Quantum statistics effects near the critical point in systems with different interparticle interactions

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Equation of state with quantum statistics corrections is derived for systems of the Fermi and Bose particles by using their van der Waals (vdW) and effective density-dependent Skyrme mean-field interactions. The first few orders of these corrections over a small quantum-statistics parameter, $\varepsilon \approx \hbar^3 n(mT)^{-3/2} g^{-1}$, where n and T are the particle number density and temperature, m and g are the mass and degeneracy factor of particles, are analytically obtained. For an interacting system of nucleon and α particles, a small impurity of α particles to a nucleon system at leading first order in both α- particle and nucleon small parameters ε does not change the basic results for the symmetric nuclear matter in the quantum vdW consideration much. Our approximate analytical results for the quantum vdW and Skyrme mean-field approaches are in a good agreement with accurate numerical calculations.

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I. INTRODUCTION

In a system of interacting hadrons, first of all, nuclear matter is an attractive subject [1–12]. Realistic versions of the nuclear matter equation of state include both the attractive and repulsive forces between particles. Thermodynamical behavior of this matter leads to the liquid-gas first-order phase transition which ends at the critical point. Experimentally, a presence of the liquid-gas phase transition in nuclear matter was reported and then analyzed in numerous papers (see, e.g., Refs. [13–19]).

Recently, the proposed van der Waals (vdW) equation of state accounting for the quantum statistics was used to describe the properties of hadronic matter [20] and was extended further to multicomponent systems [21]. Many works have presented the extensions of the phase-transition theory to the effective density-dependent Skyrme forces in terms of the potential density [22–24]; see also reviews [25]. They are especially helpful for the description of the Bose condensate in bosonic systems [23]. Starting from the pioneer works of Skyrme (Ref. [26]) and famous Skyrme self-consistent Hartree-Fock calculations by Vautherin and Brink (Ref. [27]), these forces have become very popular in nuclear physics and astrophysics; see, e.g., review articles [25,28]. In different systems of hadrons, the critical points, including the Bose condensate, for the classical and quantum approaches based on the vdW and Skyrme mean-field forces were studied in Refs. [21–23,29–37].

The role and size of the quantum statistics effects were analytically studied for nuclear matter, as well as for pure neutron and pure α-particle matter, in Ref. [38]. In this approximation, the dependence of critical point parameters on the particle mass m, the degeneracy factor g, and the vdW interparticle interaction parameters a and b was described well for each of these systems. Our consideration was restricted to small temperatures, $T \lesssim 30$ MeV, and not too large particle densities. Within these restrictions, the number of nucleons becomes a conserved number, and the chemical potential of such systems was determined by a particle number. An extension to the fully relativistic hadron resonances in a gas formulation with vdW interactions between baryons and antibaryons was considered in Ref. [39]. We do not include Coulomb forces and make no differences between protons and neutrons (both these particles are referred to as nucleons). In addition, under these restrictions, the nonrelativistic treatment becomes very accurate and, therefore, is adopted in our studies.

In the present work we apply the same analytical method as presented in Ref. [38] for systems of nucleons and α particles but with another interparticle interaction in terms of the density-dependent effective Skyrme potential. This method will be applied also to a mixed two-component system of nucleons and α particles. Another attractive subject of the application of our analytical results to the analysis of the particle number fluctuations near a critical point of nuclear matter (see, e.g., Ref. [29], and the more recent Ref. [40]) will be studied in a separate forthcoming work.

The paper is organized as follows. Equations of state for the ideal quantum gases with expansion over the small quantum-statistics parameter related to the de Broglie wavelength are considered in Sec. II. Taking into account the vdW interparticle interaction we present the quantum statistics effects in Sec. III. Section IV is devoted to the extension of our analytical results to those using the effective Skyrme potential. In Sec. V, the quantum statistics effects near the critical point are studied for a mixed system of the isotopically symmetric nuclear matter with a small impurity of α particles. The results of our calculations are discussed in Sec. VI, and are summarized in Sec. VII. Some details of our derivations are presented in the Appendix.

II. IDEAL QUANTUM GASES AND QUANTUM STATISTICS PARAMETERS

The pressure $P_i(T,\mu)$ for a system of particles (e.g., $i=N$ for nucleons, $i=\alpha$ for α particles) plays the role of the thermodynamical potential in the grand canonical ensemble (GCE) where the temperature T and chemical potential μ are independent variables [41]. The particle number density $n_i(T,\mu)$, entropy density $s_i(T,\mu)$, and energy density $\mathcal{E}_i(T,\mu)$ are given by

$$\begin{aligned} n_i &= \left(\frac{\partial P_i}{\partial \mu}\right)_T, \\ s_i &= \left(\frac{\partial P_i}{\partial T}\right)_\mu, \\ \mathcal{E}_i &= T s_i + \mu n_i - P_i. \end{aligned} \tag{1}$$

In the thermodynamic limit $V\to\infty$ considered in the present paper, all of intensive thermodynamical functions–P, n, s, and $\mathcal{E}$–depend on T and μ, rather than on the system volume V; see, for instance, Ref. [42]. We start with the GCE expressions, $\sum_i P_i^{\rm id}(T,\mu)$, for the pressure $P^{\rm id}(T,\mu)$ and particle number density, $n^{\rm id}(T,\mu)=\sum_i n_i^{\rm id}(T,\mu)$, for the ideal nonrelativistic quantum gas [41,43],

$$P_i^{\rm id} = \frac{1}{3} g_i \int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{p^2}{m_i}\left[\exp\left(\frac{p^2}{2m_iT}-\frac{\mu}{T}\right)-\theta_i\right]^{-1}, \tag{2}$$

$$n_i^{\rm id} = g_i \int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\exp\left(\frac{p^2}{2m_iT}-\frac{\mu}{T}\right)-\theta_i\right]^{-1}, \tag{3}$$

where m_i and g_i are, respectively, the particle mass and degeneracy factor of the i component. The value of $\theta_i=-1$ corresponds to the Fermi gas, $\theta_i=1$ to the Bose gas, and $\theta_i=0$ is the Boltzmann (classical) approximation when effects of the quantum statistics are neglected.[1]

Equations (2) and (3) for the pressure $P_i^{\rm id}$ and density $n_i^{\rm id}$, proportional to the famous Fermi-Dirac and Bose-Einstein integrals, can be expressed in terms of the fugacity

$$z \equiv \exp(\mu/T) \tag{4}$$

as

$$P_i^{\rm id}(T,z) \equiv \frac{g_iT}{\theta_i\lambda_i^3}\,{\rm Li}_{5/2}(\theta_i z), \tag{5}$$

$$n_i^{\rm id}(T,z) \equiv \frac{g_i}{\theta_i\lambda_i^3}\,{\rm Li}_{3/2}(\theta_i z). \tag{6}$$

