Superfluid density in disordered pasta phases in neutron star crusts

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In the inner crust of neutron stars one expects phases in which nuclei adopt rodlike and platelike forms, so-called pasta phases. For ordered phases, the superfluid density of nucleons is anisotropic and in this paper we calculate the effective superfluid density of disordered pasta phases. We use an effective medium approach which parallels that previously used for calculating the electrical conductivity of terrestrial matter. We allow for the effect of entrainment, the fact that the current density of one species of nucleon depends on the gradient of the phase of the condensate pair wave function not only of the same species but also of the other species. We find that for protons, the results of the effective medium formalism can be quite different from those of simple approximations.

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I. INTRODUCTION

The superfluid density of neutrons in the crust of the star is an important property in models of glitches in neutron star rotation rates, and the corresponding quantity for protons is an important ingredient in calculations of magnetic properties. At densities of the order of one half of nuclear saturation density it is predicted that phases with nonspherical nuclei can occur in the crust of neutron stars [1,2]. Whether or not such phases have lower energy than a uniform mixture of neutrons, protons, and electrons is still unclear, since the result depends on the nuclear interaction employed [1,3]. Further work with improved calculational methods and nuclear interactions is needed to determine whether these phases are thermodynamically stable.

In the crust at densities above that for neutron drip, neutrons and protons are predicted to be superfluid. In the phase with round nuclei, flow of neutrons can occur over large distances because of the neutrons outside nuclei. The protons, however, cannot participate in bulk flows because individual nuclei are well separated in space, and there is negligible proton tunneling between nuclei. The phases with spherical nuclei are predicted to have cubic symmetry, and consequently the neutron superfluid density tensor is isotropic. For the pasta phases, which have nonspherical nuclei, superfluid flow of protons is possible because spaghetti strands and lasagna sheets are extended. For perfect pasta phases with long-range spatial order, the superfluid density tensors for both neutrons and protons are anisotropic, with different values for directions parallel to and perpendicular to the symmetry axis of the pasta, the direction of the spaghetti strands and the normal to the lasagna sheets.

We turn now to disordered phases. On length scales large compared with the lattice spacing and the length scale on which the orientation of the crystal axes changes, the effective superfluid density tensor is isotropic if the orientation of the axes is random. The phase with spherical nuclei has cubic symmetry. For this case, the superfluid density tensor is isotropic and, to the extent that effects due to the boundary between regions with different crystal orientations are negligible, the effective superfluid density tensor is equal to that in the perfectly ordered medium.

In this paper we calculate the effective superfluid density for disordered lasagna and spaghetti phases. We adopt an effective medium approach inspired by earlier work on the effective electrical conductivity [4–7]. Such methods have also been applied to calculate elastic properties of polycrystals [8], including those in neutron star crusts [9,10]. The basic idea in these methods may be described as follows. One considers an inclusion with anisotropic properties embedded in a homogeneous, isotropic medium and then demands that, on averaging over possible orientations of the inclusion, the response of the system is the same as that of the medium outside the inclusion. Another way of expressing this in terms of scattering is that the properties of the medium are chosen so that, on average, the inclusion gives no scattering of an incident disturbance.

This article is organized as follows. Section II sets out the basic formalism, and a number of details are described in the Appendix. Applications are given in Sec. III, which begins with the case of a single component and then goes on to the two-component case with entrainment. In Sec. IV we discuss applications of our results to the spaghetti and lasagna phases. For protons, the superfluid density is expected to be very anisotropic, while for neutrons it is less so. Consequently, we predict that the effective superfluid density of protons will differ considerably from the results of simple approximations but that for neutrons the difference between the various approximations will be much less. We also consider the effects of magnetic fields on the protons. Concluding remarks are made in Sec. V.

II. BASIC FORMALISM

Consider a perfectly ordered system with a spatially periodic structure and made up of neutrons and protons (which are both superfluid), and normal electrons. In a generalization of the two-fluid model for liquid helium II we may write the current density of nucleons for long-wavelength, low-frequency phenomena in the form[1]

$$j^i_\alpha = \frac{n^{\mathrm{s},ij}_{\alpha\beta}}{m}\left(\frac{\partial\phi_\beta}{\partial x^j} - \frac{q_\alpha A^j}{c}\right) + n^{\mathrm{n},ij}_\alpha u^j. \tag{1}$$

Here the Greek subscripts α and β refer to the nucleon species (n for neutrons and p for protons), the indices i and j refer to Cartesian coordinates, and ϕ_α is equal to one half of the phase of the condensate pair wave function for species α averaged over distances large compared with the lattice spacing and other microscopic length scales but small compared with the characteristic length scale of the phenomenon in question. The components of the normal fluid velocity, which is the velocity of the periodic structure of the medium, are denoted by u^j, and those of the vector potential by A^j, and q_α is the charge of species α. The quantity $n^{\mathrm{s},ij}_{\alpha\beta}$ is the superfluid density tensor, and it is in general not diagonal in the species variables because of entrainment of the two superfluids, the fact that a gradient in the phase of one component can give rise to a flow of the other component. The normal density tensor is denoted by $n^{\mathrm{n},ij}_\alpha$. We neglect the difference between the neutron and proton masses, and denote the nucleon mass by m. We shall work in units in which $\hbar = 1$. We shall assume that the electrons are at rest in the frame moving with the normal velocity, but they will not enter in our subsequent considerations.

In most of the paper we shall neglect the effects of magnetic fields, so we shall take the vector potential to be zero. For small $\partial\phi_\beta/\partial x^j$ and small u^j, the dependence of the superfluid and normal density tensors on these variables may be neglected. However, in Sec. IV, we shall comment on the effect of magnetic fields on the protons.

