Causality and stability analysis of first-order field redefinition in relativistic hydrodynamics from kinetic theory

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In this work, the causality and stability of a first-order relativistic dissipative hydrodynamic theory, that redefines the hydrodynamic fields from a first principles microscopic estimation, are analyzed. A generic approach of gradient expansion for solving the relativistic transport equation is adopted using the Chapman-Enskog iterative method. Next, the momentum dependent relaxation time approximation is employed to quantify the collision term for analytical estimation of the field correction coefficients from kinetic theory. In the linear regime, in the local rest frame the dispersion relations are observed to produce a causal propagating mode. However, the acausality and instability reappear when a boosted background is considered for linear analysis. These facts point out relevant aspects regarding the methodology of extracting the causal and stable first-order hydrodynamics from kinetic theory and indicate the appropriate approach to construct a valid first-order theory with proper justification.

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I. INTRODUCTION

The journey of relativistic dissipative hydrodynamic theory can be traced back from the relativistic extension of the Navier-Stokes (NS) formalism introduced by Landau and Lifshitz (LL) [1] and Eckart [2]. These theories are known as the first-order theories because of the presence of first-order gradient corrections in the out of equilibrium deviations of the thermodynamic quantities such as entropy current. The problem occurs with these theories when they exhibit superluminal speed of signal propagation, causing a severe causality violation problem [3]. This undesirable feature further associates instabilities within the system such that small departures of these fluids from equilibrium lead to rapid evolution away from equilibrium [4]. These features pose a major concern for the practical applicability of these theories and make them unacceptable as a reasonable relativistic theory for fluids.

To rescue the situation, second-order theories are introduced where the dissipative fluxes are promoted as the fundamental dynamical variables and give rise to relaxation-type evolution equations. The second-order theory introduced by Israel and Stewart [5], known as Israel-Stewart (IS) theory, accepted as the standard theory of relativistic dissipative hydrodynamics, was shown to be both stable and causal in Refs. [3,4,6]. In Ref. [7] the hyperbolicity of IS theory along with subluminal signal propagation was demonstrated for linear perturbations around equilibrium. Since then, a range of second-order theories like IS [8,9] to recently developed Denicol-Niemi-Molnar-Rischke (DNMR) [10,11] and resummed Baier-Romatschke-Son-Starinets-Stephanov theory [12] have been introduced in the literature. In works like Refs. [13,14] the stability and causality were analyzed for IS theory and in Ref. [15] the same was studied for DNMR theory. These studies give conditions involving the equation of state and the transport coefficients that ensure that these theories are indeed causal and stable. Proven to be free from causality and stability related issues at least for linear perturbations around equilibrium, they have been used for a wide range of hydrodynamic numerical simulations. Constraints to ensure causality for IS-like theories in nonlinear, far-from-equilibrium regimes were recently explored in Refs. [16,17].

Recently, a new comprehensive formalism has been proposed by Bemfica, Disconzi, Noronha, and Kovtun (BDNK) to establish a causal and stable hydrodynamic theory [18–22] without incorporating extra dynamical degrees of freedom other than the fundamental ones such as temperature, hydrodynamic velocity, and charge chemical potential. In other words, they derive first-order theories in the most general way possible that prohibit superluminal signal propagation as well as retain stability criteria besides other requirements like non-negative entropy production, which are essential for an acceptable hydrodynamic theory. The basic idea is to define the out of equilibrium thermodynamic variables in a general frame other than specified either by Landau and Lifshitz or by Eckart, through their postulated constitutive relations. This is the so-called BDNK formalism, which proposed a class of stable and causal frames for the first-order relativistic hydrodynamic theory. In Refs. [18,23,24] the derivation of such a causal-stable first-order theory from the relativistic Boltzmann equation has been studied in order to establish its kinetic theory origin.

Motivated from these studies, in this work, a first-order theory is derived, where the out of equilibrium thermodynamic fields are not uniquely defined and are subject to including dissipative effects from the medium. Here, a first-order rela-

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tivistic dissipative hydrodynamic theory that includes the out of equilibrium contributions in thermodynamic fields purely from system interactions, and was recently derived from relativistic transport equation by gradient expansion technique [25], is employed to derive the dispersion equations and the associated modes. To linearize the nontrivial collision integral, the momentum dependent relaxation time approximation (MDRTA) is adopted for solving the relativistic transport equation as a model study [26–31]. The obtained results reveal interesting facts regarding the microscopic extraction of hydrodynamic field redefinition and its consequent effects on stability and causality of the theory.

