Entanglement entropy as a signature of a quantum phase transition in nuclei in the framework of the interacting boson model and interacting boson-fermion model

M. A. Jafarizadeh
Department of Theoretical Physics and Astrophysics, University of Tabriz, Tabriz 51664, Iran and Research Institute for Fundamental Sciences, Tabriz 51664, Iran

M. Ghapanvari [*]
Plasma and Nuclear Fusion Research School, Nuclear Science and Technology Research Institute, P.O. Box 14399-51113, Tehran, Iran

N. Amiri
Department of Nuclear Physics, University of Tabriz, Tabriz 51664, Iran

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Background: One of the fundamental problems of quantum information science is quantum entanglement. Recently, quantum phase transition in nuclear systems is studied in connection with quantum entanglement.
Purpose: In this paper, the use of entanglement entropy as a suitable signal for the study of quantum phase transition in the even-even and odd-*A* nuclei is investigated. The effect of the coupling of a single fermion to a boson core on entanglement entropy is studied.
Method: By use of the affine $SU(1,1)$ Lie algebra and through the Schmidt decomposition in the framework of the interacting boson model (IBM) and interacting boson-fermion model (IBFM), entanglement entropy in the even-even and odd-*A* are obtained. The entanglement entropy is used for tracking and studying the shape phase transition in these nuclei.
Results: The entanglement entropy in the IBM and the IBFM is calculated. The entanglement entropy values of the low-lying states of $^{122-134}$Xe, $^{102-110}$Pd, and $^{123-133}$Xe were calculated and analyzed. It is found that entanglement entropy is a suitable order parameter to detect shape phase transition in nuclear systems.
Conclusions: The obtained results indicate that the entropy of entanglement is sensitive to the shape-phase transition between spherical and γ-unstable regions in nuclei. The results show that entanglement entropy is a powerful tool for identifying shape phase transitions in nuclei. It is found that the coupling of the single fermion with angular momentum j to the even-even system does not change the geometry imposed by the boson core performing the transition and only the entanglement entropy values have been shifted by the addition of the odd particle with respect to the even case.

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I. INTRODUCTION

One of the basic issues of quantum information science is quantum entanglement. Quantum entanglement is a physical phenomenon that occurs when a group of particles are generated, interact, or share spatial proximity in a way such that the quantum state of each particle of the group cannot be described independently of the state of the others, including when the particles are separated by a large distance. Such a phenomenon was the subject of study by Einstein, Podolsky, and Rosen in 1935 [1]. Quantum entangled states play a key role in quantum information processes such as quantum teleportation [2], quantum cryptography [3], and quantum computing [4]. Quantum entanglement is the basis of quantum information theory and the primary source for quantum effects. Determining the amount of entanglement in the multiparticle state is a challenging task. There has been a lot of research on entanglement issues recently [5–8]. Quantum entanglement is a nonclassical phenomenon that results from the quantum correlation between separate subsystems, and in recent years quantum entanglement has played a large role in quantum information processes. One of the great goals of quantum information theory is to study and understand the entanglement properties of multiparticle systems. These properties can be a good source for quantum computing and communication. The definition of entanglement for pure states is easy to understand, while it is difficult to fully identify the entanglement properties of mixed states and remains an unsolved mathematical problem.

Recently, quantum information theory provides a new approach to the observation and study of entangled nuclear systems, such as nuclei, which are considered multiparticle systems [9]. The use of quantum information in nuclear systems provides interesting new opportunities for understanding and interpreting strong interaction phenomena [10]. Nuclear

[*]mghapanvari@aeoi.org.ir

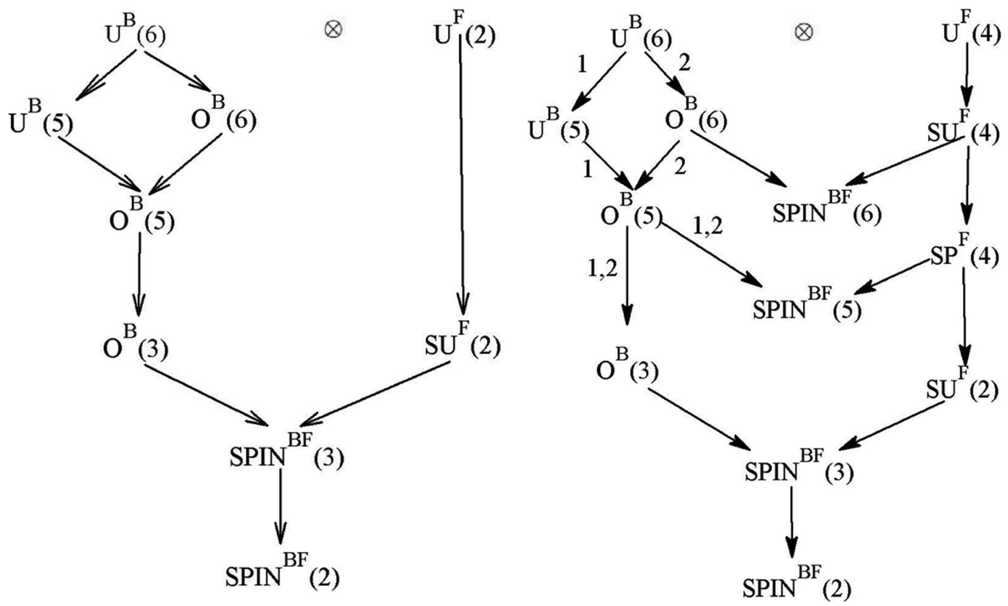

FIG. 1. The lattice of algebras in the case of a system of N bosons coupled to a fermion with angular momentum $j_f = 1/2$ (left) and $j_f = 3/2$ (right).

many-body systems are self-bound systems of four species of fermions: spin up and down protons, and neutrons. There are a variety of phenomena in many-body correlations [11]. Cluster is one of the significant spatial correlations, which is a subunit composed of strongly correlating nucleons. The α cluster is a typical cluster in nuclear systems, which is a composite particle consisting of four nucleons. All nucleons behave as independent particles if there is no correlation between nucleons in a nucleus, and the nucleus is an uncorrelated state with a clear Fermi surface written by a single Slater determinant wave function. Many-body nuclear problem is a fundamental challenge to the theory of nuclear physics. The development of new quantum algorithms for the problem of multiparticle nuclei is currently underway in nuclear physics [12–17]. Quantum entanglement is a fundamental concept of quantum physics that plays a profound role in all applications of quantum mechanics. Thus, it is clear that the concepts and tools of quantum information are entering the field of interest in nuclear physics. Consequently, it is essential that both quantum information theorists and the nuclear group be interested in the entanglement properties exhibited by nuclear systems and processes. For example, entanglement in nuclear systems and nuclear processes such as fission, etc., is being investigated [18,19].

Quantum phase transition (QPT) in nuclear systems has been studied in connection with entanglement, recently [20–29]. In order to characterize the properties of the system, one has to introduce an order parameter. Entanglement, as one of the most interesting features of quantum theory, can be regarded as a order parameter in nuclear systems. Therefore, its measurement is important for quantum phase transition. One of the best measurements for quantum entanglement in physical systems is the on Neumann entropy [30–32].

In this paper, we study quantum entanglement, given the importance of nuclear systems and quantum information top-

TABLE I. The obtained values of $S^{BF}_{a1/2}$ for some states in Eq. (41).

$J, j_f = \frac{1}{2}$	L	M	$S^{BF}_{a1/2}$
$(\frac{1}{2})_1$	0	$\frac{1}{2}, -\frac{1}{2}$	0
$(\frac{3}{2})_1$	2	$\frac{3}{2}, -\frac{3}{2}$	0.500402909459306
$(\frac{3}{2})_1$	2	$\frac{1}{2}, -\frac{1}{2}$	0.673011366800717
$(\frac{5}{2})_1$	2	$\frac{5}{2}, -\frac{5}{2}$	0
$(\frac{5}{2})_1$	2	$\frac{3}{2}, -\frac{3}{2}$	0.500402909459306
$(\frac{5}{2})_1$	2	$\frac{1}{2}, -\frac{1}{2}$	0.673011366800717
$(\frac{7}{2})_1$	4	$\frac{7}{2}, -\frac{7}{2}$	0.348831898965803
$(\frac{7}{2})_1$	4	$\frac{5}{2}, -\frac{5}{2}$	0.561670980048556
$(\frac{7}{2})_1$	4	$\frac{3}{2}, -\frac{3}{2}$	0.636513730776935
$(\frac{7}{2})_1$	4	$\frac{1}{2}, -\frac{1}{2}$	0.686961487957337
$(\frac{9}{2})_1$	4	$\frac{9}{2}, -\frac{9}{2}$	0
$(\frac{9}{2})_1$	4	$\frac{7}{2}, -\frac{7}{2}$	0.348831898965803
$(\frac{9}{2})_1$	4	$\frac{5}{2}, -\frac{5}{2}$	0.529706563726931
$(\frac{9}{2})_1$	4	$\frac{3}{2}, -\frac{3}{2}$	0.636513730776935
$(\frac{9}{2})_1$	4	$\frac{1}{2}, -\frac{1}{2}$	0.686961487957337
$(\frac{11}{2})_1$	6	$\frac{11}{2}, -\frac{11}{2}$	0
$(\frac{11}{2})_1$	6	$\frac{9}{2}, -\frac{9}{2}$	0.304635098258958
$(\frac{11}{2})_1$	6	$\frac{7}{2}, -\frac{7}{2}$	0.474139101773031
$(\frac{11}{2})_1$	6	$\frac{5}{2}, -\frac{5}{2}$	0.585952471908376
$(\frac{11}{2})_1$	6	$\frac{3}{2}, -\frac{3}{2}$	0.655481809041954
$(\frac{11}{2})_1$	6	$\frac{1}{2}, -\frac{1}{2}$	0.689009167757025
$(\frac{13}{2})_1$	6	$\frac{13}{2}, -\frac{13}{2}$	0
$(\frac{13}{2})_1$	6	$\frac{11}{2}, -\frac{11}{2}$	0.271189151457436
$(\frac{13}{2})_1$	6	$\frac{9}{2}, -\frac{9}{2}$	0.429323160436816
$(\frac{13}{2})_1$	6	$\frac{7}{2}, -\frac{7}{2}$	0.540203960636381
$(\frac{13}{2})_1$	6	$\frac{5}{2}, -\frac{5}{2}$	0.617242040531539
$(\frac{13}{2})_1$	6	$\frac{3}{2}, -\frac{3}{2}$	0.666278053500041
$(\frac{13}{2})_1$	6	$\frac{1}{2}, -\frac{1}{2}$	0.690185959613612

TABLE II. The obtained values of $S^{BF}_{a3/2}$ for some states in Eq. (48).

$J, j_f = \frac{3}{2}$	L	M	$a_{L,m_{j_f}}$	$S^{BF}_{a3/2}$
$(\frac{3}{2})_1$	0	$\frac{3}{2}, -\frac{3}{2}$	1	0
$(\frac{3}{2})_1$	0	$\frac{1}{2}, -\frac{1}{2}$	1	0
$(\frac{5}{2})_1$	2	$\frac{5}{2}, -\frac{5}{2}$	1	0.682908002922725
$(\frac{5}{2})_1$	2	$\frac{3}{2}, -\frac{3}{2}$	1	0.801402354214379
$(\frac{5}{2})_1$	2	$\frac{1}{2}, -\frac{1}{2}$	1	1.24808174892308
$(\frac{7}{2})_1$	2	$\frac{7}{2}, -\frac{7}{2}$	1	0
$(\frac{7}{2})_1$	2	$\frac{5}{2}, -\frac{5}{2}$	1	0.682908002922725
$(\frac{7}{2})_1$	2	$\frac{3}{2}, -\frac{3}{2}$	1	0.955699386213412
$(\frac{7}{2})_1$	2	$\frac{3}{2}, -\frac{3}{2}$	1	1.058469962302830
$(\frac{7}{2})_2$	2,4	$\frac{7}{2}, -\frac{7}{2}$	$a_{2,3/2} = 0.9970501385579$	0.317861458497327
			$a_{4,3/2} = 0.076752988226342$	
			$a_{2,1/2} = a_{2,-1/2} = a_{2,-3/2} = 0, a_{4,1/2} = a_{4,-1/2} = a_{4,-3/2} = 1$	
$(\frac{7}{2})_2$	2,4	$\frac{5}{2}, -\frac{5}{2}$	$a_{2,3/2} = 0.986927565048613$	0.351229512059346
			$a_{4,3/2} = 0.161164454353977$	
			$a_{2,1/2} = 0.962397155643328$	
			$a_{4,1/2} = -0.2716463046125$	
			$a_{2,-1/2} = a_{2,3/2} = 0, a_{4,-1/2} = a_{4,-3/2} = 1$	
$(\frac{7}{2})_2$	2,4	$\frac{3}{2}, -\frac{3}{2}$	$a_{2,3/2} = 0.958314847499910$	1.065877687240593
			$a_{4,3/2} = 0.285714285714286$	
			$a_{2,1/2} = 0.977107162598646$	
			$a_{4,1/2} = -0.2127477210181$	
			$a_{2,-1/2} = 0.997050134154378$	
			$a_{4,-1/2} = 0.07675304542972$	
			$a_{2,3/2} = 0, a_{4,-3/2} = 1$	
$(\frac{7}{2})_2$	2,4	$\frac{1}{2}, -\frac{1}{2}$	$a_{2,3/2} = 0.878310372739746$	1.157532567165960
			$a_{4,3/2} = 0.478090879580199$	
			$a_{2,1/2} = 0.986558491842443$	
			$a_{4,1/2} = -0.1634085131612$	
			$a_{2,-1/2} = 0.95831484749991$	
			$a_{4,-1/2} = 0.28571428571428$	
			$a_{2,3/2} = 0.654653812418035$	
			$a_{4,-3/2} = 0.75592882329392$	

ics. One of our goals is to study the quantum phase transition and the structure of the nucleus based on the degree of entanglement. Using the entanglement effect, we identify and describe the quantum phase transition and investigate the entanglement behavior near the quantum critical points. Entanglement properties are used to study different limits as well as the quantum phase transitions of multiparticle systems, which provide a set of clear formulas that can be easily compared with experience. To investigate the phase transition, we evaluate exact solutions for the entanglement entropy for the transitional region in the interacting boson model (IBM) and interacting boson-fermion model (IBFM) by using the dual algebraic structure for the two-level pairing model based on the affine $SU(1,1)$ Lie algebra and through the Schmidt decomposition. We presented experimental evidence for the U(5)–O(6) transition in even-even and odd nuclei and performed an analysis for these isotopes. The entanglement entropy values of the low-lying states of $^{122-134}$Xe, $^{102-110}$Pd, and $^{123-133}$Xe are presented. Particularly, how the presence of an odd particle can influence entanglement entropy values and phase transition is investigated.