The ith component of the current density of species α is the derivative of the energy density with respect to $(\partial\phi_\alpha/\partial x^i)/m$. Thus from Eq. (1) it follows that

$$\frac{n^{\mathrm{s},ij}_{\alpha\beta}}{m} = \frac{\partial^2 E}{(\partial\phi_\alpha/\partial x^i)(\partial\phi_\beta/\partial x^j)}. \tag{2}$$

[1]We shall use superscripts to denote spatial components and subscripts to denote species. We do not need to take into account effects of the curvature of space or scale transformations so we denote both covariant and contravariant quantities by upper indices. The superscript s denotes the superfluid density and the superscript n the normal density.

Therefore for $\alpha = \beta$, the superfluid density tensor is symmetric in the spatial coordinates i and j. However, for $\alpha \neq \beta$ it is not necessarily symmetric and consequently the principal axes are not orthogonal in general. We shall confine our attention to systems in which the principal axes of the medium are, or may be chosen to be, orthogonal. This applies to all crystal systems apart from the triclinic and monoclinic ones and is thus general enough to cover the cases of interest in neutron star crusts. For the lasagna phase with uniform sheets, the principal axes may be taken to be the normal to the sheets and two orthogonal axes lying in the plane of the sheets. For the spaghetti phase, one expects the rods to be arranged in a hexagonal pattern, and thus the superfluid and normal density tensors are isotropic in the plane perpendicular to the strands. This is a general property of second rank tensors and was demonstrated explicitly for the superfluid density tensor in Ref. [11]. Sheets in the lasagna phase may be spatially modulated in the plane of the sheets but, according to quantum molecular dynamics calculations [12], the modulation has hexagonal symmetry, and therefore also this case falls into this class. It is simplest to work in terms of components of vectors along the principal axes of the medium, in which case we may write Eq. (1) in the form

$$j^\lambda_\alpha = \frac{n^{\mathrm{s}\lambda}_{\alpha\beta}}{m}\frac{\partial\phi_\beta}{\partial x^\lambda} + n^{\mathrm{n}\lambda}_\alpha u^\lambda, \tag{3}$$

where λ denotes the principal axes and there is no sum over λ on the right side of the equation.

In a system consisting of randomly oriented domains, one expects the spatial average of the current density to be proportional to the spatial average of the gradient of the phases and to be in the same direction, and thus, omitting the magnetic field term, one may write

$$\langle j^i_\alpha\rangle = \frac{n^{\mathrm{s,e}}_{\alpha\beta}}{m}\left\langle\frac{\partial\phi_\beta}{\partial x^i}\right\rangle + n^{\mathrm{n,e}}_\alpha u^i, \tag{4}$$

where the angular brackets denote a spatial average, which we shall assume to be equivalent to an ensemble average. The objective of this article is to calculate the spatially isotropic quantity $n^{\mathrm{s,e}}_{\alpha\beta}$, the effective superfluid density tensor for the disordered medium. The effective normal density is denoted by $n^{\mathrm{n,e}}_\alpha$. Under a Galilean transformation to a reference frame moving at a velocity $-\vec{v}$ with respect the original frame, $\vec{\nabla}\phi_\beta$ changes by $m\vec{v}$ and the total current density of species α changes by $n_\alpha\vec{v}$, where n_α is the density of the species. It therefore follows from Eq. (25) that

$$\sum_\beta n^{\mathrm{s,e}}_{\alpha\beta} + n^{\mathrm{n,e}}_\alpha = n_\alpha. \tag{5}$$

As we demonstrate in the Appendix [see Eqs. (A11) and (A13)], the effective superfluid density in the effective medium approach is given by solving the matrix equation

$$\sum_{\lambda=1,2,3} 3n^{\mathrm{s,e}}(2n^{\mathrm{s,e}} + n^{\mathrm{s}\lambda})^{-1}(n^{\mathrm{s}\lambda} - n^{\mathrm{s,e}})$$
$$\equiv \sum_{\lambda=1,2,3}[\mathscr{I} + (n^{\mathrm{s}\lambda} - n^{\mathrm{s,e}})(3n^{\mathrm{s,e}})^{-1}]^{-1}(n^{\mathrm{s}\lambda} - n^{\mathrm{s,e}}) = 0, \tag{6}$$

where the symbols n with no subscripts denote 2×2 matrices in the space of nucleon species (neutrons and protons) and $\mathscr{I}$ is the unit matrix. The second form in Eq. (6) exhibits explicitly the fact that the quantity is symmetric in the species variables. It also has the form to be expected from a multiple scattering or screening problem, with the bare scattering "potential" being proportional to $n^{s\lambda} - n^{s,e}$ [6].

III. APPLICATIONS

We now apply our result to uniaxial systems, which have one principal axis along the axis of the system, for which quantities are denoted by $\|$, and two principal axes perpendicular to the axis of the system, for which quantities are denoted by $\perp$. Equation (6) then reduces to

$$n^{s,e}(2n^{s,e} + n^{s\|})^{-1}(n^{s,e} - n^{s\|}) + 2n^{s,e}(2n^{s,e} + n^{s\perp})^{-1}(n^{s,e} - n^{s\perp}) = 0. \quad (7)$$

A. No entrainment

For uniform matter, the entrainment parameter n^s_{np} is not known very well but estimates indicate that it is less than $0.1n_p$ [13], and we expect that the entrainment parameters in the pasta phases will be correspondingly small. Consequently, in calculating the pp and nn components of the effective superfluid density tensor, it is a good first approximation to neglect entrainment. In that case, Eq. (7) reduces to the following two equations:

$$\frac{n^{s,e}_{nn} - n^{s\|}_{nn}}{2n^{s,e}_{nn} + n^{s\|}_{nn}} + 2\frac{n^{s,e}_{nn} - n^{s\perp}_{nn}}{2n^{s,e}_{nn} + n^{s\perp}_{nn}} = 0, \quad (8)$$

and

$$\frac{n^{s,e}_{pp} - n^{s\|}_{pp}}{2n^{s,e}_{pp} + n^{s\|}_{pp}} + 2\frac{n^{s,e}_{pp} - n^{s\perp}_{pp}}{2n^{s,e}_{pp} + n^{s\perp}_{pp}} = 0. \quad (9)$$

