

Quantum fluctuation in an inhomogeneous background and its influence on the phase transition in a finite volume system

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(Received 30 August 2021; revised 24 December 2021; accepted 1 April 2022; published 15 April 2022)

We studied the grand potential and phase transitions of an inhomogeneous finite volume spherical quark system. First the finite volume effects are considered by applying the multiple reflection expansion method which is an approximation for the density of states of the momentum in the grand potential of the finite size system. Then the density of states of momentum is further modified by the thermal fluctuations in the thermal system with the inhomogeneous field background. The modification of the density of states is calculated by the scattering phase shift from the Dirac equation of quarks in the inhomogeneous field background. By the numerical calculation we show how the phase transition is changed by varying the configuration of the inhomogeneous field background and find that the strong first order phase transition could be weakened or even changed to a crossover as a result.

DOI: [10.1103/PhysRevC.105.045203](https://doi.org/10.1103/PhysRevC.105.045203)

I. INTRODUCTION

The physics of the phase transition in a small quantum system is very intricate, especially in a tiny droplet of quark/nuclear matter created by high energy heavy ion collisions [1,2]. Quantum chromodynamics (QCD) itself and its nonperturbative nature make it very hard to solve the problem of the phase transition from first principles [3]. Based on the lattice QCD (LQCD) simulations and effective model studies it is known that at high temperatures and low baryon densities the transition from nuclear matter to quark matter is a smooth crossover rather than a first order phase transition. At relatively high densities the phase diagram is still unclear or not confirmed [4–6]. The fireball created in heavy ion collisions is very small with an estimated radius of 2–10 fm. In a theoretical study one should consider the finite size effects [7–10], and whether the system is inhomogeneous. There are many discussions about inhomogeneous phases in chiral phase transitions or deconfinement phase transitions [11–17], such as chiral density wave (CDW) phases and quarkyonic phases. Additionally, the quantum effect is very hard to calculate in inhomogeneous systems. As for the CDW phase, a plane wave chiral condensate is assumed to simplify the complicated problem of solving the quark Dirac equation in the inhomogeneous background.

The quantum effect of the quark Dirac sea in an inhomogeneous background is important in studying the quark-nuclear matter phase transition. In this work we discuss the phase transition in an inhomogeneous spherical quark system which has a finite volume. The finite volume effect can be introduced in different ways. It could be added through the Massachusetts Institute of Technology (MIT) bag model boundary conditions or using the multiple reflection expansion (MRE) method [18–21]. The original MRE method was derived for massless particles [22]; however, it has been further extended to calculate the thermodynamic quantities for the quark system with nonzero mass [23,24]. It is suitable for studying a spherical system with a finite volume; thus, in this work the MRE scheme will be used as we discuss a finite spherical system. By the scheme there will be a correction to the density of states of the momentum in the momentum integration in the grand potential. It should be emphasized that previous studies of the grand potential of the thermal system often used the mean field approximation (MFA) in the homogeneous case to study the phase transition; however, in this study we argue that the quantum fluctuation over the spatially inhomogeneous background will have nontrivial influence on the phase transition. The inhomogeneous background means that in the MFA the expectation value of the field has spatial variation. As for a thermal quark system in the inhomogeneous background the usual density of states of the momentum in the standard MFA grand potential will receive a correction which can be derived by solving the quark Dirac equation to evaluate the scattering phase shift in the inhomogeneous field background. The method of evaluating the quantum fluctuations observed in the inhomogeneous field background through the scattering phase shift can be traced back to the work of Schwinger in the study of the energy of

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the electron's quantum fluctuations in certain nontrivial configurations of electromagnetic fields [25]. In later years Jaffe and his collaborators further completed the renormalization program and extended the method to the calculations of the quantum fluctuations in the inhomogeneous field backgrounds in electroweak systems [26,27]. The scattering phase shift method is practical for calculating quantum fluctuations in quantum field systems with spherically inhomogeneous field backgrounds.

In this paper we use the Friedberg-Lee (FL) model to address these problems and to illustrate the calculation scheme and its nontrivial influence on the phase transition. Previous studies of the FL model in the MFA indicate a first order phase transition at finite temperatures and densities [28,29]. The first order phase transition is relevant to neutron star mergers and cosmological electroweak phase transitions [30–32]. In a recent study the bubble dynamics is studied in the strong first order phase transition in the FL model [33]. In relativistic heavy ion collisions the crossover is the main feature at high temperatures and low densities. In our previous study examining the tunneling amplitude of the two vacuums in the FL model, qualitatively, we indicate that the first order phase transition may be turned into a crossover by quantum tunneling [34]. Several studies using chiral effective models demonstrate that quantum fluctuations may weaken the first order phase transition and strengthen the crossover feature [35–37]. However, in these studies the quantum fluctuations are treated in a homogeneous background. In this work we will show how quantum fluctuations in an inhomogeneous background influence the phase transition in a finite volume using the FL model.

The organization of this paper is as follows: In Sec. II the FL model is introduced and the grand potential is obtained at the MFA level. In Sec. III it is illustrated how the grand potential is modified in a finite size quark system under an inhomogeneous background. The order parameter of the phase transition is derived accordingly. In Sec. IV, we present the numerical results and discuss how the phase transition is changed by varying the spatial configuration of the inhomogeneous field background. The last section is the summary and outlook.

II. THE MODEL AND GRAND POTENTIAL AT MFA

The FL model was originally introduced to describe the properties of hadrons. It incorporates different bag models, e.g., MIT and Stanford Linear Accelerator Center (SLAC) models, into one effective model of quantum field theory which is renormalizable [38,39]. We start with the Lagrangian of the FL model as follows:

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - g\sigma)\psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - U(\sigma), \quad (1)$$

with

$$U(\sigma) = \frac{1}{2!}a\sigma^2 + \frac{1}{3!}b\sigma^3 + \frac{1}{4!}c\sigma^4 + B, \quad (2)$$

where ψ is the quark field and σ is the phenomenological scalar field which can roughly describe the nonperturbative long range effect of the gluon field. a , b , c , and g are model parameters which are fixed by fitting the general properties

of nucleons. B is the bag constant. In the MFA the σ field is treated as the classical mean field or $\sigma(\vec{r}, t) \rightarrow \bar{\sigma}$. The classical potential $U(\bar{\sigma})$ generally has two minima which correspond to two vacuums. One is the perturbative vacuum at $\bar{\sigma} = 0$ which is a local minimum. The other is the physical vacuum at $\bar{\sigma} = \sigma_v$ which is a global minimum. If there are three valence quarks at the background of the physical vacuum they will be localized and not propagate freely in space with heavy masses. As long as the vacuum is kept in the physical vacuum, the three valence quarks in the physical ground state will be confined as a soliton which corresponds to a nucleon in a physical interpretation. Phenomenologically one can regard the nucleon as a confinement state of three valence quarks through a bag. In this model in order to produce deconfinement one has to change the vacuum. One possible way is to heat the vacuum. In the FL model one can introduce a thermal grand potential through the standard method of finite temperature field theory. At finite temperature the partition function of the system is

$$Z = \int [id\psi^\dagger][d\psi][d\sigma] \exp \left[\int_0^{1/T} d\tau \int d^3\mathbf{r} \mathcal{L} \right]. \quad (3)$$

