Self-interacting particle-antiparticle system of bosons

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The thermodynamic properties of a system of interacting boson particles and antiparticles at finite temperatures are studied within the framework of the thermodynamically consistent Skyrme-like mean-field model. The mean field contains both attractive and repulsive terms. Self-consistency relations between the mean field and thermodynamic functions are derived. We assume a conservation of the isospin density for all temperatures. It is shown that, independently of the strength of the attractive mean field, at the critical temperature T_c the system undergoes the phase transition of second order to the Bose-Einstein condensate, which exists in the temperature interval $0 \leqslant T \leqslant T_c$. We obtained that the condensation represents a discontinuity of the derivative of the heat capacity at $T = T_c$, and condensate occurs only for the component with a higher particle-number density in the particle-antiparticle system.

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I. INTRODUCTION

Knowing the phase structure of the meson systems in the regime of finite temperatures and isospin densities is crucial for understanding a wide range of phenomena from nucleus-nucleus collisions to neutron stars and cosmology. This field is an essential part of investigations of hot and dense hadronic matter, which is a subject of active research [1]. Meanwhile, the meson systems' investigations have their specifics due to a possibility of the Bose-Einstein condensation of interacting bosonic particles. The problem of the Bose-Einstein condensation of π mesons has been studied previously, starting from the pioneer works of Migdal and coworkers (see Ref. [2] for references). Later this problem was investigated by many authors using different models and methods. The formation of classical pion fields in heavy-ion collisions was discussed in Refs. [3–6] and the systems of pions and K mesons with a finite isospin chemical potential have been considered in more recent studies [7–12]. First-principles lattice calculations provide a solid basis for our knowledge of the finite-temperature regime. Interesting new results concerning dense pion systems have been obtained recently using lattice methods [13–15].

In the present paper we consider interacting particle-antiparticle boson system at the conserved isospin density n_I and finite temperatures. We name the bosonic particles as "pions" just conventionally. The preference is made because the charged π mesons are the lightest hadrons that couple to the isospin chemical potential. On the other hand, the pions are the lightest nuclear boson particles, and, thus, an account for "temperature creation" of particle-antiparticle pairs is a relevant problem based on the quantum-statistical approach.

To account for the interaction between the bosons we introduce a phenomenological Skyrme-like mean field $U(n)$, which depends only on the total meson density n. This mean field rather reflects the presence of other strongly interacting particles in the system, for instance, ρ mesons and nucleon-antinucleon pairs at low temperatures or gluons and quark-antiquark pairs at high temperatures, $T > T_{\rm qgp} \approx 160$ MeV. Calculations for noninteracting hadron resonance gas show that the particle densities may reach values (0.1–0.2) fm^{-3} at temperatures 100–160 MeV, which are below the deconfinement phase transition, see e.g., Refs. [16,17].

The present study is a development of the approach proposed in Ref. [2], where the boson system was considered within the framework of the grand canonical ensemble with zero chemical potential. Meanwhile, here we investigate the thermodynamic properties of the meson system in the canonical ensemble, where the canonical variables are the temperature T and the isospin density n_I. We regard a studied self-interacting many-particle system as a toy model that can help us understand Bose-Einstein condensation and phase transitions over a wide range of temperatures and densities.

So, in this work, in the formulation of the canonical ensemble, we calculate the thermodynamic characteristics of a nonideal hot "pion" gas with a fixed isospin density $n_I = n_\pi^{(-)} - n_\pi^{(+)} > 0$, where $n_\pi^{(+)}$ and $n_\pi^{(-)}$ are the particle-number densities of the π^+ and π^- mesons, respectively.

We hope that our approach, which is physically transparent and clear enough, will help understand more complex pictures of the phase structure of mesonic systems arising in quark-meson models, for example, in the Nambu-Yona-Lasinio model and the lattice calculations.

In Sec. II we develop the formalism of the thermodynamic mean-field model [18] to describe the boson system of particles and antiparticles, which will be used in the present calculations. In Sec. III, we introduce a Skyrme-like parametrization of the mean field, and, after solving the system of self-consistent equations, we calculate the thermodynamic functions. In Sec. IV we demonstrate the possibility of Bose condensation when the attractive interaction is "weak." Furthermore, we determine that this is a second-order phase transition. Our conclusions are summarized in Sec. V.

II. THE MEAN-FIELD MODEL FOR THE SYSTEM OF BOSON PARTICLES AND ANTIPARTICLES

Our consideration of thermodynamic properties of the system of interacting bosonic particles and antiparticles at finite temperatures is carried out within the framework of the thermodynamic mean-field model, which was introduced in Refs. [19,20] and further developed in Ref. [18]. This approach is based on the representation of the free energy of the particle-antiparticle system as the sum of two parts: the first part is the free energy of the two-component system of free particles, and the second part is responsible for the interaction between all particles. Therefore, it is assumed that in general the free-energy density ϕ of the two-component system looks like

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\rm int}(n, T), \quad (1)$$

where $\phi_1^{(0)}$ and $\phi_2^{(0)}$ are the free-energy densities for the free particles of the first and second components, respectively, whereas the density of free energy $\phi_{\rm int}$ takes into account the interaction in the system, n_1 and n_2 is the particle-number density of each component and $n = n_1 + n_2$. Next, the chemical potential is calculated as the derivative $\mu_i = (\partial\phi/\partial n_i)_T$, where $i = 1, 2$, which results in $\mu_i^{(0)} = \mu_i - U(n)$, where $\mu_i^{(0)} = \partial\phi_i^{(0)}/\partial n$ and $U(n) \equiv \partial\phi_{\rm int}/\partial n$. Similarly, one can write the pressure $p = \mu_1 n_1 + \mu_2 n_2 - \phi$, dividing it into free and interacting parts,

$$p(n_1, n_2, T) = p_1^{(0)} + p_2^{(0)} + P(n, T), \quad (2)$$

where $p_i^{(0)} = \mu_i^{(0)} n_i - \phi_i^{(0)}$ is the pressure of the ideal gas created by the ith component of the system and $P(n, T) \equiv n(\partial\phi_{\rm int}/\partial n)_T - \phi_{\rm int}$ is the excess pressure. It is seen that the definitions of $U(n)$ and $P(n)$ lead to a differential correspondence: $n[\partial U(n, T)/\partial n]_T = [\partial P(n, T)/\partial n]_T$.

