Analysis of the SU(3) symmetry versus rotor model

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The SU(3) symmetry, the subgroup in the dynamic chain of the SU(6) group in the interacting boson model (IBM), corresponds to the Bohr-Mottelson rotor model for axially symmetric deformed nuclei. It is synonymous but not identical to it. The common characteristics of the two models and their contrasting features are illustrated. The effect of adding the pairing term in the IBM SU(3) Hamiltonian to bring it closer to the rotor model is highlighted. The modified SU(3) Hamiltonian expression breaks the degeneracy of the β and γ bands in the SU(3) spectrum. The different ways to produce the forbidden $E2$ transitions across the different (λ, μ)-multiplets are discussed. The related features of SL(3, R) symmetry group are illustrated. We point out that the additional pairing term affects only certain states, preserving the rotational features of the other states in the SU(3) spectrum.

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I. INTRODUCTION

The SU(3) symmetry, one of the three dynamic chains of the SU(6) group of the interacting boson model (IBM) [1], corresponds to the axially symmetric Bohr-Mottelson (BM) rotor model [2], but it is not identical to it. The BM unified collective model regards the nucleus as a liquid drop of uniform mass density with a sharp surface, and three semi-axes R_k $(k = 1, 2, 3)$, as a function of quadrupole deformation variable β and asymmetry variable γ, which is capable of rotation and vibration (RV). Besides the rotational ground g band, it allows vibration in the β and γ variables. The level energies follow the rotor formula $E(I) = AI(I+1)$, where A is related to the inverse moment of inertia θ. The interband $E2$ transitions from various spin states are allowed and $B(E2)$ refers to the Alaga rules [2], based on the angular momentum L components and band quantum number K. In the SL(3, R) symmetry (see below) these ratios also depend on the inter-state energy spacing. Small deviations from the Alaga rules are explained in terms of band mixing [2].

The SU(6) group of the IBM has three dynamic chains, SU(5), SU(3) and O(6), involving the O(3) subgroup. For chain-II, viz. SU(6) ⊃ SU(3) ⊃ O(3),

$$E^{II}(N, \lambda, \mu, \chi, L) = E_0 + (3/4\, k - k')L(L+1) - k(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu). \quad (1)$$

where $k\,(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$ corresponds to the SU(3) Casimir operator C(λ,μ). In SU(3) symmetry, the $(\lambda = 2N, \mu = 0)$ multiplet includes only the g band. The $(\lambda = 2N - 4, \mu = 2)$ multiplet includes the γ band and the β band [1]. The two excited bands are predicted to be degenerate. As distinguished from the SU(5) and O(6) symmetries, the states $(0_1^+, 2_1^+, 4_1^+, 6_1^+, \ldots)$ form the rotational g band, with $L(L+1) = I(I+1)$ spacing. The rotational bands are characterized by strong intraband $E2$ transitions, and the SU(3) nuclei have large quadrupole moments, as in the BM rotor model. The $(\lambda = 2N - 4, \mu = 2)$ multiplet lies much higher than the 4_1^+ state of the ground band but is decoupled from it. In the energy difference expression

$$E(2_\gamma) - E(2_g) = 6\beta(2N-1), \quad (2)$$

the coefficient β varies with boson number N [3]. The position of the $(\lambda = 2N-4, \mu = 2)$ multiplet is determined empirically from experiment.

The SU(3) spectra are associated with four conditions:

(a) The energies of the g band exhibit the $I(I+1)$ rotor pattern.
(b) For finite boson number N, the band has a cutoff.
(c) The β and γ bands in the $(\lambda = 2N - 4, \mu = 2)$ multiplet are predicted to be degenerate.
(d) The exact SU(3) symmetry prohibits the γ-g and β-g $E2$ transitions. In the slightly broken symmetry, the $B(E2, 2_\gamma - 0_\beta)/B(E2, 2_\gamma - 0_g)$ ratio is relatively large. ^{156}Gd and ^{170}Er are cited as good examples of SU(3) nuclei [4]. Each of the ideal symmetry applies to only a few nuclei. Most other nuclei lie in the transition classes between them.

