) = \gamma$ arise automatically, as does a relation between the two critical amplitudes. The resulting equation of state has parameters that may be inferred by hydrodynamic modeling of heavy-ion collisions in the Beam Energy Scan II at the BNL Relativistic Heavy Ion Collider or in experiments at other accelerators.

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The QCD equation of state has been a subject of intense interest ever since the discovery of asymptotic freedom. At high temperature T and baryon chemical potential μ it is a weakly interacting gas of quarks and gluons, while at low T and μ it is a strongly interacting gas of hadrons. Lattice QCD simulations have shown conclusively that the transition from one phase to the other at $T \approx 155$ MeV and $\mu = 0$ is smooth on account of the fact that the up and down quark masses, and consequently the pion mass, are not zero [1,2]. However, diverse model calculations predict the existence of a line of first-order phase transition, beginning at $T = 0$ and μ_0 and terminating in a critical point at $T_c < 155$ MeV and $\mu_c < \mu_0$ [3,4]. Such a purported critical point is presently beyond the reach of reliable lattice calculations. Existing approaches include a Taylor series expansion in powers of μ/T at $\mu = 0$ and analytic extrapolations from imaginary to real chemical potentials; recent results are reported in Refs. [5,6], respectively. Experiments during the Beam Energy Scan II at the BNL Relativistic Heavy Ion Collider (RHIC) may or may not support the existence of critical behavior [7]. The goal of this paper is to propose a general construction for the equation of state which is consistent with (i) lattice QCD for all T and small μ, (ii) perturbative QCD for large T and/or large μ, and (iii) a critical point with critical exponents and amplitude ratios from the same universality class as the liquid-gas phase transition and the three-dimensional (3D) Ising model. Parameters in this construction can be adjusted to best fit the experimental data.

The construction introduced here is different than the construtions proposed in Refs. [8,9], which are based on the work of Refs. [10,11]. These two constructions are both designed to provide a flexible description of matter near the critical point and the associated line of first-order phase transition while also being consistent with lattice QCD at $\mu = 0$. In this sense, they can perhaps be viewed as alternatives which provide some idea as to the range of uncertainty in how to build such a description. However, our construction has several strong points.

(i) Our construction has only two fundamental critical exponents, and the well-known relations $\alpha + 2\beta + \gamma = 2$ and $\beta(\delta - 1) = \gamma$ arise automatically.
(ii) Our construction requires knowledge of only one ratio of critical amplitudes, while the other ratio is predicted and consistent with known experimental observations. In contrast, the approach in Refs. [10,11] has those ratios independent of each other.
(iii) Our construction is directly in terms of the chemical potential and density. The approach of Refs. [8,9] is in terms of the magnetic field and magnetization in the Ising model. The mapping from these quantities to the QCD phase diagram introduces a significant amount of uncertainty and extra parameters with unknown values that our approach avoids.
(iv) In our approach the merging is smooth to all orders, aside from the critical point and its associated line of first-order phase transition. In contrast, Refs. [8,9] matched only to a given order of μ/T in the lattice equation-of-state Taylor expansion by equating

coefficients of the same order. That can introduce unwanted and/or unphysical phase structures.

Without loss of generality the equation of state can be expressed as

$$P(T,\mu) = P_{\rm BG}(T,\mu)R(T,\mu), \tag{1}$$

where $P_{\rm BG}(T,\mu)$ is a judiciously chosen background equation of state with no critical behavior; all critical behavior resides in the dimensionless function R. Motivated by solutions to the cubic equation, which produce S-shaped curves characteristic of the van der Waals equation of state or the nuclear liquid-gas phase transition [12,13], we consider the auxilliary functions

$$Q_\pm(T,\mu) = (\{[\Delta^2(T)]^2 + r^2(T,\mu)\}^{1/2} \pm r(T,\mu))^k, \tag{2}$$

where

$$r(T,\mu) = \frac{\mu^m - \mu_x^m(T)}{\mu^m + \mu_x^m(T)}, \tag{3}$$

with m being a positive even integer. The function $\mu_x(T)$ represents the chemical potential where the two phases are in coexistence when $T \leqslant T_c$, but it must also be a smooth function for all $T \geqslant T_c$ to avoid undesired discontinuities. Note that $-1 \leqslant r < 1$ and that it vanishes along the coexistence curve. The function $\Delta^2(T)$ is expected to have the functional form $|T/T_c - 1|^p$ near T_c. The parameters k and p will determine the four critical exponents.