We limit our consideration to the case when at a fixed temperature the interacting boson particles and boson antiparticles are in dynamic equilibrium with respect to the processes of annihilation and pair creation. Due to the opposite sign of the charge, the chemical potentials of the bosonic particles μ_1 and the bosonic antiparticles μ_2 have opposite signs (for details, see Ref. [18]):

$$\mu_1 = -\mu_2 \equiv \mu. \quad (3)$$

Therefore, the Euler relation includes only the isospin number density $n_I = n^{(-)} - n^{(+)}$ in the following way:

$$\varepsilon + p = Ts + \mu n_I, \quad (4)$$

where $n^{(-)}$ is the particle-number density of bosonic particles, and $n^{(+)}$ is the particle-number density of bosonic antiparticles. In what follows we consider the boson particle-antiparticle system with conserved isospin number density n_I, whereas in this study the total particle-number density $n = n^{(-)} + n^{(+)}$ is a thermodynamic quantity that depends on T and n_I.[1]

Roughly speaking, in such a problem the chemical potential controls the difference of particle and antiparticle numbers $\mu \to (N^{(-)} - N^{(+)})$ whereas the total number of particles is controlled by the temperature $T \to (N = N^{(-)} + N^{(+)})$. Indeed, if some amount of particle-antiparticle pairs M has been created additionally to the existing particles $N^{(-)}$ and $N^{(+)}$ in a closed system, then approximately the same value μ is in correspondence $\mu \to [(N^{(-)} + M) - (N^{(+)} + M)]$ but $T' \to (N^{(-)} + M + N^{(+)} + M)$, where $T' > T$. This qualitative consideration indicates the existence of one-to-one correspondence of independent pairs of variables $(T, \mu) \Leftrightarrow (N, N_I)$. It is an easy task to show that the latter statement is valid in ideal quantum gas of particles and antiparticles. Meanwhile, the rigorous proof of the independence of thermodynamic variables n and n_I in a more general case where the mean fields, which depend on these variables, are present in the system (see Ref. [24]), is not so simple.

In general, the mean field U depends on both independent variables n, n_I, i.e., $U(n, n_I)$. On the other hand, as proved in Ref. [24], the mean field can be separated into n-dependent and n_I-dependent pieces where then it reads respectively for particles and antiparticles as

$$U^{(-)}(n, n_I) = U(n) - U_I(n_I), \quad (5)$$

$$U^{(+)}(n, n_I) = U(n) + U_I(n_I). \quad (6)$$

The signs in Eqs. (5) and (6) are due to odd dependence on the isospin number n_I. At the first step of the investigation, we neglect the part of the mean field which depends on isospin density, i.e., we assume $U_I(n_I) = 0$. Therefore, in this approximation, the excess pressure also depends only on the total particle-number density $P(n)$. Note that, in this version of the thermodynamic mean-field model, we do not take into account the dependence of the mean field on temperature, as is the case with the Hartree approximation.

In accordance with Eq. (2) and with taking into account that $\mu_1^{(0)} = \mu - U(n)$ and $\mu_2^{(0)} = -\mu - U(n)$, one can write the total pressure in the particle-antiparticle system as

$$\begin{aligned} p = &-gT \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - \exp\left(-\frac{\sqrt{m^2+\mathbf{k}^2} + U(n) - \mu}{T}\right)\right] \\ &-gT \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - \exp\left(-\frac{\sqrt{m^2+\mathbf{k}^2} + U(n) + \mu}{T}\right)\right] \\ &+ P(n), \end{aligned} \quad (7)$$

where $P(n)$ is the excess pressure.[2]

The thermodynamic consistency of the mean-field model can be obtained by putting in correspondence of two expressions that must coincide in the result. These expressions,

[1]The dynamical conservation of the total number of pions in a pion-enriched system created on an intermediate stage of a heavy-ion collision was considered in Refs. [21–23].

[2]Here and below we adopt the system of units $\hbar = c = 1$, $k_B = 1$.

which determine the isospin density, looks like

$$n_I = \left(\frac{\partial p}{\partial \mu}\right)_T, \tag{8}$$

where pressure is given by Eq. (7), and

$$n_I = g\int \frac{d^3k}{(2\pi)^3}[f(E(k,n),\mu) - f(E(k,n),-\mu)]. \tag{9}$$

Here $E(k,n) = \omega_k + U(n)$ with $\omega_k = (m^2 + \boldsymbol{k}^2)^{1/2}$ and the Bose-Einstein distribution function reads

$$f(E,\mu) = \left[\exp\left(\frac{E-\mu}{T}\right) - 1\right]^{-1}. \tag{10}$$

In order for the expressions (8) and (9) to coincide in the result, the following relation between the mean field and the excess pressure arises:

$$n\frac{\partial U(n)}{\partial n} = \frac{\partial P(n)}{\partial n}. \tag{11}$$

It provides the thermodynamic consistency of the model. At the same time, as we have shown in the beginning of the current section, this differential correspondence results from the definitions of the mean field $U(n)$ and excess pressure $P(n)$.

When both components of π^--π^+ system are in the thermal (kinetic) phase, the pressure and energy density read

$$p = \frac{g}{3}\int \frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2}{\omega_k}[f(E(k,n),\mu) + f(E(k,n),-\mu)] + P(n), \tag{12}$$

$$\varepsilon = g\int \frac{d^3k}{(2\pi)^3}E(k,n)[f(E(k,n),\mu) + f(E(k,n),-\mu)] - P(n). \tag{13}$$

III. SKYRME-LIKE PARAMETRIZATION OF THE MEAN FIELD

The thermodynamic mean-field model has been applied for several physically interesting systems, including the hadron-resonance gas [18] and the pionic gas [25]. This approach was extended to the case of a bosonic system at $\mu = 0$, which can undergo Bose condensation [2,26]. In the present study, a generalized formalism given in Sec. II is used to describe the particle-antiparticle system of bosons when the isospin density is kept constant. As was mentioned in the previous section, the mean field in general case splits into two pieces with dependence on the total particle density n and on the isospin density n_I, respectively, see Eqs. (5) and (6). At the first stage of our investigation we assume that the interaction between particles is described by the Skyrme-like mean field, which depends only on the total particle-number density n. Loosely speaking, we take into account just a strong interaction. So, we assume that the mean field reads

$$U(n) = -An + Bn^2, \tag{14}$$

where A and B are the model parameters, which should be specified. Some additional contribution to the attractive mean field at high temperatures, ($T \propto$ 100–160 MeV), may be provided by other hadrons present in the system like ρ mesons [27] or baryon-antibaryon pairs [28]. As was mentioned in the introduction, an investigation of the properties of a dense and hot pion gas is well inspired by the formation of the medium with low baryon numbers at midrapidity what was proved in the experiments at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) [29,30].