Here, λ_i is the de Broglie thermal wavelength [41]

$$\lambda_i \equiv \hbar\sqrt{\frac{2\pi}{m_iT}}, \tag{7}$$

and ${\rm Li}_\nu$ is the polylogarithmic function of order ν. The integral representation of the polylogarithmic functions was used in these derivations; see Eqs. (2) and (3), and Refs. [44,45]. It is convenient also to use the power series for the polylogarithmic functions,

$${\rm Li}_\nu(\theta_i z) \equiv \frac{\theta_i z}{\Gamma(\nu)}\int_0^\infty \frac{{\rm d}x x^{\nu-1}}{\exp(x)-\theta_i z} = \sum_{k=1}^\infty \frac{(\theta_i z)^k}{k^\nu}, \tag{8}$$

where $\Gamma(x)$ is the gamma function. Indexes $\nu=3/2$ and $5/2$ of these functions were used in Eqs. (5) and (6). The values of $\mu>0$, i.e., $z>1$, are forbidden in the ideal Bose gas. The point $\mu=0$ corresponds to an onset of the Bose-Einstein condensation in a system of bosons. For fermions, any values of μ are possible, i.e., integrals (2) and (3) [see also Eq. (8)] exist for $\theta_i=-1$ at all real values of μ. The power series [see Eqs. (5) and (6) with Eq. (8)] is obviously convergent at $z<1$ ($z>0$) (see, e.g., Ref. [45]). For the Fermi statistics at $z\gtrsim 1$, the integral representation of the corresponding polylogarithmic function [see Eq. (8)] in Eqs. (5) and (6) can be used (see Ref. [44]). Particularly, at $z\to\infty$ one can use the asymptotic Sommerfeld expansion of ${\rm Li}_\nu(-z)$ functions over $1/\ln^2|z|$; see Ref. [46].

For the nucleon gas we take $m_N\cong 938$ MeV neglecting a small difference between proton and neutron masses. The degeneracy factor is then $g_N=4$ which takes into account two spin and two isospin states of the nucleon. For the ideal Bose gas of α nuclei, one has $g_\alpha=1$ and $m_\alpha\cong 3727$ MeV.

At $z\ll 1$, only one term, $k=1$, in series, Eq. (8), is sufficient to use in Eqs. (5) and (6) which leads to the classical ideal gas relationship $P_i=n_i\,T$. Note that this result follows automatically from Eqs. (2) and (3) at $\theta_i=0$. The classical Boltzmann approximation at $z\ll 1$ is valid for large T and/or small n region of the n-T plane. In fact, at very small n_i, one observes $z<1$ at small T too.

Inverting Eq. (6) with respect to the fugacity, $z=z(n_i)$, and substituting it into Eq. (5), one obtains the equation of state for an ideal gas through the pressure, $P_i=P_i(T,n_i)$, for any i components. Instead of the particle number density, n_i, it is convenient to introduce the dimensionless argument, $e_i\propto n_i$, of the fugacity z at a given point of the μ-T plane:

$$\begin{aligned} e_i &\equiv -\theta_i\,\varepsilon_i, \\ \varepsilon_i &= \frac{n_i\lambda_i^3}{4\sqrt{2}\,g_i} = D_i n_i, \\ D_i &= \frac{\hbar^3\,\pi^{3/2}}{2\,g_i\,(m_iT)^{3/2}}. \end{aligned} \tag{9}$$

The fugacity z as function of the quantum statistics parameter ε for its small values for nuclear matter is shown in Fig. 1. Taking thus a given component i, e.g., for nucleon matter ($\theta_i=-1$), for simplicity, we omit a subscript i in discussions of this figure. In Fig. 1, the exact fugacity $z(\varepsilon)$ was obtained by multiplying Eq. (6) by the factor $\lambda^3/(4\sqrt{2}\,g)$ to get $\varepsilon=\varepsilon(z)$ and, then, inverting this equation with respect to z.

So far, we did not use the series representation [Eq. (8)] for the polylogarithms ${\rm Li}_\nu$ in discussions of Fig. 1, in particular, for calculations of the solid curve "exact". Other different curves in this figure present the calculations for the maximal

[1]The units with Boltzmann constant $\kappa_{\rm B}=1$ are used. We keep the Planck constant in the formulas to illustrate the effects of quantum statistics, but put $\hbar=h/2\pi=1$ in all numerical calculations. For simplicity, we omitted here and below the subscript "id" for the ideal gas everywhere where it will not lead to a misunderstanding.

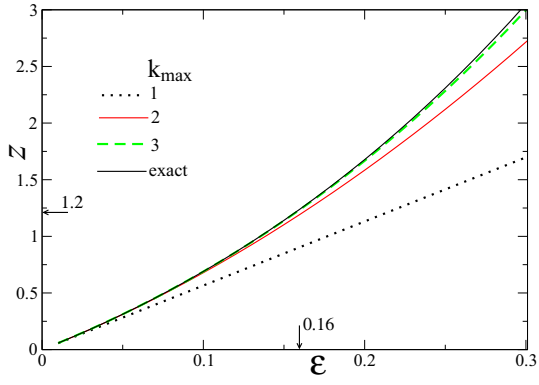

FIG. 1. Fugacities z [Eq. (4)] as functions of the quantum statistics parameter ε [Eq. (9)] for its small values where one finds the critical points $\varepsilon_c = 0.15$–0.16 [Eq. (9) at $T = T_c$ and $n = n_c$; see Ref. [38] and Table I below for nuclear matter]. Solid black curve shows the exact fugacity $z(\varepsilon)$ found by inverting Eq. (6) and using Eq. (9) at $\theta = -1$, and $k_{\max}$ is the maximal power of the cut-off series for the polylogarithm $\mathrm{Li}_{3/2}(-z)$; see Eq. (8). Arrows show approximately the maximal critical values, ε_c and corresponding z_c, under our consideration (Ref. [38]).

power, $k_{\max}$, in the partial sum of Eq. (8) over k. We multiply Eq. (6) with Eq. (8) by the same factor, $\lambda^3/(4\sqrt{2}\,g)$, and use a cutoff of the power series (8) for the polylogarithmic function $\mathrm{Li}_{3/2}(-z)$ at the power $k_{\max}$. As seen from this figure, one has the asymptotic convergence (see Ref. [44]) of $z = z(\varepsilon)$ over $k_{\max}$, with good convergence at $\varepsilon \lesssim 0.2$. Such a convergence is better with smaller ε. The first-order correction leads near the critical point (Refs. [17,19,20,38]) in the region of $\varepsilon \ll 1$. More accurately, this region is given by $\varepsilon \lesssim \varepsilon_c \approx 0.15$–$0.16$, where ε_c is the critical value of ε [see Eq. (9) at the critical values $T = T_c$ and $n = n_c$, Ref. [38], or Table I below]. This region of the variable ε is related to that of $z \lesssim z_c \approx 0.8$–$1.2$ (see Fig. 1), which covers well the corresponding critical values z_c. The first (at $k_{\max} = 2$) and, even better, second ($k_{\max} = 3$) quantum statistics corrections improve the convergence. The cut-off sum (8) for $\mathrm{Li}_{3/2}$ at the maximal power $k_{\max} = 3$ practically, with the precision of lines, achieves the exact result for the fugacity $z = z(\varepsilon)$ (Fig. 1) at $\varepsilon \lesssim 0.2$. In Fig. 1, the arrows show approximately the maximal values of the quantum statistics parameter ε and corresponding fugacity z for which one has still a very good approximation by a few first-order quantum-statistics corrections. However, for larger ε, where the fugacity z is larger or about 1.5 (e.g., in the small temperature limit), the inversion of the cut-off sum [Eq. (8)] for the polylogarithmic function $\mathrm{Li}_{3/2}$ fails: We need more terms and, then, meet a divergence of the series over k with an increasing cut-off value of $k_{\max}$. The region of larger fugacity z (and respectively, larger ε) are shown in Fig. 1 for the purpose of a contrast comparison with that of small values of $z \lesssim 1$, which are really used in our approach. As mentioned above, in a region of very large fugacity, $z \gg 1$, one has to use another asymptotic expansion, for instance, over $1/\ln^2|z|$, as suggested by Zommerfeld [46].