When $T > T_c$ we take

$$\begin{aligned} R(T,\mu) &= 1 - a(T)(\sqrt{\Delta^4+1}+1)^k \\ &\quad - a(T)\Big(\sqrt{\Delta^4+1}-1\Big)^k + a(T)(Q_+ + Q_-), \end{aligned} \tag{4}$$

where $a(T)$ is a smooth function. This has the property that $P \to P_{\rm BG}$ as $\mu \to 0$ and as $\mu \to \infty$ for any fixed value of T. It is an even function of μ. The density is

$$n(T,\mu) = n_{\rm BG}(T,\mu)R(T,\mu) + P_{\rm BG}(T,\mu)\frac{\partial R(T,\mu)}{\partial \mu}. \tag{5}$$

The critical exponent δ is determined by $P - P_c \sim {\rm sgn}(n - n_c)|n - n_c|^\delta$ as $n \to n_c$ along the critical isotherm $\Delta^2 = 0$. Assuming $1 < k < 2$, the leading behavior is

$$\begin{aligned} n - n_c &= \frac{m^k k a(T_c)}{\mu_c} P_{\rm BG}(T_c,\mu_c){\rm sgn}(\mu-\mu_c)\left|\frac{\mu-\mu_c}{\mu_c}\right|^{k-1}, \\ P - P_c &= [1 - 2^k a(T_c)] n_{\rm BG}(T_c,\mu_c)(\mu-\mu_c). \end{aligned} \tag{6}$$

Thus, the critical exponent $\delta = 1/(k-1)$ or $k = 1 + 1/\delta$.

The baryon number susceptibility is $\chi_B = \partial^2 P/\partial\mu^2 \equiv \chi_{\mu\mu}$. It diverges like $\chi_B = \chi_+(T/T_c - 1)^{-\gamma}$ as $T \to T_c^+$, with $|\mu/\mu_c - 1| \ll T/T_c - 1$. With $\Delta^2(T) = d_+(T/T_c - 1)^p$ near T_c, we find that the susceptibility diverges as

$$\chi_B \to \frac{m^2 k^2 a(T_c)}{2\mu_c^2} d_+^{k-2} P_{\rm BG}(T_c,\mu_c)\left(\frac{T}{T_c} - 1\right)^{-(2-k)p}. \tag{7}$$

Thus, the critical exponent $\gamma = (2-k)p$ or $p = \gamma\delta/(\delta - 1)$.

The heat capacity at fixed volume is

$$c_V = T\frac{\partial s}{\partial T}(T,n) = T\left(\chi_{TT} - \frac{\chi_{T\mu}^2}{\chi_{\mu\mu}}\right). \tag{8}$$

The critical behavior is $c_V \to c_+(T/T_c - 1)^{-\alpha}$. It can be shown, albeit numerically, that approaching the critical point along $r = 0$ gives the same result as approaching it at fixed n_c. The amplitude is

$$c_+ = 2p(2kp - k - p)d_+^k \frac{a(T_c)}{T_c} P_{\rm BG}(T_c,\mu_c), \tag{9}$$

with $\alpha = 2 - kp = 2 - \gamma(\delta+1)/(\delta-1)$.

When $T < T_c$ we take the pressure in the quark phase when $\mu \geqslant \mu_x$ to be

$$P_Q(T,\mu) = P_{\rm BG}(T,\mu)R_Q(T,\mu), \tag{10}$$

with

$$R_Q = 1 + a(T)Q_+(T,\mu) - a(T)(\sqrt{\Delta^4+1}+1)^k, \tag{11}$$

and in the hadron phase when $\mu \leqslant \mu_x$ to be

$$P_H(T,\mu) = P_{\rm BG}(T,\mu)R_H(T,\mu), \tag{12}$$

with

$$R_H = 1 + a(T)Q_-(T,\mu) - a(T)(\sqrt{\Delta^4+1}+1)^k. \tag{13}$$

We refer to these as quark and hadron phases because, even though the background equation of state is a crossover, one phase is predominantly composed of quarks and gluons while the other phase is predominantly composed of hadrons. Note that the pressure along the critical isotherm is

$$P(T_c,\mu) = P_{\rm BG}(T_c,\mu)\left[1 + 2^k a(T_c)\left(\left|\frac{\mu^m - \mu_c^m}{\mu^m + \mu_c^m}\right|^k - 1\right)\right] \tag{14}$$

no matter whether T_c is approached from below or above. Hence the critical exponent δ is well defined.