For this reason, in our calculations, we consider a general case of $A > 0$, to study a bosonic system with both attractive and repulsive contributions to the mean field (14). For the repulsive coefficient B we use a fixed value obtained from an estimate based on the virial expansion [31], $B = 10mv_0^2$ with v_0 equal to four times the proper volume of a particle, i.e., $v_0 = 16\pi r_0^3/3$. In our numerical calculations we take $v_0 = 0.45$ fm^3, which corresponds to a "particle radius" $r_0 \approx 0.3$ fm. The numerical calculations will be done for bosons with mass $m = 139$ MeV, which we call conventionally "pions." In this case, the repulsive coefficient is $B/m = 2.025$ fm^6, and it is kept constant through all present calculations. (For instance, in Ref. [32], the authors use the value $B/m = 21.6$ fm^6.) At the same time, the coefficient A, which determines the intensity of attraction of the mean field (14), will be varied. It is advisable to parametrize the coefficient A. We are going to do this by making use of solutions of the equation $U(n) + m = 0$, similar to the parametrization adopted in Refs. [2,26]. For the given mean field (14) there are two roots of this equation $[n_{1,2} = [A \mp (A^2 - 4mB)^{1/2}]/2B]$,

$$n_1 = \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \quad n_2 = \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right), \tag{15}$$

where

$$\kappa \equiv \frac{A}{2\sqrt{mB}}. \tag{16}$$

Then, one can parametrize the attraction coefficient as $A = \kappa A_c$ with $A_c = 2\sqrt{mB}$. As we will show below, the dimensionless parameter κ is the scale parameter of the model. When we fix the isospin density, the parameter κ determines the phase structure of the system. As it is seen from Eq. (15) for the values of parameter $\kappa < 1$ there are no real roots. The critical value A_c is obtained when both roots coincide, i.e. when $\kappa = \kappa_c = 1$, then $A = A_c = 2\sqrt{mB}$.

In general, there are two intervals of the parameter κ. (1) The first interval corresponds to $\kappa \leqslant 1$, there are no real roots of equation $U(n) + m = 0$. We associate these values of κ with a weak attractive interaction, and in the present study, we consider variations in the attraction coefficient A for values of κ only from this interval. (2) The second interval corresponds to $\kappa > 1$, there are two real roots of equation $U(n) + m = 0$. We associate this interval with a "strong" attractive interaction. This case will be considered elsewhere.

If one assumes a possibility of the Bose-Einstein condensation in the two-component system, then it is instructive to classify a phase structure of the system by two basic combinations which determine for the weak attraction the different thermodynamic states: (i) Both components, or the boson particles and boson antiparticles, i.e., π^- and π^+, are in the thermal (kinetic) phase; (ii) Particles (π^-) are in the condensate phase, and antiparticles (π^+) are in the thermal (kinetic) phase—this combination can be named the "cross" state.

Note that the expression "particles are in the condensate phase" is, of course, a conventional one because, in essence, it is a mixture phase, where at a fixed temperature, a fraction of π^- mesons is in thermal states with momentum $|\boldsymbol{k}| > 0$ and another fraction of this π^- component belongs to the Bose-Einstein condensate, where all π^- mesons have zero momentum, $\boldsymbol{k} = 0$.

We are going now to consider these basic thermodynamic states of the system using the mean field (14).

IV. THERMODYNAMIC PROPERTIES OF THE BOSON PARTICLE-ANTIPARTICLE SYSTEM UNDER WEAK ATTRACTION

In the mean-field approach, the behavior of the particle-antiparticle bosonic system in the thermal (kinetic) phase is determined by the set of two transcendental equations (we keep $n_I = \text{const.}$)

$$n = \int \frac{d^3k}{(2\pi)^3}[f(E(k,n),\mu) + f(E(k,n),-\mu)], \quad (17)$$

$$n_I = \int \frac{d^3k}{(2\pi)^3}[f(E(k,n),\mu) - f(E(k,n),-\mu)], \quad (18)$$

where the Bose-Einstein distribution function $f(E,\mu)$ is defined in (10) and $E(k,n) = \omega_k + U(n)$. Equations (17)–(18) should be solved self-consistently with respect to n and μ for a given temperature T with account for $n_I = \text{const.}$ In the present, we consider the boson system in the canonical ensemble, where the independent canonical variables are T and n_I and the particle spin equals zero. In this approach, the chemical potential μ is a thermodynamic quantity that depends on the canonical variables, i.e., $\mu(T, n_I)$.

In case of the cross state, when the particles, i.e., π^- mesons, are in the condensate phase and antiparticles are still in the thermal (kinetic) phase, Eqs. (17), (18) should be generalized to include condensate component $n^{(-)}_{\text{cond}}$. Besides this, we should take into account that the particles (π^- or high-density component) can be in the condensed state just under the necessary condition:

$$U(n) - \mu = -m. \quad (19)$$

As the temperature decreases from high values, when both π^- and π^+ are in the thermal phase, the density of the π^- component, namely, $n^{(-)}(T,\mu)$, crosses the critical curve at the temperature $T_c^{(-)}$, where the condition (19) is satisfied. The latter means that the curve $n^{(\text{id})}_{\text{lim}}(T)$, which is defined as

$$n^{(\text{id})}_{\text{lim}}(T) = \int \frac{d^3k}{(2\pi)^3} f(\omega_k,\mu)\Big|_{\mu=m}, \quad (20)$$

is the critical curve for π^- mesons or the high-density component. Here $f(\omega_k,\mu)$ is the Bose-Einstein distribution function defined in (10). As we see function (20) represents the maximal density of thermal (kinetic) boson particles of the ideal gas at temperature T when $\mu = m$. Hence, we obtain that the critical curve in the mean-field approach under consideration for the boson particles coincides with the critical curve for the ideal gas.

With account for Eqs. (19) and (20) we write the generalization of the set of Eqs. (17) and (18)

$$n = n^{(-)}_{\text{cond}}(T) + n^{(\text{id})}_{\text{lim}}(T) + \int \frac{d^3k}{(2\pi)^3} f(E(k,n),-\mu), \quad (21)$$

$$n_I = n^{(-)}_{\text{cond}}(T) + n^{(\text{id})}_{\text{lim}}(T) - \int \frac{d^3k}{(2\pi)^3} f(E(k,n),-\mu). \quad (22)$$

Meanwhile, using relation (19) between the mean field and the chemical potential, this set of equations can be reduced to just one equation for $n^{(+)}$:

$$n^{(+)} = \int \frac{d^3k}{(2\pi)^3} f(E(k,n),-\mu)\Big|_{\mu=U(n)+m}, \quad (23)$$

where $U(n) = U(2n^{(+)} + n_I)$ and $E(k,n) = \omega_k + U(2n^{(+)} + n_I)$. Solution of Eq. (23) for temperatures T from the interval $T < T_c^{(-)}$ provides the density $n^{(+)}(T)$ of π^+ mesons.