The expansion of $z(\varepsilon)$ in powers of ε is inserted, then, into Eq. (5). At small values, $\varepsilon_i \ll 1$, the expansion of the pressure over powers of ε_i is rapidly convergent. This expansion converges well to the exact (polylogarithmic) function results (5) and (6). Its convergence is the faster the smaller ε_i, such that a few first terms provide already a good approximation of the quantum statistics effects. Taking the few first terms (e.g., $k_{\max} = 4$) in the power series of Eq. (8), one obtains from Eqs. (5) and (6) a classical gas result, $P_i = n_i T$, and the leading first few-order corrections due to the quantum statistics effects:

$$P_i^{\mathrm{id}}(T, n_i) = n_i T\left[1 + e_i - c_2 e_i^2 - c_3 e_i^3 + \mathrm{O}\left(e_i^4\right)\right], \qquad (10)$$

where $c_2 = 4[16/(9\sqrt{3}) - 1] \cong 0.106$, $c_3 = 4(15 + 9\sqrt{2} - 16\sqrt{3})/3 \cong 0.0201$, and so on. For brevity, we will name the linear and quadratic ε_i terms in Eq. (10) as the first- and second-order quantum-statistics corrections.

Equation (10) demonstrates explicitly a deviation of the quantum ideal-gas pressure from its classical value: the Fermi statistics corrections lead to an increasing of the classical pressure while the Bose statistics yields its decreasing. This is often interpreted [41] as the effective Fermi 'repulsion' and Bose 'attraction' between particles.

III. QUANTUM STATISTICS EFFECTS WITH THE VAN DER WAALS INTERPARTICLE INTERACTION

Recently, the vdW equation of state was extended by taking into account the effects of quantum statistics for nuclear matter in Ref. [20]. The pressure function of the quantum vdW (QvdW) model for the one-component system was presented in this paper as

$$P(T, n) = P_{\mathrm{id}}[T, n_{\mathrm{id}}(T, \mu^*)] - an^2, \qquad (11)$$

$$n_{\mathrm{id}}(T, \mu^*) = \frac{n}{1 - bn}, \qquad (12)$$

where P_{id} and n_{id} are respectively given by Eqs. (2) and (3). The modified chemical potential, μ^*, is the solution of

TABLE I. Results for the CP parameters of the van der Waals model (second column), the symmetric nuclear matter (N) ($g_N = 4$, $m_N = 938$ MeV; third, fourth, and fifth columns), and the mixed $N + \alpha$ matter ($g_\alpha = 1$, $m_\alpha = 3737$ MeV; sixth and seventh columns). Numerical results obtained within the full QvdW model in Refs. [20,23] are shown in the fifth and seventh columns, respectively.

Critical point parameters	vdW Eq. (A2)	N 1st-order Eq. (16)	N 2st-order Eq. (17)	N numerical full QvdW	$N + \alpha$ 1st-order Eq. (38)	$N + \alpha$ numerical full QvdW
T_c [MeV]	29.2	19.0	20.0	19.7	19.4	19.9
n_c [fm^{-3}]	0.100	0.065	0.079	0.072	0.072	0.073
P_c [MeV fm^{-3}]	1.09	0.48	0.56	0.52	0.51	0.56

a transcendental equation; see more details in Ref. [20] and the Appendix, as applied for the one-component system. Following Ref. [38], we introduce a small quantum-statistics parameter δ of an expansion of the pressure $P(T,n)$ accounting for the vdW interaction in terms of parameters a and b,

$$\delta \equiv -\frac{\theta\varepsilon}{1-bn} = -\theta\,\frac{\hbar^3\,\pi^{3/2} n}{2\,g\,(1-bn)(mT)^{3/2}}, \tag{13}$$

where θ was defined above for different statistics. Both first and second quantum-statistics corrections over δ to the vdW model will be presented below.

Expanding the pressure component $P_{\rm id}[T, n_{\rm id}(T,\mu^*)]$ in Eq. (11) over the small parameter δ [Eq. (13)], and using Eqs. (10) and (12), one obtains

$$P(T,n) = \frac{nT}{1-bn}[1+\delta - c_2\delta^2 + {\rm O}(\delta^3)] - a\,n^2, \tag{14}$$

where c_2 is the same small number coefficient as in Eq. (10). We proved [38] that at small $|\delta|$ the expansion of the pressure over powers of δ becomes rapidly convergent to the exact results. Therefore, a few first terms provide already a good approximation. A new point of our consideration is the analytical estimates of the quantum statistics effects, and further study of the convergence of the results, including the second order in δ. Similar to the ideal gases, the quantum corrections in Eq. (14) increase with the particle number density n and decrease with the system temperature T, particle mass m, and degeneracy factor g. A new feature of quantum statistics effects in the system of particles with the vdW interaction is the additional factor $(1-bn)^{-1}$ in the correction δ [Eq. (13)]. Thus, the quantum statistics effects become stronger because of the repulsive interaction between particles.

The vdW model, both in its classical form and in its QvdW extension (11) and (12), describes the first order liquid-gas phase transition. The critical point (CP) of this transition satisfies the following equations [41]:

$$\left(\frac{\partial P}{\partial n}\right)_T = 0, \quad \left(\frac{\partial^2 P}{\partial n^2}\right)_T = 0. \tag{15}$$

Using Eq. (14) in the first and second approximation over δ, one derives from Eq. (15) the system of two equations for the CP parameters n_c and T_c at the same corresponding order. Solutions of this system in the same first and second order approximation over δ have the form

$$\begin{aligned} T_c^{(1)} &\cong T_c^{(0)}(1-2\delta_0), \\ n_c^{(1)} &\cong n_c^{(0)}(1-2\delta_0), \end{aligned} \tag{16}$$

and

$$\begin{aligned} T_c^{(2)} &\cong T_c^{(0)}\big(1-2\delta_0+\tfrac{4}{3}\,\delta_0^2\big), \\ n_c^{(2)} &\cong n_c^{(0)}\big(1-2\delta_0+4.62\,\delta_0^2\big). \end{aligned} \tag{17}$$