In the MFA we can write the grand potential as

$$\Omega(\bar{\sigma}, T) = -\frac{T \log Z}{V} = \Omega_\sigma + \Omega_{\text{kin}}, \quad (4)$$

where the σ field part of the grand potential has the form

$$\Omega_\sigma = \frac{1}{V} \int d^3\mathbf{r} \left[\frac{1}{2}(\nabla\bar{\sigma})^2 + U(\bar{\sigma}) \right], \quad (5)$$

and Ω_{kin} is the kinetic part of the potential which can be written as

$$\Omega_{\text{kin}} = -\frac{T}{V} \text{Tr} \log \left[\frac{1}{T} (i\partial_0 + i\gamma^0 \vec{\gamma} \cdot \vec{\nabla} - \gamma^0 g\bar{\sigma}) \right]. \quad (6)$$

In the homogeneous background field of σ the grand potential can be derived as

$$\Omega_{\text{MFA}}(\bar{\sigma}, T) = U(\bar{\sigma}) - 4N_f N_c T \int dk \rho_{\text{hom}}(k) \log(1 + e^{-E_k/T}), \quad (7)$$

where T is temperature, V is the volume, and $E_k = \sqrt{k^2 + g^2\bar{\sigma}^2}$. ρ_{hom} is the density of states of momentum for the homogeneous case, which is

$$\rho_{\text{hom}}(k) = \frac{1}{V} \frac{dN}{dk} = \frac{k^2}{2\pi^2}. \quad (8)$$

It should be noted that at $T = 0$ the grand potential will be $\Omega(\bar{\sigma}, T = 0) = U(\bar{\sigma})$. The FL model at finite temperatures has been extensively studied in the literature. The general result is that when the temperature increases, the grand potential will change its configuration. As a result, the physical vacuum at $\bar{\sigma} = \sigma_v$ rises and becomes the local minimum, while the perturbative vacuum at $\bar{\sigma} = 0$ turns into a global minimum which is the true vacuum. In the new thermal vacuum the valence quarks will propagate freely in space instead of being bounded, which means the system is deconfined. The deconfinement phase transition in the FL model is a strong first order phase transition.

III. THE FINITE SIZE THERMAL QUARK SYSTEM UNDER THE INHOMOGENEOUS BACKGROUND

In the above discussion the thermal quark system is in the homogeneous case without boundaries. However, the thermal quark/nuclear matter created in the real heavy ion collisions is always a finite size system and rather inhomogeneous. Theoretically one has to consider the corrections to the grand potential of the MFA. There are different ways to work out the corrections. In this section we will make use of the MRE method to give finite size corrections to the grand potential, while the corrections from the inhomogeneous background will be obtained through the scattering phase shift method.

First we discuss the calculation of $\log Z$ in an inhomogeneous background. For a system in a spherical box with a radius R the integration over the momentum in $\log Z$ will be changed into a summation over the discrete momentum modes. The general form of $\log Z$ by the summation over the discretized momentum modes can be written as

$$\log Z = \sum_{nlm} f(\mathbf{k}_{nlm}, T), \quad (9)$$

where n , l , and m are the quantum numbers of the radial, orbital angular momentum and its projection, respectively. Furthermore, we consider a spherically symmetric inhomogeneous background. The asymptotical eigenfunctions of the Hamiltonian of the system will be given by trigonometric functions with argument $kr + \delta_l(k)$. If we impose the condition that the eigenfunctions vanish at the boundary, it is found that $kR + \delta_l(k) = n_l\pi$ where $\delta_l(k)$ is the phase shift for the l th partial wave in the wave function at the boundary generated by the scattering of the inhomogeneous background. When the radius R of the system is large enough, the states become infinitesimally close and we have [25,26]

$$\frac{dn_l}{dk} = \frac{R}{\pi} + \frac{1}{\pi} \frac{d\delta_l(k)}{dk}. \quad (10)$$

The density of states in momentum space for the l th partial wave is defined as dn_l/dk divided by the volume V . The first term on the right hand side of Eq. (10) is related to the density of states of the homogeneous case by a summation over l with a degeneracy factor $(2l+1)$. If the system is homogeneous, the second term in Eq. (10) is zero. When the system volume goes to infinity the remaining homogeneous part will be identical as the density of states $\rho_{\text{hom}}(k)$ as in Eq. (8). Now if we assume that the volume of the homogeneous system is sufficiently large at a microscopic level but finite at a macroscopic level, then by applying the MRE method the density of states will have the following form [21,23]:

$$\rho_{\text{MRE}}(k) = \rho_{\text{hom}}(k) + \Delta\rho_{\text{MRE}}(k), \quad (11)$$

with

$$\Delta\rho_{\text{MRE}}(k) = \frac{3k}{R} f_S\left(\frac{k}{m}\right) + \frac{6}{R^2} f_C\left(\frac{k}{m}\right), \quad (12)$$

where R is the radius of the sphere and m is the mass. On the right hand side the first and second terms are the surface

and curvature contributions with the functions $f_S(k/m)$ and $f_C(k/m)$ defined as the following forms, respectively:

$$f_S\left(\frac{k}{m}\right) = -\frac{1}{8\pi} \left(1 - \frac{2}{\pi} \arctan \frac{k}{m}\right), \quad (13)$$

$$f_C\left(\frac{k}{m}\right) = \frac{1}{12\pi^2} \left[1 - \frac{3k}{2m} \left(\frac{\pi}{2} - \arctan \frac{k}{m}\right)\right]. \quad (14)$$

For the homogeneous case the second term in the density of states of Eq. (10) has been dropped. Therefore within the framework of the MRE the grand potential of a finite size spherical system in the FL model can be written as

$$\begin{aligned} \Omega_{\text{MRE}}(\bar{\sigma}, T) &= U(\bar{\sigma}) - 4N_f N_c T \\ &\times \int dk \rho_{\text{MRE}}(k) \log(1 + e^{-E_k/T}). \end{aligned} \quad (15)$$

Next we take into account the inhomogeneous field background. The second term in Eq. (10) should not be neglected. If we consider a spatial spherically symmetric σ field $\bar{\sigma}(\vec{r}) = \bar{\sigma}(r)$, the density of states of momentum will receive a further correction derived from the second term in Eq. (10), i.e.,

$$\Delta\rho_{\text{inho}}(k) = \frac{3}{4\pi^2 R^3} \sum_l (2l+1) \frac{d\delta_l(k)}{dk}, \quad (16)$$

where l is the quantum number of the angular momentum. In the FL model the density of states of momentum is changed due to the thermal quark scatterings on the inhomogeneous background field. $\delta_l(k)$ is the scattering phase shift of quarks in the inhomogeneous background field of $\bar{\sigma}(r)$. The scattering phase shift can be determined by the partial wave scattering method from the following stationary Dirac equation:

$$[-i\gamma_0 \vec{\gamma} \cdot \vec{\nabla} + \gamma_0 g \bar{\sigma}(r)]\psi = E\psi. \quad (17)$$