These equations have the same form as that for the electrical conductivity of polycrystals and binary metallic mixtures [4–6], the essential reason being that the electrostatic potential and the phase of the condensate wave function both satisfy the Laplace equation, except at the surface of the inclusion. Equations (8) and (9) may be written as

$$2n^{s,e\,2}_{\alpha\alpha} - n^{s,e}_{\alpha\alpha}n^{s\perp}_{\alpha\alpha} - n^{s\|}_{\alpha\alpha}n^{s\perp}_{\alpha\alpha} = 0, \quad (10)$$

for which the physically meaningful root is the positive one,

$$n^{s,e}_{\alpha\alpha} = \frac{n^{s\perp}_{\alpha\alpha} + \sqrt{n^{s\perp\,2}_{\alpha\alpha} + 8n^{s\|}_{\alpha\alpha}n^{s\perp}_{\alpha\alpha}}}{4}. \quad (11)$$

It is interesting to compare this result with those of simpler approaches to the problem that correspond to the Voigt and Reuss approximations in the theory of elastic properties of polycrystalline materials [14,15], which Hill demonstrated to give upper and lower bounds on the elastic constants [16]. In the Voigt approach one assumes that the strain is constant in all domains of the medium. For the superfluid density the corresponding assumption is that the gradient of the phase is the same in all domains, and the effective superfluid density is then given by the *arithmetic* mean of the superfluid density tensor over possible orientations of the domains:

$$(n^{s,e}_{\alpha\alpha})_{\text{Voigt}} = \frac{n^{s\|}_{\alpha\alpha} + 2n^{s\perp}_{\alpha\alpha}}{3}. \quad (12)$$

This is the approximation made in the calculations of the effective neutron superfluid density in Ref. [17]. In the Reuss approach, the stress is assumed to be constant in all domains, and the analogous assumption for the superfluid density is that

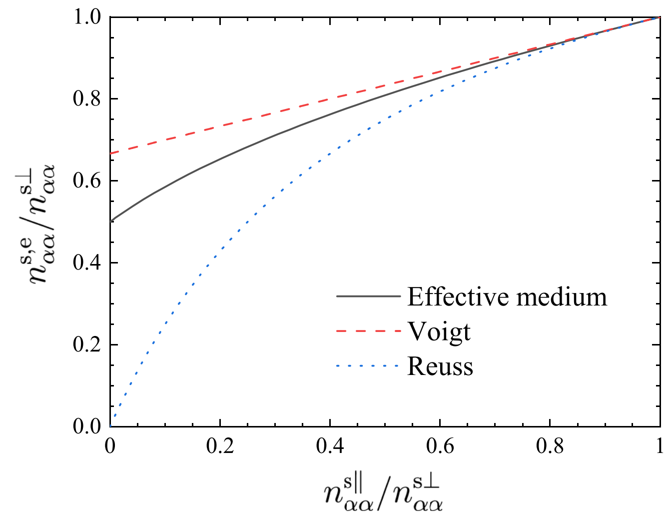

FIG. 1. Effective superfluid density component $n^{s,e}_{\alpha\alpha}$ in units of $n^{s\perp}_{\alpha\alpha}$ as a function of $n^{s\|}_{\alpha\alpha}/n^{s\perp}_{\alpha\alpha}$ for the lasagna phase in the absence of entrainment. The full line is the result of the effective medium theory, the dashed line is the Voigt approximation, and the dotted line is the Reuss approximation.

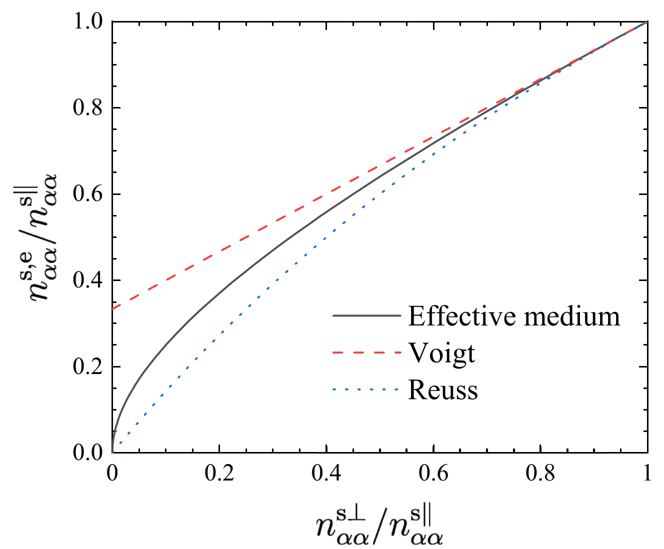

FIG. 2. Effective superfluid density component $n^{s,e}_{\alpha\alpha}$ in units of $n^{s\|}_{\alpha\alpha}$ as a function of $n^{s\perp}_{\alpha\alpha}/n^{s\|}_{\alpha\alpha}$ for the spaghetti phase in the absence of entrainment. As in Fig. 1, the full line shows the result of the effective medium theory, the dashed line the Voigt approximation, and the dotted line the Reuss approximation.

the superfluid current density is the same in all domains. The effective superfluid density is the *harmonic* mean of the superfluid density tensor over possible orientations of the domains:

$$\left(n^{\rm s,e}_{\alpha\alpha}\right)_{\rm Reuss} = \frac{3}{1/n^{\rm s\parallel}_{\alpha\alpha} + 2/n^{\rm s\perp}_{\alpha\alpha}} = \frac{3n^{\rm s\parallel}_{\alpha\alpha} n^{\rm s\perp}_{\alpha\alpha}}{2n^{\rm s\parallel}_{\alpha\alpha} + n^{\rm s\perp}_{\alpha\alpha}}. \tag{13}$$