A detailed analysis of the two models is presented here in Sec. II to further study these four conditions. Section III deals with interband transitions. The IBM study illustrates the deviations from the exact SU(3) symmetry predictions and the ways to minimize them. In some respects, our analysis differs from earlier works on this subject.

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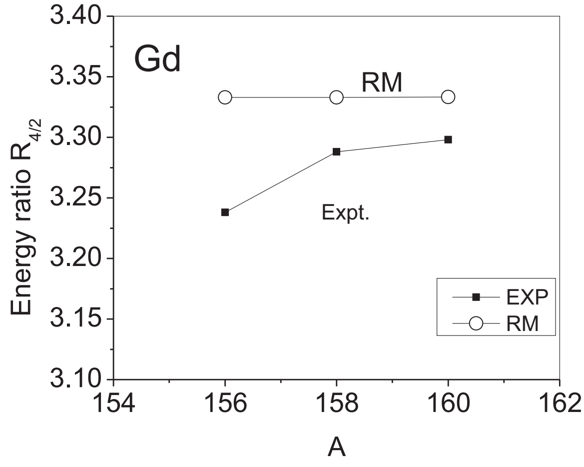

FIG. 1. Energy ratio $R_{4/2}$ in $^{156-160}$Gd isotopes.

II. ANALYSIS OF PREDICTIONS FROM SU(3) SYMMETRY

A. Four conditions

The first condition of the SU(3) symmetry, viz. the formation of the rotational $I(I+1) = L(L+1)$ spectra is well satisfied in the well-deformed nuclei, in the ground band, and also in the two excited bands. In Ref. [5], Gupta illustrated the almost equal moment of inertia of the three bands in the nuclei of the mid-mass region (in the three quadrants) of the major shell, as also predicted in the SU(3) symmetry.

The variation of rotational features with mass number of deformed nuclei is illustrated here for Gd isotopes, in comparison with rotor model predictions.

The energy ratio $R_{4/2} = E(4_1^+)/E(2_1^+)$ equals 3.24 in ^{156}Gd (Fig. 1), a good example of an SU(3) nucleus, lies within 3% of the rotor model (RM) value of 3.33. For ^{158,160}Gd it lies even closer to the 10/3 limit (within 1%).

For spin I^π up to 12^+, the energy ratio $R_{I/2}$ for $^{156-160}$Gd isotopes deviates slightly from the rotor model values (top curve in Fig. 2, crosses). Arima and Iachello [4] illustrated the effect of slight breaking of the exact SU(3) symmetry to account for the small deviations from the $I(I+1)$ limit of the ground band spectrum, as occurs in the rotor model as well. Thus, the SU(3) symmetry predicts the rotation-vibration spectra in good measure.

The second condition of the SU(3) symmetry arises due to the finite boson number N, as distinguished from the infinite number in the rotor model. For $L = 4$ g bosons, in the *sdg* IBM, the band cutoff spin increases.

The third condition, viz. degeneracy of β and γ bands, arises from the built-in K independence of the ($\lambda = 2N - 4$, $\mu = 2$) multiplet. The same degeneracy applies to the higher excited multiplet ($\lambda = 2N-8$, $\mu = 4$) as well. The real nuclei differ from the exact SU(3) symmetry in this respect. This K independence differs from the rotor model, in which K is a valid projection quantum number, which represents the projection on the intrinsic axis and which is different for the two bands [2].

The degeneracy in SU(3) arises because of the missing quantum number K, which distinguishes the degenerate β and γ bands. In the Elliot basis of SU(3) (which is not orthogonal [1]), the K quantum number was included. But in the orthogonal basis of the Vergados representation used in the *sd* IBM [1], χ is the extra quantum number used, which enables the determination of the L quantum numbers of the β and γ bands. This problem is related to the subgroup reduction problem from SU(3) to SO(3) [1]. While $\chi = 0$ allows one L, $\chi = 2$ allows two L (0 and 2), corresponding to the β and γ bands.