One can see from Eqs. (21) and (22) that the particle-number density $n^{(+)}$ is provided only by thermal π^+ mesons. Whereas, the density $n^{(-)}$ is provided by two fractions: the condensed particles (π^- mesons at $\boldsymbol{k} = 0$) with the particle-number density $n^{(-)}_{\text{cond}}(T)$, and thermal π^- mesons at $|\boldsymbol{k}| > 0$ with the particle-number density $n^{(\text{id})}_{\text{lim}}(T)$. The particle-density sum rule for these phase of π^- mesons in the interval $T < T_c^{(-)}$ reads

$$n^{(-)} = n^{(-)}_{\text{cond}}(T) + n^{(\text{id})}_{\text{lim}}(T). \quad (24)$$

Numerical results

At high temperatures, i.e., $T \geqslant T_c^{(-)}$, both components of the bosonic particle-antiparticle system are in the thermal phase and thermodynamic properties of the system are determined by the set of Eqs. (17) and (18). Solving this set for given values T and n_I we obtain the functions $\mu(T, n_I)$ and $n(T, n_I)$ and then other thermodynamic quantities.

When we decrease temperature, after crossing the value $T = T_c^{(-)}$ the particles which belong to the high-density component (or π^- mesons) start to "drop down" into the condensate state, which is characterized by the value of momentum $\boldsymbol{k} = 0$. In the limit, when $T = 0$, all particles of the high-density component, i.e., π^- mesons, are in condensed state and $n^{(-)} = n_I$. At the same time, the particles of the low-density component or π^+ mesons being in the thermal phase lose the density $n^{(+)}$ with a decrease of temperature, and it becomes rigorously zero at $T = 0$. For the temperature interval $T < T_c^{(-)}$ equations (17), (18) should be generalized and now thermodynamic properties of the system are determined by Eq. (23), where we take into account that $\mu = -U(n) + m$ for all temperatures of this interval unless the high-density component $n^{(-)}$ is in the condensed state. Otherwise it is necessary to solve the set of equations (17) and (18) for the region where $n^{(-)}$ appears again in the thermal (kinetic) phase.

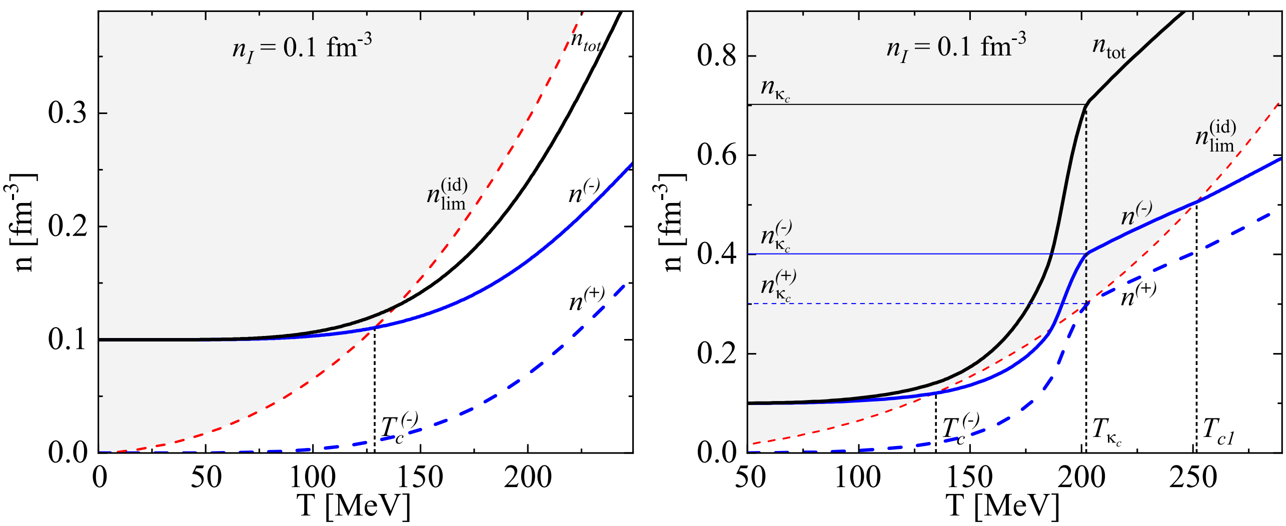

FIG. 1. (left panel) The particle-number densities $n^{(+)}$, $n^{(-)}$, and $n_{\mathrm{tot}} = n^{(+)} + n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_{\mathrm{I}} = 0.1\ \mathrm{fm}^{-3}$, and the attraction parameter is $\kappa = 0.5$. The maximum density $n_{\mathrm{lim}}^{(\mathrm{id})}$ of the ideal gas of thermal pions at $\mu = m_\pi$ is shown by the red dashed line. The shaded area shows the possible states of condensed particles. The Bose-Einstein condensation of π^- mesons occurs at the temperature $T_{\mathrm{c}} = T_{\mathrm{c}}^{(-)}$. (right panel) The same as on the left panel, but with the parameter $\kappa = \kappa_{\mathrm{c}} = 1$. Here $n_1 = n_2 \equiv n_{\kappa_{\mathrm{c}}}$ [see Eq. (15)], $n_{\kappa_{\mathrm{c}}}^{(-)} = (n_{\mathrm{tot}} + n_I)/2$, $n_{\kappa_{\mathrm{c}}}^{(+)} = (n_{\mathrm{tot}} - n_I)/2$ and $T_{\kappa_{\mathrm{c}}}$ is the temperature at which the curve $n^{(+)}(T)$ touches the critical curve $n_{\mathrm{lim}}^{(\mathrm{id})}$.

For parameters $n_{\mathrm{I}} = 0.1\ \mathrm{fm}^{-3}$, $\kappa = 0.5$, and $\kappa = 1.0$ we solve the set of equations (17) and (18) for the thermal phase and Eq. (23) for the "cross" thermodynamic state. The behavior of the density $n^{(+)}$ of π^+ mesons and the density $n^{(-)}$ of π^- mesons are depicted in Fig. 1. In this figure, we also depicted the behavior of the total density of mesons $n = n^{(+)} + n^{(-)}$ depending on temperature (in the field of the figure, this density is denoted as n_{tot}).