In the usual NS theory, the macroscopic thermodynamic quantities such as energy density and particle number density in the conservation equations are usually set to their equilibrium values even in the dissipative medium by imposing certain matching or fitting conditions. The resulting dispersion relation, at the limit of large wave number (k), gives rise to nonpropagating modes with k^2 dependence [13,32], identical to that of the diffusion process which is acausal with an infinite propagation speed. This behavior ($\omega(k)$ is growing faster than k) is a consequence of the acausal nature of the equations; when linearized around equilibrium, the resulting modes become superluminal. The intention of the current analysis is to observe if there are any changes in the linear modes after the first-order field redefinition is introduced. Here, first the equations of motion have been linearized around a hydrostatic equilibrium at local rest frame (LRF). Next, the equations have been tested with a more general state of equilibrium where the background is Lorentz boosted with an arbitrary velocity. This generalization from zero to arbitrary background velocity is necessary, since this could result in a whole new set of modes. In this context, the precedence of LL theory can be remembered. At zero chemical potential the LL theory gives two modes with zero background velocity, keeping the stability of the theory intact. It is the new mode that appears with a boosted background, and drives the instability in the theory [13]. A well defined relativistic theory cannot depend on whether one sets the background velocity to zero or not, and hence a consistency check between the boosted and nonboosted results is essential.

The paper is organized as follows: In Sec. II the first-order relativistic hydrodynamics with thermodynamic field redefinition is derived from relativistic transport equation of kinetic theory. In Sec. III the dispersion relations and the modes are analyzed with a hydrostatic background at local rest frame giving the asymptotic causality condition. Section IV studies the same dispersion relations but with a Lorentz boosted background and demonstrates the additional acausal modes. Finally in Sec. V the work is summarized with prior conclusions and useful remarks regarding the choice of hydrodynamic field redefinition and its consequence on the causality and stability of a first-order theory.

Throughout the paper I use natural units ($\hbar = c = k_B = 1$) and consider flat space-time with mostly negative metric signature $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The timelike fluid four-velocity, u^μ, satisfies the normalization condition $u^\mu u_\mu = 1$. The projection operator orthogonal to u^μ is defined as $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. The space-time partial derivative can be decomposed as $\partial_\mu = u_\mu D + \nabla_\mu$, with a temporal part $D = u^\mu \partial_\mu$ and a spatial part $\nabla_\mu = \Delta_{\mu\nu}\partial^\nu$. The traceless irreducible tensors of rank 1 and rank 2 are defined as $A_{\langle\mu\rangle} = \Delta_{\mu\nu}A^\nu$ and $A_{\langle\mu}B_{\nu\rangle} = \Delta^{\alpha\beta}_{\mu\nu}A_\alpha B_\beta$, respectively, with $\Delta^{\alpha\beta}_{\mu\nu} = \frac{1}{2}(\Delta^\alpha_\mu \Delta^\beta_\nu + \Delta^\beta_\mu \Delta^\alpha_\nu) - \frac{1}{3}\Delta_{\mu\nu}\Delta^{\alpha\beta}$.

II. FIELD REDEFINITION IN RELATIVISTIC HYDRODYNAMICS

The basic problem is to estimate the first-order out of equilibrium correction of the thermodynamic fields needed to define the particle four-flow N^μ and energy-momentum tensor $T^{\mu\nu}$. Here, a relativistic transport equation serves the purpose by providing the first-order correction in the single particle distribution function via gradient expansion technique. The first-order Chapman-Enskog (CE) method gives the following integro-differential equation over the single particle distribution function [33]:

$$p^\mu \partial_\mu f^{(0)}(x, p) = -\mathcal{L}[\phi]. \tag{1}$$

Here, the first-order particle distribution function is decomposed as $f = f^{(0)} + f^{(0)}(1 \pm f^{(0)})\phi$ with $f^{(0)} = [\exp(\frac{p\cdot u}{T} - \frac{\mu}{T}) \mp 1]^{-1}$ as the equilibrium distribution for bosons and fermions, respectively, and ϕ denotes the distribution deviation from equilibrium. The linearized collision term ($\mathcal{L}[\phi]$) over the deviation of the first-order distribution function is given by

$$\mathcal{L}[\phi] = \int d\Gamma_{p_1} d\Gamma_{p'} d\Gamma_{p'_1} f^{(0)} f_1^{(0)} (1 \pm f'^{(0)})(1 \pm f_1'^{(0)}) \times (\phi + \phi_1 - \phi' - \phi'_1) W(p' p'_1 | p p_1), \tag{2}$$

with $d\Gamma_p = \frac{d^3p}{(2\pi)^3 p^0}$ as the phase space factor and W as the microscopic interaction rate. The equilibrium temperature, chemical potential, and hydrodynamic four-velocity of the system are denoted by T, μ, and u^μ, respectively.

In order to solve Eq. (1), I am adopting here one of the most conventional techniques. In transport equation (1), the time derivatives on the left hand side are eliminated by the spatial gradients using the first-order thermodynamic identities such as $\frac{DT}{T} = -(\frac{\partial P_0}{\partial \epsilon_0})_{\rho_0}(\partial \cdot u)$, $D\tilde{\mu} = -\frac{1}{T}(\frac{\partial P_0}{\partial \rho_0})_{\epsilon_0}(\partial \cdot u)$, and $(\epsilon_0 + P_0)Du^\mu = \nabla^\mu P_0$. Here, ρ_0, ϵ_0, and P_0 are the equilibrium values of particle number density, energy density, and hydrodynamic pressure of the system, respectively. It is to be noted that the spatial gradients over field variables contribute to the thermodynamic forces and that is why in conventional methods to extract the single particle distribution function from transport equation the time derivatives are eliminated by spatial gradients using first-order thermodynamic identities.