TABLE III. The obtained values of $S(m_{j_f}, L)$ according to Eq. (54) and Table II.

$J, j_f = \frac{3}{2}$	M	$S(L\|m_{j_f} = m_{j_f\,\mathrm{fix}})$	$\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} P(m_{j_f}) S(L\|m_{j_f} = m_{j_f\,\mathrm{fix}})$	$S(m_{j_f}, L)$
$(\frac{3}{2})_1$	$\frac{3}{2}, -\frac{3}{2}$	0	0	0
$(\frac{3}{2})_1$	$\frac{1}{2}, -\frac{1}{2}$	0	0	0
$(\frac{5}{2})_1$	$\frac{5}{2}, -\frac{5}{2}$	0	0	0.682908002922725
$(\frac{5}{2})_1$	$\frac{3}{2}, -\frac{3}{2}$	0	0	0.801402354214379
$(\frac{5}{2})_1$	$\frac{1}{2}, -\frac{1}{2}$	0	0	1.248081748923086
$(\frac{7}{2})_1$	$\frac{7}{2}, -\frac{7}{2}$	0	0	0
$(\frac{7}{2})_1$	$\frac{5}{2}, -\frac{5}{2}$	0	0	0.682908002922725
$(\frac{7}{2})_1$	$\frac{3}{2}, -\frac{3}{2}$	0	0	0.955699386213412
$(\frac{7}{2})_1$	$\frac{3}{2}, -\frac{3}{2}$	0	0	1.058469962302830
$(\frac{7}{2})_2$	$\frac{7}{2}, -\frac{7}{2}$	$S(2\|m_{j_f} = 3/2) = 0.005873634961758$	0.033368053562019	0.351229512059346
		$S(4\|m_{j_f} = 3/2) = 0.03024642283602$		
$(\frac{7}{2})_2$	$\frac{5}{2}, -\frac{5}{2}$	$S(2\|m_{j_f} = 3/2) = 0.025633698415217$	0.176435682081098	1.018053623664356
		$S(4\|m_{j_f} = 3/2) = 0.094822173691193$		
		$S(2\|m_{j_f} = 1/2) = 0.070999551624644$		
		$S(4\|m_{j_f} = 1/2) = 0.192338755430037$		
$(\frac{7}{2})_2$	$\frac{3}{2}, -\frac{3}{2}$	$S(2\|m_{j_f} = 3/2) = 0.078206150516608$	0.203947080003536	1.2527401342420873
		$S(4\|m_{j_f} = 3/2) = 0.204532729550264$		
		$S(2\|m_{j_f} = 1/2) = 0.044221473473632$		
		$S(4\|m_{j_f} = 1/2) = 0.140098047383937$		
		$S(2\|m_{j_f} = -1/2) = 0.00587364369094$		
		$S(4\|m_{j_f} = -1/2) = 0.030246459139805$		
$(\frac{7}{2})_2$	$\frac{1}{2}, -\frac{1}{2}$	$S(2\|m_{j_f} = 3/2) = 0.200193951358005$	0.176309328850992	1.119592176907048
		$S(4\|m_{j_f} = 3/2) = 0.337349804950128$		
		$S(2\|m_{j_f} = 1/2) = 0.026342618382404$		
		$S(4\|m_{j_f} = 1/2) = 0.09674269238582$		
		$S(2\|m_{j_f} = -1/2) = 0.0782061505166$		
		$S(4\|m_{j_f} = -1/2) = 0.204532729550264$		
		$S(2\|m_{j_f} = -3/2) = 0.36312762611895$		
		$S(4\|m_{j_f} = -3/2) = 0.31978053195856$		

The paper is organized as follows. In Sec. II, we focus our attention on the theoretical aspects. The results are presented in Sec. III. The final section is devoted to summary and conclusion.

II. THEORETICAL FRAMEWORK

In order to clarify the entanglement entropy investigation in nuclear systems, the used models in even-even and odd-A nuclei must be introduced based on the $SU(1,1)$ algebraic technique. An alternative solvable description of the quantum shape phase transition in the interacting boson model within the framework of $SU(1,1)$ Lie algebra is reported in [33,34]. Recently, exact solutions of the solvable Hamiltonian regarding the extension of IBM using the vector boson [35,36], IBFM [37–40], and the interacting boson-fermion-fermion model (IBFFM) [41] were obtained.

A. The interacting boson model along the $U^B(5)$–$O^B(6)$ line based on the affine $SU(1,1)$ algebra

The quasispin algebras have been explained in detail in Refs. [42–45]. First, a set of three operators that satisfies the following commutation relations is considered [44,45]:

$$[S^0, S^{\pm}] = \pm S^{\pm}, \qquad [S^+, S^-] = -2\theta S^0. \tag{1}$$

In relation (2.1), algebra is equivalent to $SU(1,1)$ when $\theta = +$, and with $SU(2)$ algebra if $\theta = -$. The $SU(1,1)$ or $SU(2)$

TABLE IV. The obtained values of $S(L, m_{j_f})$ according to Eq. (56).

$J, j_f = \frac{3}{2}$	M	$S(L)$	$\sum_L P(L)S(m_{j_f}\|L = L_{\rm fix})$	$S(L, m_{j_f})$
$(\frac{3}{2})_1$	$\frac{3}{2}, -\frac{3}{2}$	$\eta^{0,0}_{\frac{1}{2},3/2} = 1, S(L) = 0$	0	0
$(\frac{3}{2})_1$	$\frac{1}{2}, -\frac{1}{2}$	$\eta^{0,0}_{\frac{1}{2},3/2} = 1, S(L) = 0$	0	0
$(\frac{5}{2})_1$	$\frac{5}{2}, -\frac{5}{2}$	$\eta^{0,0}_{\frac{3}{2},3/2} = 1, S(L) = 0$	0.682908002922725	0.682908002922725
$(\frac{5}{2})_1$	$\frac{3}{2}, -\frac{3}{2}$	$\eta^{0,0}_{\frac{3}{2},3/2} = 1, S(L) = 0$	0.801402354214379	0.801402354214379
$(\frac{5}{2})_1$	$\frac{1}{2}, -\frac{1}{2}$	$\eta^{0,0}_{\frac{3}{2},3/2} = 1, S(L) = 0$	1.248081748923086	1.248081748923086
$(\frac{7}{2})_1$	$\frac{7}{2}, -\frac{7}{2}$	$\eta^{1,2}_{\frac{3}{2},7/2} = 1, S(L) = 0$	0	0
$(\frac{7}{2})_1$	$\frac{5}{2}, -\frac{5}{2}$	$\eta^{1,2}_{\frac{3}{2},7/2} = 1, S(L) = 0$	0.682908002922725	0.682908002922725
$(\frac{7}{2})_1$	$\frac{3}{2}, -\frac{3}{2}$	$\eta^{1,2}_{\frac{3}{2},7/2} = 1, S(L) = 0$	0.955699386213412	0.955699386213412
$(\frac{7}{2})_1$	$\frac{1}{2}, -\frac{1}{2}$	$\eta^{1,2}_{\frac{3}{2},7/2} = 1, S(L) = 0$	1.058469962302830	1.058469962302830
$(\frac{7}{2})_2$	$\frac{7}{2}, -\frac{7}{2}$	$\eta^{2,4}_{\frac{5}{2},7/2} = \frac{2}{7}, \eta^{2,2}_{\frac{5}{2},7/2} = \frac{3\sqrt{5}}{7}$, $S(L) = 0.282738880066872$	0.068490583548693	0.351229463615565
$(\frac{7}{2})_2$	$\frac{5}{2}, -\frac{5}{2}$	$\eta^{2,4}_{\frac{5}{2},7/2} = \frac{2}{7}, \eta^{2,2}_{\frac{5}{2},7/2} = \frac{3\sqrt{5}}{7}$, $S(L) = 0.282738880066872$	0.735681623527116	1.018420503593988
$(\frac{7}{2})_2$	$\frac{3}{2}, -\frac{3}{2}$	$\eta^{2,4}_{\frac{5}{2},7/2} = \frac{2}{7}, \eta^{2,2}_{\frac{5}{2},7/2} = \frac{3\sqrt{5}}{7}$, $S(L) = 0.282738880066872$	0.970001254175214	1.2527401342420873
$(\frac{7}{2})_2$	$\frac{1}{2}, -\frac{1}{2}$	$\eta^{2,4}_{\frac{5}{2},7/2} = \frac{2}{7}, \eta^{2,2}_{\frac{5}{2},7/2} = \frac{3\sqrt{5}}{7}$, $S(L) = 0.282738880066872$	0.836853296840176	1.119592176907048

quadratic Casimir operator can be obtained as [35,36]

$$\hat{C}_2 = (S^0)^2 - \frac{\theta}{2}(S^+S^- + S^-S^+) = S^0(S^0 - 1) - \theta S^+S^+. \tag{2}$$

For an irrep of $SU(1,1)$, quadratic Casimir operator takes on eigenvalues $k(k-1)$, and the possible eigenvalues of S^0 are given by $\mu = k, k+1, k+2, \ldots$ [44,45].

Then, we investigate a system, involving several j shells with angular momentum $j_i (i = 1, 2, \ldots)$. The scalar pair creation and annihilation operators and also the number operator in this form can be defined as [44,45]

$$S^+_{j_i} = \frac{1}{2}\sum_{m_i} a^+_{j_i m_i} a^+_{j_i \bar{m}_i}, \quad S^-_{j_i} = \frac{1}{2}\sum_{m_i} a_{j_i \bar{m}_i} a_{j_i m_i},$$
$$S^0_{j_i} = \frac{1}{4}\sum_{m_i}\left(a^+_{j_i m_i} a_{j_i m_i} + \theta a_{j_i m_i} a^+_{j_i m_i}\right). \tag{3}$$

These operators form an $SU(1,1)$ quasispin algebra for bosons that present by $SU^{j_i}(1,1)$. The operator $S^0_{j_i}$ is depended to the number operator $N_{j_i} = \sum_{m_i} a^+_{j_i m_i} a_{j_i m_i} = (a^+_{j_i} \times \tilde{a}_{j_i})^{(0)}_0$ as [44,45]

$$S^0_{j_i} = \tfrac{1}{2}\left(N_{j_i} + \theta\Omega_i\right). \tag{4}$$

The $\Omega_i = \frac{1}{2}(2j_i + 1)$ show the pair degeneracy of the level i. Using the duality relations, we can relate the quasispin S_i to the ν_i seniority quantum number. So, the relation between the quasispin quantum numbers k_i and μ_i, for each level, and occupation-seniority quantum numbers N_i and ν_i, is according to [44,45]

$$k_i = \tfrac{1}{2}(\Omega_i + \theta\nu_i), \quad \mu_i = \tfrac{1}{2}(N_i + \theta\Omega_i). \tag{5}$$

The seniority ν_i takes on values $\nu_i = 0, 1, 2, \ldots$ with the constraint $N_i \preceq \nu_i$ [32,33]. For the two-level system, a quasispin algebra $SU^{j_1 j_2}(1,1)$ for the bosonic case is given by the sum operators according to [44,45]

$$S^+ = S^+_{j_1} + \sigma S^+_{j_2}, \quad S^- = S^-_{j_1} + \sigma S^-_{j_2}, \quad S^0 = S^0_{j_1} + S^0_{j_2}. \tag{6}$$

The well-known bosonic $SU^b(1,1)$ quasispin algebra generators are given by

$$S^+_{j_b} = \frac{1}{2}\sum_{m_b}(-1)^{j_b \mp m_b} b^+_{j_b m_b} b^+_{j_b - m_b} = \frac{1}{2}(b^+_{j_b} b^+_{j_b}), \tag{7}$$

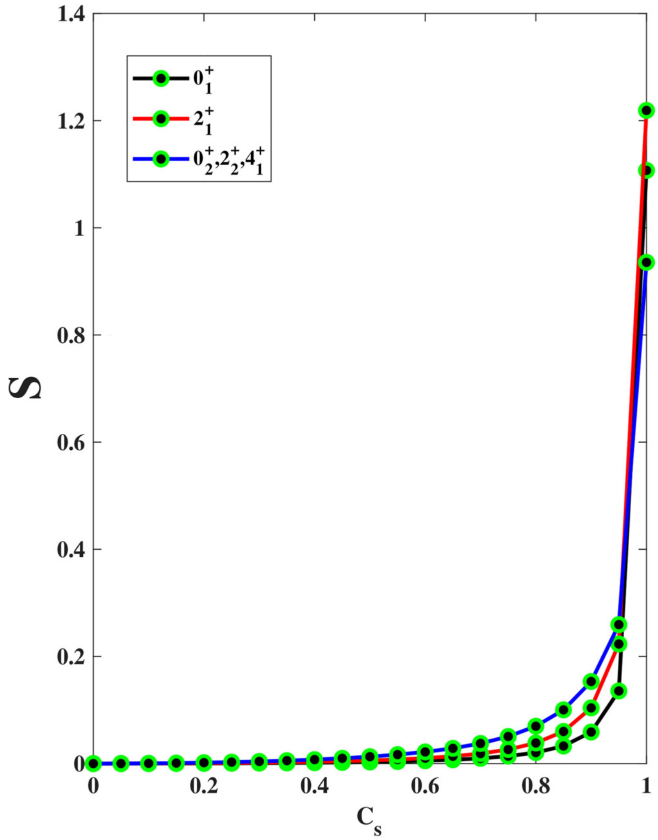

FIG. 2. The entropy of the ground and two lowest excited states calculated for the interacting boson model as a function of the control parameter C_s.

$$S^-_{j_b} = \frac{1}{2}\sum_{m_b}(-1)^{j_b \mp m_b}\widetilde{b}_{j_b m_b}\widetilde{b}_{j_b - m_b} = \frac{1}{2}(\widetilde{b}_{j_b}\widetilde{b}_{j_b}), \quad (8)$$

$$S^0_{j_b} = \frac{N_B}{2} + \frac{2j_b+1}{4} = \frac{N_B}{2} + \frac{n_b}{4}. \quad (9)$$

n_b is the dimension of the bosonic system space that is given by $n_b = \sum_{j_b}(2j_b+1)$.