Analyzing the behavior of the condensate creation (see Fig. 1), it is necessary to note that just the high-density component of the particle-antiparticle gas undergoes the phase transition to the Bose-Einstein condensate. If we apply our consideration to pion gas with $n_I = n_\pi^{(-)} - n_\pi^{(+)} > 0$ this means that the π^- component undergoes the phase transition to the Bose-Einstein condensate and the low-density component or π^+ mesons exist only in the thermal phase for the whole range of temperatures. Hence, it makes sense to look at $T_{\mathrm{c}} = T_{\mathrm{c}}^{(-)}$ for the Bose-Einstein condensate of π^- mesons only in the lattice calculations and in an experiment, for instance, in heavy-ion collisions.

At the same time, the temperature behavior of the particle-number density $n^{(+)}$ (see Fig. 1) is very similar to the behavior of the pion density $n(T)$ for $\kappa \leqslant 1$ obtained in Ref. [2], where the pion system at $\mu = 0$ was investigated. Note that we consider the system of pions only for weak attraction in the present study, i.e., at $\kappa \leqslant 1$. As was shown in Ref. [2] the behavior of the pion system at $\kappa > 1$ is drastically different. In this case, with an increase in temperature at $T = T_{\mathrm{cd}} < T_{\mathrm{c}}$, the system undergoes the first-order phase transition.

1. The critical temperature

Equation (21) can be used to determine the critical temperature $T_{\mathrm{c}}^{(-)}$. Indeed, let us take into account that, at the crossing point with the critical curve, the density of condensate is zero so far, $n_{\mathrm{cond}}^{(-)}(T_{\mathrm{c}}^{(-)}) = 0$, and the density of thermal π^- particles becomes equal to $n^{(-)}(T_{\mathrm{c}}^{(-)}) = n_{\mathrm{lim}}^{(\mathrm{id})}(T_{\mathrm{c}}^{(-)})$. Then, at this temperature $T = T_{\mathrm{c}}^{(-)}$ on the left-hand side (l.h.s.) of Eq. (21) we have $n = 2n_{\mathrm{lim}}^{(\mathrm{id})}(T_{\mathrm{c}}^{(-)}) - n_I$, and now at this temperature point on the critical curve Eq. (21) with respect to T reads

$$n_{\mathrm{lim}}^{(\mathrm{id})}(T) - n_I = \int \frac{d^3k}{(2\pi)^3} f(E(k,n), -\mu)\Big|_{\mu = U(n)+m}$$
$$\text{with} \qquad E(k,n) = \omega_k + U\big(2n_{\mathrm{lim}}^{(\mathrm{id})} - n_I\big). \quad (25)$$

Solving Eq. (25) at $n_I = 0.1\ \mathrm{fm}^{-3}$, for $\kappa = 0.5$ and $\kappa = \kappa_{\mathrm{c}} = 1$ we obtained $T_{\mathrm{c}}^{(-)} = 129$ MeV and $T_{\mathrm{c1}}^{(-)} = 251$ MeV, respectively. These results are depicted in Fig. 1 in the left and right panels, respectively.

It turns out that $T_{\mathrm{c}}^{(-)}$ is the critical temperature, which determines the phase transition with the formation of a BEC for the entire pion system since antiparticles or π^+ mesons, which represent the low-density component $n^{(+)}(T)$, are entirely in a thermal state for all temperatures. Thus, condensate is created only by particles or by π^- mesons, i.e., $n_{\mathrm{cond}} = n_{\mathrm{cond}}^{(-)}$, and this particle-number density plays the role of the order parameter.

The condensate densities as functions of temperature obtained in the framework of our model for three values of the attraction parameter, $\kappa = 0.0,\ 0.5,\ 1.0$, and for three values of the isospin density, $n_{\mathrm{I}} = 0.04,\ 0.07,\ 0.1\ \mathrm{fm}^{-3}$, are depicted in Fig. 2, left panel. We record a minimal difference in the critical temperature $T_{\mathrm{c}}^{(-)}$ when the attraction parameter κ changes, the difference does not exceed 4 MeV when $n_{\mathrm{I}} = 0.1\ \mathrm{fm}^{-3}$. This difference is much less, as we can see in Fig. 2 for smaller isospin densities. Then it would be helpful to define only one average value of $T_{\mathrm{c}}^{(-)}$ as