Following this prescription, the left hand side of Eq. (1) turns out to be a linear combination of thermodynamic forces as the following [34]:

$$f^{(0)}(1 \pm f^{(0)}) \left[\hat{Q} \partial \cdot u + \left(\frac{\tau_p}{\hat{h}} - 1 \right) \tilde{p}^\mu \nabla_\mu \tilde{\mu} + \tilde{p}^\mu \tilde{p}^\nu \sigma_{\mu\nu} \right] = \frac{1}{T} \mathcal{L}[\phi], \tag{3}$$

with $\hat{Q} = \frac{z^2}{3} + \tau_p^2((\frac{\partial P_0}{\partial \epsilon_0})_{\rho_0} - \frac{1}{3}) + \tau_p \frac{1}{T}(\frac{\partial P_0}{\partial \rho_0})_{\epsilon_0}$. $\sigma_{\mu\nu} = \nabla_{\langle\mu} u_{\nu\rangle}$ is the traceless, symmetric velocity gradient and $\hat{h} = (\epsilon_0 + P_0)/\rho_0 T$ is the scaled enthalpy per particle at equilibrium. The other used notations denote $\tilde{p}^\mu = p^\mu/T$ as the scaled particle 4-momenta, $\tau_p = (p \cdot u)/T$ as the scaled particle energy at local rest frame, $z = m/T$ as the scaled particle mass, and $\tilde{\mu} = \mu/T$.

Since the thermodynamic forces are independent, in order to be a solution of Eq. (3), ϕ must be a linear combination of the thermodynamic forces as

$$\phi = A(\partial \cdot u) + B^\nu \nabla_\nu \tilde{\mu} + C^{\mu\nu} \sigma_{\mu\nu}, \tag{4}$$

with $B^\mu = B\tilde{p}^{\langle\mu\rangle}$ and $C^{\mu\nu} = C\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}$. It is customary to expand the unknown coefficients in the particle momentum basis as $A = \sum_{s=0}^{p} A^s(z,x)\tau_p^s$, $B = \sum_{s=0}^{p} B^s(z,x)\tau_p^s$, and $C = \sum_{s=0}^{p} C^s(z,x)\tau_p^s$, with the series expanded up to any desired degree of accuracy.

The next job is to estimate the out of equilibrium dissipative correction in the thermodynamic fields. For this purpose, the two most general field variables, namely, the particle four-flow (N^μ) and the energy-momentum tensor ($T^{\mu\nu}$), are given respectively in their integral forms as the following:

$$N^\mu = \int d\Gamma_p p^\mu f, \quad T^{\mu\nu} = \int d\Gamma_p p^\mu p^\nu f. \tag{5}$$

The out of equilibrium part of the distribution function f in Eqs. (5) gives the necessary field corrections. The correction in particle number density ($\delta\rho$), energy density ($\delta\epsilon$), pressure (δP), energy flow or momentum density (W^α), and particle flux (V^α) are given by

$$\delta\rho = u_\mu \delta N^\mu = \int d\Gamma_p f^{(0)}(1 \pm f^{(0)})(p \cdot u)\phi, \tag{6}$$

$$\delta\epsilon = u_\mu u_\nu \delta T^{\mu\nu} = \int d\Gamma_p f^{(0)}(1 \pm f^{(0)})(p \cdot u)^2\phi, \tag{7}$$

$$\begin{aligned}\delta P &= -\frac{1}{3}\Delta_{\mu\nu}\delta T^{\mu\nu},\\ &= \frac{1}{3}\int d\Gamma_p f^{(0)}(1 \pm f^{(0)})\big[(p \cdot u)^2 - m^2\big]\phi,\end{aligned} \tag{8}$$

$$\begin{aligned}W^\mu &= \Delta^\alpha_\mu u_\nu \delta T^{\mu\nu},\\ &= \int d\Gamma_p f^{(0)}(1 \pm f^{(0)})p^{\langle\mu\rangle}(p \cdot u)\phi,\end{aligned} \tag{9}$$

$$V^\mu = \Delta^\alpha_\mu N^\mu = \int d\Gamma_p f^{(0)}(1 \pm f^{(0)})p^{\langle\mu\rangle}\phi. \tag{10}$$

Here, δN^μ and $\delta T^{\mu\nu}$ are the first-order dissipative corrections in particle four-flow and energy-momentum tensor, respectively.