Arima and Iachello [4] noted that the eigenvalues of the SU(3) Hamiltonian do not depend on K and M but depend only on angular quantum number L. While in the BM model expression

$$E(n_\beta, n_\gamma,\ K, L, M) = \alpha L(L+1) + \beta n_\beta + \gamma n_\gamma, \quad (3)$$

the values of the coefficients α, β, and γ are arbitrary, in SU(3), the coefficients β and γ are equal on account of the K independence of the SU(3) Casimir operator $C(\lambda, \mu)$ [see Eq. (1)] in the same L state.

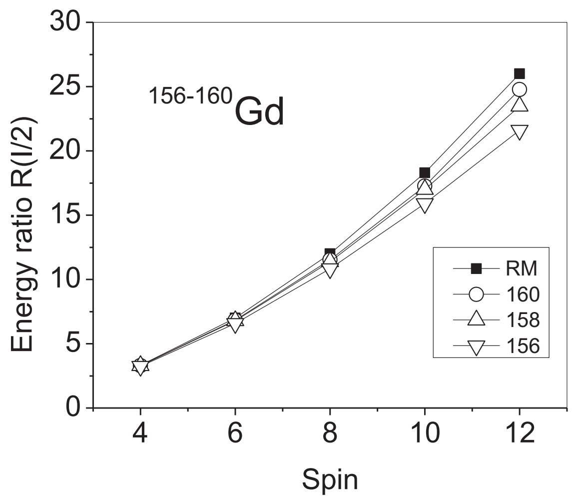

FIG. 2. Energy ratio $R_{I/2}$ for $^{156-160}$Gd isotopes.

B. Application of IBM-1 and use of PAIR term

In the multipole representation of the phenomenological IBM-1 calculation [1], one often employs the four-term Hamiltonian

$$H_{IBM} = \varepsilon\, n_d + kQ \cdot Q + k'L \cdot L + k''P \cdot P. \quad (4)$$

The first term is the boson-energy term, second term is the quadrupole interaction, third term is angular momentum, and the fourth term is the pairing interaction. In the exact SU(3) limit, only the $Q \cdot Q$ and $L \cdot L$ terms are required. The PHINT Program of Scholten [6] solves Eq. (4). The coefficients of the four terms in Eq. (4) are evaluated by a fit to the experimental spectrum.

As an illustration, we applied the IBM-1 program for $^{154-160}$Gd isotopes. See Table I for the parameters. Scholten *et al.* (1978) [7] retained the εn_d term and varied it with boson number N to reproduce the varying spectra (with mass number A) of $^{148-156}$Sm isotopes. Here we are dealing only with

TABLE I. IBM-1 Hamiltonian parameters (keV) used for Gd isotopes. Energies are in keV and $B(E2)$ are in e^2b^2, $\chi = (-)\sqrt{7}/2$.

Item	^{160}Gd	^{158}Gd	^{156}Gd	^{154}Gd
EPS = ε	0.0	0.0	0.0	349.5
QQ = $2k$	−22.9	−29.6	−31.1	−26.7
ELL = $2k'$	17.2	14.9	16.3	12.7
PAIR = $k''/2$	17.1	3.9	−1.7	1.9
$E(2_\beta) - E(2_\gamma)$				
EXP	389	73.7	−26	−180
IBM	407	90	−36	−150
$B(E2, 2_1^+ - 0_1^+)$	1.05	0.89	0.756	0.765
EXP	1.04(3)	1.02(2)	0.94(3)	0.77(1)
$B(E2.2_\gamma - 0_1^+)$	0.023	0.019	0.016	0.027
EXP	0.020(1)	0.017(2)	0.024(1)	0.028(2)
$B(E2, 2_\beta - 0_1^+)$	0.0009	0.0022	0.0027	0.0005
EXP		0.0012 (2)	0.0032 (3)	0.0045(5)
$B(E2.2_\gamma - 2_1^+)$	0.035	0.029	0.026	0.043
EXP	0.020 (1)	0.029 (4)	0.025 (2)	0.061(5)
$B(E2, 2_\beta - 2_1^+)$	0.0014	0.0035	0.0045	0.0127
$B(E2, 2_\beta - 0_2^+)$	0.81	0.67	0.56	0.51
Ratio $(2_\gamma - 0_1^+/2_1^+)$	0.65	0.65	0.62	0.63
EXP	0.60(2)	0.59(6)	0.67(3)	0.46(1)
$Q(2_1^+)$ eb	−2.07	−1.91	−1.75‘	−1.76
$Q(2_2^+)$ eb	+1.92	+1.75	−1.51	−1.25
$Q(2_3^+)$ eb	−1.81	−1.66	+1.60	+1.35

well-deformed rotor SU(3) nuclei, so we set the boson-energy term εn_d to zero (except for $N = 90$ ^{154}Gd).