The background field $\bar{\sigma}(r)$ can be decomposed into two parts, the homogeneous vacuum part and the inhomogeneous spatial part, as the following:

$$\bar{\sigma}(r) = \sigma_{v0} + \tilde{\sigma}(r). \quad (18)$$

By this decomposition we need to ensure that when $r \rightarrow \infty$, the background field $\bar{\sigma}(r)$ goes to its vacuum value σ_{v0} , and $\tilde{\sigma}(r) \rightarrow 0$. In this work $\tilde{\sigma}(r)$ is set to be a spatial configuration described using a Woods-Saxon form, which can be written as

$$\tilde{\sigma}(r) = -\frac{\alpha}{1 + e^{\beta(r-r_0)}}, \quad (19)$$

where α , β , and r_0 are the adjustable parameters. The calculation of the scattering phase shift from Eq. (17) was discussed in our previous work [40,41], where the scattering phase shift was derived and regularized. The whole calculation of the renormalization for the phase shift is highly nontrivial. However, in the MFA limit the zero-point energy will be neglected and we only preserve the regularized phase shift in this work. To make the calculation self-contained, a brief derivation of the phase shift is presented in the Appendix. The final result is the regularized phase shift $\tilde{\delta}_l(k)$ which can be numerically evaluated. By substituting the regularized scattering phase shift into Eq. (16) one can further determine the correction of the density of states due to the inhomogeneous background.

Compared to that of the homogeneous system the density of states for a finite volume inhomogeneous system will have the correction term of $\Delta\rho_{\text{inho}}$. The final result of the density of states for the finite size inhomogeneous system is

$$\begin{aligned}\rho(k) &= \rho_{\text{hom}}(k) + \Delta\rho_{\text{MRE}}(k) + \Delta\rho_{\text{inho}}(k) \\ &= \frac{k^2}{2\pi^2} + \frac{3k}{R}f_S\left(\frac{k}{m}\right) + \frac{6}{R^2}f_C\left(\frac{k}{m}\right) \\ &\quad + \frac{3}{4\pi^2 R^3} \sum_l (2l+1) \frac{d\bar{\delta}_l(k)}{dk}.\end{aligned}\quad (20)$$

Then the kinetic part of the grand potential for an inhomogeneous sphere can be written as

$$\Omega_{\text{kin}} = -4N_f N_c T \int dk \rho(k) \log(1 + e^{-E_k/T}), \quad (21)$$

where $E_k = \sqrt{k^2 + g^2\bar{\sigma}^2}$. In contrast, the σ field part of the grand potential for the inhomogeneous sphere has the following form:

$$\Omega_\sigma = \frac{4\pi}{V} \int dr r^2 \left[\frac{1}{2} \left(\frac{d\bar{\sigma}(r)}{dr} \right)^2 + U(\bar{\sigma}(r)) \right]. \quad (22)$$

According to the decomposition of the σ field the homogeneous part can be separated from the integration and the result is

$$\Omega_\sigma = U(\bar{\sigma}) + \frac{4\pi}{V} \int dr r^2 \left[\frac{1}{2} \left(\frac{d\tilde{\sigma}(r)}{dr} \right)^2 + \tilde{U}(\tilde{\sigma}(r)) \right], \quad (23)$$

where the potential $\tilde{U}(\tilde{\sigma})$ satisfies that $\tilde{U}(\tilde{\sigma}) \rightarrow 0$ when $r \rightarrow \infty$ and $\tilde{\sigma} \rightarrow 0$ and its form is

$$\begin{aligned}\tilde{U}(\tilde{\sigma}(r)) &= \frac{1}{2!} \left(a + b\sigma_{v0} + \frac{1}{2}c\sigma_{v0}^2 \right) \tilde{\sigma}(r)^2 \\ &\quad + \frac{1}{3!} (b + c\sigma_{v0}) \tilde{\sigma}(r)^3 + \frac{1}{4!} c \tilde{\sigma}(r)^4,\end{aligned}\quad (24)$$

in which σ_{v0} is fixed by the absolute minimum of the $U(\bar{\sigma})$ as

$$\sigma_{v0} = \frac{3|b|}{2c} \left[1 + \left(1 - \frac{8ac}{3b^2} \right)^{1/2} \right]. \quad (25)$$

The bag constant B in $U(\bar{\sigma})$ is thus fixed as

$$B = - \left(\frac{a}{2!} \sigma_{v0}^2 + \frac{b}{3!} \sigma_{v0}^3 + \frac{c}{4!} \sigma_{v0}^4 \right). \quad (26)$$

Finally we obtain the grand potential of the finite volume spherical inhomogeneous system as

$$\begin{aligned}\Omega_{\text{inhom}}(\bar{\sigma}, \tilde{\sigma}, T) &= U(\bar{\sigma}) + \frac{4\pi}{V} \int dr r^2 \left[\frac{1}{2} \left(\frac{d\tilde{\sigma}(r)}{dr} \right)^2 \right. \\ &\quad \left. + \tilde{U}(\tilde{\sigma}(r)) \right] - 4N_f N_c T \int dk \rho(k) \\ &\quad \times \log(1 + e^{-E_k/T}).\end{aligned}\quad (27)$$

One should notice that the calculation of $\rho(k)$ is nontrivial. In principle $\rho(k)$ should vary with the quark mass $m = g\sigma_v$ which is temperature dependent. However, by our numerical study we find that the variation of the mass in $\rho(k)$ at finite

temperatures for a given system size R only gives a small modification to the grand potential, which is specifically discussed at the end of the next section. Therefore, as an assumption in the following discussion the quark mass in $\rho(k)$ could be taken as a fixed value at $m = g\sigma_{v0}$ and this assumption is also addressed at the end of the next section. As the $\rho(k)$ is only dependent on the spatial configuration of the σ background field $\bar{\sigma}(r)$ and the system size R , the homogeneous mean field $\bar{\sigma}$ in the grand potential could be treated as a variable. At a certain temperature the $\bar{\sigma}$ can be determined by the thermodynamic equilibrium condition as

$$\left. \frac{\partial \Omega_{\text{inhom}}(\bar{\sigma}, \tilde{\sigma}, T)}{\partial \bar{\sigma}} \right|_{\bar{\sigma}=\sigma_v} = 0. \quad (28)$$

Thus, the σ mean field value $\bar{\sigma} = \sigma_v$ at the global minimum of the grand potential $\Omega(\bar{\sigma}, \tilde{\sigma}, T)$ is an order parameter of the phase transition in the finite volume inhomogeneous system at finite temperatures.

IV. NUMERICAL RESULTS AND DISCUSSIONS

Now we present the numerical results of the systems with the MFA and the finite volume effect by MRE, and the system with inhomogeneous background in a finite volume sphere, respectively. The grand potential and the order parameter are focused on in order to illustrate the phase transition. We give the variation curves of the grand potential with respect to $\bar{\sigma}$ at different temperatures in different conditions, and the corresponding curves of its global minimum, or the order parameter σ_v with respect to T , is also illustrated. The model parameters are set as $a = 17.7 \text{ fm}^{-2}$, $b = -1457.4 \text{ fm}^{-1}$, $c = 20000$, and $g = 12.16$, which are widely used in the literature [29,42]. Thus the quark mass $m = g\sigma_{v0}$ and the bag constant B are determined according to Eqs. (25) and (26). The quark flavors and colors are set to be $N_f = 2$ and $N_c = 3$. The following numerical results are based on the assumption that the quark mass in $\rho(k)$ is fixed. However, at the end of this section we give some numerical evaluations on the cases of the temperature-dependent quark mass in $\rho(k)$ and discuss its influence on our results.