The density difference along the coexistence curve is

$$\Delta n(T) = \frac{mka(T)}{\mu_x(T)} P_{\rm BG}[T,\mu_x(T)]\big[\Delta^2(T)\big]^{k-1}. \tag{15}$$

The critical exponent β is defined via $\Delta n \sim (1 - T/T_c)^\beta$. With $\Delta^2(T) = d_-(1 - T/T_c)^p$ near T_c, we find that $\beta = p(k-1) = \gamma/(\delta - 1)$.

The susceptibility along the coexistence curve is

$$\begin{aligned} \chi_B(T) &= P_{\rm BG}(T,\mu_x)\left[\frac{m^2k^2a}{4\mu_x^2}\big(\Delta^2\big)^{k-2} \mp \frac{mka}{2\mu_x^2}(\Delta^2)^{k-1}\right] \\ &\quad + \chi_{B,\rm BG}(T,\mu_x)[1 + a(\Delta^2)^k - a(\sqrt{\Delta^4+1}+1)^k] \\ &\quad \pm \frac{mka}{\mu_x} n_{\rm BG}(T,\mu_x)(\Delta^2)^{k-1}. \end{aligned} \tag{16}$$

Recalling that $\gamma = (2-k)p$, we write the critical part as $\chi_B(T) = \chi_-(1 - T/T_c)^{-\gamma}$. When T_c is approached from above at n_c, the critical part is $\chi_B(T) = \chi_+(T/T_c - 1)^{-\gamma}$. From the above equations the ratio of critical amplitudes is

$$\frac{\chi_+}{\chi_-} = 2\left(\frac{d_-}{d_+}\right)^{2-k}. \tag{17}$$

Mathematically the 2 arises because above T_c the sum $Q_+ + Q_-$ enters, whereas below T_c only Q_+ or Q_- does. For the universality class which includes the liquid-gas phase transition and the 3D Ising model, the exact result is $\chi_+/\chi_- \approx 5$, whereas in the mean-field approximation $\chi_+/\chi_- = 2$. For the latter then $d_+ = d_-$.

The last quantity to examine when $T < T_c$ is the heat capacity. The critical behavior is $c_V(T) = c_-(1 - T/T_c)^{-\alpha}$ when the critical point is approached along the coexistence curve. We find that

$$c_- = p(2kp - k - p)d_-^k \frac{a(T_c)}{T_c} P_{\rm BG}(T_c, \mu_c). \tag{18}$$

The critical exponent α is the same above and below T_c as it should be. The ratio of critical amplitudes is

$$\frac{c_+}{c_-} = 2\left(\frac{d_+}{d_-}\right)^k. \tag{19}$$

This model has two independent exponents, k and p, in terms of which the critical exponents α, β, γ, and δ are expressed. They obey the known relations $\alpha + 2\beta + \gamma = 2$ and $\gamma = \beta(\delta - 1)$. The true critical exponents for the universality class that includes the liquid-gas phase transition and the 3D Ising model are $\alpha \approx 0.1101$, $\beta \approx 0.3264$, $\gamma \approx 1.2371$, and $\delta \approx 4.7898$ [14,15], which results in $p \approx 1.564$ and $k \approx 1.209$. In the mean-field approximation $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, and $\delta = 3$, which results in $p = 3/2$ and $k = 4/3$. In mean-field approximation there is a discontinuity in c_V but no divergence. Using the true critical exponents and assuming $d_+/d_- = 1/3$ yields the ratios of critical amplitudes $c_+/c_- = 0.530$ and $\chi_+/\chi_- = 4.769$. Given that the amplitude ratios have an uncertainty of a few percent these numbers are entirely consistent with published results [10,11,16,17].