The quadrupole operator in IBM-1 is expressed as

$$Q^{(2)} = \alpha(d^+s + s^+d) - \beta(d^+d)^{(2)}, \tag{5}$$

and the $E2$ transition operator $T(E2) = e_b\, Q^{(2)}$, for $e_b = \alpha$, and $\beta/\alpha = \chi$.

For the IBM Hamiltonian, the $|\chi|$ coefficient ($= \beta/\alpha$) is taken equal to $(-)\sqrt{7}/2$ in the exact SU(3) limit. In the slightly broken SU(3) symmetry, or the vibrational or shape transitional nuclei, it may be reduced. The value of χ is set to zero for the O(6) nuclei. The coefficient k in the $k\, Q{\cdot}Q$ term in Eq. (4) determines the position of the excited bands. The $L{\cdot}L$ term affects the band spread with spin L.

The pairing term is small for ^{156}Gd (Table I), with almost degenerate β and γ bands. It is larger for ^{160}Gd, with larger $\Delta E = [E(2_\beta) - E(2_\gamma)]$ (see rows 4 and 5 of Table I). It illustrates the role of the pairing term for the splitting of the K degeneracy in the SU(3) symmetry, for the well-deformed nuclei.

The increased boson number N in Gd isotopes leads to the increased split of the degeneracy of the β and γ bands. The $B(E2, 2_\beta - 0_1^+)$ is increasingly small in the more-deformed isotopes. The pairing term does not affect the rotational structure of the β band, and of the ground band and γ band. This is an interesting aspect of the use of the pairing term in the SU(3) IBM. Further, the features of the two bands are also illustrated below.

The same features, viz. E, $B(E2)$, and Q, M, occur in the BM model, as illustrated in Table II.

TABLE II. DPPQ model Hamiltonian parameters (keV) used for Gd isotopes. Energies (theory) are in keV and $B(E2)$ are in e^2b^2,

Item	160	158	156	154
$E(2_1^+)$	76.8	76.5	87.2	126
$E(2_2^+)$	1.491	1.429	1.420	1.179
$E(0_2^+)$	1.596	1.383	1.235	0.984
$E(2_3^+)$	1.717	1.583	1.525	1.505
Quad param	66.0	66.0	68.0	70.0
$B(E2, 2_1^+ - 0_1^+)$	1.04	0.92	0.868	0.772
$B(E2, 2_\gamma - 0_1^+)$	0.0222	0.0197	0.0295	0.0278
$B(E2, 2_\beta - 0_1^+)$	0.0103	0.0141	0.0039	0.0037
$B(E2, 2_\gamma - 2_1)$	0,0456	0.0598	0.0432	0.0494
$B(E2, 2_\beta - 2_1)$	0.0135	0.0112	0.0133	0.0330
$B(E2, 2_\beta - 0_2)$	0.99	0.74	0.82	0.76
$B(E2, 2_\gamma - 0_1^+/2_1^+)$	0.48	0.33	0.68	0.56
$B(E2, 0_2^+ - 2_1^+)$	0.051	0.084	0.113	0.219
$Q(2_1^+)$	−2.05	−1.93	−1.88	−1.79
$Q(2_2^+)$	1.97	1.81	−1.76	−1.93
$Q(2_3^+)$	−2.015	−1.73	1.626	1.57

C. Necessity of pairing term

In the collective model, the pairing interaction promotes sphericity, and for the axially symmetric deformed nucleus, it induces the axially symmetric β vibration. For the moment of inertia (MoI), the θ expression in the shell-model cranking approach [8] is

$$\theta = \hbar^2 \sum_n [\langle n|l|0\rangle]^2 \big/ (E_n - E_0), \tag{6}$$

and one needs to incorporate the pairing interaction to obtain the realistic value of the moment of inertia θ for a deformed rotor, as illustrated by Beleav [9].