For the lasagna phase, one expects physically that superfluid flow is impeded in the parallel direction, and therefore $n^{\rm s\perp}_{\alpha\alpha} > n^{\rm s\parallel}_{\alpha\alpha}$ and we may write Eq. (11) as

$$\frac{n^{\rm s,e}_{\alpha\alpha}}{n^{\rm s\perp}_{\alpha\alpha}} = \frac{1+\sqrt{1+8n^{\rm s\parallel}_{\alpha\alpha}/n^{\rm s\perp}_{\alpha\alpha}}}{4}, \tag{14}$$

which is plotted in Fig. 1. Remarkably, even if there is no superfluid flow perpendicular to the lasagna sheets, the predicted effective superfluid density falls to only one half of the value it would have if the parallel and perpendicular superfluid densities were equal.

In the case of the spaghetti phase, one expects flow to be impeded in the perpendicular direction ($n^{\rm s\perp}_{\alpha\alpha} < n^{\rm s\parallel}_{\alpha\alpha}$), and it is therefore convenient to write Eq. (11) in the form

$$\frac{n^{\rm s,e}_{\alpha\alpha}}{n^{\rm s\parallel}_{\alpha\alpha}} = \left[\frac{1}{2}\frac{n^{\rm s\perp}_{\alpha\alpha}}{n^{\rm s\parallel}_{\alpha\alpha}} + \frac{1}{16}\left(\frac{n^{\rm s\perp}_{\alpha\alpha}}{n^{\rm s\parallel}_{\alpha\alpha}}\right)^2\right]^{1/2} + \frac{1}{4}\frac{n^{\rm s\perp}_{\alpha\alpha}}{n^{\rm s\parallel}_{\alpha\alpha}}, \tag{15}$$

which we plot in Fig. 2. For small $n^{\rm s\perp}_{\alpha\alpha}$, $n^{\rm s,e}_{\alpha\alpha} \simeq \sqrt{n^{\rm s\perp}_{\alpha\alpha} n^{\rm s\parallel}_{\alpha\alpha}/2}$, which vanishes for $n^{\rm s\perp}_{\alpha\alpha} \to 0$.

B. Effects of entrainment

On writing out the matrix products in Eq. (A11) explicitly, one finds

$$f^\lambda_{\rm nn} = \frac{-n^{\rm s,e}_{\rm nn}\left[n^{\rm s\lambda}_{\rm nn}n^{\rm s\lambda}_{\rm pp} - \left(n^{\rm s\lambda}_{\rm np}\right)^2\right] - 2n^{\rm s,e}_{\rm nn}\left(n^{\rm s,e}_{\rm pp}n^{\rm s\lambda}_{\rm nn} + n^{\rm s,e}_{\rm np}n^{\rm s\lambda}_{\rm np}\right) + \left(n^{\rm s,e}_{\rm nn}\right)^2 n^{\rm s\lambda}_{\rm pp} + 3\left(n^{\rm s,e}_{\rm np}\right)^2 n^{\rm s\lambda}_{\rm nn} + 2n^{\rm s,e}_{\rm nn}\left[n^{\rm s,e}_{\rm nn}n^{\rm s,e}_{\rm pp} - \left(n^{\rm s,e}_{\rm np}\right)^2\right]}{\left(n^{\rm s\lambda}_{\rm nn} + 2n^{\rm s,e}_{\rm nn}\right)\left(n^{\rm s\lambda}_{\rm pp} + 2n^{\rm s,e}_{\rm pp}\right) - \left(n^{\rm s\lambda}_{\rm np} + 2n^{\rm s,e}_{\rm np}\right)^2}, \tag{16}$$

$$f^\lambda_{\rm pp} = \frac{-n^{\rm s,e}_{\rm pp}\left[n^{\rm s\lambda}_{\rm nn}n^{\rm s\lambda}_{\rm pp} - \left(n^{\rm s\lambda}_{\rm np}\right)^2\right] - 2n^{\rm s,e}_{\rm pp}\left(n^{\rm s,e}_{\rm nn}n^{\rm s\lambda}_{\rm pp} + n^{\rm s,e}_{\rm np}n^{\rm s\lambda}_{\rm np}\right) + \left(n^{\rm s,e}_{\rm pp}\right)^2 n^{\rm s\lambda}_{\rm nn} + 3\left(n^{\rm s,e}_{\rm np}\right)^2 n^{\rm s\lambda}_{\rm pp} + 2n^{\rm s,e}_{\rm pp}\left[n^{\rm s,e}_{\rm nn}n^{\rm s,e}_{\rm pp} - \left(n^{\rm s,e}_{\rm np}\right)^2\right]}{\left(n^{\rm s\lambda}_{\rm nn} + 2n^{\rm s,e}_{\rm nn}\right)\left(n^{\rm s\lambda}_{\rm pp} + 2n^{\rm s,e}_{\rm pp}\right) - \left(n^{\rm s\lambda}_{\rm np} + 2n^{\rm s,e}_{\rm np}\right)^2}, \tag{17}$$