As the deconfinement phase transition in the FL model at MFA has been extensively studied in the literature, here we just discuss the modification to the phase transition of the MFA due to the effect of MRE for the given system size R . From the result of the grand potential (15) with the density of states ρ_{MRE} as shown in Eq. (11), the variation curves of the grand potentials $\Omega_{\text{MRE}}(\bar{\sigma})$ as functions of $\bar{\sigma}$ at different temperatures at $R = 3 \text{ fm}$ are shown in Fig. 1(a). To compare the potential curves at different temperatures the curves have been shifted to have the same fixed zero value of the grand potential at $\bar{\sigma} = 0$. One can see there are two minima corresponding to the two vacua. At relatively low temperatures the physical vacuum is at the minimum with nonzero value of $\bar{\sigma}$. With the temperature increasing, the physical vacuum rises. At a certain temperature $T_c = 136.5 \text{ MeV}$ the two minima degenerate. The temperature T_c is the critical temperature of the deconfinement phase transition at the system size of $R = 3 \text{ fm}$. When the temperature is further increased the physical vacuum is changed to be at the minimum with zero value of $\bar{\sigma}$

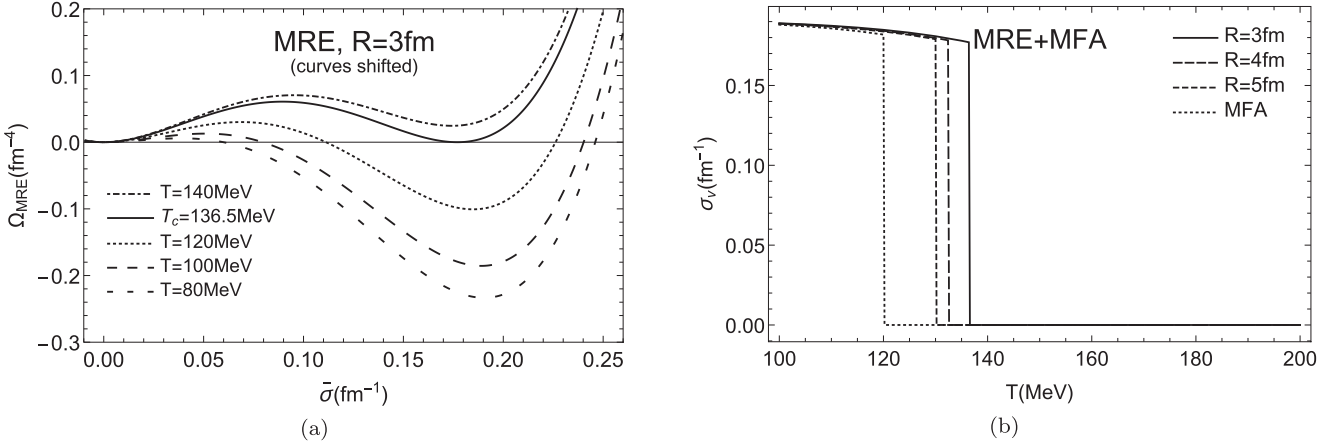


FIG. 1. (a) The variation curves of the grand potentials as functions of $\bar{\sigma}$ at $R = 3$ fm for different temperatures. (b) The variation curve of the order parameter σ_v as a function of temperature T for $R = 3, 4,$ and 5 fm and the MFA, respectively.

which becomes the global minimum of the grand potential. The σ field value σ_v at the physical vacuum is the order parameter, so Fig. 1(b) illustrates how the order parameter varies with the temperature. The solid line corresponds to the case of $R = 3$ fm. One can see that the order parameter has a sudden jump at the critical temperature T_c which represents a first order phase transition. If the system size R is increased as $R = 4$ fm and $R = 5$ fm, one can see that the critical temperature T_c becomes lower. If it approaches the limiting case as $R \rightarrow \infty$, the MFA result which has a critical temperature $T_c = 120.1$ MeV will be recovered. Thus, one can see that the description of the first order phase transition at the MFA is not much changed by the finite volume effect via MRE except for a higher critical temperature at a smaller system size.

Next, we examine the numerical results of cases with the inhomogeneous background field $\bar{\sigma}(r)$ for which the spatial configuration is determined by the parameters α , β , and r_0 as shown in Eq. (19). As we know at a semiclassical level there is a soliton solution in the FL model. The σ field in the soliton solution has a spatial configuration of the Woods-Saxon type in which the values of the parameters of α , β , and r_0 can be fitted approximately as $\alpha \approx 0.26 \text{ fm}^{-1}$, $\beta \approx 8 \text{ fm}^{-1}$, and $r_0 \approx 1$ fm. In this study we take the configuration of the σ field in the soliton solution as an initial configuration. The parameters α , β , and r_0 can vary. Our purpose is to study how the phase transition is influenced by varying the configuration of the inhomogeneous background field of the σ . The parameter α describes the depth of the potential well of the σ field. The parameters β and r_0 determine the spatial varying rate of the potential wall and the width of the potential well of $\bar{\sigma}(r)$. It should be emphasized that in our study $\bar{\sigma}(r)$ is not the meaning of the hadron but describes the spatially inhomogeneous fireball in the heavy ion collision. In the following study the vacuum value of the σ field outside the fireball is fixed at σ_{v0} . The parameter α is fixed at $\alpha \approx 0.26 \text{ fm}^{-1}$ which corresponds to a fixed depth of the potential well in $\bar{\sigma}(r)$, while β and r_0 vary by hand. The reason for this treatment is that we assume in the heavy ion collisions that the difference between the vacuum values of the σ field inside the fireball and outside the fireball will change relatively slowly during

the phase transition while the size and shape of the fireball will change more remarkably.

In the calculation of the grand potential one may notice that there is a summation over angular momentum quantum number l in the evaluation of the inhomogeneous part of the density of states [Eq. (20)]. In principle the summation index l goes to infinity. However, the contribution to the density of states from the scattering of the partial wave will decrease with the angular momentum quantum number l increasing. As an approximation in the real numerical calculation we only do the summation over $l = 0, 1$ which means only the s and p partial waves are included in the summation.