In the microscopic theories based on the mean-field approximation, besides the one-body interaction term, for the residual two-body interaction, one adds the quadrupole interaction, as well as the pairing interaction terms. The quadrupole interaction produces the deformation of the nucleus. In the application of the pairing plus quadrupole interaction, Kumar and Baranger [10,11] treated the two residual interactions on equal footing. Kumar [11] pointed out that, in the absence of pairing, the quadrupole force will make all nuclei deformed, near or far from the closed shell.

The inclusion of the $P{\cdot}P$ term in Eq. (4) affects primarily the β band in the ($\lambda = 2N{-}4$, $\mu = 2$) representation, effecting a split of the degeneracy from the γ band. It also removes some of the deviations from the rotor model. In the microscopic approach, pairing leads to the concept of quasiparticles. In the IBM, the P operator ($= 1/2\{dd{-}ss\}$) [1] changes the s boson−d boson level separation.

Unlike the inclusion of g bosons or the f and p bosons to the sd boson IBM, which lead to supersymmetries for odd-A and odd-odd nuclei, the inclusion of the pairing term selectively modifies the character of some rotational excited states in the SU(3) spectrum. Here it is an additional essential term, which preserves the rotational character of the three bands.

Besides the degeneracy of β and γ bands, the quadrupole moment $Q(2^+)$ differs in sign for the two excited bands. Also, their decay characteristics differ. In the above-cited example of Gd isotopes, we get different $Q(2^+)$ for the two bands, their signs being different (Table I). Here we set the χ value slightly smaller for the $Q^{(2)}$ operator for the FBEM part of the PHINT package [6]. If $|\chi|$ is increased to $\sqrt{7}/2$, the signs of $Q(2^+)$ still differ for the 2_γ and 2_β states in the IBM-1 calculation. The same variation of $Q(2^+)$ is observed in the BM model (Table II).

III. INTERBAND $E2$ TRANSITIONS

A. Fourth condition

The perturbation to SU(3) symmetry may be done in two ways:

(a) By modifying $\beta/\alpha = \chi$ for the quadrupole operator. In the perturbed SU(3) symmetry, the coefficient $|\chi|$ may be reduced slightly.
(b) The second way is to add other terms to the SU(3) Hamiltonian, which may be the $\varepsilon n_{\rm d}$ term of SU(5) symmetry as used in Ref. [7], or/and the pairing term of O(5), as suggested here.

For the Hamiltonian, we set boson energy term $\varepsilon n_{\rm d} = 0$. The $|\chi|$ is kept equal to $\sqrt{7}/2 = 1.322$ for energies. For the $B(E2)$ values, we use the adjustable parameter χ. Normally we kept $\beta/\alpha = 12/14 = 0.96$ to get the $B(E2)$ values, the same for the four isotopes. For ^{160}Gd, it yields $B(E2, 2_\gamma - 0_1^+) = 0.023\ e^2b^2$.

The $B(E2, 2_\beta - 0_1^+)/B(E2, 2_\gamma - 0_1^+)$ ratio varies from 3/16 to 1/23 in $^{156-160}$Gd in the IBM (Table I) compared with the prediction of $\approx 1/6$ in the slightly broken SU(3) limit [4], in general. In a review of the validity of the β band representing the axially symmetric vibration, generally (and not in exceptional cases only), Gupta and Hamilton [12] demonstrated that the intraband $E2$ transition $(2_\beta - 0_\beta)$ is as strong as $B(E2, 2_g - 0_g)$ (see Table I in Ref. [12]). The same is also illustrated in Table II here. In the IBM also for the Gd isotopes $B(E2, 2_\beta - 0_\beta)$ is nearly as strong as $B(E2, 2_g - 0_g)$ (Table I). It implies that the intraband features in SU(3) are similar to the rotor model. The pairing term preserves it.