$$T_{\mathrm{c}} = \langle T_{\mathrm{c}}^{(-)} \rangle. \quad (26)$$

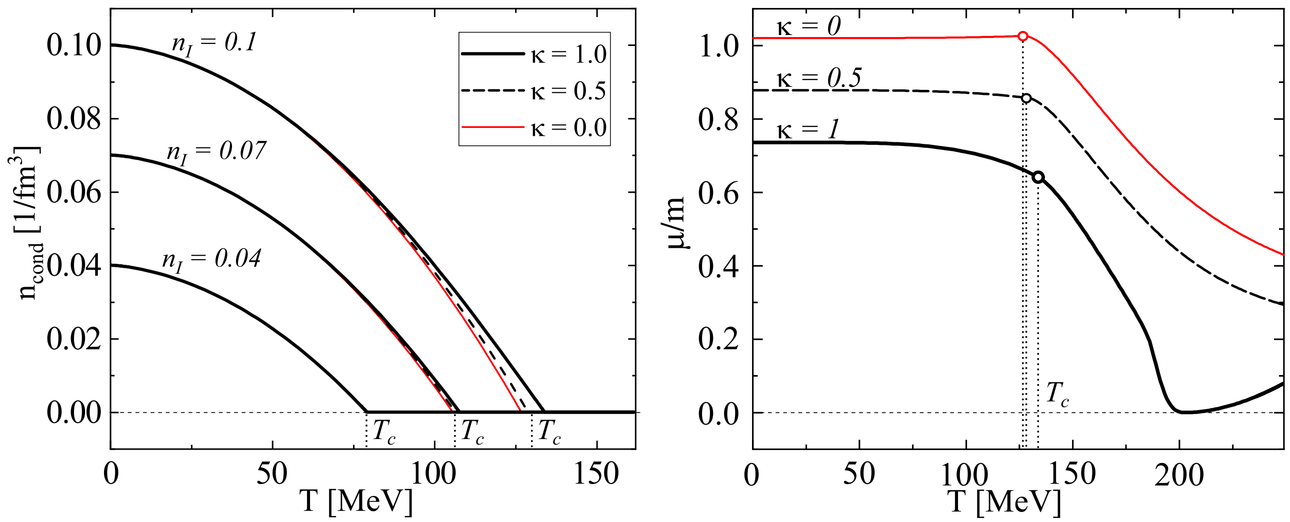

FIG. 2. (left panel) The density of condensate versus temperature in the particle-antiparticle self-interacting system for three values of the isospin density, $n_\mathrm{I} = 0.04,\ 0.07,\ 0.1\ \mathrm{fm}^{-3}$.(right panel) The chemical potential versus temperature at values of the attraction parameter $\kappa = 0.0,\ 0.1,\ 0.5,\ 1.0$ and the isospin density $n_\mathrm{I} = 0.1\ \mathrm{fm}^{-3}$. The marked points on the curves correspond to the critical temperature $T_\mathrm{c}^{(-)}$. In both panels we set $T_\mathrm{c} = \langle T_\mathrm{c}^{(-)} \rangle$.

For example for $n_\mathrm{I} = 0.1\ \mathrm{fm}^{-3}$ the averaging gives $T_\mathrm{c} \approx 129$ MeV. The temperature T_c "signals" the creation of condensate when temperature decreases and crosses this value. Note that the critical temperature T_c is practically independent of the attraction parameter A of the mean field (14). In other words, the average attraction between particles in the system has little effect on the critical temperature.

The dependence of the chemical potential on temperature is depicted in Fig. 2 in the right panel for three values of the attraction parameter, $\kappa = 0.0,\ 0.1,\ 0.5,\ 1.0$. First of all, we notice that the chemical potential is almost independent of temperature when condensate exists in the system, i.e., in the interval $0 < T \leqslant T_\mathrm{c}$. The value of μ changes from $1.02 m_\pi$ at the absence of attraction, $\kappa = 0.0$, to $\mu = 0.74 m_\pi$ for the critical attraction parameter $\kappa = 1.0$. Hence, for $0 \leqslant \kappa \leqslant 1$ the chemical potential is in the range $103 \leqslant \mu \leqslant 142$ MeV. It is intriguing to remind that already first attempts to fit the p_T spectra of π^- mesons in O + Au collisions at 200 AGeV/nucleon (at midrapidity) by the ideal-gas Bose-Einstein distribution results in the values $\mu \approx 126$ MeV, $T \approx 167$ MeV and in S + S collisions at 200 AGeV/nucleon it results in the values $\mu \approx 118$ MeV, $T \approx 164$ MeV [33]. So, the fit of data required the pion chemical potential in the range $\mu \approx 115$–130 MeV, which we can formally compare with the values of the chemical potential obtained in our model.

2. The heat capacity

The derivative of the chemical potential on temperature has a jump in points marked on the curves as small black circles, see Fig. 2, right panel. These points on the curves $\mu(T)$ correspond to $T_\mathrm{c}^{(-)}$, which values differ from one another not more than $\Delta T = 4$ MeV. As we concluded before, this is the temperature of phase transition, see Eq. (26), which practically does not depend on the intensity of attraction. To prove that this is indeed a phase transition of the second order, we first calculate the heat capacity c_v as[3]

$$c_\mathrm{v} = -T\frac{\partial^2 \phi}{\partial T^2}, \tag{27}$$

where $\phi(T, n_I) = -p(T, n_I) + n_I \mu(T, n_I)$ is the density of free energy. We are going to calculate $\phi(T, n_I)$ for two thermodynamic scenarios, when $T \geqslant T_\mathrm{c}$ and when $T < T_\mathrm{c}$.

Having solved Eqs. (17) and (18) then, using Eq. (12) one can calculate pressure for the case when particles and antiparticles are both in the thermal phase, i.e., $T \geqslant T_\mathrm{c}$. In this case, the density of free energy looks like

$$\begin{aligned}\phi = n_I \mu(T, n_I) - \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2}{\omega_k}[f(E(k,n),\mu) \\ + f(E(k,n),-\mu)] - P(n),\end{aligned} \tag{28}$$

where functions $n(T, n_I)$ and $\mu(T, n_I)$ are known. The excess pressure $P(n)$ is obtained by integrating Eq. (11) for the Skyrme-like parametrization of the mean field (14):

$$P(n) = -\frac{A}{2}n^2 + \frac{2B}{3}n^3, \tag{29}$$

where $P(n = 0) = 0$ is taken into account.

For temperatures less than T_c, when the high-density component of the pion gas (π^- mesons) is in the condensate phase, and the low-density component (π^+ mesons) is in the thermal

[3] As a matter of fact, here we calculate the volumetric heat capacity, which is the heat capacity C_V of a system divided by the volume V, i.e., $c_\mathrm{v} = C_\mathrm{V}/V$.

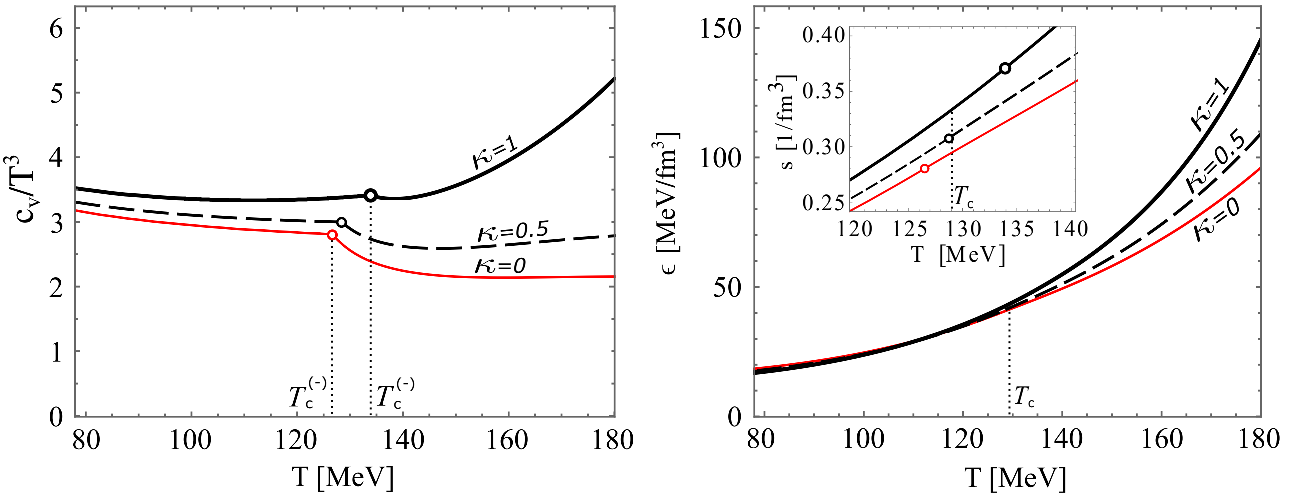

FIG. 3. (left panel) Heat capacity normalized to T^3 as a function of temperature in a self-interacting π^--π^+ meson system. The isospin density is kept constant, $n_I = 0.1$ fm^{-3}. The curves are marked with the attraction parameter κ. (right panel) Energy density versus temperature for the same meson system and the same conditions as in the left panel. The entropy density versus temperature in the vicinity of T_c is shown in a small window for attraction parameters $\kappa = 0,\ 0.5,\ 1.0$. We set $T_c = \langle T_c^{(-)} \rangle$.