Keeping up to the first nonvanishing contribution from the collision operator, the respective corrections in the thermodynamic fields listed in Eqs. (6)–(10) are given by

$$\delta\rho = c_\Gamma(\partial \cdot u), \quad \delta\epsilon = c_\Lambda(\partial \cdot u), \quad \delta P = c_\Omega(\partial \cdot u), \tag{11}$$

$$W^\alpha = c_\Sigma \nabla^\alpha \tilde{\mu}, \quad V^\alpha = c_\Xi \nabla^\alpha \tilde{\mu}, \tag{12}$$

with

$$c_\Gamma = T(A^0 a_1 + A^1 a_2 + A^2 a_3), \tag{13}$$

$$c_\Lambda = T^2(A^0 a_2 + A^1 a_3 + A^2 a_4), \tag{14}$$

$$\begin{aligned}c_\Omega = \frac{T^2}{3}\big[&A^0(a_2 - z^2 a_0) + A^1(a_3 - z^2 a_1)\\ &+A^2(a_4 - z^2 a_2)\big],\end{aligned} \tag{15}$$

$$c_\Sigma = T^2(B^0 b_1 + B^1 b_2), \tag{16}$$

$$c_\Xi = T(B^0 b_0 + B^1 b_1). \tag{17}$$

The moment integrals are defined here as $a_n = \int dF_p \tau_p^n$, $\Delta^{\mu\nu} b_n = \int dF_p \tilde{p}^{\langle\mu\rangle}\tilde{p}^{\langle\nu\rangle}\tau_p^n$, ad $\Delta^{\alpha\beta\mu\nu} c_n = \int dF_p \tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tau_p^n$, with $dF_p = d\Gamma_p f^{(0)}(1 \pm f^{(0)})$.

It is to be noted here that, by the virtue of the collision integral properties $\mathcal{L}[p^\mu] = 0$ and $\mathcal{L}[1] = 0$ which follow from the energy-momentum and particle number conservation, the coefficients A^0, A^1, and B^0 cannot be determined from the transport equation (3) and hence they are called homogeneous solutions. Beyond that, A^s, B^s, and C^s can be fully estimated from the transport equation and can be called interaction solutions. In the present case they are estimated to be

$$A^2 = \frac{T}{[\tau_p^2, \tau_p^2]}\int dF_p \tau_p^2 \hat{Q}, \tag{18}$$

$$B^1 = \frac{T}{[\tau_p \tilde{p}^{\langle\mu\rangle}, \tau_p \tilde{p}^{\langle\nu\rangle}]}\int dF_p \tilde{p}^{\langle\mu\rangle}\tilde{p}^{\langle\nu\rangle}\tau_p\left(\frac{\tau_p}{\hat{h}} - 1\right), \tag{19}$$

$$C^0 = \frac{T}{[\tilde{p}^{\langle\alpha}p^{\beta\rangle}, \tilde{p}^{\langle\mu}p^{\nu\rangle}]}\int dF_p \tilde{p}^{\langle\alpha}\tilde{p}^{\beta\rangle}\tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}. \tag{20}$$

The bracketed quantities are defined as $[\phi, \phi] = \int d\Gamma_p \phi \mathcal{L}[\phi]$, which are always non-negative. The homogeneous solutions are fully arbitrary and the field corrections in Eqs. (13)–(17) due to them are attributed solely to the hydrodynamic frame choice. In certain situations the frame is so chosen that the homogeneous part exactly cancels the interaction part, giving rise to field correction zero such that the field can be identified with its equilibrium value even in the dissipative medium. In current analysis, the nonequilibrium field corrections will be kept nonzero to generate the equations of motion that give rise to the dispersion relations and finally the frequency modes.

However, these field corrections are not independent but constrained to give the dissipative flux of the same tensorial rank. The coefficients are shown to follow

$$c_\Omega - c_\Lambda\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - c_\Gamma\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0} = -\zeta, \tag{21}$$

$$c_\Sigma - \hat{h}T c_\Xi = -\frac{\lambda T}{\hat{h}}, \tag{22}$$

such that the field corrections add up to produce dissipative fluxes as

$$\delta P - \left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0}\delta\epsilon - \left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0}\delta\rho = \Pi, \tag{23}$$

$$W^\alpha - \hat{h}TV^\mu = q^\alpha. \tag{24}$$

Here, $\Pi = -T^2\int d\Gamma_p f^{(0)}(1 \pm f^{(0)})\hat{Q}\phi = -\zeta(\partial \cdot u)$ and $q^\alpha = T^2\int d\Gamma_p f^{(0)}(1 \pm f^{(0)})\tilde{p}^{\langle\alpha\rangle}(\tau_p - \hat{h})\phi = -\frac{\lambda T}{\hat{h}}\nabla^\alpha\tilde{\mu}$ are respectively the first-order bulk viscous and diffusion flow. The coefficient of bulk viscosity (ζ) and thermal conductivity

(λ) in this theory are respectively given by

$$\zeta = T^2 \int d\Gamma_p f^{(0)}(1 \pm f^{(0)})\hat{Q}A, \quad (25)$$

$$\lambda = -\frac{T}{3}\hat{h} \int d\Gamma_p f^{(0)}(1 \pm f^{(0)})\tilde{p}^\mu \tilde{p}_\mu(\tau_p - \hat{h})B. \quad (26)$$