In Eqs. (16) and (17), the values $T_c^{(0)}$ and $n_c^{(0)}$ are the CP parameters of the classical vdW model; see Eq. (A2). They are defined by Eq. (15) and the vdW equation of state at the zero approximation [see Eq. (14) at $\delta = 0$]. The parameter δ_0 in Eqs. (16) and (17) is given by Eq. (13), taken at the CP of the zero-order approximation (A2), i.e., at $n = n_c^{(0)}$ and $T = T_c^{(0)}$. For simplicity, we present approximately the number 4.62 in Eq. (17) for a cumbersome expression. Substituting Eqs. (16) and (17) for the results of the corresponding critical temperature, $T_c^{(j)}$, and density, $n_c^{(j)}$, where $j = 1$ and 2, into equation of state [Eq. (14)], at a given perturbation order, one can calculate the CP pressure $P_c^{(j)}$ at the same order. Notice that the temperature $T_c^{(1)}$ and density $n_c^{(1)}$ are decreased for Fermi and increased for Bose particles with respect to $T_c^{(0)}$ and $n_c^{(0)}$.

IV. THE SKYRME POTENTIAL MODEL WITH QUANTUM STATISTICS CORRECTIONS

The pressure function of the quantum Skyrme mean-field (QSMF) model [23], after some transformations, can be presented as

$$P_{sk,i}(T,n_i) = P_i^{\rm id}(T,n_i) - a_{sk,i}n_i^2 + b_{sk,i}n_i^{\gamma+2}, \tag{18}$$

where $P_i^{\rm id}$ is given by Eq. (2); $a_{sk,i}$, $b_{sk,i}$, and γ are parameters of the QSMF parametrization. The index i means, e.g., nucleons N or α particles ($i = \{N,\alpha\}$). The QSMF parameters are chosen by fitting properties of one-component nucleon or α matter at temperature $T = 0$.

Within the QSMF model, one can consider the critical points for a first-order liquid-gas phase transition for pure nucleon ($i = N$) or α ($i = \alpha$) matter, separately. The critical point (CP) for the QSMF model obeys the same equation (15) but with the quantum Skyrme mean-field pressure, $P_i = P_{sk,i}(T,n_i)$ [Eq. (18)] for each of components i,

$$\left(\frac{\partial P_i}{\partial n_i}\right)_T = 0, \quad \left(\frac{\partial^2 P_i}{\partial n_i^2}\right)_T = 0. \tag{19}$$

For calculations of the first-order quantum-statistics corrections over the small parameter $|e_i|$ [see Eq. (9)] to the QSMF pressure $P_{sk,i}(T,n_i)$ [Eq. (18)], one obtains approximately from Eq. (10) the following expression for the pressure component $P_i^{\rm id}(T,n_i)$ of Eq. (18):

$$P_i^{\rm id}(T,n_i) = n_i\,T\,(1+e_i). \tag{20}$$

Then, the system of two equations [Eq. (19) for a given i] for the CP density and temperature values, $n_{sk,c}$ and $T_{sk,c}$, up to the same first order over e_i, is reduced to

$$\begin{aligned} T_{sk}(1+2e_i) - 2a_{sk,i}n_{sk} + (\gamma+2)b_{sk,i}n_{sk}^{\gamma+1} &= 0, \\ 2T_{sk}e_i - 2a_{sk,i}n_{sk} + (\gamma+2)(\gamma+1)b_{sk,i}n_{sk}^{\gamma+1} &= 0. \end{aligned} \tag{21}$$

Solving the system [Eq. (21)] of equations for the CP parameters, in the first-order approximation over e_i, one obtains

$$\begin{aligned} T_{sk,c}^{(1)} &\cong T_{sk,c}^{(0)}\big(1-2e_{i,0}\big), \\ n_{sk,c}^{(1)} &\cong n_{sk,c}^{(0)}\left(1-\frac{2e_{i,0}T_{sk,c}^{(0)}}{\gamma(\gamma+1)(\gamma+2)b_{sk,i}\left[n_{sk,c}^{(0)}\right]^{\gamma+1}}\right). \end{aligned} \tag{22}$$

TABLE II. Results for the CP parameters of the symmetric nuclear matter in the quantum Skyrme mean-field (QSMF) model ($g = 4$, $m = 938$ MeV, $\gamma = 1/6$, $a_{sk,N} = 1.167$ GeV fm^3, $b_{sk,N} = 1.475$ GeV fm$^{3+3\gamma}$). Numerical results obtained within the full QSMF model in Ref. [23] are shown in the fourth column.

Critical point parameters	0th-order Eq. (23)	1st-order Eq. (22)	numerical full QSMF
$T_{sk,c}$ [MeV]	20.06	15.1	15.3
$n_{sk,c}$ [fm^{-3}]	0.06	0.047	0.048
$P_{sk,c}$ [MeV fm^{-3}]	0.325	0.194	–

In Eq. (22), the temperature $T^{(0)}_{sk,c}$ and density $n^{(0)}_{sk,c}$ are the solutions of equations [see Eq. (21)] at zeroth-order perturbation, $e_i = 0$,

$$T^{(0)}_{sk,c} = \frac{2\gamma a_{sk,i}}{\gamma+1}\left[\frac{2a_{sk,i}}{b_{sk,i}(\gamma+1)(\gamma+2)}\right]^{1/\gamma},$$
$$n^{(0)}_{sk,c} = \left[\frac{2a_{sk,i}}{b_{sk,i}(\gamma+1)(\gamma+2)}\right]^{1/\gamma}; \quad (23)$$

see also Ref. [22] where another Skyrme parametrization for the critical temperature and particle number density at zero quantum statistics corrections was used. The parameters of Skyrme parametrization, $a_{sk,i}$ and $b_{sk,i}$, and their dimensions are given in the captions of Tables II and III. The value $e_{i,0}$ in Eq. (22) is defined by Eq. (9) at $T = T^{(0)}_{sk,c}$ and $n = n^{(0)}_{sk,c}$ [Eq. (23)]. For the CP pressure at $e_i = 0$, from Eqs. (18), (20), and (23) one finds

$$P^{(0)}_{sk,c} = n^{(0)}_{sk,c}T^{(0)}_{sk,c} - a_{sk,i}\left[n^{(0)}_{sk,c}\right]^2 + b_{sk,i}\left[n^{(0)}_{sk,c}\right]^{\gamma+2}. \quad (24)$$

The first-order pressure, $P^{(1)}_{sk,c}$, can be straightforwardly calculated from Eq. (18) by using Eq. (20) and expressions for $T^{(1)}_{sk,c}$ and $n^{(1)}_{sk,c}$ [Eq. (22)].

V. QUANTUM STATISTICS EFFECTS IN THE QVDW MODEL FOR *N* AND *α* PARTICLES SYSTEMS

For the infinite system of a mixture of different Fermi and Bose particles, e.g., nucleons and α particles, one can present a more simple model based on the vdW forces as a continuation of Sec. III. For this aim, we present the results

TABLE III. Results for the CP parameters of pure α matter in the QSMF model ($g = 1$, $m = 3727$ MeV, $\gamma = 1/6$, $a_{sk,\alpha} = 3.831$ GeV fm^3, $b_{sk,\alpha} = 6.667$ GeV fm$^{3+3\gamma}$). Numerical results obtained within the full QSMF model in Ref. [23] are shown in the fourth column.