Now we are in a position to study how the phase transition in a finite volume is influenced by varying the spatial configuration of the background field $\bar{\sigma}(r)$ through changing the parameters β and r_0 . In the following calculation the system size is taken as $R = 3$ fm. First let β be fixed at $\beta = 8 \text{ fm}^{-1}$ and vary r_0 . From Fig. 2(a) one can see with r_0 increasing the width of the potential well becomes larger. By Eq. (17) the scattering phase shift can be numerically calculated in the potential background with different r_0 , and the order parameter σ_v can be determined at different temperatures through Eq. (28) at certain r_0 . In Fig. 2(b) the temperature dependence of the σ_v for different r_0 has been plotted. When r_0 is very small ($r_0 \lesssim 0.3$ fm), the spatial distribution of the inhomogeneous area is very small, and the quantum fluctuation over this tiny inhomogeneous background is also very small. As a result the phase transition is still a strong first order phase transition, and the case is very similar to that of the MRE result. When r_0 gets larger it is clear that quantum corrections from the inhomogeneous background become notable. In Fig. 3(a) at $r_0 = 1.5$ fm it is shown that besides the global minimum of the physical vacuum a new local minimum emerges near the zero value of the σ field in the grand potential at relatively low temperatures before the phase transition. When the temperature is greater than the critical temperature this local minimum becomes the global minimum which corresponds to the physical vacuum. The corresponding variation of the order parameter σ_v with the temperature can be seen in Fig. 2(b). It is clear that the first order phase transition has been weakened.

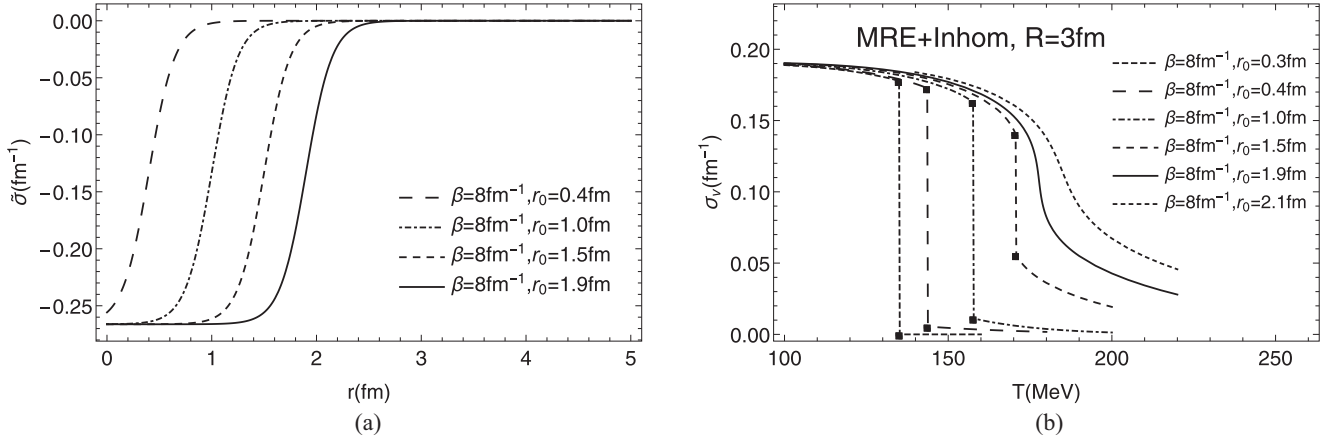


FIG. 2. (a) The spatial configurations of $\tilde{\sigma}(r)$ for the fixed $\beta = 8 \text{ fm}^{-1}$ and different r_0 . (b) The σ_v as functions of temperature for different inhomogeneous backgrounds represented by different r_0 with fixed β and $R = 3 \text{ fm}$.

When r_0 is further increased ($r_0 \gtrsim 1.9 \text{ fm}$) it is found that the first order phase transition has been changed into a crossover. One can see the order parameter continuously decreasing with temperature increasing in Fig. 2(b). From Fig. 3(b) at $r_0 = 2.1 \text{ fm}$ one can see that there is only one minimum in the grand potential which is corresponding to a unique vacuum in this system and with temperature increasing it continuously moves towards a small value of the σ field which displays a crossover.

In the above analysis one can see that with the spatial size of the inhomogeneous background increasing the quantum fluctuations become more and more remarkable, which gradually changes the first order phase transition into the crossover. Further, it is found that the critical temperature of the phase transition does not always increase with r_0 increasing. There is a subtle competition and balance between the contribution from the s -wave scattering and p -wave scattering, as follows. When $r_0 \lesssim 0.3 \text{ fm}$ the quantum corrections from both s and p waves are small. The critical temperature of the phase transition almost stays unchanged with r_0 increasing. When $r_0 \approx 0.4 \text{ fm}$ there emerges an s -wave bound state energy level in the energy spectrum of the Dirac Eq. (17) which makes a

relatively notable enhancement of the critical temperature of the phase transition as shown in Fig. 2(b). At relatively small r_0 the quantum correction from the s wave is larger than that of the p wave. When an s -wave bound orbital appears the first order phase transition is modified remarkably which means the critical temperature of the phase transition is increased and the first order phase transition is weakened. However, when r_0 is further increased from 0.4 to 0.9 fm one will find that the critical temperature of the phase transition decreases and the first order phase transition strengthens as shown in Fig. 4(b). This abnormal varying trend in the above interval of varying r_0 is because the s -wave contribution is decreasing while the p -wave contribution is increasing. The contribution from the p wave counteracts that from the s wave as long as there is no bound state energy level in the p -wave state. According to the Levinson index theorem the number of bound states n_l with the angular momentum l in our case is given by $n_l \pi = \bar{\delta}_l(k=0)$ [43,44]. From Fig. 4(a) one can see that the p -wave phase shift at zero momentum remains at zero when r_0 changes from 0.4 to 0.9 fm. As a result there is no bound state level in the p -wave state and the p -wave contribution plays the role of strengthening the first order phase transition

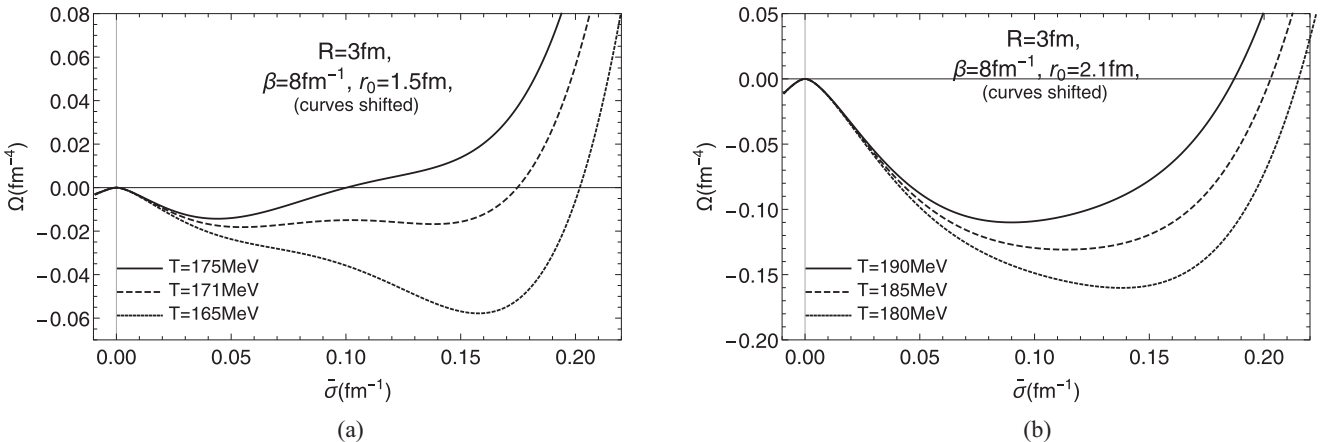


FIG. 3. (a) The grand potentials as functions of $\tilde{\sigma}$ for $\beta = 8 \text{ fm}^{-1}$ and $r_0 = 1.5 \text{ fm}$ at $R = 3 \text{ fm}$ for different temperatures. (b) The grand potentials as functions of $\tilde{\sigma}$ for $\beta = 8 \text{ fm}^{-1}$ and $r_0 = 2.1 \text{ fm}$ at $R = 3 \text{ fm}$ for different temperatures.