For the background equation of state we choose the one described in Ref. [18]. It uses a switching function to transition smoothly from a hadron resonance gas, with excluded volume interactions, to a perturbative quark-gluon plasma. The switching function is

$$S(T, \mu) = \exp\left[-\left(T^2/T_s^2 + \mu^2/\mu_s^2\right)^{-2}\right]. \tag{20}$$

It ranges between 0 and 1 as μ and T increase. It is an even function of μ and infinitely differentiable so as not to introduce an artificial phase transition of any order. The parameters T_s and μ_s are adjusted to give a good representation of lattice results for the pressure, interaction measure/trace anomaly, speed of sound, and baryon susceptibility at $\mu = 0$ [19–21], which results in $T_s = 195$ MeV and $\mu_s = 1300$ MeV. Approximate chiral symmetry, due to the small but nonzero quark masses, is automatically included in the lattice calculations and therefore also in the background equation of state.

Only behavior in the immediate vicinity of a critical point is universal. For heavy-ion collisions it is not even clear whether one can probe it closely enough to reveal the true critical exponents or whether mean-field values are more appropriate. Choosing the functions appearing in $R(T, \mu)$ is informed guesswork. Here we choose the function $\mu_x(T)$ to follow a curve of constant density. Other choices are possible, but this one produces an inverted U-shape in the T versus n plane. It is determined implicitly via the relation

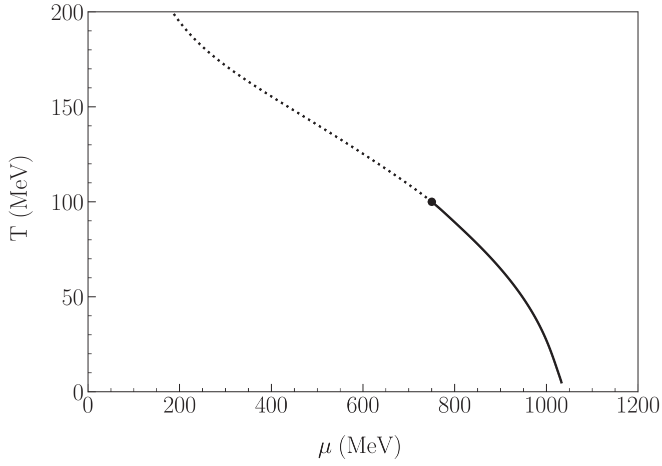

FIG. 1. The choice of critical curve as described in the text. The critical temperature is taken to be 100 MeV and the critical chemical potential to be 750 MeV.

$$R[T, \mu_x(T)]n_{\rm BG}[T, \mu_x(T)] = n_c. \tag{21}$$

This function is displayed in Fig. 1. The critical point lies along this curve. For illustration we choose $T_c = 100$ MeV and $\mu_c = 750$ MeV.

In general, our approach treats T_c, μ_c, and n_c as independent parameters. The requirement of an inverted U-shape curve results in only two of them being independent, as they are related by the formula

$$R(T_c, \mu_c)n_{\rm BG}(T_c, \mu_c) = n_c. \tag{22}$$

As can be seen from Eq. (15), the strength of the transition is directly proportional to the exponent m. We choose $m = 4$ for illustration. The remaining functions are parametrized as

$$\begin{aligned} a(T) &= a_0 \exp(-T/T_a), \\ \Delta^2(T) &= d_+(T/T_c - 1)^p \exp(-T/T_d) \quad \text{when} \quad T \geqslant T_c, \\ \Delta^2(T) &= d_-(1 - T/T_c)^p \exp(-T/T_d) \quad \text{when} \quad T \leqslant T_c. \end{aligned} \tag{23}$$

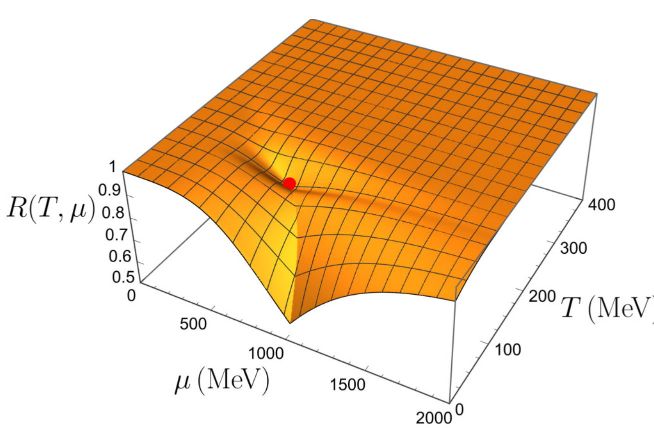

FIG. 2. The dimensionless function R with parameters given in the text. The critical point is indicated by the solid dot.