Critical point parameters	0th-order Eq. (23)	1st-order Eq. (22)	numerical full QSMF
$T_{sk,c}$ [MeV]	9.667	10.198	10.200
$4n_{sk,c}$ [fm^{-3}]	0.0353	0.037	0.037
$P_{sk,c}$ [MeV fm^{-3}]	0.023	0.025	–

for the pressure function of the vdW model with quantum statistics ingredients of the QvdW model [21],

$$P(T,n) = P^{\rm id}_N(T,\mu^*_N) + P^{\rm id}_\alpha(T,\mu^*_\alpha) - a_{NN}n_N^2 - 2a_{N\alpha}n_Nn_\alpha - a_{\alpha\alpha}n_\alpha^2, \quad (25)$$

where $P^{\rm id}_i$ is the pressure of an ideal ($i = N, \alpha$) gas [Eq. (A5)]. The chemical potential, μ^*_i, in Eq. (25) is modified, as shown in the Appendix, through the transcendent system of equations (A3) and (A4) within the QvdW model in terms of the particle number densities n_i. Following Ref. [21], one can fix the interparticle interaction parameters a_{ij} and b_{ij}. Then, it is convenient to introduce the new volume-exclusion parameters, $\tilde{b}_{ij} = 2b_{ii}b_{ij}/(b_{ii}+b_{jj})$, where $b_{ij} = 2\pi(R_i+R_j)^3/3$, and R_i is the hard-core radius for the ith hard-core particle of a multicomponent system; see Refs. [11,21]. Using the ground state properties of the corresponding system components (see, e.g., Ref. [20]), one has

$$a = a_{NN} = 329.8\,\text{MeV fm}^3,$$
$$b = b_{NN} = \tilde{b}_{NN} = 3.35\,\text{fm}^3. \quad (26)$$

Again, these values are very close to those found in Refs. [20,21]. Small differences appear because of the nonrelativistic formulation used in the present studies. For simplicity, for other attractive interparticle-interaction components we will put [21]

$$a_{N\alpha} = a_{\alpha N} = a_{\alpha\alpha} = 0. \quad (27)$$

For the repulsive interaction components $\tilde{b}_{ij}$ of the vdW exclusion-volume constants we will use those of Ref. [21]:

$$\tilde{b}_{\alpha\alpha} = 16.76\ \text{fm}^3,$$
$$\tilde{b}_{\alpha N} = 13.95\ \text{fm}^3, \quad (28)$$
$$\tilde{b}_{N\alpha} = 2.85\,\text{fm}^3.$$

Notice that the system of N and α particles was studied in Ref. [23] within the quantum Skyrme mean-field model (Sec. IV). However, the authors of this article criticized the QvdW approach because the Bose condensation cannot be described within the QvdW model. This phenomenon is out of scope of the present study, and will be worked out within our analytical approach based on the QSMF model of the previous section in the forthcoming work.

In the Boltzmann approximation, i.e., at $\theta = 0$ in Eqs. (2) and (3), the quantum vdW model is reduced to the classical vdW one [21],

$$P_i = \frac{n_iT}{1-n_j\tilde{b}_{ij}} - a_{ij}n_in_j, \quad (29)$$

where the summations over double repeated subscripts j are implied. Note that the classical vdW approach (29) is further reduced to the ideal classical gas, $P_i = n_iT$, at $a_{ij} = 0$ and $b_{ij} = 0$. At $a_{ij} = 0$ and $b_{ij} = 0$, the QvdW approach turns into that of the quantum ideal gas [Eqs. (2) and (3)].

As mentioned above, a few first quantum-statistics corrections of the QvdW model will be considered. Expanding the pressure component $P^{\rm id}_i(T,\mu^*_i)$ [Eq. (A5)] over small parameters $|e^*_i|$ ($i = N, \alpha$), given by Eq. (9) with replacing n_i by n^*_i,

which will be used below, at the first order in e_i^* one obtains

$$P_i^{\rm id}(T, n_i^*) = n_i^*\, T\, (1 + e_i^*). \tag{30}$$

This expression is similar to those of Eq. (10) at the first order over e_i. As found below, at small e_i^*, the expansion of the pressure over powers of e_i^* becomes rapidly convergent to the exact results, and even the first term provides already a good approximation. Our results of Eqs. (25) and (14) for equations of state, in contrast to Eq. (10), as discussed in Refs. [41,47], take into account the particle interaction effects (cf. with Sec. III). A new point of our consideration now is the analytical estimates of the quantum statistics effects in a mixed system of interacting fermions and bosons. Similar to the ideal gases, the quantum statistics corrections in Eqs. (25) and (14) are increased for Fermi or decreased for Bose particles with the particle number density n_i. They are decreased (or relatively, increased) with the system temperature T, particle mass m_i, and degeneracy factor g_i.

As in Ref. [21], we introduce the "mass fraction" for the α particles impurity,

$$X_\alpha = \frac{4n_\alpha}{n_N + 4n_\alpha} \equiv \frac{4n_\alpha}{n}, \tag{31}$$

where n is the baryon particle-number density, $n = n_N + 4n_\alpha$. According to the numerical solutions in Ref. [21], for the parameters of Eqs. (26)–(28), the value of X_α [Eq. (31)] has been approximately obtained from a thermodynamical equilibrium of our mixed system, $X_\alpha \approx 0.013$. As shown in Ref. [32], the critical point in a similar two-component (neutron-proton) system, as a function of X_i (i is protons), converges with decreasing X_i smoothly to the one-component (neutron) system. In line with these results, one may assume similarly a small change of the critical point with a small α particle impurity, X_α, mentioned above. Therefore, for simplicity, we will use below a smallness of the α particle contribution, X_α, in our approximate CP calculations by Eq. (15) which was applied in Secs. III and IV for one-component systems. Taking this estimate for a simple exemplary case, one can easily find n_N^* and n_α^* from Eq. (A4). Then, using Eqs. (31) and (28), one can present them in the following approximate form:

$$n_N^* \approx \frac{r_1 n}{1 - b_1 n}, \quad n_\alpha^* \approx \frac{r_2 n}{1 - b_2 n}, \tag{32}$$

where

$$r_1 \approx 1 - X_\alpha \approx 0.987, \quad r_2 \approx \frac{X_\alpha}{4} \approx 0.0033. \tag{33}$$