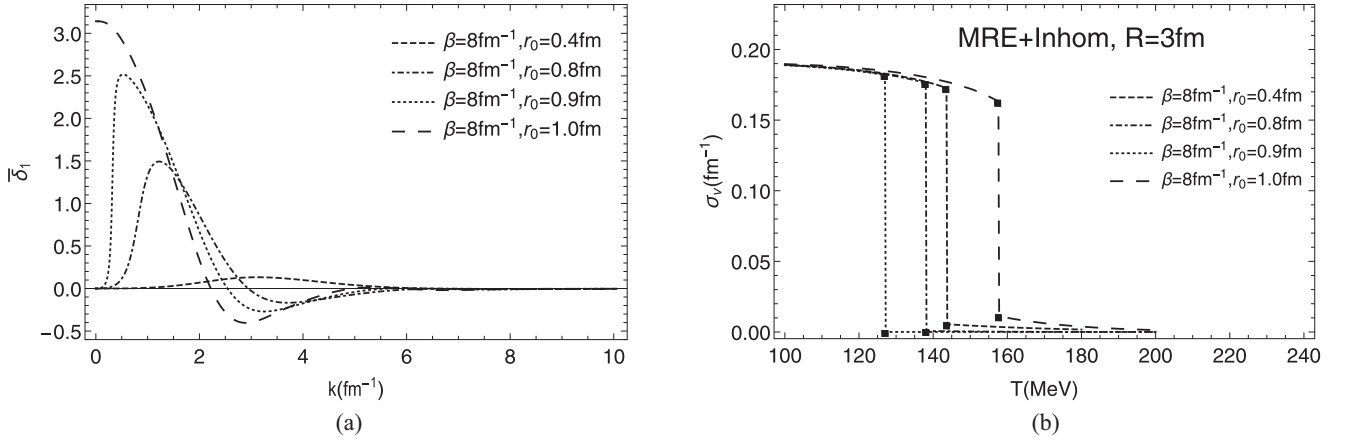


FIG. 4. The regulated p -wave phase shifts as functions of momentum k for fixed β and different r_0 . (b) The σ_v as functions of temperature for different inhomogeneous backgrounds represented by different r_0 with fixed β and $R = 3 \text{ fm}$.

in the above varying interval of r_0 . However, the situation will change until $r_0 \gtrsim 1.0 \text{ fm}$ when there emerges a bound state energy level in the p -wave state which can be seen from Fig. 4(a) that $\delta_1(k=0) = \pi$. There is a sign flipping of the p -wave contribution such that its influence becomes greater than that of the s -wave. As a result the critical temperature of the phase transition is suddenly increased dramatically and the first order phase transition is remarkably weakened which can be seen in Fig. 4(b). After that the p -wave contribution increases with r_0 increasing and plays the dominant role in modifying the phase transition. Therefore, the first order phase transition is further weakened with increasing r_0 and gradually turned into a crossover at $r_0 \gtrsim 1.9 \text{ fm}$ as shown in Fig. 2(b).

Second let us fix $r_0 = 1.5 \text{ fm}$ and let β vary. In Fig. 5(a) it is shown that with β decreased the variation of the σ field over radial distance near the potential wall becomes milder. The parameter β determines the steep level of the wall of the potential well of the σ field. It is found that with β decreased the first order phase transition is weakened and the critical temperature of the phase transition is increased as shown in Fig. 5(b). When $\beta \gtrsim 4 \text{ fm}^{-1}$ there is one bound state energy level in the s -wave state and one in the p -wave state. The quantum correction from the p -wave scattering is

more remarkable than that of the s wave. With β decreased the first order phase transition is weakened and the critical temperature of the phase transition is increased. However, it should be noted that there is a nontrivial variation of the modification to the phase transition between $\beta = 4 \text{ fm}^{-1}$ and $\beta = 3 \text{ fm}^{-1}$. At $\beta \approx 3 \text{ fm}^{-1}$ there emerges a second bound state level in the s -wave state which can be observed from the s -wave phase shift. From our numerical calculation it is found that the s -wave phase shift value at zero momentum jumps from $\bar{\delta}_0(k=0) = \pi$ to $\bar{\delta}_0(k=0) = 2\pi$. The correction from the s -wave scattering is once again greater than that of the p wave and becomes the dominant part in the contribution to the density of states. As a result the first order phase transition is turned into a crossover. After that the pseudocritical temperature of the crossover will slowly increase with β decreasing as long as there is no additional bound state level appearing in the s -wave or p -wave state.

Here we want to make some comments on the bound state energy level and how it influences the phase transition. In the inhomogeneous background field of σ the bound state may be formed in the thermal quark system. In principle these discrete bound state energies may directly contribute additional isolated terms in the grand potential. However, as these

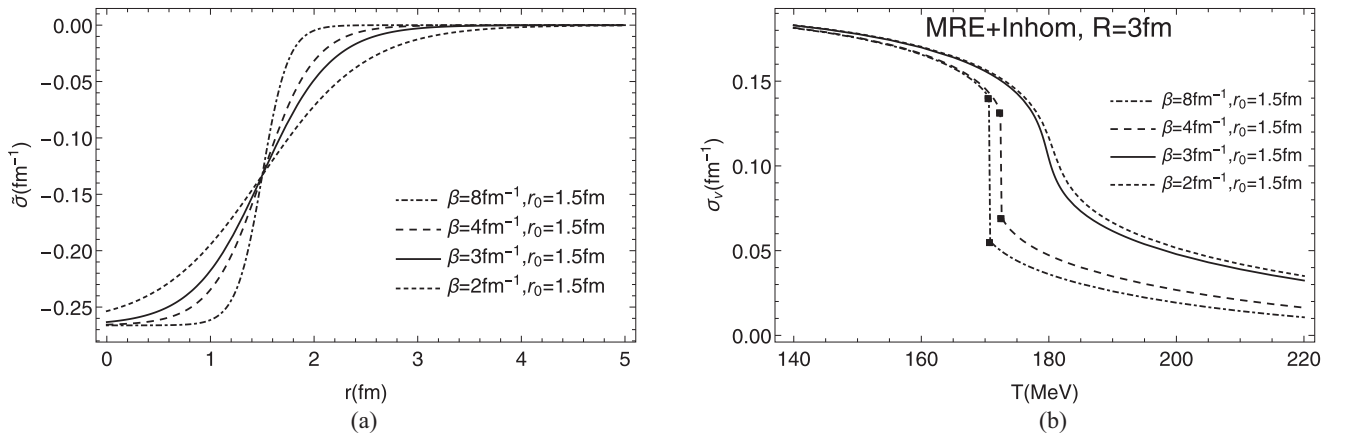


FIG. 5. (a) The spatial configurations of $\bar{\sigma}(r)$ for fixed $r_0 = 1.5 \text{ fm}$ and different β . (b) The σ_v as functions of temperature for different inhomogeneous backgrounds represented by different β with fixed r_0 and $R = 3 \text{ fm}$.