In this study, the QPTs between vibrational dynamical symmetry and γ-unstable dynamical symmetry are investigated. In order to investigate the phase transition in the atomic nuclei according to IBM, we have considered two kinds of bosons with $j_b = 0, 2$ (s, d bosons) [30,31].

The infinite dimensional $SU^{sd}(1,1)$ algebra that is generated by use of [33,34]

$$S^\pm_n = c_s^{2n+1}S^\pm_s + c_d^{2n+1}S^\pm_d, \quad S^0_n = c_s^{2n}S^0_s + c_d^{2n}S^0_d, \quad (10)$$

where c_s and c_d are real parameters and n can be $0, \pm1, \pm2, \ldots$. These generators satisfy the commutation relations

$$[S^0_p, S^\pm_q] = \pm S^\pm_{p+q}, \quad [S^+_p, S^-_q] = -2S^0_{p+q+1}. \quad (11)$$

Then, S^μ_p, $\mu = 0, +, -$; $p = 0, \pm1, \pm2, \ldots$ generate an affine Lie algebra $\widehat{SU(1,1)}$ without central extension. Calculations for this study were performed using the $\widehat{SU(1,1)}$ formalism of IBM with a Hamiltonian in the SO(6) $\leftrightarrow$ U(5) transitional region given by [33,34]

$$H = gS^+_0S^-_0 + \alpha S^0_1 + \beta C_2(SO(5)) + \gamma C_2(SO(3)), \quad (12)$$

where g, α, β, and γ are the real parameters. $C_2(SO(5))$ and $C_2(SO(3))$ denote the Casimir operators of these groups. For evaluating the eigenvalues of Hamiltonian Eq. (12) the eigenstates are considered as [30,31]

$$|\psi\rangle^B_{\nu,L,M_L} = |m;\nu_s\nu_d n_\Delta LM_L\rangle = \sum_{n_i\in Z} a_{n_1n_2\ldots n_m}x_1^{n_1}x_2^{n_2}x_3^{n_3}\ldots \times x_m^{n_m}S^+_{n_1}S^+_{n_2}S^+_{n_3}\ldots S^+_{n_m}|lw\rangle^B_{M_L}. \quad (13)$$

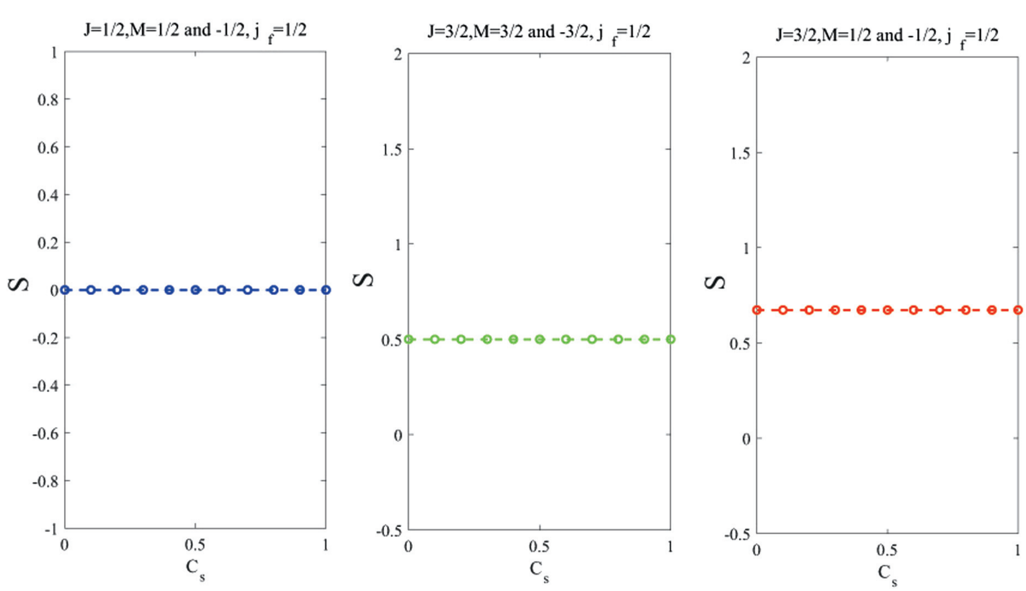

FIG. 3. The changes of the entanglement entropy of boson states and a single fermion in odd-A nuclei with $j_f = 1/2$ as a function of the control parameter C_s.

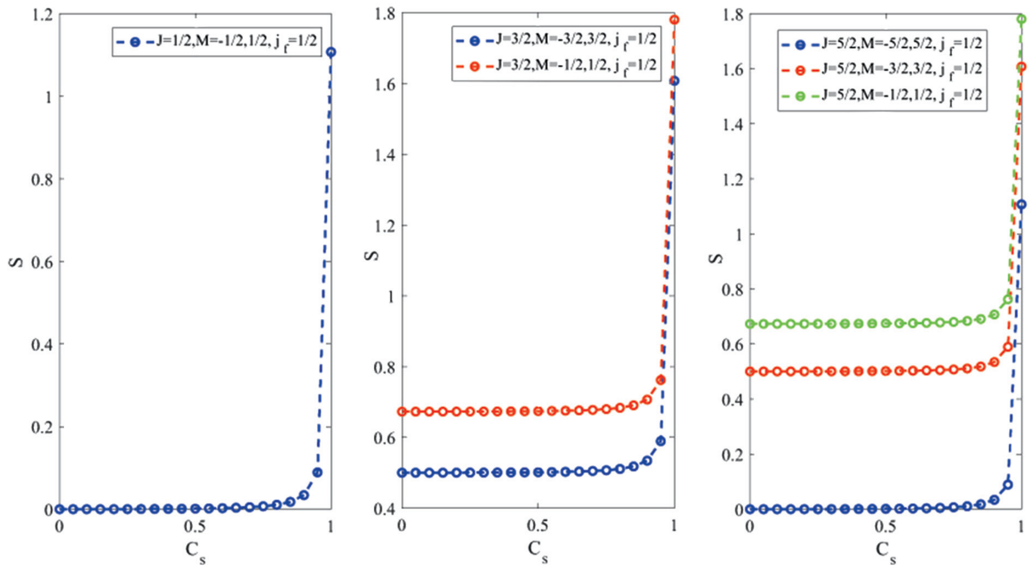

FIG. 4. The changes of the entanglement entropy of s boson with the rest of the system components in odd-A nuclei with $j_f = 1/2$ as a function of the control parameter C_s.

Eigenstates Eq. (13) can be obtained with using the Fourier-Laurent expansion of eigenstates and $SU(1,1)$ generators in terms of c-unknown number parameters x_i with $i = 1, 2, \dots, m$. It means, one can consider the eigenstates as [33,34]

$$|\psi\rangle^B_{\nu,L,M_L} = |m; \nu_s \nu_d n_\Delta L M_L\rangle = \Theta S^+_{x_1} S^+_{x_2} S^+_{x_3} \dots S^+_{x_m} |lw\rangle^B_{M_L}, \tag{14}$$

where Θ is the normalization factor and

$$S^+_{x_i} = \frac{c_s}{1 - c_s^2 x_i} S^+(s) + \frac{c_d}{1 - c_d^2 x_i} S^+(d). \tag{15}$$

The lowest weight state, $|lw\rangle^B$ was considered as [33,34]

$$|lw\rangle^B_{M_L} = \big|N_B, k_d = \tfrac{1}{2}\big(\nu_d + \tfrac{5}{2}\big), \mu_d = \tfrac{1}{2} + \big(n_d + \tfrac{5}{2}\big), k_s = \tfrac{1}{2}\big(\nu_s + \tfrac{1}{2}\big), \mu_s = \tfrac{1}{2}\big(n_s + \tfrac{1}{2}\big), L, M_L\big\rangle. \tag{16}$$

The quantum number (m) is related to the total boson number N_B by $N_B = 2m + \nu_s + \nu_d$. The eigenvalues of Hamiltonian (12) can then be expressed as [30,31]

$$E^{(m)} = h^{(m)} + \alpha \Lambda_1^0 + \beta \nu_d(\nu_d + 3) + \gamma L(L+1), \tag{17}$$

$$h^{(m)} = \sum_{i=1}^{m} \frac{\alpha}{x_i}, \quad \frac{\alpha}{x_i} = \frac{g c_s^2\big(\nu_s + \frac{1}{2}\big)}{1 - c_s^2 x_i} + \frac{g c_d^2\big(\nu_d + \frac{5}{2}\big)}{1 - c_d^2 x_i},$$

$$\Lambda_1^0 = c_s^2\left(\nu_s + \frac{1}{2}\right)\frac{1}{2} + c_d^2\left(\nu_d + \frac{5}{2}\right)\frac{1}{2}. \tag{18}$$

In order to obtain the numerical results for energy spectra ($E^{(m)}$) of considered nuclei, a set of nonlinear Bethe-Ansatz equations (BAE) with m unknowns for m-pair excitations must be solved [33–41] also constants of Hamiltonian with least square fitting processes to experimental data are obtained. To this aim, we have changed variables as

$$C_s = \frac{c_s}{c_d} \leqslant 1, \quad g = 1, \quad y_i = c_d^2 x_i.$$

The new form of Eq. (18) would be

$$\frac{\alpha}{y_i} = \frac{{C_s}^2\big(\nu_s + \frac{1}{2}\big)}{1 - {C_s}^2 y_i} + \frac{\big(\nu_d + \frac{5}{2}\big)}{1 - y_i} - \sum_{j \neq i} \frac{2}{y_i - y_j},$$

for $i = 1, 2, \dots, m$.

C_s is the control parameter that have values between the 0–1 region. To calculate the roots of Bethe-Ansatz equations (BAE) with specified values of ν_s and ν_d, we have solved this equation with definite values of C_s and α [33–41]. Then, we carry out this procedure with different values of C_s and α to give energy spectra with minimum variation in compare to experimental values [33,34]:

$$\sigma = \left(\frac{1}{N_{\text{tot}}} \sum_{i,\text{tot}} |E_{\text{exp}}(i) - E_{\text{cal}}(i)|^2\right)^{\frac{1}{2}}$$

(N_{tot} is the number of energy levels where included in the fitting processes).

The method for optimizing the set of parameters in the Hamiltonian (β, γ, δ) includes carrying out a least-square fit (LSF) of the excitation energies of selected states.

B. The interacting boson-fermion model along the $U^{BF}(5)$–$O^{BF}(6)$ line based on the affine $SU(1,1)$ algebra

In this section, we study the coupling of an even core that undergoes a transition from spherical U(5) to γ-unstable O(6) situation to a particle moving in the j orbitals by using the affine $\widehat{SU(1,1)}$ Lie algebra in the framework of the interacting boson-fermion model [46]. To this aim we have used

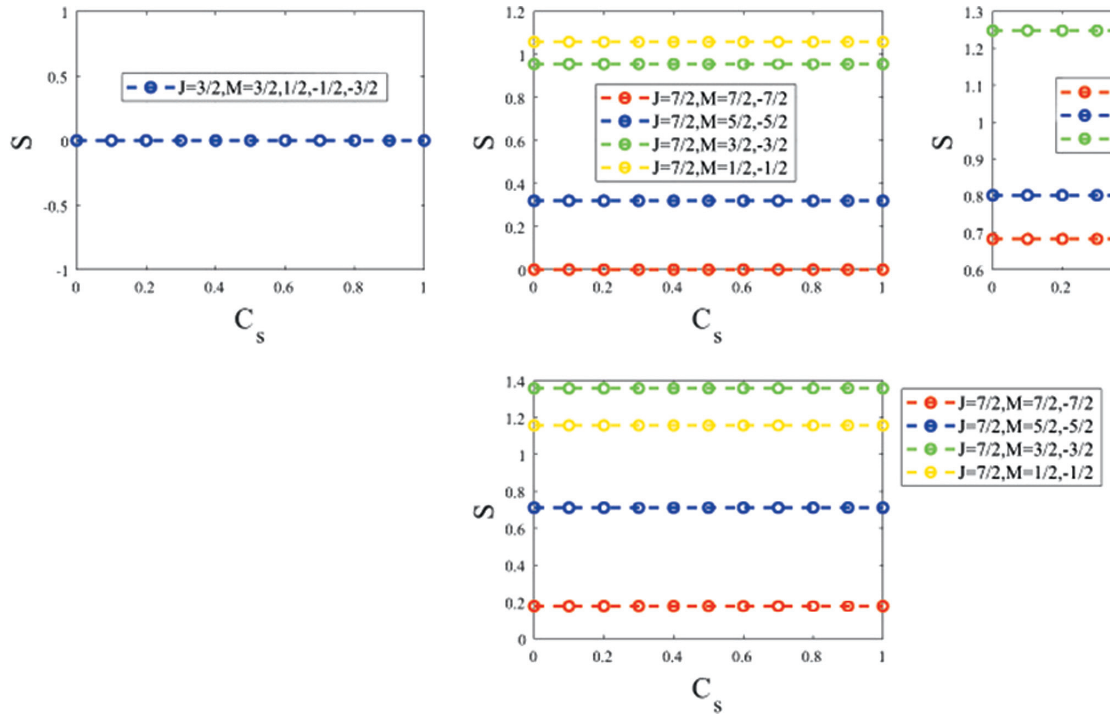

FIG. 5. The changes of the entanglement entropy of boson states and a single fermion in odd-A nuclei with $j_f = 3/2$ as a function of the control parameter C_s.