In Eq. (32), b_1 and b_2 are the coefficients which are related approximately to the repulsive interaction constants $\tilde{b}_{ij}$ [$i, j = N, \alpha$; see Eq. (28)]. These coefficients, as functions of $\tilde{b}_{ij}$, can be evaluated as

$$b_1 \approx 3.29\,{\rm fm}^3, \quad b_2 \approx 2.81\,{\rm fm}^3. \tag{34}$$

For another modified attractive-interaction parameter a_1, one can use

$$a_1 = r_1^2\, a_{NN} \approx 321.3\,{\rm MeV\ \ fm}^3. \tag{35}$$

Using Eqs. (11), (30), and (32), for the total system pressure $P(T, n)$ one arrives at

$$P(T, n) = T\,\frac{r_1 n(1+\rho_1)}{1 - b_1 n} + T\,\frac{r_2 n(1 - \rho_2)}{1 - b_2 n} - a_1 n^2. \tag{36}$$

Here,

$$\rho_1 = \frac{D_N r_1 n}{1 - b_1 n}, \quad \rho_2 = \frac{D_\alpha r_2 n}{1 - b_2 n}, \tag{37}$$

D_N and D_α are the constants given by Eq. (9), r_1 and r_2 are given by Eq. (33). Note that the expression for the pressure, Eq. (36), in the case of $r_2 = 0$ and $r_1 = 1$ is exactly the same as for a pure nuclear matter presented in Ref. [38] (see Sec. III). A new feature of the quantum statistics effects in the system of particles with the vdW interactions is the additional factors $(1 - b_1 n)^{-1}$ and $(1 - b_2 n)^{-1}$ in the perturbation parameters. Thus, the quantum statistics effects become stronger because of the repulsive interactions between particles.

The vdW model, both in its classical form (29) and in its QvdW extension [Eqs. (11) and (36)], describes the first-order liquid-gas phase transition. As the value of X_α in our derivations is very small, the critical points in the considered approach can be determined approximately (see above) by the same equations given by Eq. (15). Using Eq. (36) in the first approximation over quantum statistics corrections, one derives from Eq. (15) the system of two equations for the CP parameters n_c and T_c at the same first order:

$$\begin{aligned} 2na_1 &= \frac{{\rm Tr}_1\,(1 + 2\rho_1)}{(1 - b_1 n)^2} + \frac{{\rm Tr}_2\,(1 - 2\rho_2)}{(1 - b_2 n)^2}, \\ a_1 &= \frac{{\rm Tr}_1 b_1}{(1 - b_1 n)^3}\left[1 + \rho_1 \frac{(1 + 2b_1 n)}{b_1 n}\right] \\ &\quad + \frac{{\rm Tr}_2 b_2}{(1 - b_2 n)^3}\left[1 - \rho_2 \frac{(1 + 2b_2 n)}{b_2 n}\right]. \end{aligned} \tag{38}$$

Note that Eq. (38) for the CP in the case of $r_1 = 1$, $r_2 = 0$, and $b_1 = b_{NN}$ is exactly the same as that for a pure nucleon matter, which was derived in Ref. [38] (see Sec. III).

VI. DISCUSSION OF THE RESULTS

A summary of the results for CP parameters are presented for the QvdW model and Skyrme mean-field parametrization in Figs. 2 and 3 and Tables I–III. Figure 2 shows the contour graphics in the n-T plane where black lines mean $z(n, T) =$ const [see Fig. 1 and Eqs. (6) and (4)] in the left and $\varepsilon(n, T) =$ const [Eq. (9)] in the right panels with the constant values written in white squares. As seen from these plots, all values of $z \lesssim 1$ ($z \lesssim 1.2$) correspond to $\varepsilon \ll 1$ ($\varepsilon \lesssim 0.2$) above blue regions. Therefore, together with Fig. 1, this explains the reason for using the expansion over a small parameter ε, even when the fugacity z is of the order of one, having a little larger values. In particular, such a region of $\varepsilon \ll 1$ and $z \sim 1$ contains the critical points, which were obtained in Ref. [38] and shown now in Fig. 2 and Table I.

Figure 3 shows the isotherms of the pressure P as a function of the reduced volume, $v = 1/n$, and the particle number density n for an isotopically symmetric nuclear matter. The first- (and second-)order quantum-statistics corrections are

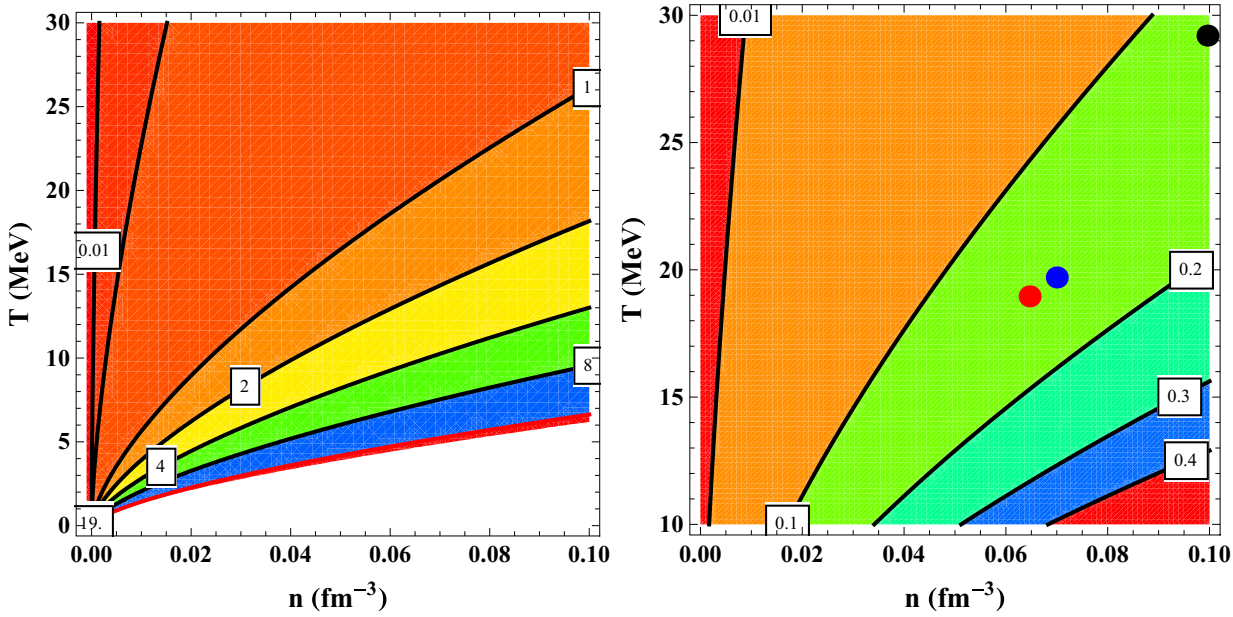

FIG. 2. Contour plots for the first-order fugacity $z(n, T)$ ($k_{\max} = 2$) and corresponding parameter $\varepsilon(n, T)$ for nucleon matter in the plane of density n and temperature T are shown in the left and right panels, respectively. The red line (left) shows the zero entropy line, such that the white area is related to a nonphysical region where the entropy of the ideal gas is negative. The critical point for our first-order and the zero-order (standard vdW) approximations for nuclear matter at the parameters a and b [Eq. (26)] are shown in the right panel by the red [see Eq. (16), Table I, and Ref. [38]] and the black (vdW) point, respectively. The blue point in the same plot presents the numerical result for the critical point (Ref. [20] and Table I).

presented in this figure and Table I. The critical point is shown by the closed circle found from the accurate solution to equations of Eq. (15) [see Eq. (16)] for nucleon matter; see also the closed red circle in Fig. 2. The dotted line shows the second-order approximation [see Eqs. (17) and (14)] at the same nuclear matter parameters. Dotted line presents schematically a binodal boundary for the two-phase coexistence curve in the transition from the two- to one-phase range [38].