Now, since $\hat{Q}f^{(0)}(1 \pm f^{(0)}) = \frac{1}{T}\mathcal{L}[A]$ and $\tilde{p}^{\langle\mu\rangle}(\frac{\tau_p}{\hat{h}} - 1)f^{(0)}(1 \pm f^{(0)}) = \frac{1}{T}\mathcal{L}[B^\mu]$, then by virtue of the self-adjoint property of the collision integral $\int d\Gamma_p \psi \mathcal{L}[\phi] = \int d\Gamma_p \phi \mathcal{L}[\psi]$ with $\psi = \psi(x, p^\mu)$, ζ and λ do not include the homogeneous solutions and purely depend upon interactions. Equations (21) and (22) reveal that these combinations are frame invariant as suggested by Ref. [20], and retain only the interaction part of the field corrections through the physical transport coefficients associated with dissipative fluxes. Detailed discussion of this derivation for any order of gradient expansion are available in Ref. [25]. Including field corrections, the expressions for particle four-flow and the energy-momentum tensor are respectively given as follows:

$$N^\mu = (\rho_0 + \delta\rho)u^\mu + V^\mu, \quad (27)$$

$$T^{\mu\nu} = (\epsilon_0 + \delta\epsilon)u^\mu u^\nu - (P_0 + \delta P)\Delta^{\mu\nu} + (W^\mu u^\nu + W^\nu u^\mu) + \pi^{\mu\nu}. \quad (28)$$

Here, $\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\delta T^{\alpha\beta} = 2\eta\sigma^{\mu\nu}$ is the first-order shear stress tensor with η as the shear viscous coefficient.

The next job is to implement a microscopic model that can explicitly determine the field correction coefficients from Eqs. (13)–(17). Since the homogeneous part in the field correction of Eqs. (13)–(17) can be chosen arbitrarily, here I consider only the interaction correction provided by the transport equation itself. For the same, I propose here solving the relativistic transport equation (1) in the MDRTA. The idea is just to replace $\mathcal{L}[\phi]$ in Eq. (1) with the help of relaxation time τ_R of the single particle distribution function as follows:

$$\tilde{p}^\mu \partial_\mu f = -\frac{\tau_p}{\tau_R} f^{(0)}(1 \pm f^{(0)})\phi, \quad \tau_R(x, p) = \tau_R^0(x)\tau_p^n, \quad (29)$$

where the momentum dependence of τ_R is expressed as a power law of the scaled particle energy τ_p in the comoving frame, with τ_R^0 as the momentum independent part and n as the exponent specifying the power of the scaled energy. In Refs. [25,29] the interaction part of the out of equilibrium field corrections were estimated using the MDRTA technique from the relativistic transport equation. Here, the first-order field correction coefficients are listed below:

$$\frac{c_\Lambda}{\tau_R^0} = T^2\left[\frac{z^2}{3}a_{n+1} + \left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - \frac{1}{3}\right\}a_{n+3} + \frac{1}{T}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0} a_{n+2}\right], \quad (30)$$

$$\frac{c_\Gamma}{\tau_R^0} = T\left[\frac{z^2}{3}a_n + \left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - \frac{1}{3}\right\}a_{n+2} + \frac{1}{T}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0} a_{n+1}\right], \quad (31)$$

$$\frac{c_\Omega}{\tau_R^0} = T^2\left[\frac{z^2}{9}a_{n+1} + \frac{1}{3}\left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - \frac{1}{3}\right\}a_{n+3} + \frac{1}{3T}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0} a_{n+2} - \frac{z^4}{9}a_{n-1} - \frac{z^2}{3}\left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - \frac{1}{3}\right\}a_{n+1} - \frac{z^2}{3T}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0} a_n\right], \quad (32)$$

$$\frac{c_\Sigma}{\tau_R^0} = T^2\left[\frac{1}{\hat{h}}b_{n+1} - b_n\right], \quad (33)$$

$$\frac{c_\Xi}{\tau_R^0} = T\left[\frac{1}{\hat{h}}b_n - b_{n-1}\right]. \quad (34)$$

The corresponding first-order transport coefficients bulk viscosity (ζ), thermal conductivity (λ), and shear viscosity (η) in the MDRTA are given by

$$\frac{\zeta}{T^2\tau_R^0} = \frac{z^4}{9}a_{n-1} + \left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - \frac{1}{3}\right\}^2 a_{n+3} + \frac{2z^2}{3T}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0} a_n + \frac{2}{T}\left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - \frac{1}{3}\right\}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0} a_{n+2} + \frac{1}{T^2}\left(\frac{\partial P_0}{\partial \rho_0}\right)^2_{\epsilon_0} a_{n+1} + \frac{2z^2}{3}\left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} - \frac{1}{3}\right\}a_{n+1}, \quad (35)$$

$$\frac{\lambda T}{T^2\tau_R^0} = -\left\{b_{n+1} - 2\hat{h}b_n + \hat{h}^2 b_{n-1}\right\}, \quad (36)$$

$$\frac{\eta}{T^2\tau_R^0} = \frac{1}{2}c_{n-1}. \quad (37)$$

The conservation of particle four-flow and energy-momentum tensor along with the non-negativity of entropy production rate have been confirmed within the present theory in Ref. [25].