and

$$f^\lambda_{\rm np} = \frac{-n^{\rm s,e}_{\rm np}\left[n^{\rm s\lambda}_{\rm nn}n^{\rm s\lambda}_{\rm pp} - \left(n^{\rm s\lambda}_{\rm np}\right)^2\right] - 3n^{\rm s,e}_{\rm nn}n^{\rm s,e}_{\rm pp}n^{\rm s\lambda}_{\rm np} + n^{\rm s,e}_{\rm np}\left(n^{\rm s,e}_{\rm nn}n^{\rm s\lambda}_{\rm pp} + n^{\rm s,e}_{\rm pp}n^{\rm s\lambda}_{\rm nn}\right) + \left(n^{\rm s,e}_{\rm np}\right)^2 n^{\rm s\lambda}_{\rm np} + 2n^{\rm s,e}_{\rm np}\left[n^{\rm s,e}_{\rm nn}n^{\rm s,e}_{\rm pp} - \left(n^{\rm s,e}_{\rm np}\right)^2\right]}{\left(n^{\rm s\lambda}_{\rm nn} + 2n^{\rm s,e}_{\rm nn}\right)\left(n^{\rm s\lambda}_{\rm pp} + 2n^{\rm s,e}_{\rm pp}\right) - \left(n^{\rm s\lambda}_{\rm np} + 2n^{\rm s,e}_{\rm np}\right)^2}. \tag{18}$$

For the uniaxial case, the effective superfluid densities $n^{\rm e}_{\alpha\beta}$ are the solutions of the three simultaneous equations

$$f^\parallel_{\alpha\beta} + 2f^\perp_{\alpha\beta} = 0. \tag{19}$$

To leading order in $n^{\rm s\lambda}_{\rm np}$, the nn and pp components of the effective superfluid density tensor are given by Eq. (11) with the densities $n^{\rm s\lambda}$ and $n^{\rm s,e}$ put equal to the nn and pp components:

$$n^{\rm s,e}_{\rm nn} \simeq \frac{n^{\rm s\perp}_{\rm nn} + \sqrt{\left(n^{\rm s\perp}_{\rm nn}\right)^2 + 8n^{\rm s\parallel}_{\rm nn}n^{\rm s\perp}_{\rm nn}}}{4}, \tag{20}$$

$$n^{\rm s,e}_{\rm pp} \simeq \frac{n^{\rm s,\perp}_{\rm pp} + \sqrt{\left(n^{\rm s\perp}_{\rm pp}\right)^2 + 8n^{\rm s\parallel}_{\rm pp}n^{\rm s\perp}_{\rm pp}}}{4}, \tag{21}$$

and

$$n^{\rm s,e}_{\rm np} \simeq 3n^{\rm s,e}_{\rm nn}n^{\rm s,e}_{\rm pp}\frac{\Delta^\perp n^{\rm s\parallel}_{\rm np} + 2\Delta^\parallel n^{\rm s\perp}_{\rm np}}{\Delta^\perp\Gamma^\parallel + 2\Delta^\parallel\Gamma^\perp}. \tag{22}$$

Here

$$\Delta^\lambda = \left(n^{\rm s\lambda}_{\rm nn} + 2n^{\rm s,e}_{\rm nn}\right)\left(n^{\rm s\lambda}_{\rm pp} + 2n^{\rm s,e}_{\rm pp}\right), \quad \text{and} \tag{23}$$

$$\Gamma^\lambda = -n^{\rm s\lambda}_{\rm nn}n^{\rm s\lambda}_{\rm pp} + n^{\rm s\lambda}_{\rm nn}n^{\rm s,e}_{\rm pp} + n^{\rm s,e}_{\rm nn}n^{\rm s\lambda}_{\rm pp} + 2n^{\rm s,e}_{\rm nn}n^{\rm s,e}_{\rm pp}. \tag{24}$$

In Eqs. (22)–(24), the $n^{\rm s,e}_{\alpha\alpha}$ should be taken to be the values (20) and (21) in the absence of entrainment. For an isotropic inclusion, $n^{\rm s,e}_{\rm np}$ is equal to $n^{\rm s\parallel}_{\rm np} = n^{\rm s\perp}_{\rm np}$, as one would expect physically.

Once the effective superfluid density tensor has been determined, the effective normal densities of neutrons and protons may be found from the condition for Galilean invariance, Eq. (5).

It is interesting to consider the case of no normal currents ($\vec{u} = 0$) and no bulk flow of protons. The neutron current density is then given by

$$\left\langle j^i_{\rm n}\right\rangle = \left[n^{\rm s,e}_{\rm nn} - \frac{\left(n^{\rm s,e}_{\rm np}\right)^2}{n^{\rm s,e}_{\rm pp}}\right]\frac{\langle\nabla^i\phi_{\rm n}\rangle}{m}. \tag{25}$$

The second term in parentheses reflects the fact that backflow of neutrons around an inclusion results in backflow of protons. To achieve no average proton current, a nonzero value of $\langle\nabla^i\phi_{\rm p}\rangle$ is required.

IV. DISCUSSION

We begin with some general remarks. Because entrainment of the two superfluid motions, which are expressed in the formalism via the np components of the superfluid density tensor, is not expected to be large, we shall consider the case when it is absent. Our calculations predict a qualitative difference between the lasagna and spaghetti phases, since even if the superfluid density for flow perpendicular to lasagna sheets is zero, the effective superfluid density drops to only one half of the value for flow in directions in the plane of the lasagna sheets. For spaghetti, when the superfluid density is zero for flow perpendicular to the strands, the effective superfluid density is zero. This difference is due to the fact that, while the lasagna phase has two "easy" axes (those lying in the plane of the sheets) and one "hard" axis, the spaghetti phase has one "easy" axis (along the strands) and two "hard" axes. We therefore conclude that for the lasagna phase, the effective superfluid density for the disordered state will be greater than one half of the superfluid density tensor for flow in the plane of the sheets. However, the effective superfluid density for the disordered spaghetti state depends more sensitively on the component of the superfluid density tensor for flow perpendicular to the strands.