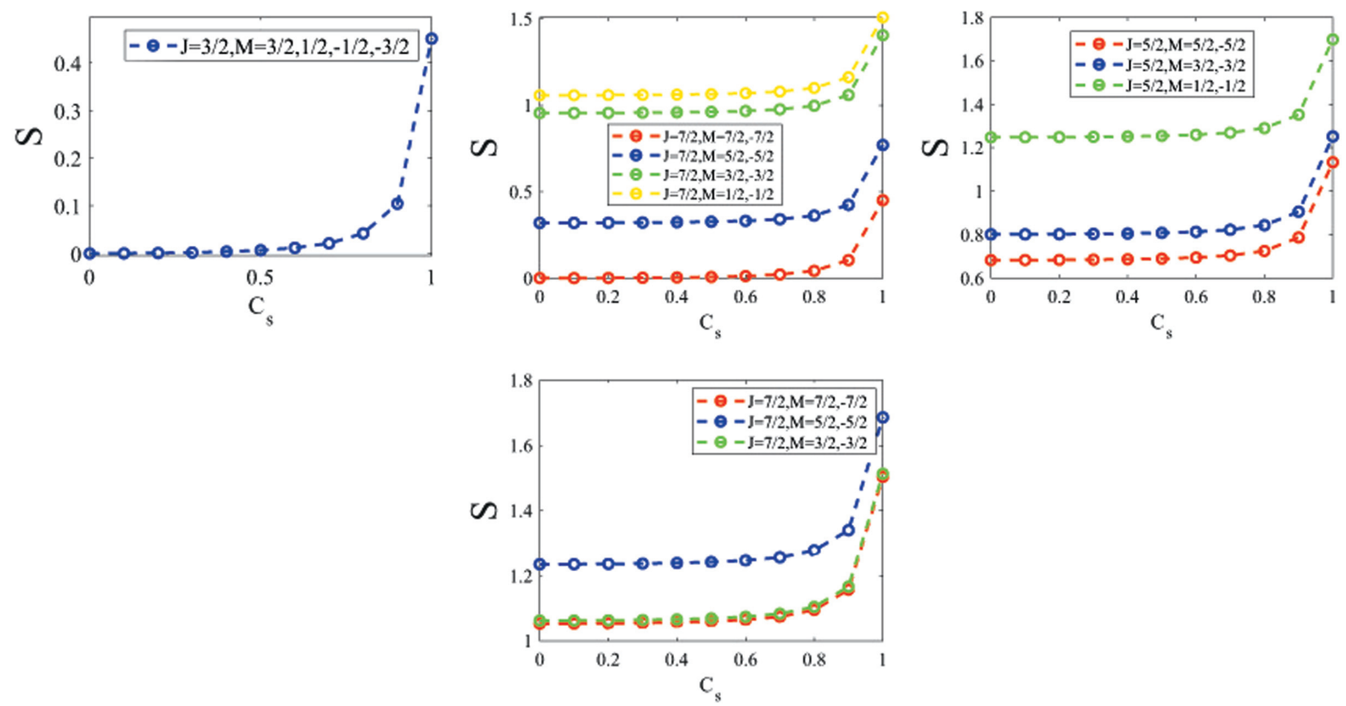

FIG. 6. The changes of the entanglement entropy of s boson with the rest of the system components in odd-A nuclei with $j_f = 3/2$ as a function of the control parameter C_s.

TABLE V. The calculated entanglement entropy for ^{102}Pd isotope with $N_B = 5$, $\alpha = 1100$, $\beta = -590.86$, $\gamma = 45.08$, and $C_s = 0$.

^{102}Pd	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	2	0	1	0	0	9.4741×10^{-17}	(0.9977,0.999)
	$(2)_1^+$	2	1	0	1	653.84	5.0613×10^{-17}	(0.9968,0.9985)
	$(0)_2^+$	2	0	1	2	1102.50	4.9428×10^{-16}	(0.9977,0.9946)
	$(2)_2^+$	1	2	1	2	1264.56	2.2258×10^{-16}	(0.9959)
	$(4)_1^+$	1	2	1	2	1380	2.2258×10^{-16}	(0.9959)
	$(0)_3^+$	1	3	0	3	1832.18	9.2331×10^{-17}	(0.995)
	$(3)_1^+$	1	3	0	3	1958.89	9.2331×10^{-17}	(0.995)
	$(2)_3^+$	2	1	0	3	1757.34	2.294×10^{-16}	(0.9968,0.993)
	$(4)_2^+$	1	3	0	3	2043.36	9.2331×10^{-17}	(0.995)
	$(6)_1^+$	1	3	0	3	2275.67	9.2331×10^{-17}	(0.995)
	$(0)_4^+$	2	0	1	4	2207	4.9428×10^{-16}	(0.9977,0.9946)
	$(2)_4^+$	1	2	1	4	2369.06	2.2258×10^{-16}	(0.9959)

TABLE VI. The calculated entanglement entropy for ^{104}Pd isotope with $N_B = 6$, $\alpha = 1100$, $\beta = -980.86$, $\gamma = 41.32$, and $C_s = 0.1$.

^{104}Pd	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	3	0	0	0	0	2.1951×10^{-8}	(99.9547, 99.9922,99.9986)
	$(2)_1^+$	2	1	1	1	617.03	4.3546×10^{-7}	(99.8643,99.9547)
	$(o)_2^+$	3	0	0	2	1091.46	2.8268×10^{-6}	(99.9547,99.9922,0.9977)
	$(2)_2^+$	2	2	0	2	1189.34	8.7186×10^{-8}	(99.9548,99.9922)
	$(4)_1^+$	2	2	0	2	1297.32	2.7633×10^{-6}	(99.9548,0.9959)
	$(0)_3^+$	1	3	1	3	1716.96	1.2067×10^{-6}	(99.8644)
	$(3)_1^+$	1	3	1	3	1824.94	8.362×10^{-7}	(0.9959)
	$(2)_3^+$	2	1	1	3	1709.46	1.2174×10^{-6}	(0.9968,0.9968)
	$(4)_2^+$	1	3	1	3	1896.92	8.362×10^{-7}	(0.9959)
	$(6)_1^+$	1	3	1	3	2094.88	8.362×10^{-7}	(0.9959)
	$(0)_4^+$	3	0	0	4	2184.94	3.3882×10^{-7}	(0.9977,0.9977,0.9977)
	$(2)_4^+$	2	2	0	4	2282.82	5.7011×10^{-7}	(0.9959,0.9959)

TABLE VII. The calculated entanglement entropy for ^{106}Pd isotope with $N_B = 7$, $\alpha = 1100$, $\beta = -1370.86$, $\gamma = 37.56$, and $C_s = 0.25$.

^{106}Pd	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	3	0	1	0	0	1.1466×10^{-6}	(15.9783,15.9928,15.9975)
	$(2)_1^+$	3	1	0	1	569.95	1.8674×10^{-7}	(15.9928,15.9988,15.9998)
	$(0)_2^+$	3	0	1	2	1033.40	2.7868×10^{-5}	(15.9783,15.9928,0.9977)
	$(2)_2^+$	2	2	1	2	1098.18	3.4283×10^{-6}	(15.9783,15.9928)
	$(4)_1^+$	2	2	1	2	1190.19	3.3581×10^{-5}	(15.9783,0.9959)
	$(0)_3^+$	2	3	0	3	1584.69	6.5241×10^{-7}	(15.9928,15.9988)
	$(3)_1^+$	2	3	0	3	1676.70	2.0307×10^{-5}	(15.9928,0.995)
	$(2)_3^+$	3	1	0	3	1604.42	3.262×10^{-6}	(0.9968,0.9968,0.9968)
	$(4)_2^+$	2	3	0	3	1738.04	1.4589×10^{-5}	(0.995, 15.9988)
	$(6)_1^+$	2	3	0	3	1906.73	4.4039×10^{-6}	(0.995,0.995)
	$(0)_4^+$	3	0	1	4	2068.94	5.9393×10^{-6}	(0.9977,0.9977,0.9977)
	$(2)_4^+$	2	2	1	4	2133.71	9.9339×10^{-6}	(0.9959,0.9959)

TABLE VIII. The calculated entanglement entropy for ^{108}Pd isotope with $N_B = 8$, $\alpha = 1100$, $\beta = -1760.86$, $\gamma = 33.8$, and $C_s = 0.38$.

^{108}Pd	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	4	0	0	0	0	1.985×10^{-7}	(6.9221,6.9547,6.9251,6.9252)
	$(2)_1^+$	3	1	1	1	513.17	4.1276×10^{-6}	(6.9158,6.9221,6.9241)
	$(0)_2^+$	4	0	0	2	942.72	7.7148×10^{-5}	(6.9221,6.9247,6.9251,0.9977)
	$(2)_2^+$	3	2	0	2	989.17	6.2284×10^{-7}	(6.9221,6.9247,6.9251)
	$(4)_1^+$	3	2	0	2	1067.58	7.558×10^{-5}	(6.9221,6.9247,0.9959)
	$(0)_3^+$	2	3	1	3	1427.99	1.077×10^{-5}	(6.9158,6.9221)
	$(3)_1^+$	2	3	1	3	1506.40	8.3157×10^{-5}	(6.9158,6.8971)
	$(2)_3^+$	3	1	1	3	1457.05	2.3631×10^{-5}	(0.9968,0.9968,0.9968)
	$(4)_2^+$	2	3	1	3	1558.67	6.5072×10^{-5}	(0.995, 6.9221)
	$(6)_1^+$	2	3	1	3	1702.41	3.1799×10^{-5}	(0.995,0.995)
	$(0)_4^+$	4	0	0	4	1887.74	5.985×10^{-6}	(0.9977,0.9977,0.9977,0.9977)
	$(2)_4^+$	3	2	0	4	1934.19	1.1527×10^{-5}	(0.9959,0.9959,0.9959)

the same formalism to extend IBFM calculation to the case that a $j_f(1/2, 3/2)$ fermion coupled to a boson core.

Solving sd-IBFM with $j_f = 1/2$ and $j_f = 3/2$ by quasispin algebra is completely explained in our previous works [37,38]. In the following, a brief description of the IBFM model that will be used is mentioned. The chains of subalgebras $U^B(6) \otimes U^F(2j_f + 1)$ unitary superalgebras in $j_f = 1/2$ (left) and $j_f = 3/2$ (right) cases are shown in Fig. 1.

By employing the generators of algebra $\widehat{SU(1,1)}$ and Casimir operators of subalgebras, the following Hamiltonian for state that odd nucleon being in a $j_f = 1/2$ shell for transitional region between $U^{BF}(5)$-$O^{BF}(6)$ limits is prepared:

$$H = gS_0^+S_0^- + \alpha S_1^0 + \eta C_2(SO^B(5)) + \delta C_2(SO^B(3)) + \gamma C_2(\text{spin}^{BF}(3)). \quad (19)$$

The following Hamiltonian for a state with an odd nucleon being in a $j_f = 3/2$ shell for the transitional region between $U^{BF}(5)$-$O^{BF}(6)$ limits is prepared:

$$H = gS_0^+S_0^- + \alpha S_1^0 + \eta C_2(\text{spin}^{BF}(5)) + \gamma C_2(\text{spin}^{BF}(3)). \quad (20)$$

Equations (19) and (20) are the suggested Hamiltonians for boson-fermion systems with $j_f = 1/2, 3/2$, respectively, and α, η, δ, γ are real parameters. For calculating the eigenvalues of Hamiltonians in odd-A nuclei with $j_f = 1/2$, the eigenstate is considered as

$$|\psi\rangle_{JM}^{BF} = |m; \nu_s \nu_d n_\Delta JM\rangle = \Theta S_{x_1}^+ S_{x_2}^+ S_{x_3}^+ \dots S_{x_m}^+ |lw\rangle_{JM}^{BF} \quad (21)$$

with Clebsch-Gordan (CG) coefficient, we can calculate the lowest weight state, $|lw\rangle_{JM}^{BF}$, in terms of boson and fermion part as

$$|lw\rangle_{JM}^{BF} = \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2} C_{M,M_L,m_{j_f}}^{J,L,1/2} |lw\rangle_{M_L}^B |1/2, m_{j_f}\rangle. \quad (22)$$

TABLE IX. The calculated entanglement entropy for ^{110}Pd isotope with $N_B = 9$, $\alpha = 1100$, $\beta = -2150.86$, $\gamma = 30.04$ and $C_s = 0.48$.