III. CAUSALITY AND STABILITY ANALYSIS IN LOCAL REST FRAME

To analyze the modes, first small perturbations of the hydrodynamic variables are considered around a hydrostatic equilibrium state of the fluid which is in the local rest frame such as

$$T = T_0 + \delta T(t, x), \quad \tilde{\mu} = \tilde{\mu}_0 + \delta\tilde{\mu}(t, x), \quad u^\mu = (1, \vec{0}) + \delta u^\mu(t, x). \quad (38)$$

In linear approximation, the velocity perturbation has only spatial components $\delta u^\mu = (0, \delta u^x, \delta u^y, \delta u^z)$, since one needs $u_0^\mu \delta u_\mu = 0$ to retain the normalization condition. It is convenient to express these fluctuations in their plane wave solutions via a Fourier transformation $\delta\psi(t, x) \to e^{i(\omega t - kx)}\delta\psi(\omega, k)$, with wave 4-vector $k^\mu = (\omega, k, 0, 0)$. Following this prescription, the conservation equations $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$, over Eqs. (27) and (28), give the dispersion relations. Following the convention of Ref. [12], retaining the component of δu^μ parallel to k^μ, the dispersion relation for the longitudinal or sound mode is obtained as the following:

$$\omega^3(1 + Ak^2) - iB\omega^2 k^2 - \omega(Ck^2 + Dk^4) + iEk^4 = 0, \quad (39)$$

with

$$A = \hat{h}\tilde{c}_\Sigma(\tilde{c}_\Lambda - \tilde{c}_\Gamma), \tag{40}$$

$$B = (4\eta/3 + \zeta + \lambda T)/(\epsilon_0 + P_0), \tag{41}$$

$$C = c_s^2, \tag{42}$$

$$D = (4\eta/3 + \zeta)\lambda T/(\epsilon_0 + P_0)^2 + \hat{h}(\tilde{c}_\Lambda - \tilde{c}_\Gamma)\left[\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0}\tilde{c}_\Sigma + \frac{1}{\hat{h}}\frac{1}{T}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0}\tilde{c}_\Xi\right], \tag{43}$$

$$E = c_s^2\lambda T/(\epsilon_0 + P_0). \tag{44}$$

The used notations read $\tilde{c}_\Lambda = c_\Lambda/(\epsilon_0 + P_0)$, $\tilde{c}_\Sigma = c_\Sigma/(\epsilon_0 + P_0)$, $\tilde{c}_\Gamma = c_\Gamma/\rho_0$, $\tilde{c}_\Xi = c_\Xi/\rho_0$, and $c_s^2 = (\frac{\partial P_0}{\partial \epsilon_0})_{\rho_0} + \frac{1}{\hat{h}}\frac{1}{T}(\frac{\partial P_0}{\partial \rho_0})_{\epsilon_0}$ is the velocity of sound squared [35]. The coefficients B, C, E, and the first part of D being the function of physical transport coefficients associated with dissipative fluxes only (which are independent of hydrodynamic field corrections), they will be present in the usual NS theory as well, i.e., without field redefinition in the out of equilibrium scenario. However, as mentioned earlier, in most of the studies c_Λ and c_Γ are set to zero, employing a certain frame choice in order to keep the energy density and particle number density at their equilibrium values even in a dissipative medium. In such cases, A and the second part of D in Eq. (39) vanish. In such situations, propagating modes appear only at small k values with a propagation speed of usual sound velocity c_s. The problem occurs at the large k limit, where the propagating modes are changed to nonpropagating modes with k^2 dependence which indicate acausality [13]. Here the sound channel dispersion relation [Eq. (39)] is analyzed in the presence of all the field corrections.

At small k values the dispersion relation gives

$$\omega_{1,2}^{\parallel} = \frac{i}{2}\left[\frac{4\eta/3 + \zeta}{(\epsilon_0 + P_0)}\right]k^2 \pm kc_s, \tag{45}$$

$$\omega_3^{\parallel} = i\left[\frac{\lambda T}{(\epsilon_0 + P_0)}\right]k^2, \tag{46}$$

which is identical to the usual NS theory without field redefinition. $\omega_{1,2}^{\parallel}$ is the conventional propagating sound mode with a propagation velocity of the speed of sound. $\omega_3^{\parallel}$ is the purely nonpropagating heat-diffusion mode. Because of the imaginary part of all the modes being always positive by virtue of positive physical transport coefficients, the modes are always stable.

It is the large k limit that differs from the conventional NS theory. At large k, the dispersion relation renders

$$\omega_{1,2}^{\parallel} = \frac{i}{2}\left\{\frac{B}{A} - \frac{E}{D}\right\} \pm k\sqrt{\frac{D}{A}}, \tag{47}$$

$$\omega_3^{\parallel} = i\frac{E}{D}, \tag{48}$$

where positive values of D/A give two propagating modes via the real part of frequency $\omega_{1,2}^{\parallel}$. Equation (40) shows that in the absence of field redefinition in energy density and particle number density, A (as well as the second term of D) vanishes and consequently Eq. (39) produces only the nonpropagating modes at the large k limit. Since propagation speed of the fluid is characterized by the group velocity of the propagating mode, in order to analyze the causality of the mode, here the asymptotic value of group velocity (v_g) is defined as follows:

$$v_g = \lim_{k\to\infty}\left|\frac{\partial \ \mathrm{Re}(\omega)}{\partial k}\right| = \sqrt{\frac{D}{A}}. \tag{49}$$