Another prediction of our calculations is that for the spaghetti phase, the situation for protons will be different from that for neutrons. For neutrons, superfluid flow perpendicular to the strands is relatively easy because of the neutron fluid between strands, but for protons in the perfectly ordered spaghetti phase, flow between strands occurs by tunneling and is consequently small. The effective proton superfluid density of the disordered spaghetti phase is therefore sensitive to imperfections such as bridges between adjacent strands.

To make use of the results derived above, one needs as input the superfluid density tensors of the pasta phases. There have been a number of calculations of these tensors for neutrons in phases with no spatial modulations of the spaghetti and lasagna elements. In one class of calculations, one assumes that the superfluid densities are equal to the response of a system to an applied vector potential in the absence of both pairing and scattering between excitations. These calculations lead to values of $n_{\rm nn}^{\rm s\parallel}/n_{\rm nn}^{\rm s\perp}$ for spaghetti in the range 0.71–0.86 [18,19]. For lasagna $n_{\rm nn}^{\rm s\parallel}/n_{\rm nn}^{\rm s\perp}$ is found to lie in the range 0.93–0.95. More recent calculations for lasagna based on the Hartree-Fock method give values in approximately the same range [20]. Time-dependent density-functional theory has been applied to calculate neutron superfluid densities for the lasagna phase, but only for proton fractions higher than those expected in pasta phases in the inner crust of neutron stars, where matter is in beta equilibrium [21].

In a second class of calculation, the problem was treated by use of quantum hydrodynamics, in which it is assumed implicitly that the superfluid coherence length of the neutrons is much less than other lengths in the problem [17]. Simple analytical results may be obtained for the lasagna phase. For a neutron superfluid flow perpendicular to the sheets, which we take to be in the z direction, the kinetic energy may be written as

$$\mathscr{E}_{\rm kin} = \frac{1}{2}\int_V d^3r\,\frac{\tilde{n}_{\rm n}(\mathbf{r})}{m}\left(\frac{\partial\tilde{\phi}}{\partial z}\right)^2, \tag{26}$$

since in the hydrodynamic model, the current density of neutrons is given by $[\tilde{j}_{\rm n}(\mathbf{r})]_z = \tilde{n}_{\rm n}(\mathbf{r})\partial\tilde{\phi}/\partial z$. The tilde over a quantity indicates that it is the local value, not the coarse-grained average, and V is the volume of the system. For a steady flow, particle number conservation requires that $[\tilde{j}_{\rm n}(\mathbf{r})]_z$ is independent of z and may be written as $(j_{\rm n})_z$ and the kinetic energy is therefore

$$\mathscr{E}_{\rm kin} = \frac{m}{2}(j_{\rm n})_z^2\int_V d^3r\frac{1}{\tilde{n}_{\rm n}(\mathbf{r})}, \tag{27}$$

from which one concludes that

$$\frac{1}{n_{\rm nn}^{\rm s\parallel}} = \int_V \frac{d^3r}{V}\,\frac{1}{\tilde{n}_{\rm n}(\mathbf{r})}. \tag{28}$$

Thus for lasagna, the neutron superfluid density for flow perpendicular to the sheets is the *harmonic* mean of the density of neutrons, while the mean neutron density is the *arithmetic* mean of the neutron density. For the case of a sharp boundary between the nuclear matter and the neutron matter, the result is

$$\frac{n_{\rm nn}^{\rm s\parallel}}{n_{\rm n}} = \left[1 + \frac{(n_{\rm n,2}-n_{\rm n,1})^2}{n_{\rm n,1}n_{\rm n,2}}u(1-u)\right]^{-1}, \tag{29}$$

in agreement with Ref. [17]. Here $n_{\rm n,2}$ is the density of neutrons in the nuclear matter regions, $n_{\rm n,1}$ is the neutron density in the pure neutron regions, and u is the fraction of space occupied by nuclear matter. For the pasta phases, the numerical values for $n_{\rm nn}^{\rm s\parallel}/n_{\rm n}$ in the hydrodynamic approximation are found to be greater than 0.97 [11,17].

As argued in Ref. [17], the calculations that neglect pairing and the those that make the hydrodynamic approximation represent extreme cases, and the true value of the superfluid density likely lies between them. That the superfluid density for flow parallel to the axis of the lasagna phase is reduced when pairing is taken into account is supported by the calculations of Ref. [22], which include both the periodic potential and pairing.

With respect to disordered pasta phases, even if the anisotropy of the neutron superfluid density tensor is as large as predicted by the calculations neglecting pairing, our calculations show that the (Voigt) approximation of taking the effective superfluid density tensor of the disordered medium to be the average of the superfluid density tensor over directions, which has been employed previously [17,18], is very good; indeed, the result of the effective medium approach is close to those of the Voigt and Reuss approximations.

The effective medium approach has been very successful in accounting for the elastic and electrical properties of terrestrial materials, as may be seen from, e.g., Ref. [5] for binary metallic mixtures, but it would be useful to explore how well it works for very anisotropic phases.

A basic assumption of the effective medium approach is that the system may be regarded as a collection of "domains" which are oriented randomly, a situation explored in the case of elastic constants in the simulations of Ref. [23]. An important question is whether or not this assumption is realistic, since it could well be that the orientation of the pasta varies continuously in space, rather than having sharp boundaries between regions with different orientations. It would be valuable to make simulations to investigate the disorder of pasta structure that results when matter is cooled below the critical temperature for formation of pasta structures.

In the calculations above we have largely neglected the effects of magnetic fields. In a charged superfluid, the length scale for variations of the magnetic field is of order the London length, $\lambda = (4\pi n_{\rm p}e^2/mc^2)^{-1/2}$, which is of order 110 fm for a proton fraction of 5% and a density of half nuclear saturation density. A spatially independent vector potential $\vec{A}$ has no physical effect, and in the formalism can be taken into account by shifting the proton phase by an amount $e\vec{A}\cdot\vec{r}/c$. Only the spatially dependent part of the vector potential is physically relevant, and in order to be able to neglect this, the size of the inclusion, which corresponds to the spatial scale of the disorder of the orientation of the pasta elements, must be less than the London length. This requires that the pasta be rather highly disordered, since, e.g., for the lasagna phase the lattice spacing is estimated to be about 44 fm [24].