^{110}Pd	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	4	0	1	0	0	3.6424×10^{-6}	(4.3344,4.3383,4.3396,4.34)
	$(2)_1^+$	4	1	0	1	457.45	5.5039×10^{-7}	(4.3383,4.3399,4.3402,4.3403)
	$(0)_2^+$	4	0	1	2	847.34	1.9423×10^{-4}	(4.3344,4.3383,4.3396,0.9977)
	$(2)_2^+$	3	2	1	2	882.77	9.8559×10^{-6}	(4.3344,4.3383,4.3396)
	$(4)_1^+$	3	2	1	2	949.54	2.2281×10^{-4}	(4.3344,4.3383,0.9959)
	$(0)_3^+$	3	3	0	3	1275.47	1.5461×10^{-6}	(4.3383,4.3399,4.3402)
	$(3)_1^+$	3	3	0	3	1342.28	1.6946×10^{-4}	(4.3383,4.3399,0.995)
	$(2)_3^+$	4	1	0	3	1306.13	1.7877×10^{-5}	(0.9968,0.9968,0.9968,0.9968)
	$(4)_2^+$	3	3	0	3	1386.83	1.622×10^{-4}	(4.3383,0.995, 0.955)
	$(6)_1^+$	3	3	0	3	1509.32	2.7446×10^{-5}	(0.995,0.995,0.995)
	$(0)_4^+$	4	0	1	4	1697.11	3.1404×10^{-5}	(0.9977,0.9977,0.9977,0.9977)
	$(2)_4^+$	3	2	1	4	1732.52	6.0098×10^{-5}	(0.9959,0.9959,0.9959)

TABLE X. The calculated entanglement entropy for ^{122}Xe isotope with $N_B = 9$, $\alpha = 540$, $\beta = -4.198$, $\gamma = 15.33$ and $C_s = 0.95$.

^{122}Xe	J^{π}	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	4	0	1	0	0	0.002	(1.1079,1.1074,1.1077,1.1079)
	$(2)_1^+$	4	1	0	1	753.9	2.9182×10^{-4}	(1.1071,1.1079,1.108,1.108)
	$(0)_2^+$	4	0	1	2	1080.7	0.0167	(0.9953,0.9951,0.9949,0.9947)
	$(2)_2^+$	3	2	1	2	891.2	0.0277	(0.9915,0.9913,0.991)
	$(4)_1^+$	3	2	1	2	779.7	0.0049	(0.9898,0.995, 0.955)
	$(3)_1^+$	3	3	0	3	1101.2	7.3602×10^{-4}	(1.1071,1.1079,1.108)
	$(4)_2^+$	3	3	0	3	1239.4	0.0128	(0.9898,0.995, 0.955)
	$(6)_1^+$	3	3	0	3	1319.5	0.0128	(0.9898,0.9895,0.9892)
	$(2)_3^+$	4	1	0	3	1365.7	0.0093	(0.9959,0.9933,0.9931,0.9928)

The $C^{J,L,j_f}_{M,M_L,m_{j_f}}$ symbols represent Clebsch-Gordan coefficients [46]. For calculating the eigenvalues of Hamiltonians in odd-A nuclei with $j_f = 3/2$, the eigenstate is considered as

$$\begin{aligned}|\psi\rangle^{BF}_{\nu_1 JM} &= |N, n_d, \nu, (\nu_1, \nu_2), L, J, M\rangle \\ &= \sum_{m_{j_f}=-3/2}^{m_{j_f}=+3/2} \left(\sum_L C^{J,L,3/2}_{M,M_L,m_{j_f}} \eta^{\nu,L}_{\nu_1,J} |\psi\rangle^B_{\nu,L,M_L}\right) \\ &\quad \times |\psi\rangle^F_{(\nu_1=1/2,\nu_2=-1/2),3/2,m_{j_f}} \\ &= \sum_{m_{j_f}=-3/2}^{m_{j_f}=+3/2} N_{j_f,m_{j_f}} |\psi\rangle^B_{\nu,L,M_L} \\ &\quad \times |\psi\rangle^F_{(\nu_1=1/2,\nu_2=-1/2),3/2,m_{j_f}},\end{aligned} \tag{23}$$

where $\eta^{\nu,L}_{\nu_1,J}$ symbols represent isoscalar factors and $N_{j_f,m_{j_f}}$ is the normalization constant [46]:

$$N_{j_f,m_{j_f}} = \sqrt{\sum_L \left(C^{J,L,3/2}_{M,M_L,m_{j_f}} \eta^{\nu,L}_{\nu_1,J}\right)^2}. \tag{24}$$

The eigenvalues of Hamiltonians (19) and (20) can then be expressed:

$$E^{(m)} = h^{(m)} + \alpha \Lambda_1^0 + \eta \nu_d(\nu_d + 3) + \delta L(L+1) + \gamma J(J+1), \tag{25}$$

$$\begin{aligned}E^{(m)} = h^{(m)} + \alpha \Lambda_1^0 + \eta(\nu_1(\nu_1 + 3) \\ + \nu_2(\nu_2 + 1)) + \gamma J(J+1).\end{aligned} \tag{26}$$

The fitting procedure for odd nuclei is similar to even-even nuclei.

C. Effective order parameter of entanglement entropy in IBM-1

In the following, we focus on the entanglement in the transition region and present how to determine the entropy entanglement in IBM-1. We will only consider von Neumann entropy as a measure suitable for entanglement. One of the simplest quantities to measure the entanglement between two bodies is the Von Neumann entropy. The entanglement entropy can be represented using the singular values of the Schmidt decomposition of the state. Any bipartite pure state can be expressed as [47]

$$|\psi\rangle_{AB} = \sum_{i=1}^{m} a_i |\upsilon_i\rangle_A \otimes |\upsilon_i\rangle_B, \tag{27}$$

TABLE XI. The calculated entanglement entropy for ^{124}Xe isotope with $N_B = 8$, $\alpha = 680$, $\beta = -4.3283$, $\gamma = 18.91425$ and $C_s = 0.89$.

^{124}Xe	J^{π}	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	4	0	0	0	0	3.2844×10^{-5}	(1.2616,1.2623,1.2624,1.2625)
	$(2)_1^+$	3	1	1	1	882.7	6.0673×10^{-4}	(1.2598,1.2616,1.2622)
	$(0)_2^+$	4	0	0	2	1081.2	9.3871×10^{-4}	(0.9963, 0.9913,0.9887, 0.9806)
	$(2)_2^+$	3	2	0	2	875.2	0.0017	(0.9934, 1.2572,1.2539)
	$(4)_1^+$	3	2	0	2	1140.0	1.066×10^{-4}	(0.9934, 1.2572,1.2539)
	$(3)_1^+$	2	3	1	3	10429	2.6496×10^{-4}	(0.9919,0.9955)
	$(2)_3^+$	3	1	1	3	1156.1	0.0035	(0.9948,0.9892,0.9961)
	$(4)_2^+$	2	3	1	3	1194.2	0.0052	(0.9919,0.9955)
	$(6)_1^+$	2	3	1	3	1610.3	0.0017	(0.9919,0.9955)

TABLE XII. The calculated entanglement entropy for ^{126}Xe isotope with $N_B = 7$, $\alpha = 695$, $\beta = -3.1441$, $\gamma = 16.9279$ and $C_s = 0.83$.

^{126}Xe	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	3	0	1	0	0	1.9524×10^{-5}	(1.4485,1.4506,1.4512)
	$(2)_1^+$	3	1	0	1	908.1	3.0347×10^{-5}	(1.4506,1.4514,1.4516)
	$(0)_2^+$	3	0	1	2	1121.2	9.3586×10^{-4}	(1.4485, 0.9985,0.9974)
	$(2)_2^+$	2	2	1	2	921.4	0.0016	(1.4485, 1.4506)
	$(4)_1^+$	2	2	1	2	1158.4	5.5671×10^{-4}	(1.4485, 1.4506)
	$(3)_1^+$	2	3	0	3	1161.7	1.0939×10^{-4}	(0.9921, 0.9858)
	$(2)_3^+$	3	1	0	3	1308.9	5.3266×10^{-5}	(0.995,0.9895,0.9962)
	$(4)_2^+$	2	3	0	3	1297.1	0.0022	(0.9921, 0.9858)
	$(6)_1^+$	2	3	0	3	1669.5	7.0147×10^{-4}	(0.9921, 0.9858)

where $|\upsilon_i\rangle_A$ and $|\nu_i\rangle_B$ are orthonormal states in subsystem A and subsystem B, respectively. So, the Von Neumann entropy can be written by using Schmidt decomposition as [48–50]

$$S_A = S_B = -\sum_{i=1}^{m} |a_i|^2 \log(|a_i|^2). \tag{28}$$

To formulate the entanglement of entropy in interacting boson model-1, we consider two quantum systems s and d with respective Hilbert space H_s and H_d. The Hilbert space of the composite system is the tensor product $H = H_s \otimes H_d$, the single-particle is the Hilbert space of s and d bosons. We calculated the entanglement entropy of the s boson and d boson. We start to define with $S_{x_i}^+$ as

$$S_{x_i}^+ = \frac{c_s}{1 - c_s^2 x_i} S_s^+ + \frac{c_d}{1 - c_d^2 x_i} S_d^+ = \alpha_i S_s^+ + \beta_i S_d^+, \tag{29}$$

where $\alpha_i = \frac{c_s}{1-c_s^2 x_i}$ and $\beta_i = \frac{c_d}{1-c_d^2 x_i}$, by using the following relation [51]:

$$(S^+)^m |k, k\rangle = \sqrt{\frac{m!\Gamma(m + 2k)}{\Gamma(2k)}} |k, k + m\rangle. \tag{30}$$

We can write Eq. (2.13) as the Schmidt decomposition:

$$\begin{aligned} |\psi\rangle^B_{\nu,L,M_L} &= \Theta \prod_{i=1}^{m} (\alpha_i S_s^+ + \beta_i S_d^+) |k_s, k_s, L, M_L\rangle |k_d, k_d, L, M_L\rangle \\ &= \Theta \sum_{l=0}^{m} A_l \sqrt{\frac{l!\Gamma(l + 2k_s)}{\Gamma(2k_s)} \times \frac{(m - l)!\Gamma((m - l) + 2k_d)}{\Gamma(2k_d)}}, \\ &\quad \times |k_s, k_s + l, L, M_L\rangle |k_d, k_d + (m - l), L, M_L\rangle \\ &= \sum_{l=0}^{m} b_{l,k_s,k_d} |k_s, k_s + l, L, M_L\rangle |k_d, k_d + (m - l), L, M_L\rangle, \end{aligned} \tag{31}$$

where normalization factors Θ, A_l, and b_{l,k_s,k_d} are defined as

$$\Theta = \frac{1}{\sqrt{\sum_{r=0}^{m} A_l^2 \left(\frac{r!\Gamma(r+2k_s)}{\Gamma(2k_s)}\right) \times \left(\frac{(m-r)!\Gamma((m-r)+2k_d)}{\Gamma(2k_d)}\right)}}, \tag{32}$$

$$A_l = \frac{1}{l!(m - l)!} \sum_{\pi \in S_m} \alpha_{\pi(1)}\alpha_{\pi(2)} \ldots \alpha_{\pi(l)}\beta_{\pi(l+1)} \ldots \beta_{\pi(m)}, \tag{33}$$

TABLE XIII. The calculated entanglement entropy for ^{128}Xe isotope with $N_B = 6$, $\alpha = 1570$, $\beta = -5.2829$, $\gamma = 3.76$ and $C_s = 0.65$.

^{128}Xe	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	3	0	0	0	0	1.2924×10^{-6}	(2.3661,2.3667,2.3668)
	$(2)_1^+$	2	1	1	1	969.6	2.1014×10^{-5}	(2.3646,2.3661)
	$(0)_2^+$	3	0	0	2	1980.2	1.5897×10^{-5}	(0.9984,0.9984,0.9984)
	$(2)_2^+$	2	2	0	2	1357.5	4.6898×10^{-6}	(0.9971,0.9971)
	$(4)_1^+$	2	2	0	2	1270.1	2.7311×10^{-6}	(2.3661,2.3667)
	$(3)_1^+$	1	3	1	3	1538.2	5.6988×10^{-5}	(2.3646)
	$(2)_3^+$	2	1	1	3	1902.1	5.4643×10^{-5}	(0.9978,0.9978)
	$(4)_2^+$	1	3	1	3	1648.2	5.6331×10^{-5}	(0.9965)
	$(6)_1^+$	1	3	1	3	1670.1	5.6331×10^{-5}	(0.9965)

TABLE XIV. The calculated entanglement entropy for ^{130}Xe isotope with $N_B = 5$, $\alpha = 1100$, $\beta = 53.92$, $\gamma = 34.75$ and $C_s = 0.46$.