phase, the density of the free energy reads

$$\phi = n_I[U(n)+m] - \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2}{\omega_k} f(\omega_k,\mu)\Big|_{\mu=m} - \frac{1}{3}\int \frac{d^3k}{(2\pi)^3}\frac{\mathbf{k}^2}{\omega_k} f(E(k,n),-\mu)\Big|_{\mu=U(n)+m} - P(n). \tag{30}$$

Here the total pion density is $n = 2n^{(+)} + n_I$, $\mu = U(n) + m$ as in Eq. (23), $E(k,n) = \omega_k + U(n)$ and $n^{(+)}(T, n_I)$ is the solution of Eq. (23).

Using the density of free energy (28) to the right of T_c and Eq. (30) to the left of T_c, respectively, we calculate the heat capacity normalized to T^3, as function of temperature at $n_I = 0.1$ fm^{-3} for three values of the attraction parameter $\kappa = 0,\ 0.5,\ 1.0$. These dependencies are depicted in Fig. 3 in the left panel. The temperature dependence of the heat capacity is a continuous function. However, the derivative of this function has a finite discontinuity, which indicates a second-order phase transition, where the condensate density is an order parameter (strictly speaking, this is a third-order phase transition). To make sure that this is indeed a second-order phase transition without the release of latent heat at the temperature T_c, we calculate the energy density ε for the same set of parameters κ, and the functions $\varepsilon(T)$ are shown in Fig. 3 in the right panel. To be sure that the first derivative of the free energy is a smooth function, we calculate the entropy density $s = -\partial\phi(T, n_I)/\partial T$, and its dependence on temperature in the vicinity of T_c is shown in a small window in Fig. 3 in the right panel for three values of the attraction parameter $\kappa = 0,\ 0.5,\ 1.0$. Indeed, one can see that the temperature dependencies of the energy density and entropy density are continuous and smooth at $T = T_c$, which proves that the system undergoes a second-order phase transition at this temperature. It is also interesting to note that the energy density in the temperature interval $0 < T \leqslant T_c$ is practically independent of the weak attraction ($0 \leqslant \kappa \leqslant 1$) between the particles.

We will now fix some similarities between the picture obtained above for the interacting two-component particle-antiparticle system when $n_I =$ const. and the single-component ideal gas, where we keep constant the particle-number density n. First, the behavior of the high-density component in the "condensate" temperature interval $T \leqslant T_c$ in the system with interaction is similar to the behavior of the single-component ideal gas ($m = m_\pi$) when $n_I = n =$ const. Indeed, that is seen when one compares the dependance $n = n^{(-)}(T)$ in Fig. 1 and dependence $n =$ const. in Fig. 4 in the left panel. Next, we compare the heat capacities in these two-boson systems, a question of particular interest is the behavior of the heat capacity at the critical temperature. For the ideal gas, it is natural to treat the problem in the canonical ensemble, where the canonical variables are T and n. The critical temperature T_c is the starting point for the onset of condensation when the temperature is decreasing. For a given density n the critical temperature can be determined as solution of the transcendental equation $n = n_{\rm lim}^{\rm (id)}(T_c)$, where $n_{\rm lim}^{\rm (id)}(T)$ is defined in (20).

The energy density in the condensate phase consists of two contributions: for the ideal gas it reads

$$\varepsilon = mn_{\rm cond}(T) + \int \frac{d^3k}{(2\pi)^3}\omega_k f(\omega_k,\mu)\Big|_{\mu=m}, \tag{31}$$

where $n_{\rm cond}(T) = n - n_{\rm lim}^{\rm (id)}(T)$. We calculate the heat capacity $c_v = \partial\varepsilon(T,n)/\partial T$, which is attributed to the condensate phase, and obtain

$$c_v^{\rm (cond)} = \frac{1}{4T^2}\int \frac{d^3k}{(2\pi)^3}\left[\frac{E_{\rm kin}}{\sinh(E_{\rm kin}/T)}\right]^2, \tag{32}$$

where $E_{\rm kin} = \omega_k - m$ is the single-particle kinetic energy. It is seen that the dependence of the heat capacity $c_v^{\rm (cond)}(T)$ in the condensate phase is of a universal character. In this phase, the

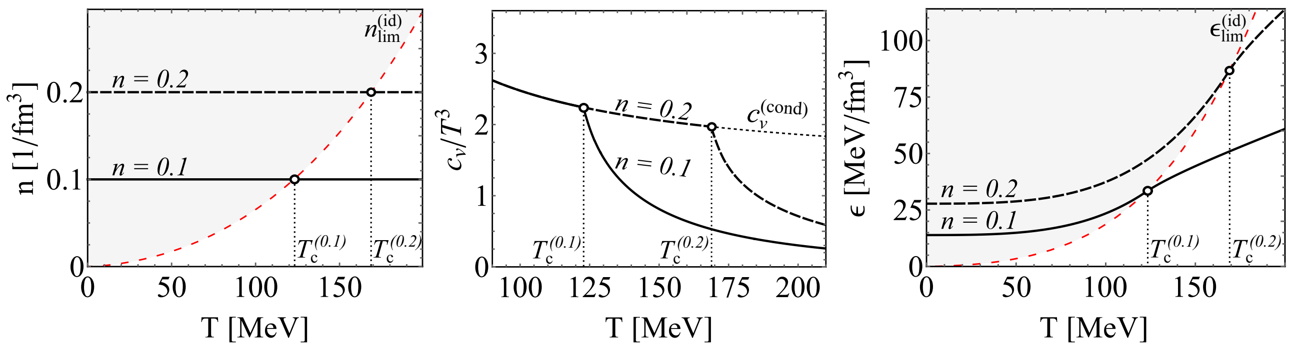

FIG. 4. (left panel) Particle-number density versus temperature in the ideal single-component gas. The horizontal lines represent two constant particle density samples, $n = 0.1,\ 0.2\ \text{fm}^{-3}$, which correspond to critical temperatures $T_c^{(0.1)}$, $T_c^{(0.2)}$, respectively. (middle panel) Heat capacity normalized to T^3 as a function of temperature in the ideal single-component gas where the particle-number density is kept constant. (right panel) Energy density versus temperature for the same system and conditions as in the left panel. The red dashed line marked as $\epsilon_{\text{lim}}^{(\text{id})}$ represents the energy density of the states that belong to the critical curve $n_{\text{lim}}^{(\text{id})}$ depicted in the left panel.