Results for the CP parameters obtained by Eqs. (16) and (17), and by solving the system of equations, Eq. (38), are presented in Table I. These analytical results are close to the numerical results obtained in Refs. [20,21]. For the same nucleon matter case, a difference of the results for the vdW [Eq. (A2)] and QvdW [Eqs. (16) and (17)] models in Table I demonstrates a significant role of the effects of quantum statistics for the CP of the symmetric nuclear matter. Table I shows also a good convergence of expansion over the quantum

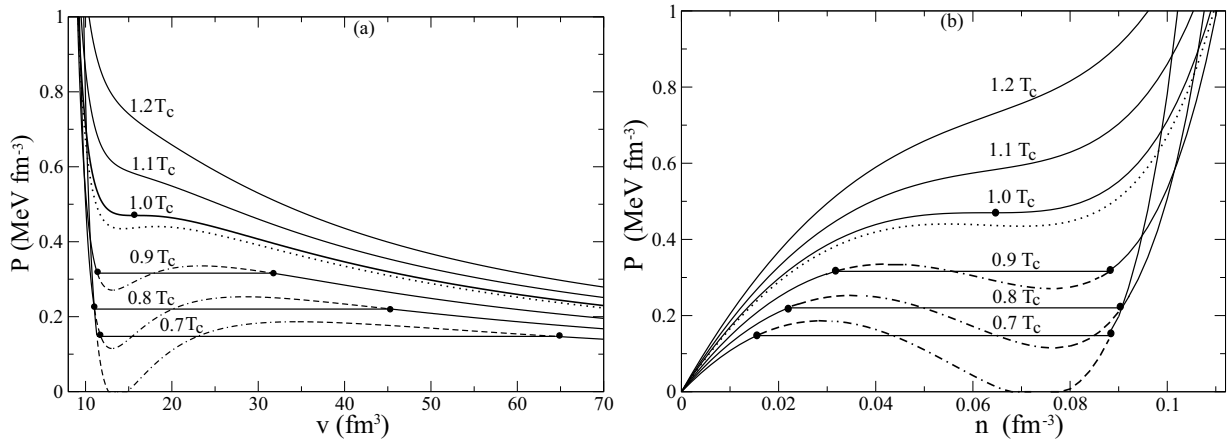

FIG. 3. Pressures P as functions of the reduced volume v (a) and particle number density n (b) at different temperatures T (in units of the critical value T_c) at first order in the quantum statistics expansion over δ [Eq. (13)] for the simplest case of the symmetric nucleon matter. The critical point is shown by the closed circle, see text for details. The dotted line shows the second-order approximation over δ; see Ref. [38] and Eq. (14). The horizontal lines are plotted by using the Maxwell area law in (a) and correspondingly in (b). The unstable and metastable parts of the isothermal lines are presented by dashed and dash-dotted lines, respectively. Other closed dots show schematically a binodal boundary for the two-phase coexistence curve.

statistics parameter for the case of the mixed N-α system with a small X_α, given by Eq. (31). Even the first-order corrections are in good agreement with the exact numerical QvdW results; see Refs. [20,21,38]. In addition, many other examples were recently considered in Ref. [48]. As seen from Table I, the quantum statistics effects of the α-particle impurity are, in fact, small because, first of all, of too small of a relative concentration X_α of this impurity, according to Eq. (31), which was suggested in Ref. [21]. The size of these effects appears to be rather different for the case of impurity contributions $X_\alpha \cong 1$ of α particles into the nucleon matter.

As stated above, our analysis can be applied beyond the vdW model. In fact, similar estimates of the quantum statistic effects have been done also for one of the mean-field models (Ref. [24] and references therein) in Sec. IV; see Tables II and III. The QSMF calculations were performed for $\gamma = 1/6$ and other corresponding parameters from Ref. [23] and presented in these tables. We found very good agreement between the analytical results of calculations (22) up to the first-order corrections over e_i and numerical results obtained in Ref. [23]; see Table II for nucleon matter and Table III for α matter. A similar good agreement with the results of Ref. [23] takes place also for another parameter, $\gamma = 1$. This value was found in the derivations of the SMF approach [25] from the original Skyrme forces [26,27]. Thus, the first order over small parameter e_i in the expansion of the pressure within the quantum Skyrme mean-field approach, as well in the QvdW model, turns out to be sufficient for a very good agreement with numerical calculations [23] of the CP parameters.

We should emphasize also that it is remarkable that the results obtained up to the first-order corrections reproduce the quantum statistics effects with a high accuracy (see Tables I–III). The contribution of high-order (e.g., second-order) corrections in an expansion over δ_i for the QvdW model, or over e_i for the Skyrme-mean field parametrization, is much smaller than the first-order correction, that shows a fast convergence in δ_i, or e_i, by accounting for the first-order terms. Notice that a smallness of the parameters δ_i is associated with those of e_i because of $\delta_i \propto e_i$. Therefore, high-order corrections due to the quantum statistics effects can be neglected for main evaluations of the critical point values.

VII. CONCLUSIONS

We derived the critical-point temperature, particle number density, and pressure for the nucleon and α-particle matter within the quantum van der Waals (QvdW) and quantum Skyrme mean-field (QSMF) models taking into account the Fermi and Bose statistics corrections. We found their analytical dependence on the system parameters of particles, such as their mass, degeneracy, and interparticle interaction constants. In order to determine the equation of state and the critical point while accounting for the quantum statistics and interaction effects, it is sufficient to keep only the first term in the pressure expansion basically over the small quantum-statistics parameter e_i, where i denotes the baryon system under consideration. The specific properties of particles (their mass and degeneracy factor) appear in the CP values through this small parameter e_i.

Our derivations were carried out for systems of Fermi or Bose particles in two cases: for the QvdW model and the QSMF parametrization. In both cases, taking into account already the first-order terms in the expansion of the pressure over e_i greatly simplifies the form of the equation of state and solution of this equation for the critical values of temperature, density, and pressure. The values of these critical quantities at leading first order turn out to be very close to those obtained in the accurate numerical calculations within the full QvdW and QSMF models. For relatively small temperatures T and/or large particle-number densities n, the quantum statistics parameter, $|e_i| \propto n_i T^{-3/2}$, becomes large. In this region of the phase diagram, the perturbation expansion diverges and, therefore, the QvdW and QSMF approaches should be treated within the full quantum statistical formulation. However, as is well known [41], for the limit of small temperatures T and/or large particle densities n, the vdW approach fails. In particular, as shown earlier (Ref. [23]), the Bose condensation phenomenon should be treated within the QSMF model, in contrast to the vdW approach.