^{130}Xe	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	2	0	1	0	0	8.5014×10^{-6}	(4.7195,4.7238)
	$(2)_1^+$	2	1	0	1	541	1.8866×10^{-6}	(4.7238,4.7255)
	$(0)_2^+$	2	0	1	2	1745.1	2.2173×10^{-5}	(0.9977,0.9977)
	$(2)_2^+$	1	2	1	2	1297	2.6643×10^{-5}	(0.9959)
	$(4)_1^+$	1	2	1	2	1199.7	2.6974×10^{-5}	(4.7195)
	$(3)_1^+$	1	3	0	3	1634	1.1163×10^{-5}	(4.7238)
	$(2)_3^+$	2	1	0	3	2033.8	1.1655×10^{-5}	(0.9968,0.9968)
	$(4)_2^+$	1	3	0	3	1755.2	1.1632×10^{-5}	(0.995)
	$(6)_1^+$	1	3	0	3	1935	1.1632×10^{-5}	(0.995)

$$b_{l,k_s,k_d} = \frac{A_l\sqrt{\frac{l!\Gamma(l+2k_s)}{\Gamma(2k_s)} \times \frac{(m-l)!\Gamma((m-l)+2k_d)}{\Gamma(2k_d)}}}{\sqrt{\sum_{r=0}^{m} A_r^2\left(\frac{r!\Gamma(r+2k_s)}{\Gamma(2k_s)}\right) \times \left(\frac{(m-r)!\Gamma((m-r)+2k_d)}{\Gamma(2k_d)}\right)}}. \quad (34)$$

$|k_s, k_s + l, L, M_L\rangle$ and $|k_d, k_d + (m - l), L, M_L\rangle$ are s boson and d boson bases, respectively. Usually, the choice of Hilbert space bases does not affect the entanglement calculation process. Any base chosen should eventually be replaced with Schmidt bases. Fortunately, by chance, in the boson part, the selected bases are Schmidt bases. Accordingly, Schmidt numbers (b_{l,k_s,k_d}) through the parameters α_i and β_i are dependent on the control parameter and therefore the entropy is a function of the control parameter. So, we achieve the entanglement entropy in interacting boson model-1 as

$$S^B = S_s = S_d = -\sum_{l=0}^{m} \left|b_{l,k_s,k_d}\right|^2 \log\left(\left|b_{l,k_s,k_d}\right|^2\right). \quad (35)$$

D. Investigation of entanglement entropy in odd-*A* nuclei

For calculating the entanglement entropy of odd-A nuclei in two cases $j_f = 1/2, 3/2$, is acted as follows.

1. The case of $j_f = 1/2$

The eigenstate of Eq. (21) can be rewritten as the Schmidt decomposition

$$|\psi\rangle_{JM}^{BF} = \Theta \prod_{i=1}^{m} (\alpha_i S_s^+ + \beta_i S_d^+) \times \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2} C_{M,M_L,m_{j_f}}^{J,L,1/2} |lw\rangle_{M_L}^{B} |1/2, m_{j_f}\rangle. \quad (36)$$

So, the general form of the wave function in the suggested interacting boson-fermion is prepared as

$$|\psi\rangle_{JM}^{BF} = \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2} \sum_{l=0}^{m} b_{l,k_s,k_d} C_{M,M_L,m_{j_f}}^{J,L,1/2} |k_s, k_s + l, L, M_L\rangle |k_d, k_d + (m - l), L, M_L\rangle |1/2, m_{j_f}\rangle, \quad (37)$$

$${}_{JM}^{BF}\langle\psi| = \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2} \sum_{l=0}^{m} b_{l,k_s,k_d}^{*} C_{M,M_{L'},m_{j_f}}^{J,L',1/2} \langle k_s, k_s + l, L', M_{L'}| \langle k_d, k_d + (m - l), L', M_{L'}\big| \langle 1/2, m_{j_f}\big|. \quad (38)$$

TABLE XV. The calculated entanglement entropy for ^{132}Xe isotope with $N_B = 4$, $\alpha = 1680$, $\beta = 40.31$, $\gamma = 63.5$ and $C_s = 0.37$.

^{132}Xe	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	2	0	0	0	0	3.6516×10^{-7}	(7.3024,7.3042)
	$(2)_1^+$	1	1	1	1	672.3	5.4106×10^{-6}	(7.2981)
	$(0)_2^+$	2	0	0	2	2000.5	2.0017×10^{-6}	(0.9985,0.9985)
	$(2)_2^+$	1	2	1	2	1300.7	6.8465×10^{-6}	(0.9973)
	$(4)_1^+$	1	2	1	2	1485.2	9.9705×10^{-6}	(7.2981)
	$(3)_1^+$	0	3	1	3	1903.4	-	(-)
	$(2)_3^+$	1	1	1	3	1823.5	5.5097×10^{-6}	(0.9979)
	$(4)_2^+$	0	3	1	3	1845	-	(-)
	$(6)_1^+$	0	3	1	3	2124.1	-	(-)

TABLE XVI. The calculated entanglement entropy for ^{134}Xe isotope with $N_B = 3$, $\alpha = 670$, $\beta = 79.2$, $\gamma = 49.9$ and $C_s = 0.06$.

^{134}Xe	J^π	m	ν_d	ν_s	n_d	E (ke V)	S	x_i
	$(0)_1^+$	1	0	1	0	0	5.5078×10^{-7}	(277.1596)
	$(2)_1^+$	1	1	0	1	842.5	2.6702×10^{-7}	(277.5717)
	$(0)_2^+$	1	0	1	2	1623.4	5.4591×10^{-7}	(0.9963)
	$(2)_2^+$	0	2	1	2	1553.4	-	(-)
	$(4)_1^+$	0	2	1	2	1740	-	(-)
	$(3)_1^+$	0	3	0	3	1824.5	-	(-)
	$(2)_3^+$	1	1	0	3	2175	2.673×10^{-7}	(0.9948)
	$(4)_2^+$	0	3	0	3	2006.3	-	(-)
	$(6)_1^+$	0	3	0	3	1981.1	-	(-)

The total density matrix is ρ and is defined as follows:

$$\begin{aligned}\rho &= |\psi\rangle^{BF}_{JMJM}{}^{BF}\langle\psi| \\ &= \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2}\sum_{l=0}^{m}\left|b_{l,k_sk_d}\right|^2 C^{J,L',1/2}_{M,M_{L'},m_{j_f}} C^{J,L,1/2}_{M,M_{L'},m_{j_f}}|k_s,k_s+l,L,M_L\rangle \\ &\quad\times\langle k_s,k_s+l,L',M_{L'}||k_d,k_d+(m-l),L,M_L\rangle \\ &\quad\times\langle k_d,k_d+(m-l),L',M_{L'}|\big|1/2,m_{j_f}\big\rangle\big\langle 1/2,m_{j_f}\big|. \end{aligned} \tag{39}$$

In examining entanglement in the boson-fermion interaction model, we consider two states:

(a) The first case is when we want to determine the amount of boson-fermion entanglement in the A-odd nuclei. To do this, the trace is performed over bosons, or a single fermion. With trace over bosons states, the following relation is obtained:

$$Tr^B(\rho) = \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2}\left|C^{J,L,1/2}_{M,M_L,m_{j_f}}\right|^2|1/2,m_{j_f}\rangle\langle 1/2,m_{j_f}|. \tag{40}$$

Therefore, the entanglement entropy of bosons and a single fermion is obtained:

$$S^{BF}_{a1/2} = -\sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2}\left|C^{J,L,1/2}_{M,M_L,m_{j_f}}\right|^2\log\left(\left|C^{J,L,1/2}_{M,M_L,m_{j_f}}\right|^2\right). \tag{41}$$

As can be seen from Eq. (41), entropy depends only on the values of Clebsch-Gordan coefficients and gives a constant quantity that does not change with the control parameter. For example, the values of $S^{BF}_{a1/2}$ for some states given in Table I.

(b) In the second case, we want to determine how the entanglement entropy changes based on the control parameter. For investigation entanglement s boson with the rest of the system components (d boson and a single fermion), the trace is performed over d boson and single fermion. $Tr^{dF}(\rho)$ is obtained as follows:

$$\begin{aligned}Tr^{dF}(\rho) &= \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2}\sum_{l=0}^{m}|b_{l,k_sk_d}|^2\left|C^{J,L,1/2}_{M,M_d,m_{j_f}}\right|^2|k_s,k_s+l,L,M_L\rangle \\ &\quad\times\langle k_s,k_s+l,L,M_L|. \end{aligned} \tag{42}$$

Entropy of entanglement is given as

$$C_{l,m_{j_f}} = C^{J,L,1/2}_{M,M_L,m_{j_f}} b_{l,k_s,k_d}, \tag{43}$$

$$\begin{aligned}S^{BF}_{b1/2} &= -\sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2}\sum_{l=0}^{m}\left|C_{l,m_{j_f}}\right|^2\log\left(\left|C_{l,m_{j_f}}\right|^2\right) \\ &= \sum_{m_{j_f}=-1/2}^{m_{j_f}=+1/2}\left|C^{J,L,1/2}_{M,M_d,m_{j_f}}\right|^2\log\left(\left|C^{J,L,1/2}_{M,M_d,m_{j_f}}\right|^2\right) \\ &\quad-\sum_{l=0}^{m}\left|b_{l,k_s,k_d}\right|^2\log\left(\left|b_{l,k_s,k_d}\right|^2\right) \\ &= S^{BF}_{a1/2} + S^B. \end{aligned} \tag{44}$$

As can be seen from the above equation, changes of entanglement entropy based on the control parameter in boson-fermion systems are similar to entanglement entropy of the boson system, with this difference entropy values are shifted.

2. The case of $j_f = 3/2$

In examining entanglement in the boson-fermion interaction model in odd-A nuclei with $j_f = 3/2$ similar to the case of $j_f = 1/2$, we consider two states:

(a) In order to investigate entanglement entropy, the eigenstate of Eq. (23) can be rewritten as the Schmidt decomposition:

$$|\psi\rangle^{BF}_{\nu_1 JM} = \sum_{m_{j_f}=-3/2}^{m_{j_f}=+3/2} N_{j_f,m_{j_f}}|\psi\rangle^B_{\nu,m_{j_f}}|\psi\rangle^F_{(\nu_1=1/2,\nu_2=-1/2),j_f,m_{j_f}}, \tag{45}$$

where

$$|\psi\rangle^B_{\nu,m_{j_f}} = \sum_L \frac{C^{J,L,3/2}_{M,M_L,m_{j_f}}\eta^{\nu,L}_{\nu_1,J}}{N_{j_f,m_{j_f}}}|\psi\rangle^B_{\nu,L,M_L}. \tag{46}$$

With trace over single fermion states, the following relation is obtained:

$$Tr^F(\rho) = \sum_{m_{j_f}=-3/2}^{m_{j_f}=+3/2} N_{j_f,m_{j_f}}{}^2|\psi\rangle\langle\psi|. \tag{47}$$

TABLE XVII. The calculated entanglement entropy for ^{123}Xe isotope with $N_B = 8$, $\alpha = 3.4401$, $\eta = -1.0681$, $\delta = 0.0949$, $\gamma = 9.58$ and $C_s = 0.78$.

^{123}Xe	J	m_J	m	ν_s	ν_d	E (ke V)	S
	$(\frac{1}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	4	0	0	0	3.2844×10^{-5}
	$(\frac{3}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	3	1	1	83.24	0.501
	$(\frac{3}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	1	1	83.24	0.6736
	$(\frac{5}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	3	1	1	131.175	6.0673×10^{-4}
	$(\frac{5}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	3	1	1	131.175	0.501
	$(\frac{5}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	1	1	131.175	0.6736
	$(\frac{7}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	3	0	2	366.034	0.3489
	$(\frac{7}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	3	0	2	366.034	0.5618
	$(\frac{7}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	3	0	2	366.034	0.6366
	$(\frac{7}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	0	2	366.034	0.6871
	$(\frac{9}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	3	0	2	405.83	1.066×10^{-4}
	$(\frac{9}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	3	0	2	405.83	0.3489
	$(\frac{9}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	3	0	2	405.83	0.5298
	$(\frac{9}{2})^+$	$\frac{3}{2}, -\frac{3}{2}$	3	0	2	405.83	0.6366
	$(\frac{9}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	0	2	405.83	0.6871
	$(\frac{11}{2})_1^+$	$\frac{11}{2}, -\frac{11}{2}$	2	1	3	791.353	0.0017
	$(\frac{11}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	2	1	3	791.353	0.3063
	$(\frac{11}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	2	1	3	791.353	0.4758
	$(\frac{11}{2})_1^+$	$\frac{5}{2}, -\frac{7}{2}$	2	1	3	791.353	0.5877
	$(\frac{11}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	1	3	791.353	0.6572
	$(\frac{11}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	3	791.353	0.6907

So, the entanglement entropy is obtained as

$$S_{a3/2}^{BF} = -\sum_{m_{j_f}=-3/2}^{m_{j_f}=+3/2} \left|N_{j_f,m_{j_f}}\right|^2 \log\left(\left|N_{j_f,m_{j_f}}\right|^2\right). \tag{48}$$

As can be seen from Eq. (48), entropy depends on the values of the normalization constant ($N_{j_f,m_{j_f}}$), it depends on isoscalar factors and Clebsch-Gordan coefficients and gives a constant quantity that does not change with the control parameter. The values of $S_{a3/2}^{BF}$ for some states are given in Table II.

(b) To display how the entanglement entropy changes as a function of the control parameter, we introduce the symbol $a_{L,m_{j_f}}$, which leads to

$$a_{L,m_{j_f}} = \frac{C_{M,M_L,m_{j_f}}^{J,L,3/2}\,\eta_{\nu_1,J}^{\nu,L}}{N_{j_f,m_{j_f}}}, \quad |\psi\rangle_{\nu,m_{j_f}}^{B} = \sum_L a_{L,m_{j_f}} |\psi\rangle_{\nu,L,M_L}^{B}. \tag{49}$$

To describe the entanglement entropy, we perform a trace operation over d boson and single fermion states, leading to

$$Tr^{dF}(\rho) = \sum_{m_{j_f}=-3/2}^{m_{j_f}=+3/2} \sum_{l=0}^{m} \sum_L N_{j_f,m_{j_f}}{}^2 \left|a_{L,m_{j_f}}\right|^2 \times \left|b_{l,k_s k_d}\right|^2 |k_s, k_s+l, L, M_L\rangle\langle k_s, k_s+l, L, M_L|. \tag{50}$$

By using Eq. (50), the entanglement entropy yields

$$S^{dF} = -\sum_{m_{j_f}=-3/2}^{m_{j_f}=+3/2} \sum_L \left|C_{M,M_L,m_{j_f}}^{J,L,3/2}\eta_{\nu_1,J}^{\nu,L}\right|^2 \times \log\left(\left|C_{M,M_L,m_{j_f}}^{J,L,3/2}\eta_{\nu_1,J}^{\nu,L}\right|^2\right) - \sum_{l=0}^{m} |b_l|^2 \log(|b_l|^2), \tag{51}$$

$$N_{j_f,m_{j_f}}{}^2 \left|a_{L,m_{j_f}}\right|^2 = \left|C_{M,M_L,m_{j_f}}^{J,L,3/2}\eta_{\nu_1,J}^{\nu,L}\right|^2 = P(L, m_{j_f}), \tag{52}$$

$$P(L, m_{j_f}) = P(m_{j_f})P(L|m_{j_f}) = P(L)P(m_{j_f}|L). \tag{53}$$

TABLE XVIII. The calculated entanglement entropy for ^{125}Xe isotope with $N_B = 7$, $\alpha = 35.0834$, $\eta = -0.0209$, $\delta = -18.9978$, $\gamma = 30.1158$ and $C_s = 0.9$.