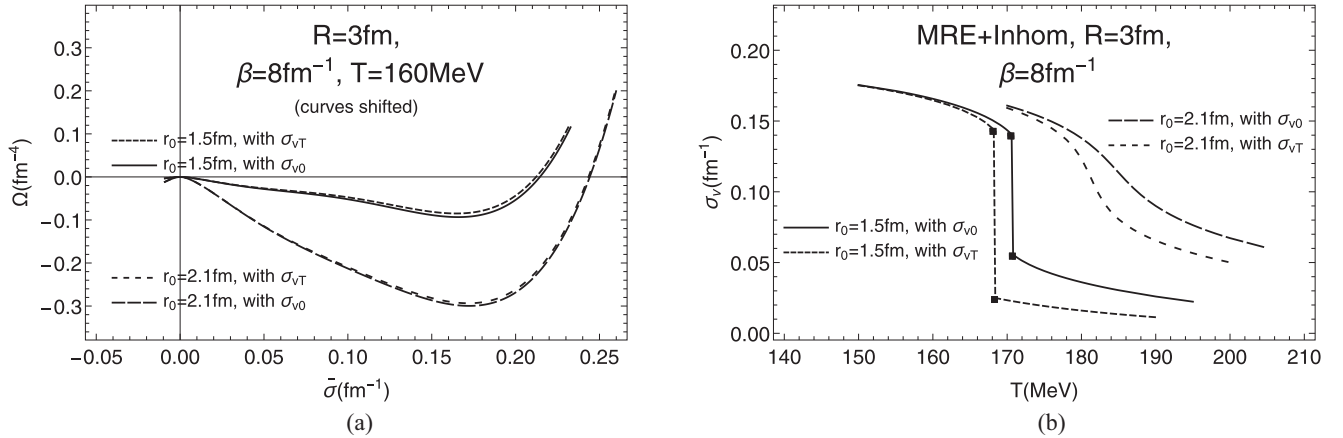


FIG. 6. (a) The grand potentials as functions of $\bar{\sigma}$ with σ_{v0} vs with σ_{vT} at $\beta = 8 \text{ fm}^{-1}$ for $r_0 = 1.5 \text{ fm}$ and $r_0 = 2.1 \text{ fm}$ at the temperature of 160 MeV and $R = 3 \text{ fm}$. (b) The σ_v as functions of temperatures with σ_{v0} vs with σ_{vT} at $\beta = 8 \text{ fm}^{-1}$ for $r_0 = 1.5 \text{ fm}$ and $r_0 = 2.1 \text{ fm}$ at $R = 3 \text{ fm}$.

energy terms are constant energies and averaged by the space volume, they give a small constant background energy which only shifts the whole grand potential upward or downward. Therefore, these isolated terms are ignored in our calculation. However, the discrete bound state can also influence the grand potential by the density of states. Through the scattering mode of the continuum over the inhomogeneous background, the bound state indirectly modifies the phase transition. The phase shift of the different scattering partial wave will be changed dramatically when the bound state level emerges, so it will give remarkable modification to the density of states. As a result the grand potential will be substantially modified accordingly. Our study shows that the possible bound state in the special partial wave state in the thermal quark system will largely enhance the quantum scattering effect of the corresponding partial wave which results in weakening the first order phase transition or even turning the first order phase transition into a crossover.

We will also discuss the volume dependence of the quantum scattering effect in the inhomogeneous background. From the expression of the density of states [Eq. (20)] one can see that the size R of the system appears in the denominator; specifically, the surface and curvature terms are an order of $1/R$ and $1/R^2$, respectively, and the inhomogeneous term is order of $1/R^3$. Therefore, by a qualitative analysis it can be deduced that when the system size is large enough the quantum fluctuations of the inhomogeneous background can be neglected and the MRE result will be recovered, and if the system size goes to infinity the MFA result will be recovered. In our case if the system size R is greater than about 10 fm , with r_0 increasing or β decreasing, the first order phase transition may be weakened in some degree while the crossover will not appear. Therefore, the MRE result is nearly recovered. When the system size R is greater than about 20 fm the quantum scattering effect over the inhomogeneous background could be almost neglected and the MRE effect also becomes very small. Hence the result is very close to the case of the MFA.

Finally we analyze the effects on our results of using the temperature-dependent quark mass instead of the fixed mass in $\rho(k)$. One can let the quark mass $m = g\sigma_v$ vary with

temperature which means the σ_v will not be fixed at σ_{v0} but determined by the absolute minimum σ_{vT} of the grand potential Ω at finite temperatures for the given system size R . This requires that the σ_v in $\rho(k)$ should be iterated when solving Eq. (28) at a given temperature. The numerical iteration process can be described, as follows. First we use σ_{v0} as an initial value in $\rho(k)$ to solve Eq. (28) at the given temperature. One can obtain the solution σ_{v1} which is the absolute minimum of the potential Ω . Then σ_{v1} will be substituted into $\rho(k)$ for the next iteration, and so on. Once the difference between σ_{vi} and $\sigma_{v,i+1}$ is very small we obtain the real stable vacuum value σ_{vT} and the temperature-dependent quark mass $m = g\sigma_{vT}$ in $\rho(k)$. For an illustration of the results, we numerically evaluated two cases at the fixed $\beta = 8 \text{ fm}^{-1}$ for $r_0 = 1.5 \text{ fm}$ and $r_0 = 2.1 \text{ fm}$, respectively, for the given system size $R = 3 \text{ fm}$. The numerical results are shown in Fig. 6. In Fig. 6(a) the grand potentials are plotted at given temperature $T = 160 \text{ MeV}$ with the fixed mass $m = g\sigma_{v0}$ and with the iterated mass $m = g\sigma_{vT}$ for comparisons. One can see that the grand potentials are modified very slightly due to the substitution of σ_{v0} with σ_{vT} in $\rho(k)$. The modified degree of the grand potential will gradually increase with temperature increasing, especially near the phase transition or crossover region. In Fig. 6(b) it could be seen that the order parameter determined by the absolute minimum of the grand potential has been modified accordingly. The modifications of the order parameter are more remarkable near the phase transition or crossover region than those in the regions before and after the transition or crossover. However, generally speaking the quantitative modifications at finite temperatures for the finite size system are moderate and they will not qualitatively change the physical conclusions which we have made above.