A simple and explicit dependence on the system parameters, such as the particle mass m_i and degeneracy factor g_i, is demonstrated at the leading few first orders of expansion over e_i. Such a dependence is absent within the classical van der Waals and Skyrme mean-field models. The quantum statistics parameter e_i, is proportional to $m_i^{-3/2} g_i^{-1}$. Therefore, the effects of quantum statistics become smaller for more heavy particles and/or for larger values of their degeneracy factor.

The quantum statistics corrections to the CP parameters of the symmetric nuclear matter appear to be quite significant. For a pure nuclear matter, the value of $T_c^{(0)} = 29.2$ MeV in the classical vdW model is decreased dramatically to the QvdW value $T_c^{(1)} = 19.0$ MeV at the first-order approximation in the quantum statistics expansion. On the other hand, this approximate analytical result within the first-order quantum-statistics approach is already close to the accurate numerical value of $T_c = 19.7$ MeV, which was obtained by numerical calculations within the full QvdW model. For the Skyrme mean-field parametrization, the quantum statistics effect is smaller than that for the quantum van der Waals model. This improves the foundation of the perturbation approach used with respect to the small parameter e_i. The agreement of the first-order QSMF approach with full numerical calculations [23] is even better than that within the QvdW model. The nuclear matter value of $T_c^{(0)} = 20.6$ MeV in the classical SMF case is decreased to the quantum SMF value $T_c^{(1)} = 15.1$ MeV. This result is obviously very close to that of numerical calculations, $T_c = 15.3$ MeV, obtained in Ref. [23].

The QwdW equation of state has been derived analytically and used to study the quantum statistics effects in a vicinity of the critical point of the two-component system of nucleon and α-particle matter. The expressions for the pressure of the equation of state were obtained by using the quantum statistics expansion over two small parameters e_i^* ($i = \{N, \alpha\}$) near the vdW approach. The CP parameters are somewhat increased as compared to those for a pure nucleon system. These analytical results are in good agreement with those of more accurate numerical calculations. A very small impurity of α particles to the nucleon matter leads to very small corrections to the

equation of state, and to the critical point of the nuclear matter.

Finally, one can conclude that our derivations within the quantum van der Waals and Skyrme mean-field parametrizations are straightforwardly extended to other types of interparticle interactions. In particular, it is expected to be the case for a more general mean-field approach.

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APPENDIX: FORMULATION OF THE QVDW MODEL

The constants of the QvdW model, $a > 0$ and $b > 0$, are responsible for, respectively, attractive and repulsive interactions between particles. We fix the model parameters a and b using the ground state properties of the symmetric nuclear matter (see, e.g., Ref. [49]): at $T = 0$ and $n = n_0 = 0.16\,\mathrm{fm}^{-3}$, one requires $P = 0$ and the binding energy per nucleon $\varepsilon(T = 0, n = n_0)/n_0 = -16$ MeV. From the above requirements,[2] one finds

$$a = 329.8\,\mathrm{MeV\ fm}^3, \quad b = 3.35\,\mathrm{fm}^3. \tag{A1}$$

Therefore, in Eqs. (16) and (17), one has the CP parameters of the classical vdW model

$$\begin{aligned} T_c^{(0)} &= \frac{8a}{27b} \cong 29.2\ \mathrm{MeV}, \\ n_c^{(0)} &= \frac{1}{3b} \cong 0.100\mathrm{fm}^{-3}, \\ P_c^{(0)} &= \frac{a}{27b^2} \cong 1.09\,\mathrm{MeV\,fm}^{-3}. \end{aligned} \tag{A2}$$

They were found from Eq. (15) for the equation of states in the case of $\delta = 0$.

Following the formulation of the QvdW model [21], one can present the equation of state (25) in terms of the pressures of the ideal two-component gas in the grand canonical ensemble, but in terms of the modified chemical potentials μ_i^* ($i = N, \alpha$). These potentials, μ_i^*, were determined through the particle number densities n_i by a system of transcendent equations. First, they are found as functions of the densities n_i^* by solving the following equations:

$$\begin{aligned} n_N^* &= n_N^{\mathrm{id}}(T, \mu_N^*) \equiv \frac{2g_N}{\sqrt{\pi}\lambda_N^3} \int_0^\infty d\eta \frac{\eta^{1/2}}{\exp\left(\eta - \frac{\mu_N^*}{T}\right) + 1}, \\ n_\alpha^* &= n_\alpha^{\mathrm{id}}(T, \mu_\alpha^*) \equiv \frac{2g_\alpha}{\sqrt{\pi}\lambda_\alpha^3} \int_0^\infty d\eta \frac{\eta^{1/2}}{\exp\left(\eta - \frac{\mu_\alpha^*}{T}\right) - 1}, \end{aligned} \tag{A3}$$

where n_i^{id} is defined by Eq. (3); see more details in Refs. [20,21]. Then, n_i^* can be obtained in terms of the true particle-number densities n_i of an interacting particles system where interaction is described by the vdW exclusion-volume constants $\tilde{b}_{ij}$, Eq. (28),

$$\begin{aligned} n_N &= \frac{n_N^*[1 + (\tilde{b}_{\alpha\alpha} - \tilde{b}_{\alpha N})n_\alpha^*]}{1 + \tilde{b}_{NN}n_N^* + \tilde{b}_{\alpha\alpha}n_\alpha^* + \left(\tilde{b}_{NN}\tilde{b}_{\alpha\alpha} - \tilde{b}_{N\alpha}\tilde{b}_{\alpha N}\right)n_N^* n_\alpha^*}, \\ n_\alpha &= \frac{n_\alpha^*[1 + (\tilde{b}_{NN} - \tilde{b}_{N\alpha})n_N^*]}{1 + \tilde{b}_{NN}n_N^* + \tilde{b}_{\alpha\alpha}n_\alpha^* + \left(\tilde{b}_{NN}\tilde{b}_{\alpha\alpha} - \tilde{b}_{N\alpha}\tilde{b}_{\alpha N}\right)n_N^* n_\alpha^*}. \end{aligned} \tag{A4}$$

Finally, one can obtain the pressure P_i^{id} given by Eq. (2) but with the modified chemical potential μ_i^*, found from Eqs. (A3) and (A4), as

$$\begin{aligned} P_N^{\mathrm{id}}(T, \mu_N^*) &= \frac{4g_N T}{3\sqrt{\pi}\lambda_N^3} \int_0^\infty d\eta \frac{\eta^{3/2}}{\exp\left(\eta - \frac{\mu_N^*}{T}\right) + 1}, \\ P_\alpha^{\mathrm{id}}(T, \mu_\alpha^*) &= \frac{4g_\alpha T}{3\sqrt{\pi}\lambda_\alpha^3} \int_0^\infty d\eta \frac{\eta^{3/2}}{\exp\left(\eta - \frac{\mu_\alpha^*}{T}\right) - 1}. \end{aligned} \tag{A5}$$

Equations (A4) and (A5) are used for derivations of the equation of state (25) in the main text.

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