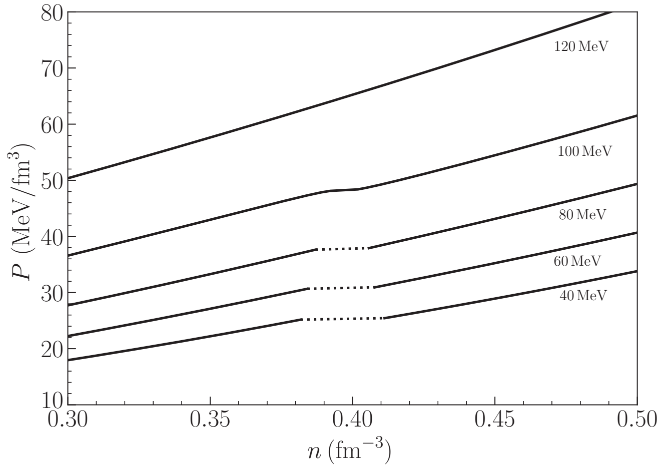

FIG. 3. Isotherms of pressure versus density. The numbers label the temperature with $T_c = 100$ MeV.

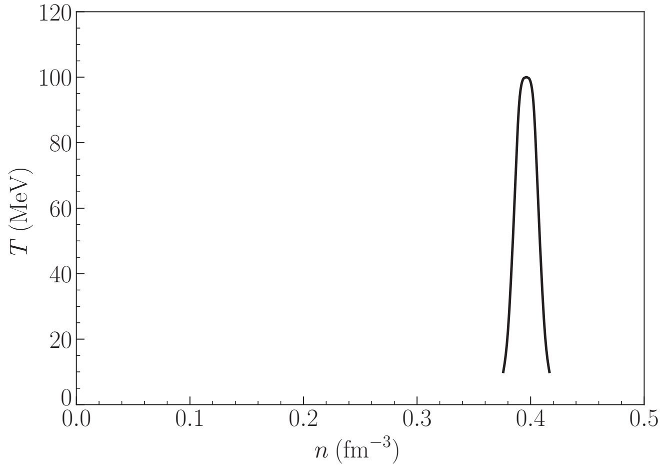

FIG. 4. Temperature versus density separating the two phases.

The parameters T_a and T_d determine how fast $R \to 1$ as T increases beyond T_c. We use the true critical exponents and amplitude ratios with parameters $d_- = 50$, $a_0 = 0.15$, $T_a =$ 80 MeV, and $T_d = 200$ MeV. Figure 2 is a contour plot of the function $R(T, \mu)$. Note that $R \to 1$ as $\mu \to 0$ and for large T and/or μ.

Isotherms of pressure versus density are shown in Fig. 3. Even at T_c there is almost a plateau on account of the large critical exponent $\delta \approx 4.79$ relative to the mean-field value of 3.

Figure 4 shows the phase transition in the temperature versus density plane. It has the shape of an inverted U. Other shapes are possible using different functions $\mu_x(T)$.

In conclusion, we have proposed a novel way to embed critical behavior in background equations of state that exhibit a smooth crossover from hadrons to quarks and gluons. The goal is to use these equations of state in hydrodynamic simulations of heavy-ion collisions in order to infer whether there is critical behavior. The approach has flexibility in selecting the location of the critical point, the coexistence curve, and the reach of these into the background equation of state. This flexibility is an advantage, as it allows parameters to be adjusted to best fit experimental data. Examples were provided; further details and exploration, such as using a background equation of state that includes more realistic attractive and repulsive nuclear interactions at low temperature, will be published elsewhere.

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