B. Role of the wave function

Gupta, Kumar, and Hamilton (1977) [13] pointed out the important contribution of the state wave functions. Here we illustrate it for ^{156}Gd (with near-degenerate β and γ bands), the wave function for the 2^+ states of the three bands (taken from Ref. [13]) (Figs. 3, 4, 5) obtained by the dynamic pairing plus quadrupole model [10,11]. From these plots, it is apparent that the overlap of the β wave function (which has a node in the middle) with the ground-state wave function is the least, which explains the smallness of $B(E2, 0_g - 2_\beta)$ compared with $B(E2, 0_g - 2_\gamma)$. See Table II for BM predictions and Table I for the SU(3) IBM predictions.

In the IBM, the distribution of the $n_{\rm d}$ components in the IBM-SU(3) wave functions are also obtained, which enables

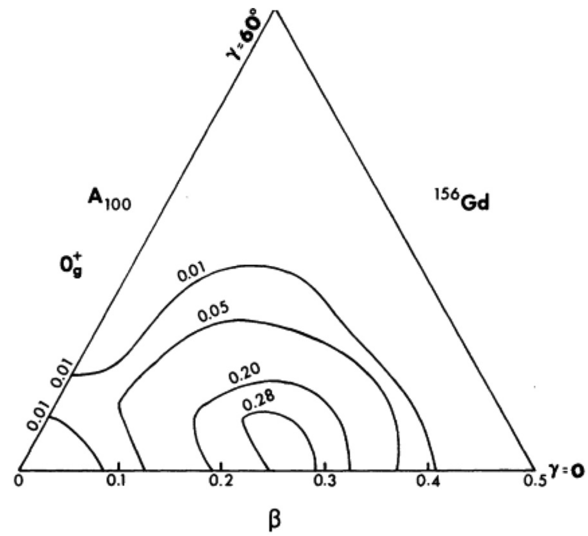

FIG. 3. Wave function of ground state in ^{156}Gd from DPPQ model.

the distinction between the two excited bands. This explains the different $E2$ transition rates in the SU(3) spectrum as well.

C. Role of softness parameter

Here we note the role of the softness parameter, which may correspond to the σ parameter of the soft rotor formula (SRF) [14],

$$E(I) = AI(I+1)/(1+\sigma I). \tag{7}$$

From the plot of band-head energy $E(0_2^+)$ versus σ (Fig. 6), it is apparent that smaller the σ, higher lies the 0_2^+ band. The smaller σ signifies the β hardness of the nucleus, raising the band-head energy. It indicates a dichotomy whereby the larger deformation corresponds to the larger β hardness (saturation). This feature of a nucleus is missing in the SU(3) symmetry. The pairing term fulfills this deficiency. This agrees with the illustration in the above for Gd isotopes. Thus, the separation here of the β band from the γ band for $N \geqslant 92$ occurs naturally in the nucleus.

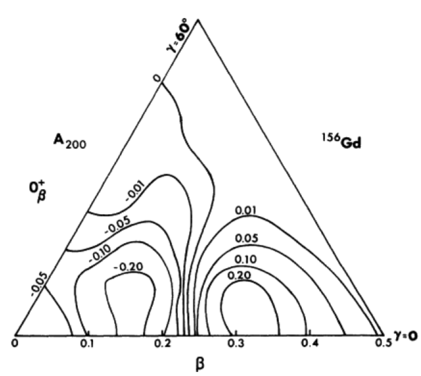

FIG. 4. Wave function of 0_β state in ^{156}Gd from DPPQ model.

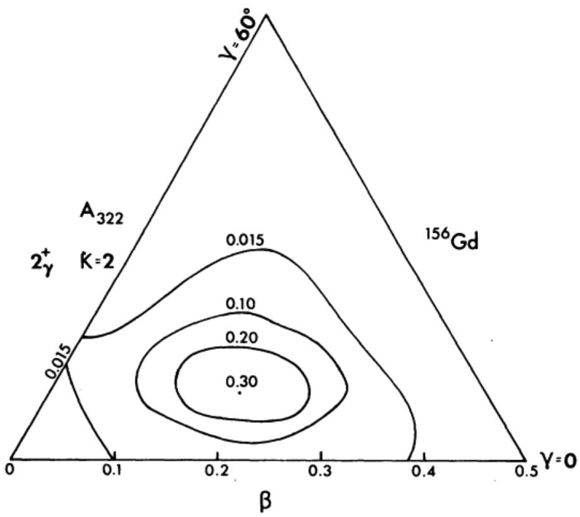

FIG. 5. Wave function of 2_γ state in ^{156}Gd from DPPQ model.