To be subluminal, the theory must satisfy $D/A < 1$ along with $D/A > 0$. Equations (40) and (43) show that A and D explicitly depend upon the field correction coefficients. So it can be derived that, in order to preserve causality of the propagating mode at local rest frame, the coefficients must satisfy the following relation:

$$\frac{(4\eta/3 + \zeta)\lambda T/(\epsilon_0 + P_0)^2}{\hat{h}\tilde{c}_\Sigma(\tilde{c}_\Lambda - \tilde{c}_\Gamma)} < \left[1 - \left\{\left(\frac{\partial P_0}{\partial \epsilon_0}\right)_{\rho_0} + \frac{1}{\hat{h}}\frac{1}{T}\left(\frac{\partial P_0}{\partial \rho_0}\right)_{\epsilon_0}\frac{\tilde{c}_\Xi}{\tilde{c}_\Sigma}\right\}\right]. \tag{50}$$

Equation (50) is the asymptotic causality condition of the theory at local rest frame. Under the MDRTA formalism it can be shown that with small mass (typically z values below 0.25) and nonzero values of the exponent n in Eq. (29) (particularly on the negative n side), the asymptotic causality condition $0 < v_g < 1$ is indeed satisfied with linearized perturbations around LRF equilibrium.

The causal propagating mode of Eq. (47) with thermodynamic field redefinition is, however, interesting, but certainly not conclusive for verifying the causality of the theory as a whole. Since the local rest frame is not the most general equilibrium state, it is crucial to check also the situation with a more general state of equilibrium, i.e., to consider linear disturbances around the background with hydrodynamic velocity $u_0^\mu = \gamma(1, \mathbf{v})$, where the velocity $\mathbf{v}$ is nonzero and $\gamma = 1/\sqrt{1 - \mathbf{v}^2}$. Anyway, with field redefinition and at LRF background, the shear channel does not improve and gives the same mode as in NS theory, $\omega^\perp = i[\eta/(\epsilon_0 + P_0)]k^2$. Though this mode is stable, it is certainly not causal. That is why, in order to have a more rigorous study, in the next section a more general equilibrium with a boosted background is considered for linear stability and causality analysis.

IV. CAUSALITY AND STABILITY ANALYSIS IN LORENTZ-BOOSTED FRAME

The background fluid is now considered to be boosted along the x axis with a constant velocity $\mathbf{v}$, $u_0^\mu = \gamma(1, \mathbf{v}, 0, 0)$. The corresponding velocity fluctuation is $\delta u^\mu = (\gamma\mathbf{v}\delta u^x, \gamma\delta u^x, \delta u^y, \delta u^z)$ which again gives $u_0^\mu\delta u_\mu = 0$ to maintain velocity normalization. The dispersion relations can be obtained in the boosted frame by giving the transformations $\omega \to \gamma(\omega - k\mathbf{v})$ and $k^2 \to \gamma^2(\omega - kv)^2 - \omega^2 + k^2$ to the local rest frame [19].

The dispersion relation for the shear channel with boosted background turns out to be a quadratic equation of ω. Here, I address the shear modes in two limiting cases.

At the small k limit, the shear modes are

$$\omega_1^\perp = \mathbf{v}k + \mathcal{O}(k^2), \tag{51}$$

$$\omega_2^\perp = -\frac{i}{\gamma\Gamma\mathbf{v}^2} + \frac{(2 - \mathbf{v}^2)}{\mathbf{v}}k + \mathcal{O}(k^2), \tag{52}$$

with $\Gamma = \eta/(\epsilon_0 + P_0)$. In the small k limit, it is clear that the mode $\omega_1^\perp$ is a propagating mode with just the background velocity itself. The imaginary part of the other shear mode $\omega_2^\perp$ is always negative since η is a positive quantity, indicating the mode is unstable. For a background velocity $0 < \mathbf{v} < 1$, the mode is acausal as well.

At the large k limit, the shear modes become

$$\omega_{1,2}^\perp = \frac{1}{\mathbf{v}}k, \tag{53}$$

which can be readily seen as acausal for any acceptable background velocity $\mathbf{v}$. So, with a boosted background velocity, the causality and stability both are violated in the shear channel of the fluid.

With boosted background, the dispersion relation for the sound channel becomes an extremely complicated fifth-order polynomial which is not possible to solve analytically. For this difficulty, here I again address the numerical solution of the modes in two limiting cases.