^{125}Xe	J	m_J	m	ν_s	ν_d	E (ke V)	S
	$(\frac{1}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	1	0	0	1.9524×10^{-5}
	$(\frac{3}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	3	0	1	123.7	0.5004
	$(\frac{3}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	0	1	123.7	0.673
	$(\frac{7}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	2	1	1	309.8	0.3494
	$(\frac{7}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	2	1	2	309.8	0.5622
	$(\frac{7}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	1	2	309.8	0.6371
	$(\frac{7}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	1	309.8	0.6875
	$(\frac{5}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	3	0	1	322	3.0347×10^{-5}
	$(\frac{5}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	3	0	1	322	0.5004
	$(\frac{5}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	0	1	322	0.673
	$(\frac{9}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	2	1	1	567.5	5.5671×10^{-4}
	$(\frac{9}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	2	1	2	567.5	0.3494
	$(\frac{9}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	2	1	2	567.5	0.5303
	$(\frac{9}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	1	2	567.5	0.6371
	$(\frac{9}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	1	567.5	0.6875
	$(\frac{11}{2})_1^+$	$\frac{11}{2}, -\frac{11}{2}$	2	0	3	984.8	7.0147×10^{-4}
	$(\frac{11}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	2	0	3	984.8	0.3053
	$(\frac{11}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	2	0	3	984.8	0.4748
	$(\frac{11}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	2	0	3	984.8	0.5867
	$(\frac{11}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	0	3	984.8	0.6562
	$(\frac{11}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	0	3	984.8	0.6897

As seen in Eq. (52), a sufficient condition establishes a joint probability distribution for m_{j_f} and L with probability $P(m_{j_f})$ and $P(L)$. The joint probability distribution shows a probability distribution for two (or more) random variables. The joint distribution of the two random variables takes values (m_{j_f}, L) with probability $P(m_{j_f}, L)$. Equation (53) is Bayes rule, where $P(L|m_{j_f})$ is the conditional probability distribution of L given m_{j_f}, that means the probability distribution of L when m_{j_f} is known to be a particular value and also $P(m_{j_f}|L)$ is the conditional probability distribution of m_{j_f} given L. Therefore, two cases can be examined.

(b1) The first case is when m_{j_f} has a certain value:

The entropy of joint two random m_{j_f} and L (see Appendix) in this case is defined as

$$\begin{aligned}
S(m_{j_f}, L) &= -\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L N_{j_f,m_{j_f}}{}^2 |a_{L,m_{j_f}}|^2 \\
&\quad \times \log\left(N_{j_f,m_{j_f}}{}^2 |a_{L,m_{j_f}}|^2\right) \\
&= -\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L P(m_{j_f}, L) \log\left(P(m_{j_f}, L)\right) \\
&= -\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L P(m_{j_f}) P(L|m_{j_f}) \log\left(P(m_{j_f})\right) \\
&\quad - \sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L P(m_{j_f}) P(L|m_{j_f}) \log\left(P(L|m_{j_f})\right) \\
&= -\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} P(m_{j_f}) \log\left(P(m_{j_f})\right) \\
&\quad - \sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} P(m_{j_f}) P(L|m_{j_f}) \log\left(P(L|m_{j_f})\right) \\
&= S_{a3/2}^{BF} + \sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} P(m_{j_f}) S(L|m_{j_f} = m_{j_f\,\mathrm{fix}}).
\end{aligned} \tag{54}$$

The values of $S(m_{j_f}, L)$ according to Eq. (54) and Table II are calculated and presented in Table III. Accordingly, the total entanglement entropy of the s boson with the rest of the

TABLE XIX. The calculated entanglement entropy for ^{127}Xe isotope with $N_B = 6$, $\alpha = 6.6672$, $\eta = -2.8253$, $\delta = -10.3085$, $\gamma = 20.788$ and $C_s = 0.7$.

^{127}Xe	J	m_J	m	v_s	v_d	E (keV)	S
	$(\frac{1}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	3	0	0	0	1.2924×10^{-6}
	$(\frac{3}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	1	1	134.0378	0.5004
	$(\frac{3}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	1	134.0378	0.673
	$(\frac{3}{2})_2^+$	$\frac{3}{2}, -\frac{3}{2}$	2	1	1	274.946	0.5004
	$(\frac{3}{2})_2^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	1	274.946	0.673
	$(\frac{5}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	2	1	1	382.83	2.1014×10^{-5}
	$(\frac{5}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	1	1	382.83	0.5004
	$(\frac{5}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	1	382.83	0.673
	$(\frac{1}{2})_2^+$	$\frac{1}{2}, -\frac{1}{2}$	3	0	0	509.38	1.5897×10^{-5}
	$(\frac{9}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	2	0	2	591.06	2.7311×10^{-6}
	$(\frac{9}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	2	0	2	591.06	0.3488
	$(\frac{9}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	2	0	2	591.06	0.5297
	$(\frac{9}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	2	0	2	591.06	0.6365
	$(\frac{9}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	2	0	2	591.06	0.687
	$(\frac{11}{2})_1^+$	$\frac{11}{2}, -\frac{11}{2}$	1	1	3	792.43	5.6331×10^{-5}
	$(\frac{11}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	1	1	3	792.43	0.3047
	$(\frac{11}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	1	1	3	792.43	0.4742
	$(\frac{11}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	1	1	3	792.43	0.586
	$(\frac{11}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	1	3	792.43	0.6555
	$(\frac{11}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	1	3	792.43	0.6891

system components (d boson and single fermion) is given as

$$S = S^{BF}_{a3/2} + \sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} P(m_{j_f}) S(L|m_{j_f} = m_{j_f\,\mathrm{fix}}) + S(B)$$
$$= S(m_{j_f}, L) + S(B). \quad (55)$$

(b2) The second case is when L has a certain value: The entropy in this condition is expressed as

$$S(L, m_{j_f}) = -\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L \left|C^{J,L,j_f}_{M,M_L,m_{j_f}} \eta^{\nu,L}_{\nu_1,J}\right|^2 \times \log\left(\left|C^{J,L,j_f}_{M,M_L,m_{j_f}} \eta^{\nu,L}_{\nu_1,J}\right|^2\right)$$
$$= -\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L P(L, m_{j_f}) \log(P(L, m_{j_f}))$$
$$= -\sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L P(L)P(m_{j_f}|L) \log(P(L)) - \sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L P(L)P(m_{j_f}|L) \log(P(m_{j_f}|L))$$
$$= -\sum_L P(L) \log(P(L)) - \sum_{m_{j_f}=-j_f}^{m_{j_f}=+j_f} \sum_L P(L)P(m_{j_f}|L) \log(P(m_{j_f}|L))$$
$$= S(L) + \sum_L P(L) S(m_{j_f}|L = L_{\mathrm{fix}}). \quad (56)$$

The values of $S(L, m_{j_f})$ according to Eq. (56) and Table III are calculated and presented in Table IV. The total entanglement entropy is obtained as

$$S = S(L) + \sum_L P(L) S(m_{j_f}|L = L_{\mathrm{fix}}) + S(B)$$
$$= S(L, m_{j_f}) + S(B). \quad (57)$$

So, the entropy values of $S(L, m_{j_f})$ or $S(m_{j_f}, L)$ give a constant quantity that does not change with the control parameter

TABLE XX. The calculated entanglement entropy for ^{129}Xe isotope with $N_B = 5$, $\alpha = 40.5045$, $\eta = -0.7259$, $\delta = -17.3238$, $\gamma = 37.4465$ and $C_s = 0.52$.

^{129}Xe	J	m_J	m	ν_s	ν_d	E (ke V)	S
	$(\frac{1}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	0	0	8.5014×10^{-6}
	$(\frac{3}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	0	1	96.9	0.500404796059306
	$(\frac{3}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	0	1	96.9	0.673013253400717
	$(\frac{5}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	2	0	1	400.3	1.8866×10^{-6}
	$(\frac{5}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	0	1	400.3	0.500404796059306
	$(\frac{5}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	0	1	400.3	0.673501306800717
	$(\frac{7}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	1	1	2	459.3	0.349088198965803
	$(\frac{7}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	1	1	2	459.3	0.561927280048556
	$(\frac{7}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	1	2	459.3	0.636770030776935
	$(\frac{7}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	1	2	459.3	0.687217787957337
	$(\frac{3}{2})_2^+$	$\frac{3}{2}, -\frac{3}{2}$	2	0	1	252.4	0.500429552459306
	$(\frac{3}{2})_2^+$	$\frac{1}{2}, -\frac{1}{2}$	2	0	1	252.4	0.673013253400717
	$(\frac{9}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	1	1	2	796.3	2.6974×10^{-5}
	$(\frac{9}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	1	1	2	796.3	0.348858872965803
	$(\frac{9}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	1	1	2	796.3	0.529706563726931
	$(\frac{9}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	1	2	796.3	0.636540704776935
	$(\frac{9}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	1	2	796.3	0.686988461957337
	$(\frac{1}{2})_2^+$	$\frac{1}{2}, -\frac{1}{2}$	2	1	0	348.5	2.2173×10^{-5}
	$(\frac{11}{2})_1^+$	$\frac{11}{2}, -\frac{11}{2}$	1	0	3	1075	1.1632×10^{-5}
	$(\frac{11}{2})_1^+$	$\frac{9}{2}, -\frac{9}{2}$	1	0	3	1075	0.304646730258958
	$(\frac{11}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	1	0	3	1075	0.474150733773031
	$(\frac{11}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	1	0	3	1075	0.585964103908376
	$(\frac{11}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	0	3	1075	0.655493441041954
	$(\frac{11}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	0	3	1075	0.689020799757025

as shown in Tables III and IV, respectively. As can be seen from the above equation, changes of entanglement entropy based on the control parameter in boson-fermion systems are similar to entanglement entropy of the boson system, with this difference entropy values are shifted. Thus, it was found that the coupling of the single fermion to the even-even system does not change the geometry imposed by the boson core performing the transition and only the entanglement entropy values have been shifted.

E. Quadrupole moments

The observables such as electric quadrupole transition probabilities, $B(E2)$, as well as electric quadrupole moment within the low-lying state provide important information about the nuclear structure and QPTs. A measure of the deviation of a charge distribution from a spherical shape is the electric quadrupole moment of the distribution. The quadrupole moment is an important property for nuclei. In this section we discuss the calculation of quadruple moments for even-even nuclei. The electric quadrupole moments of the nuclei are defined according to Refs. [37,52] as

$$Q_L = \langle L, M_L|\sqrt{\frac{16\pi}{5}}T_0^{(E_2)}|L, M_L\rangle = \sqrt{\frac{16\pi}{5}}\begin{pmatrix} L & 2 & L \\ -L & 0 & L \end{pmatrix}\langle L||T^{(E_2)}||L\rangle, \tag{58}$$

$$T_\mu^{(E_2)} = q_2[s^+ \times \tilde{d} + d^+ \times \tilde{s}]_\mu^{(2)} + q_2'[d^+ \times \tilde{d}]_\mu^{(2)}, \tag{59}$$

where q_2 and q_2' are parameters specifying the magnitude and scale of the $T^{(E_2)}$ operator [52]. For evaluating Q_L, we consider eigenstates Eq. (14) where the normalization factor was obtained from Eq. (34) in Ref. [37]. We have calculated the matrix elements of $T_0^{(E_2)}$ operators between the eigenstates of Eq. (14). To determine the coefficients (q_2, q_2'), these

TABLE XXI. The calculated entanglement entropy for ^{131}Xe isotope with $N_B = 4$, $\alpha = 1.054$, $\eta = -8.0227$, $\delta = -$, $\gamma = 26.7727$ and $C_s = 0.55$.

^{131}Xe	J	m_J	m	ν_s	ν_d	E (ke V)	S
	$(\frac{3}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	2	0	0	0	3.6516×10^{-7}
	$(\frac{3}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	2	0	0	0	3.6516×10^{-7}
	$(\frac{5}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	1	1	1	504.8	0.682958115922725
	$(\frac{5}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	1	1	504.8	0.801452467214379
	$(\frac{5}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	1	1	504.8	1.248131861923086
	$(\frac{7}{2})_1^+$	$\frac{7}{2}, -\frac{7}{2}$	1	1	1	692.2	9.9705×10^{-6}
	$(\frac{7}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	1	1	1	692.2	0.682958115922725
	$(\frac{7}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	1	1	692.2	0.955749499213412
	$(\frac{7}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	1	1	692.2	1.05852007530283
	$(\frac{3}{2})_2^+$	$\frac{3}{2}, -\frac{3}{2}$	2	0	0	411.1	2.0017×10^{-6}
	$(\frac{3}{2})_2^+$	$\frac{1}{2}, -\frac{1}{2}$	2	0	0	411.1	2.0017×10^{-6}

quantities are extracted from the empirical $B(E2)$ values via to least square technique by MATLAB software.