Superconducting vortices in ordered pasta phases have been discussed in Ref. [24], and the vortex energy depends on its direction with respect to the principal axes. In disordered phases, the properties of vortices will become independent of direction if the length scale of the disorder becomes small compared with the superconducting penetration depth.

V. CONCLUSION

We have developed an effective medium approach to calculating the properties of disordered pasta phases that allows for the effects of entrainment. For the lasagna phase, the calculations predict that, with the neglect of entrainment, the diagonal (nn and pp) components of the effective superfluid density are greater than one half of the corresponding value of the superfluid density tensor for flow in directions lying in the plane of the sheets. Because flow of protons perpendicular to the direction of the strands is suppressed in the spaghetti phase, the pp component of the effective superfluid density tensor can be very much less than the corresponding value of the superfluid density tensor for flow in the direction of the strands. However, since neutrons can move relatively easily between strands, the reduction of the nn component of the effective superfluid density tensor is expected to be much less.

In our calculations we have not taken into account possible effects of the interface, e.g., the existence of weak links between neighboring regions with different orientations of the pasta structures. Such effects are important for laboratory superconductors because of the existence of grain boundaries [25] but are expected to be less important in neutron star crusts where on length scales comparable to the internucleon spacing the structure is that of a liquid rather than a solid. One situation in which interface effects could be important for protons is if there is no path for percolation within the proton-rich phase. However, for neutrons, which are present everywhere, such effects are unlikely to be important.

To improve estimates of the effective superfluid density of the disordered pasta phases, it is necessary to have better values for the superfluid density tensor for the ordered phases. One possible approach would be to extend the calculations of Ref. [22] for fermions in a one-dimensional sinusoidal potential to potentials that better reflect the structure of the pasta phases.

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APPENDIX: SIMPLE DERIVATION OF THE BASIC RESULT

Consider an inclusion consisting of the pasta phases with superfluid density tensor $n^{s,ij}_{\alpha\beta}$ immersed in an isotropic medium with superfluid density tensor $n^{s,e}_{\alpha\beta}$. For simplicity, we shall take the inclusion to be spherical, but the final results we derive also apply for an ellipsoidal inclusion when the principal axes of the ellipsoid coincide with the principal axes of the superfluid density tensor; this may be demonstrated by working in terms of ellipsoidal coordinates rather than spherical polar coordinates, just as in the analogous problem in electrostatics [26]. In the uniform medium without the inclusion, the current density in the presence of uniform gradients of the phases given by $\nabla^i\phi_\alpha = k^i_\alpha$ may be written as $j^i_\alpha = n^{s,e}_{\alpha\beta}k^i_\beta/m$. The inclusion induces backflow around it and changes the current density within the inclusion. Because, by symmetry, the current density integrated over space for $\vec{k}$ lying along one of the principal axes also lies in the direction of that axis, it is convenient to work in terms of the components of $\vec{k}$ along the principal axes of the inclusion, which we denote by the label $\lambda = 1, 2, 3$ and we denote the components of the superfluid density tensor by $n^{s\lambda}_{\alpha\beta}$.

We consider a spherical inclusion of a pasta phase with superfluid density tensor $n^{s,ij}_{\alpha\beta}$ embedded in a homogeneous medium with effective superfluid density tensor $n^{s,e}_{\alpha\beta}$. Since the problem we are considering is linear, we may superimpose solutions for $\vec{k}$ along the three principal directions of the inclusion. In a homogeneous medium, $\phi = \vec{k}\cdot\vec{r}$, the wave vector of the flow $\vec{\nabla}\phi = \vec{k}$ is uniform, and we shall take it to be in the z direction. The current density is given by Eq. (1) and therefore in regions where the superfluid density tensor is isotropic, conservation of particle number demands that

$$\nabla^i j^i_\alpha = 0, \tag{A1}$$

and thus $\nabla^2\phi_\alpha = 0$. We shall seek a solution of the form

$$\phi_\alpha = \begin{cases} \left(k^\lambda_\alpha + A^\lambda_\alpha\right)z & \text{for } r \leqslant a, \\ k^\lambda_\alpha z + C^\lambda_\alpha z/r^3 & \text{for } r \geqslant a, \end{cases} \tag{A2}$$

where the origin of the coordinate system is at the center of the inclusion, r is the radial coordinate, and z is the coordinate in the direction of the principal axis λ. From the condition that ϕ_α be continuous at the surface of the inclusion ($r = a$), it follows that

$$A^\lambda_\alpha = \frac{C^\lambda_\alpha}{a^3}. \tag{A3}$$

Within the inclusion, the current density is constant in space, and thus the solution (A2) satisfies the condition for particle conservation (A1) everywhere except at the surface of the inclusion. To ensure conservation of particle number there, the component of the current density, Eq. (1), normal to the surface must be continuous, which gives

$$n^{s\lambda}_{\alpha\beta}\left(k^\lambda_\beta + A^\lambda_\beta\right) = n^{s,e}_{\alpha\beta}\left(k^\lambda_\beta - 2A^\lambda_\beta\right), \tag{A4}$$

where we use the summation convention for subscripts but not for the superscript λ. Thus in a compact matrix notation,

$$A^\lambda = (2n^{s,e} + n^{s\lambda})^{-1}(n^{s,e} - n^{s\lambda})k^\lambda, \tag{A5}$$

where A^λ and k^λ are vectors and $n^{s\lambda}$ and $n^{s,e}$ are matrices in the two-dimensional species space. The current density integrated over the whole of space is given in this matrix notation by