V. SUMMARY AND OUTLOOK

In summary we studied the grand potential and the phase transition in the thermal quark system with a finite volume under the spatially inhomogeneous background field. By varying the spatial configuration of the inhomogeneous background the scattering phase shift for the different partial wave was

numerically calculated in the corresponding inhomogeneous background at a certain small spatial volume of the system. It is found that when the bound state is formed in the thermal system the density of states of momentum will be substantially modified by the scattering phase shift of the corresponding partial wave. As a result the first order phase transition in the system with the given volume has been weakened remarkably and even turned into the crossover. However, with the system volume increasing, the quantum corrections in the inhomogeneous background will decrease. At sufficiently large volume of the system the quantum corrections from the inhomogeneous background can be neglected.

It should be noted that the result here may be model dependent. The different model parameters or approximation schemes may quantitatively modify the numerical results of our discussion. However, the basic physical features and conclusions here will not be changed. Additionally the FL model is only a simple phenomenological model and lacks chiral symmetry. The chiral phase transition could not be studied in this model. Some outstanding questions after this study include how the scattering phase shift in the chiral models can be calculated in a spherical system with an inhomogeneous background and how the chiral phase transition will be modified by the inhomogeneous background with the possible bound states emerged. Future studies will use chiral models like the linear σ model or the Nambu-Jona-Lasinio model to address these issues.

APPENDIX: DERIVATIONS OF THE PHASE SHIFT

For the spherical symmetrical system one can assume the form of the quark field as

$$\psi(\vec{r}) = \frac{1}{r} \begin{pmatrix} F(r) \\ i\vec{\sigma} \cdot \vec{e}_r G(r) \end{pmatrix} y_{jm}^l, \quad (\text{A1})$$

where y_{jm}^l is the two-component Pauli spinor harmonic and \vec{e}_r is the spatial unit vector. $F(r)$ and $G(r)$ are the upper and lower components of the Dirac spinor. From the Dirac equation (17) one can further derive the first order coupled differential equations of $F(r)$ and $G(r)$ as

$$\left(\frac{d}{dr} - \frac{l+1}{r} \right) F(r) + (g\bar{\sigma}(r) + E)G(r) = 0, \quad (\text{A2})$$

$$\left(\frac{d}{dr} + \frac{l+1}{r} \right) G(r) + (g\bar{\sigma}(r) - E)F(r) = 0. \quad (\text{A3})$$

$$i\beta_1'' r h_l + 2i\beta_1'(h_l + r h_l') - \beta_1'^2 r h_l - \frac{g\bar{\sigma}(r)'}{E + g\bar{\sigma}(r)} (i\beta_1' r h_l + h_l + r h_l') + \left[\frac{l+1}{r} \frac{g\bar{\sigma}(r)'}{E + g\bar{\sigma}(r)} - g^2(\bar{\sigma}(r)^2 - \sigma_{v0}^2) \right] r h_l = 0. \quad (\text{A11})$$

In the fixed background field of $\bar{\sigma}(r)$ this equation could be numerically solved to obtain the phase shift $\delta_l(k)$.

To regularize the phase shift one has to make a Born expansion which means β_l should be expanded in powers of g as

$$\beta_l = g\beta_{l1} + g^2\beta_{l2} + \dots \quad (\text{A12})$$

Substituting the expansion (A12) into Eq. (A11) and neglecting the higher order terms $O(g^3)$ one can obtain a set of coupled differential equations about β_{l1} and β_{l2} as

$$i\beta_{l1}'' r h_l + \left(2i\beta_{l1}' - \frac{\bar{\sigma}(r)'}{E} \right) (h_l + r h_l') + (l+1) \frac{\bar{\sigma}(r)'}{E} h_l = 0, \quad (\text{A13})$$

Then the coupled equations (A2) and (A3) can be rewritten into the following decoupled second order differential equations as

$$F'' - \frac{g\bar{\sigma}(r)'}{E + g\bar{\sigma}(r)} F' + \left[\frac{l+1}{r} \frac{g\bar{\sigma}(r)'}{E + g\bar{\sigma}(r)} - \frac{l(l+1)}{r^2} + (E^2 - g^2\bar{\sigma}(r)^2) \right] F = 0, \quad (\text{A4})$$

$$G'' + \frac{g\bar{\sigma}(r)'}{E - g\bar{\sigma}(r)} G' + \left[\frac{l+1}{r} \frac{g\bar{\sigma}(r)'}{E - g\bar{\sigma}(r)} - \frac{(l+1)(l+2)}{r^2} + (E^2 - g^2\bar{\sigma}(r)^2) \right] G = 0, \quad (\text{A5})$$

where the prime denotes the differentiation with respect to r . For the evaluation of the phase shift both equations will give the identical result. In the following discussion Eq. (A4) is used for the calculation of the phase shift. When $r \gg R$ the asymptotic form of Eq. (A4) is

$$F'' - \left[\frac{l(l+1)}{r^2} - k^2 \right] F = 0, \quad (\text{A6})$$

where $k^2 = E^2 - g^2\sigma_{v0}^2$. The solutions will be spherical Hankel functions. Meanwhile it should satisfy that $F(r) \rightarrow 0$ with $r \rightarrow 0$. There are two linearly independent solutions,

$$F_l^{(1)}(r) = e^{i\beta_l(k,r)} r h_l^{(1)}(kr), \quad (\text{A7})$$

$$F_l^{(2)}(r) = e^{-i\beta_l^*(k,r)} r h_l^{(2)}(kr), \quad (\text{A8})$$

where $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$ are the Hankel functions of the first and second kinds and $h_l^{(2)}(kr) = h_l^{(1)*}(kr)$. The function $\beta_l(k, r)$ should satisfy $\beta_l(k, r) \rightarrow 0$ as $r \rightarrow \infty$. Then the scattering solution is

$$F_l(r) = F_l^{(2)}(r) + e^{i2\delta_l(k)} F_l^{(1)}(r) \quad (\text{A9})$$

and obeys $F_l(0) = 0$, which leads to the result of the scattering phase shift,

$$\delta_l(k) = -\text{Re}\beta_l(k, 0), \quad (\text{A10})$$

where Re means the real part. It is obvious that the phase shift could be evaluated from β_l . By substituting $F_l^{(1)}$ into Eq. (A4) one could obtain the equation of β_l ,

$$i\beta_{l2}''rh_l - \beta_{l1}^2rh_l - \frac{i\bar{\sigma}(r)'}{E}\beta_{l1}'rh_l + \left(2i\beta_{l2}' + \frac{\bar{\sigma}(r)'\bar{\sigma}(r)}{E^2}\right)(h_l + rh_l) - \left[\frac{(l+1)\bar{\sigma}(r)'\bar{\sigma}(r)}{rE^2} + (\bar{\sigma}(r)^2 - \sigma_{v0}^2)\right]rh_l = 0. \quad (\text{A14})$$

These equations could be numerically solved to obtain the first and second Born approximations of the phase shifts, namely, $\delta_l^{(1)}$ and $\delta_l^{(2)}$, as

$$\delta_l^{(1)} = -g\text{Re}\beta_{l1}(k, r=0), \quad \delta_l^{(2)} = -g^2\text{Re}\beta_{l2}(k, r=0). \quad (\text{A15})$$

Finally the regularized phase shift can be defined as

$$\bar{\delta}_l(k) \equiv \delta_l(k) - \delta_l^{(1)}(k) - \delta_l^{(2)}(k), \quad (\text{A16})$$

which can be numerically calculated.

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