III. NUMERICAL RESULT

The entanglement entropy tool to investigate shape transitional of spherical U(5) to γ-unstable O(6) is applied in even-even and odd-A nuclei. We start with the interacting boson model in even-even nuclei, where $C_s = \frac{c_s}{c_d} = 0$, $c_d = 1$ is in spherical limit, and $C_s = \frac{c_s}{c_d} = 1$, $c_s = c_d = 1$ is in γ-unstable limit. For each state, one can calculate the entanglement entropy as a function of the control parameter C_s. For example, the results for the ground and two lowest states are calculated and shown in Fig. 2, where in Fig. 2 other fixed parameters are $\alpha = 1000$ keV, $x_i = (0.9941, 0.9977, 0.999, 0.9946, 0.9962)$ for ground state (0_1^+), $x_i = (0.9941, 0.9977, 0.999, 0.9946)$ for first state (2_1^+) and $x_i = (0.9941, 0.9977, 0.999)$ for second state $(0_2^+, 2_2^+, 4_1^+)$. Figure 2 depicts how the entanglement entropy of s and d bosons as a function of the control parameter C_s evolve from one dynamical symmetry limit [U(5)] to the other [O(6)]. It can be seen from Fig. 2 that the entanglement entropy values of the s and d bosons for each state are zero in one phase for $C_s < C_{s,\text{critical}}$ and begin to make larger for $C_s > C_{s,\text{critical}}$, entanglement entropy reaches its maximum in $C_s = 1$. Therefore, in U(5) limit there is no entanglement between s and d bosons and maximum entanglement in the O(6) limit is observed.

As previously explained in Sec. II D, it is possible to recognize the signature of the phase transition in the odd-A nuclei by using entanglement entropy. The results for the odd-A nuclei, that a single fermion has angular momentum $j_f = 1/2$ and $j_f = 3/2$, are given in Figs. 3–6. The entanglement entropy according to Eq. (41) depends only on the values of Clebsch-Gordan coefficients and in Eq. (2.48) depends on isoscalar factors and Clebsch-Gordan coefficients. So, the entanglement entropy values give a constant quantity that does not change with the control parameter as shown in Figs. 3, 5 and Tables I, II.

In Fig. 4 the entanglement entropy of the s boson with the rest of the system components (d bosons and a single fermion) is shown. It is found from Fig. 4 that the changes of ground-state entanglement entropy based on the control parameter are similar to the boson system and the entanglement entropy values in the exited states have been shifted with respect to the even case according to Eq. (44).

To display how the entanglement entropy changes as a function of the control parameter in case $j_f = 3/2$, we used Eq. (55) or Eq. (57) which leads to Fig. 6. In analyzing Fig. 6 it can be said that, as can be seen in Table III, the entropy values of $S(m_{j_f}, L)$ for the ground state ($J = 3/2(M = -3/2, -1/2, 1/2, 3/2)$) and $J = 7/2(M = 7/2, -7/2)$ states are zero. So, the entropy of boson-fermion systems is obtained only from the entropy values of boson system according Eq. (55) or Eq. (57). In the other graphs drawn in Fig. 6, the entropy values of $S(m_{j_f}, L)$ are not zero. Also, these values have no dependence on the parameter control (C_s). Therefore, the entropy changes in terms of parameter control (C_s) in these states are similar to the boson system, except that the entropy values are shifted by $S(m_{j_f}, L)$. By comparing the states with $S(m_{j_f}, L) = 0$ of this figure with the states with $S(m_{j_f}, L) \neq 0$, one can see that the effect of the fermionic impurity is to increase the entropy values. In other words, we can see in Fig. 6 that the modification induced by the presence of a fermion shift the entanglement entropy value by $S(m_{j_f}, L)$.

In the following, the possible occurrence of transitional characteristics intermediate between spherical and γ-unstable shapes in $^{102-110}$Pd, $^{122-134}$Xe, and $^{123-133}$Xe isotopic chains is investigated by using entanglement entropy. The theoretical and experimental studies of energy spectra done in Refs. [49–55] show Pd and Xe isotopes have U(5) $\leftrightarrow$ O(6) transitional characteristics. In this study, we analyze the entanglement entropy values of the low-lying states for these

TABLE XXII. The calculated entanglement entropy for ^{133}Xe isotope with $N_B = 3$, $\alpha = 245.99$, $\eta = -17.59$, $\delta = -$, $\gamma = 30.62$ and $C_s = 0.2$.

^{133}Xe	J	m_J	m	ν_s	ν_d	E (ke V)	S
	$(\frac{3}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	1	0	0	5.5078×10^{-7}
	$(\frac{3}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	1	0	0	5.5078×10^{-7}
	$(\frac{5}{2})_1^+$	$\frac{5}{2}, -\frac{5}{2}$	1	0	1	815.9	0.682908269942725
	$(\frac{5}{2})_1^+$	$\frac{3}{2}, -\frac{3}{2}$	1	0	1	815.9	0.801402621234379
	$(\frac{5}{2})_1^+$	$\frac{1}{2}, -\frac{1}{2}$	1	0	1	815.9	1.248082015943086

nuclei. In order to obtain entanglement entropy values and realistic calculations for these nuclei, we need to specify quantum numbers $(m, \nu_d, \nu_s, n_d, x_i)$. The best sets of transitional Hamiltonian parameters have been achieved via the least square fit to the available experimental. The experimental data for the $^{102-110}$Pd, $^{122-134}$Xe, and $^{123-133}$Xe nuclei are taken from Refs. [56–68], respectively. The calculated entanglement entropy values and the used quantum numbers are shown in Tables V–XXII. Figure 7 also shows the predictions of our results for these isotopes in the low-lying states. Control parameter values, C_s, suggest structural changes in nuclear deformation and shape-phase transitions in Pd and Xe chains isotopes. It can be seen from Fig. 7(left) and Table X that the entanglement entropy values of the s and d bosons for Pd vibrational isotopes are zero and begin to become larger in other isotopes.

As can be seen in Table X, the value of control parameter, C_s, decrease from ^{122}Xe($C_s = 0.95$) to ^{134}Xe($C_s = 0.06$). Therefore, the Xe isotopes show a phase transition from γ-unstable limit to spherical limit. As seen in Fig. 7(right), the entropy values reach from the maximum value in isotope ^{122}Xe to the minimum value in isotope ^{134}Xe. Also in odd-A isotopes of xenon, like even-even isotopes, the control parameters and the calculated entropy values are reached from a maximum value to a minimum value. Therefore, in the investigation entropy between s bosons and the rest (d bosons and fermion), the structure of odd-A isotopes of xenon is similar to that of even-even isotopes. Except for that, a shift in entropy values of excited states in odd-A isotopes is found.

Table XXIII shows the experimental and calculated values of $Q_{2_1^+}$ and extracted values for effective charge parameters for $^{102-110}$Pd and $^{122-134}$Xe nuclei. The calculations of the electric quadrupole moment $Q_{2_1^+}$ are within the framework of IBM-1. As can be seen, the values of $Q_{2_1^+}$, increase similar to calculated entanglement entropy values from ^{102}Pd to ^{110}Pd. We noticed that the 2_1^+ state corresponds to a well deformed shape for the heavier isotopes, which is smoothly changing into a rather spherical shape for the lightest isotopes. And also the value of $Q_{2_1^+}$, decreases from ^{122}Xe to ^{134}Xe like the calculated entanglement entropy values for Xe isotopes. However, the 2_1^+ state in Xe isotopes correspond to a deformed shape for the lightest isotopes, changing into a spherical shape for heavier isotopes. So, the obtained values of the quadrupole moment in $^{102-110}$Pd and $^{122-134}$Xe nuclei confirm the results of the entanglement entropy in these nuclei.

Our results confirmed a phase transition between two different phases in Xe and Pd isotopes. And also from the

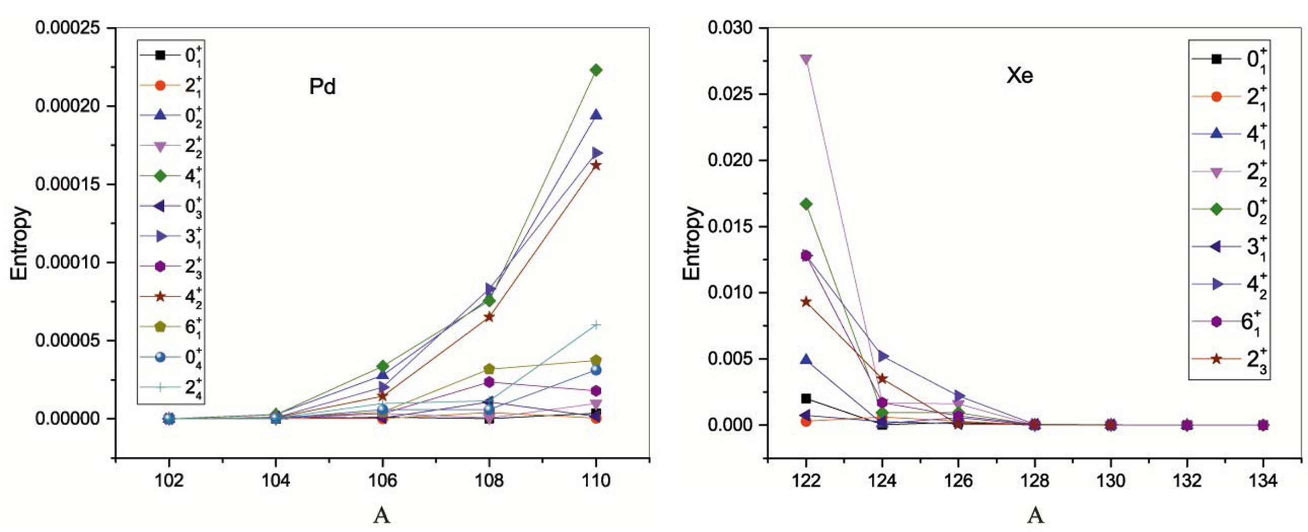

FIG. 7. The calculated entanglement entropy values of the low-lying states of $^{122-134}$Xe and $^{102-110}$Pd isotopes.

TABLE XXIII. The calculated Q_2 and the coefficients of $T^{(E_2)}$ for $^{102-110}$Pd and $^{122-134}$Xe nuclei. The experimental data for the $^{102-110}$Pd and $^{122-134}$Xe nuclei are taken from Refs. [56–60] and [61–67], respectively. The units of Q_2 and the coefficients are (eb).

nucleus	q_2	q_2'	$Q_{2,Exp}$	$Q_{2,Th}$
^{102}Pd	1.19734	−0.103295	−0.20	−0.2401
^{104}Pd	1.31162	−0.123954	−0.45	−0.4202
^{106}Pd	1.4167	−0.1446126	−0.51	−0.49
^{108}Pd	1.51453	−0.16527	−0.58	−0.5602
^{110}Pd	1.6064	−0.1859305	−0.72	−0.6302
^{122}Xe	0.9771	0.1563	–	0.5298
^{124}Xe	6.6280	0.1458	–	0.4942
^{126}Xe	2.7863	0.1365	–	0.4627
^{128}Xe	4.5965	0.1287	–	0.4362
^{130}Xe	2.5900	0.1243	–	0.4213
^{132}Xe	2.8923	0.1215	–	0.4118
^{134}Xe	7.9181	0.1188	–	0.4027

entropy values, we found that the evolution of the control parameter has corresponded to an enhancement of entropy and vice versa.

IV. CONCLUSION

In this paper, we discussed how entropy can be used to detect quantum phase transitions. We examined the entanglement entropy in the interacting boson model (IBM) and the interacting boson fermion model (IBFM). A theoretical technique for tracking and quantitatively studying the phase transitions in even-even and odd-A nuclei was presented. It is found that entanglement entropy is a suitable order parameter to detect shape phase transition in nuclear systems. The entanglement entropy values of the low-lying states of $^{122-134}$Xe, $^{102-110}$Pd, and $^{123-133}$Xe were calculated and analyzed. The results show that no entanglement between s and d bosons in the U(5) limit exists and the maximum entanglement value is in the O(6) limit. In the boson-fermion systems, the entanglement of bosons and a single fermion is a constant quantity that does not change with the control parameter. The changes of entanglement entropy based on the control parameter in boson-fermion systems are similar to entanglement entropy of boson system. We found the coupling of the single fermion with angular momentum j to the even-even system does not change the nature of the system with respect to the even-even case and only the entanglement increase. Our study confirmed the importance of the entanglement entropy as necessary signatures to characterize the occurrence of the shape phase transition.

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APPENDIX: JOINT PROBABILITY DISTRIBUTION

Probability $p(a, b)$ of variables A and B takes values (a, b) which are defined as

$$p(a,b) = p(a)p(b|a), \quad p(a,b) = p(b)p(a|b). \tag{A1}$$

The below equation is Bayes rule:

$$p(a)p(b|a) = p(b)p(a|b). \tag{A2}$$

The above equation has a very simple derivation that directly leads from the relationship between joint and conditional probabilities.

The entropy of joint two random is defined as

$$\begin{aligned} S(A,B) &= \sum_{a,b} p(a,b)\log\frac{1}{p(a,b)} \\ &= \sum_{a,b} p(a)p(b|a)\log\frac{1}{p(a)} + \sum_{a,b} p(a)p(b|a)\log\frac{1}{p(b|a)} \\ &= \sum_{a} p(a)\log\frac{1}{p(a)}p(b|a) + \sum_{a,b} p(a)p(b|a)\log\frac{1}{p(b|a)} \\ &= S(A) + \sum_{a} p(a)S(B|A=a) \\ &= S(A) + S(B|A) = S(B) + S(A|B). \end{aligned} \tag{A3}$$

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