$$\int_V d^3r\, j^\lambda(\vec{r}) = V n^{s,e}\frac{k^\lambda}{m} + \frac{4\pi a^3}{3}\left[n^{s\lambda}\frac{A^\lambda}{m} - (n^{s,e} - n^{s\lambda})\frac{k^\lambda}{m}\right], \tag{A6}$$

where V is the volume of the system. The change in the integrated current density due to the presence of the inclusion is therefore

$$\Delta\int_V d^3r\, j^\lambda(\vec{r}) = \frac{4\pi a^3}{3}\left[n^{s\lambda}\frac{A^\lambda}{m} - (n^{s,e} - n^{s\lambda})\frac{k^\lambda}{m}\right] \tag{A7}$$

$$= -6V_i n^{s,e}(2n^{s,e} + n^{s\lambda})^{-1}(n^{s,e} - n^{s\lambda})\frac{k^\lambda}{m}, \tag{A8}$$

where

$$V_i = \frac{4\pi a^3}{3} \tag{A9}$$

is the volume of the inclusion. While locally the current density has components in directions other than λ, the integrals over space of these components vanish because of the dipolar character of the flow outside the inclusion.

For $\vec{k}$ in an arbitrary direction with respect to the principal axes of the inclusion, the change in the integrated current density is given by

$$\Delta\int_V d^3r\, \vec{j}(\vec{r}) = -6V_i \sum_{\lambda=1,2,3} f^\lambda \frac{\vec{k}\cdot\vec{\varepsilon}^\lambda}{m}\vec{\varepsilon}^\lambda, \tag{A10}$$

where

$$f^\lambda = n^{s,e}(2n^{s,e} + n^{s\lambda})^{-1}(n^{s,e} - n^{s\lambda}) \tag{A11}$$

and $\vec{\varepsilon}^\lambda$ is a unit vector in the λ direction. On averaging over possible orientations of the principal axes of the inclusion, the current density integrated over space is in the direction of $\vec{k}$ and

$$\left\langle \Delta\int_V d^3r\, \vec{j}(\vec{r})\cdot\vec{k}\right\rangle = \frac{6V_i}{m}\sum_{\lambda=1,2,3} f^\lambda\langle(\vec{k}\cdot\vec{\varepsilon}^\lambda)^2\rangle. \tag{A12}$$

For a random orientation of the inclusion, $\langle(\vec{k}\cdot\vec{\varepsilon}^\lambda)^2\rangle$ is equal to $k^2/3$ and the integrated current density vanishes when

$$\sum_{\lambda=1,2,3} f^\lambda = 0, \tag{A13}$$

which is the condition determining $n^{s,e}$ in the self-consistent effective medium approach. For the two-component case, f is a 2×2 symmetric matrix, and therefore this equation is equivalent to three scalar equations.

[1] C. P. Lorenz, D. G. Ravenhall, and C. J. Pethick, Phys. Rev. Lett. **70**, 379 (1993).
[2] K. Oyamatsu, Nucl. Phys. A **561**, 431 (1993).
[3] F. Douchin and P. Haensel, Phys. Lett. B **485**, 107 (2000).
[4] D. A. G. Bruggeman, Ann. Phys. (Leipzig) **416**, 636 (1935).
[5] R. Landauer, J. Appl. Phys. **23**, 779 (1952).
[6] D. Stroud, Phys. Rev. B **12**, 3368 (1975).
[7] For a review and references to earlier work, see D. J. Bergman and D. Stroud, in *Solid State Physics*, Vol. 46, edited by H. Ehrenreich and D. Turnbull (Academic Press, New York, 1992), p. 165.
[8] See, e.g., J. G. Berryman, J. Mech. Phys. Solids **53**, 2141 (2005).
[9] D. Kobyakov and C. J. Pethick, Mon. Not. R. Astron. Soc. **449**, L110 (2015).
[10] C. J. Pethick, Z.-W. Zhang, and D. N. Kobyakov, Phys. Rev. C **101**, 055802 (2020).
[11] D. Durel and M. Urban, Phys. Rev. C **97**, 065805 (2018).
[12] A. S. Schneider, D. K. Berry, C. M. Briggs, M. E. Caplan, and C. J. Horowitz, Phys. Rev. C **90**, 055805 (2014).
[13] D. N. Kobyakov, C. J. Pethick, S. Reddy, and A. Schwenk, Phys. Rev. C **96**, 025805 (2017).
[14] W. Voigt, Abh. Kgl. Ges. Wis. Göttingen, Math. Kl. **34**, 1 (1887).
[15] A. Reuss, Z. Angew. Math. Mech. **9**, 49 (1929).
[16] R. Hill, Proc. Phys. Soc. A **65**, 349 (1952).
[17] N. Martin and M. Urban, Phys. Rev. C **94**, 065801 (2016).
[18] N. Chamel, Nucl. Phys. A **747**, 109 (2005).
[19] B. Carter, N. Chamel, and P. Haensel, Nucl. Phys. A **748**, 675 (2005).
[20] Y. Kashiwaba and T. Nakatsukasa, Phys. Rev. C **100**, 035804 (2019).
[21] K. Sekizawa, S. Kobayashi, and M. Matsuo, arXiv:2112.14350.
[22] G. Watanabe and C. J. Pethick, Phys. Rev. Lett. **119**, 062701 (2017).
[23] M. E. Caplan, A. S. Schneider, and C. J. Horowitz, Phys. Rev. Lett. **121**, 132701 (2018).
[24] Z.-W. Zhang and C. J. Pethick, Phys. Rev. C **103**, 055807 (2021).
[25] J. R. Clem and V. G. Kogan, Jpn. J. Appl. Phys. **26** (Suppl. 26–3), 1161 (1987).
[26] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960), Secs. 4 and 8.