heat capacity of an ideal gas does not depend on the particle density n. For each specific value n, the curve $c_{\text{v}}^{(\text{cond})}(T)$, defined in (32), is bounded at the right end by the value of T_c, which in turn, depends on the given particle-number density n. This feature is seen in Fig. 4 in the middle panel, where we consider two samples of the density $n = 0.1,\ 0.2\ \text{fm}^{-3}$. It is evidently seen that the derivative of the heat capacity has a finite discontinuity, which can indicate a third-order phase transition. To be sure about that, we calculate the energy density for the same samples of the particle-number densities, the functions $\varepsilon(T)$ are shown in Fig. 4 in the right panel. We see that these functions are continuous and smooth at $T = T_c$ and this proves an absence of latent-heat release at the critical temperature.

Let us briefly summarize the results obtained for an interacting particle-antiparticle boson system, where the isospin (charge) density n_I is conserved, and for a single-component ideal gas, where the particle-number density n remains constant. First of all, we claim that they both have the same critical curve $n_{\text{lim}}^{(\text{id})}(T)$. Furthermore, when $n^{(-)}(T)$, obtained for an interacting system, and $n(T)$, obtained for an ideal gas, intersects the critical curve $n_{\text{lim}}^{(\text{id})}(T)$, respectively, both systems undergo a phase transition of the second-order or following the Ehrenfest classification of the third order.

It has long been known, see Ref. [34], that the Bose-Einstein condensation is indeed a third-order phase transition according to the first classification of general types of transitions between phases of matter, introduced by Paul Ehrenfest in 1933 [35,36]. Therefore, the obtained temperature T_c is really the temperature of the phase transition of the second order (according to modern terminology) and the density of condensate $n_{\text{cond}} = n_{\text{cond}}^{(-)}$ provided by π^- mesons is the order parameter.

V. CONCLUDING REMARKS

In this paper, we have presented a thermodynamically consistent method to describe at finite temperatures a dense bosonic system that consists of interacting particles and antiparticles at a fixed isospin density n_I. We considered the system of meson particles with $m = m_\pi$ and zero spin, which we named conventionally as "pions" because the charged π-mesons are the lightest nuclear particle and the lightest hadrons that couple to the isospin chemical potential.

It turns out that the introduced dimensionless quantity $\kappa = A/2\sqrt{mB}$, which is itself a combination of the mean-field parameters A, B and the value of a particle mass, is the scale parameter of the model. Furthermore, it determines the different possible phase scenarios which occur in the particle-antiparticle boson system. The attraction coefficient $A = \kappa A_c$, where $A_c \equiv 2\sqrt{mB}$, was parametrized by κ with $\kappa = 1$ as the critical value that separates the regime of a weak attraction ($\kappa \leqslant 1$) from the regime of a strong attraction ($\kappa > 1$). In this paper, we only looked at the weak-attraction case.

It was shown that in the particle-antiparticle meson system, where the isospin density n_I is conserved, there is a Bose-Einstein condensate in the system in the temperature interval $0 \leqslant T \leqslant T_c$, which is the result of a second-order phase transition that occurs at a temperature T_c and condensate density is an order parameter.[4] This statement is in contrast with the conclusion given in Refs. [2,26,32,37], where the system with zero chemical potential, $\mu = 0$, was investigated. Indeed, in these works it was shown that in the case of a sufficiently strong attractive mean field ($\kappa > 1$), the multibosonic system undergoes a first-order phase transition and, as a result, develops a Bose condensate, starting from a finite temperature.

So, we obtained that, independently of parameters of the mean field, the multiboson system develops the Bose condensate for particles of the high-density component only. This means that, in the pion gas where $n_I = n_\pi^{(-)} - n_\pi^{(+)} > 0$, the π^- mesons only undergo the phase transition to the Bose-Einstein condensate. At the same time, the π^+ mesons exist only in the thermal phase for the whole range of temperatures. Then, for the experimental efforts, it makes sense to look for the Bose condensate, which is created just by π^- mesons.

For the description of the system's thermodynamic properties, we use the canonical ensemble formulation, where

[4] Note that the chiral perturbation theory predicts that the transition between the vacuum and the BEC state is of the second order with universality class $O(2)$ [7].

the chemical potential μ is a thermodynamic quantity that depends on the canonical variables (T, n_I). We calculated dependence of the chemical potential on temperature for different attraction parameters κ which show that $\mu \approx$ const. in the "condensate" interval of temperatures $0 \leqslant T \leqslant T_c$, where these constant values depend on the intensity of attraction. Meanwhile, the temperature $T_c^{(-)}$ of the phase transition to the Bose-Einstein condensate of π^- mesons (high-density component) exhibits a weak dependence on κ, as one can see in Fig. 2 in the left panel. For all values $0 \leqslant \kappa \leqslant 1$ that we have considered, these critical temperatures differ from one another by not more than 4 MeV, which inspires an introduction of the mean value $T_c = \langle T_c^{(-)} \rangle$ of the phase transition to the Bose-Einstein condensate.

The results obtained are in correspondence with known peculiar property of the ideal Bose gas: the Bose-Einstein condensation represents the third-order phase transition or a discontinuity of the derivative of the specific heat [34]. In the framework of the presented model, we obtained that, in the same way, the derivative of the specific heat undergoes a break at the temperature T_c, as one can see in the left panel of Fig. 3. The smooth dependencies of the energy density and entropy density on temperature and the absence of latent-heat release at T_c can be seen in the right panel of Fig. 3, which proves that the system actually undergoes a second-order phase transition at this temperature.

The role of neutral pions is beyond the scope of the present paper. The present analysis can be improved by addressing these issues in more detail and generalizing the calculation to nonzero contribution to the mean field that depends on n_I. The authors plan to consider these problems elsewhere.

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