D. SL(3,R) symmetry

Reflecting an improvement over the Alaga rules for the γ-g $E2$ transition ratios in the rotor model, Weaver and Biedenharn [15] proposed the SL(3,R) symmetry group.

It is a group of symmetry transformations based on the following assumptions:

(i) The electric quadrupole moment operator Q_{el} is proportional to the mass quadrupole moment operator Q.
(ii) The time derivatives of the components of Q generate the SL(3,R) group, the group of real (3 × 3) matrices, with unit determinant, with the SO(3) subgroup.
(iii) The excited states of a nucleus form the basis for a single unitary representation of SL(3, R).

It represents a noncompact special linear symmetry group, in contrast with the compact SU(3) symmetry group. The SL(3, R) symmetry was first suggested by Gell-Mann, Ne'eman, and Dothan. The generators of the group are the five components of the quadrupole operator and the three components of the angular moment operator, just as for the SU(3) group. The first derivatives of mass quadrupole moment Q_{ij} are the transition operators T_{ij}.

They applied it to the limited problem of the interband $E2$ transition ratios from the $K^\pi = 2^+$ γ band to the ground band in the well-deformed rotational nuclei.

For $\Delta K = 2$ $E2$ transitions,

$$\langle L'\,K-2||T|||L\;K\rangle = (-)^{L'-K}[(2\,L+1)(2L'+1)]^{1/2} \times[-i(1-K)+\delta] \times\begin{pmatrix} 2 & L & L' \\ -2 & K & -K+2\end{pmatrix}, \tag{8}$$

which gives

$$B(E2,\;L\,K\to L'K-2) = \{\text{constant}/(E_\gamma)^2\}[(1-K)^2-\delta^2] \times|\langle 2L-2K|L'\,K-2\rangle|^2. \tag{9}$$

The model includes the K quantum number exclusively in its expressions, unlike the SU(3) symmetry of IBM, besides $E_{\gamma 1}$ and $E_{\gamma 2}$. It yields the $B(E2)$ ratio

$$\text{Ratio } R = B(E2)\,(LK\to L_1\;K-2)/(LK-L_2K-2) = (E_{\gamma 2}/E_{\gamma 1})^2\times F, \tag{10}$$

where

$$F = [\langle 2L-2\,K|L_1\,K-2\rangle/\langle 2L-2K|L_2K-2\rangle]^2. \tag{11}$$

In a symposium presentation in 1980 [16] Jain and Gupta illustrated the application of the SL(3, R) model for the γ-g $E2$ transition ratios in a few rare-earth nuclei and noted that the model predictions represent an improvement on the Alaga rules which include the dependence on L and K of the two levels. The SL(3, R) ratio also includes the energies E_γ, between the two levels involved in the transition. The inclusion of the E_γ, improves the calculated value of the ratio, bringing it closer to the experimental value.

As state above, the model is based purely on the rotor model, with no connection to the shell model as in the IBM SU(3) symmetry. In the latter, the γ-g energy difference is taken into account through the experimental data and the Z, N difference through the dependence on the total boson number $N_{\text{B}} = N_{\text{p}} + N_{\text{n}}$.

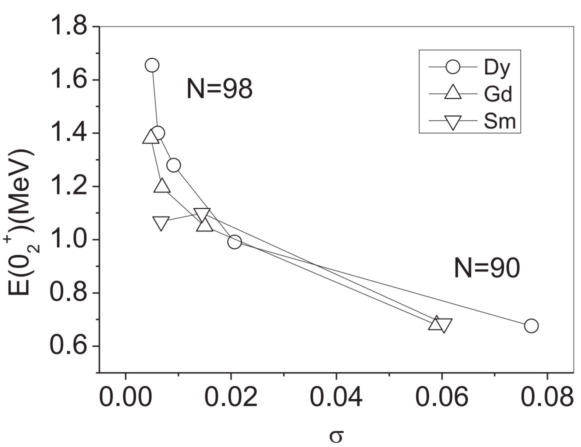

FIG. 6. Level energy of the 0_2^+ state in Dy, Gd, and Sm isotopes versus softness parameter sigma σ.