In the small k limit, the sound modes become

$$\omega_1^\parallel = \mathbf{v}k + \mathcal{O}(k^2), \tag{54}$$

$$\omega_{2,3}^\parallel = \frac{1}{2}\Big[M \pm \sqrt{M^2 - 4N}\Big]k + \mathcal{O}(k^2), \tag{55}$$

$$\omega_{4,5}^\parallel = \frac{i}{2}\big[Q \pm \sqrt{Q^2 + 4R}\big] + \mathcal{O}(k), \tag{56}$$

with

$$M = \frac{2\mathbf{v}\big(c_s^2 - 1\big)}{\big(c_s^2\mathbf{v}^2 - 1\big)}, \quad N = \frac{c_s^2 - \mathbf{v}^2}{\big(c_s^2\mathbf{v}^2 - 1\big)}, \tag{57}$$

$$Q = \frac{E\mathbf{v}^2 - B}{\gamma(D\mathbf{v}^2 - A)}, \quad R = \frac{c_s^2\mathbf{v}^2 - 1}{\gamma^2\mathbf{v}^2(D\mathbf{v}^2 - A)}. \tag{58}$$

Here, the coefficients A, B, D, and E are listed in Eqs. (40)–(44). The $\mathcal{O}(k^2)$ terms of $\omega_1^\parallel$ and $\omega_{2,3}^\parallel$ are complicated functions of field correction coefficients which at $\mathbf{v} \to 0$ reduce to the same nonpropagating parts of the LRF modes of Eqs. (46) and (45), respectively. So with that, at vanishing background velocity, the modes in Eqs. (54) and (55) boil down to LRF sound modes with small wave number. It is the $\omega_{4,5}^\parallel$ modes which were not present in the local rest frame. It has been tested that no combination of the field correction coefficients can produce a positive imaginary part for $\omega_{4,5}^\parallel$ and the modes become unstable.

In the limit of large wave numbers, an expansion of the form $\omega^\parallel = v_g^\parallel k + \sum_{n=0}^{\infty} c_n k^{-n}$ can be used as a solution [15]. The roots of $v_g^\parallel$ are obtained as

$$v_{g,1}^\parallel = \mathbf{v}, \tag{59}$$

$$v_{g,2,3}^\parallel = \frac{\big[\mathbf{v}(A - D) \pm \sqrt{AD - 2AD\mathbf{v}^2 + AD\mathbf{v}^4}\big]}{A - D\mathbf{v}^2}, \tag{60}$$

$$v_{g,4,5}^\parallel = \pm\frac{1}{\mathbf{v}}. \tag{61}$$

At $\mathbf{v} \to 0$, $v_{g,1}^\parallel$ vanishes and $v_{g,2,3}^\parallel$ reduce to Eq. (49) with asymptotic group velocity $v_g = \sqrt{D/A}$ of the local rest frame. With $0 < \mathbf{v} < 1$, these three modes remain always subluminal as long as the causal parameters of the field correction coefficients are being used from condition (50). It is the two new roots $v_{g,4,5}^\parallel$ that are always acausal for $0 < \mathbf{v} < 1$. So finally it can be concluded that, although at the local rest frame the asymptotic causality condition and stability criteria are maintained, the new modes of shear and sound channels due to the boosted background are conclusively showing that the theory is acausal and unstable, irrespective of whatever values of the field correction coefficients are taken.

V. SUMMARY AND CONCLUSION

In this work, I analyze the causality and stability of a relativistic hydrodynamic theory, including the out of equilibrium field redefinition estimated from the relativistic kinetic equation. In the local rest frame, when linearized around an equilibrium, the equations of motion give a propagating mode which was previously absent for the usual NS theory, along with the asymptotic causality condition [Eq. (50)] obeyed for certain constrained values of the field correction coefficients. However, when the background fluid is boosted with an arbitrary velocity $\mathbf{v}$, it is observed that new modes appear on the top of the LRF modes, which are both acausal and unstable. This observation reveals two important points here. The first one is quite straightforward. To analyze the causality and stability of a theory, observing Fourier modes in the local rest frame is not only insufficient, but sometimes can be misleading (like the present case) as well. To have the correct conclusions, the most general equilibrium state is needed to be implemented. In fact causality is a general property of the equations of motion, and the ability of doing a Fourier analysis is not requisite. Considering this fact, in future a full nonlinear analysis with thermodynamic field redefinition is in order to study the causality and stability of the theory in a more general way.

The second point is somewhat more significant. The derivation of a first-order theory, introducing nonequilibrium field corrections in fundamental macroscopic quantities, is a recent venture which is establishing itself as an authentic framework for the relativistic hydrodynamics. In their analysis, the contribution from the homogeneous part of the nonequilibrium distribution function is attributed to generate new terms that give rise to causality and stability of the theory [18]. Motivated by these works, in the current analysis the effects of the purely interacting or inhomogeneous contribution from the distribution function have been tested in the hydrodynamic field corrections. To do that, the most general approach of gradient expansion, the Chapman-Enskog method, has been used that expresses the out of equilibrium distribution function purely in terms of spatial gradients (eliminating the time derivatives imposing conservation equations). The resulting field corrections lack the time derivatives of the general constitutive relations of the BDNK formalism and only include the spatial gradients resembling thermodynamic forces. The theory turns out to be causal and stable at the local rest frame background; however, it shows an anomaly in a more general boosted background. This is the limitation of the general Chapman-Enskog methodology which replaces the time derivatives with pure spatial gradients. The presence of terms including comoving derivatives is fundamental for causality and stability in any first-order formulation. So, an alternate microscopic approach of solving the transport equation is required [24] that retains the comoving derivatives, because this work singularly proves that they are the crucial counterparts in the hydrodynamic field corrections that definitively make the theory causal and stable.

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