E. Role of triaxiality

In the rigid triaxial rotor (RTR) model of Davydov and Filippov [17], the γ-g transitions are allowed transitions. The $B(E2)$ transitions are a function of the asymmetry parameter γ. In a recent study [18] of its application to the deformed rotors, it is shown that, for γ less than 20°, the absolute $B(E2, 2_\gamma - 0_1^+)$ value increases with increasing γ (starting from zero, for γ = zero). This is an interesting result, vis-à-vis the SU(3) symmetry in the IBM. In the application of IBM-SU(3), one takes level energies from experiment, which affects the coefficient of the quadrupole term in the IBM Hamiltonian and affects the (γ-g) $E2$ transition in the IBM.

IV. DISCUSSION AND SUMMARY

In Sec. I, we cite the important features of the IBM-SU(3) symmetry and of the BM rotor model. In SU(3) symmetry, the position of the excited band depends on boson number N and the β, γ bands are predicted to be degenerate. Interband γ-g, β-g $E2$ transitions are prohibited in SU(3) symmetry. This differs from the rotor model and real nuclei.

In Sec. II, we illustrate the increased validity of the $I(I+1)$ energy formula for ground band of Gd isotopes with increasing mass number, which are closer to SU(3), symmetry. The β, γ band degeneracy is predicted in SU(3) symmetry on account of K independence, which is related to the reduction from SU(3) to the SO(3) subgroup. In Sec. II B the application of IBM-SU(3) + the PAIR term to the Gd isotopes is described. Table I cites the IBM predictions. Table II cites the results from BM model based DPPQ model calculation for the Gd isotopes. Details of the DPPQ model are available in Refs. [19,20].

In both cases, the main features of the Gd spectra are illustrated, including the increase in PAIR term leading to larger split of β band from the γ band. The effect on the interband transitions is cited for the two excited bands. The signs of the quadrupole moment of the two $I = 2$ states in $^{154-160}$Gd are presented. Comparison with the BM model–based calculation given in Table II displays similar features for the four isotopes of Gd. The available experimental data are included.

In Sec. II C the necessity of including the PAIR term in the SU(3) Hamiltonian expression (4) and of the pairing interaction in the BM model is discussed. Warner *et al.* [21] illustrated the application of IBM-SU (3) to the well-deformed nucleus of ^{168}Er.

Later, in Ref. [22], the relation of wave functions, calculated in the SU(5) basis, was studied with the inclusion of the pairing term for boson number equal to 16 nuclei. Bohr and Mottelson [23] illustrated the use of the pairing term in their Eq. (3) for reproducing the spectra of ^{168}Er. We illustrate that, even for the well-deformed symmetric nucleus ^{160}Gd (see Table I) one also needs the pairing term to reproduce the lowest three bands along with their distinguishing features.

Section III deals with the interband $E2$ transitions. We explain that the εn_d term is required for the shape transition for varying Z or N. The reduction of $|\chi|$ in the quadrupole operator leads to relaxation of prohibited $E2$ transitions. Both of these alternatives affect the SU(3) rotational symmetry for the whole nucleus. The PAIR term preserves the rotational symmetry and affects selectively the β band. Section III B cites the role of wave functions difference of the three bands. Section III C illustrates the role of the reduction in the β softness of the nucleus for increasing deformation in raising the β band head, cited for Sm, Gd, and Dy isotopes. Section III D cites the simpler SL(3) symmetry proposed long ago.

In the SL(3,R) symmetry, the separation of bands is taken into account by including the $(E_\gamma)^2$ factor. In Sec. III E we cite the role of the asymmetry parameter γ related to the γ-g band separation and affecting (γ-g) $E2$ transitions. The same effect is achieved in the IBM by fitting the band-head energies.

In the above, the emphasis is on the stated unique features of the SU(3) symmetry Hamiltonian. Our work should be helpful for further study of these aspects of SU(3) symmetry in the IBM.

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