


Symmetry projection to coupled-cluster singles plus doubles wave function through the Monte Carlo method

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A method for calculating the symmetry-projected energy of coupled-cluster singles plus doubles (CCSD) wave function through the Monte Carlo method is proposed. We present benchmark calculations in considering the three-level Lipkin model which is a simple and minimal model with two phases: spherical and deformed. It is demonstrated that this method gives good ground-state energy and low-lying spectra.

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I. INTRODUCTION

To solve many-body problems, the use of an intricate trial wave function is absolutely desirable, while computability limits its functional form. To enlarge the power of trial wave functions has been one of the central issues of many-body physics. Let us look back at some well-established theories. In the Rayleigh-Ritz variational approach, Hartree-Fock (HF) and Hartree-Fock-Bogoliubov (HFB) wave functions correspond to mean-field theories and have been well developed [1] in the past. Beyond mean field, symmetry restoration is essential and is well compatible with HF and HFB approaches. Symmetry projection has also been extensively reviewed in Ref. [2]. On the other hand, as a nonvariational approach, the coupled-cluster method [3–11] is quite prominent and has been well developed with a broad field of applications. Its trial wave function is a mean-field wave function multiplied by an exponential with many-body correlations e^Z , which has an elaborate structure. The integration of the coupled-cluster method and symmetry projection is a challenging issue, and very recently, such a study has been actively pursued [12–20].

In the present study, we propose an integration of coupled-cluster wave functions with symmetry projection via the Monte Carlo method. We start with the coupled-cluster singles plus doubles (CCSD) wave function, and optimize it by the coupled-cluster method. Then, we apply the symmetry projection operator to it and evaluate the expectation value of the Hamiltonian. Of course, due to the e^Z correlation factor, the straightforward utilization inevitably leads to huge and mostly intractable computational difficulties. Therefore, we introduce the Monte Carlo approach.

To evaluate this proposed method, we use the three-level Lipkin model [6,21,22], which is a generalization of the Lipkin-Meshkov-Glick model [23]. This model is simple but exhibits a spherical-deformed phase transition. For the case of degenerate single-particle energies of the upper two levels, there is an exchange symmetry, which can be handled by

the symmetry projection method. Thus, the three-level Lipkin model is a minimal model for our purpose. With numerical investigations, we will test the performance of our method.

In the symmetry projection, we use spherical and deformed bases in a mixed way. For the spherical basis, the creation and annihilation operators are denoted by c_i^\dagger and c_i , while for deformed basis, they are denoted by a_i^\dagger and a_i . The deformed operators are canonically related to the spherical operators as $a_i = \sum_j D_{ij} c_j$. The true vacuum is represented as $|-\rangle$. Hereinafter, a state defined by spherical operators is denoted $|i_1, \dots, i_m\rangle = c_{i_1}^\dagger \cdots c_{i_m}^\dagger |-\rangle$, while a state defined by deformed operators is denoted $|k_1, \dots, k_m\rangle = a_{k_1}^\dagger \cdots a_{k_m}^\dagger |-\rangle$.

II. METHOD

Coupled-cluster singles plus doubles and coupled-cluster doubles wave functions. We begin by considering the coupled-cluster (CC) method [3–6]. Based on a HF wave function $|\psi_0\rangle$, we define the CC wave function $|\psi_{cc}\rangle$ with the many-body Z operator as $|\psi_{cc}\rangle = e^Z |\psi_0\rangle$. Z is defined as $\sum \chi_{p_1, p_2, \dots, h_1, h_2, \dots} a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{h_1} a_{h_2} \cdots$, where p (h) stands for unoccupied (occupied) orbits. The summation is arranged as $Z = Z_1 + Z_2 + \cdots$, where Z_k are defined as $Z_k = \sum_i x_i^{(k)} Z_i^{(k)}$ and the $Z_i^{(k)}$ are the different k -body operators and $x_i^{(k)}$ are their parameters. We limit Z by up to two-body terms ($k = 2$) due to the increasing numerical complexity in the applications for higher values of k . We call this the CCSD approximation.

The parameters x are determined in the following way: If $|\psi_{cc}\rangle$ is an exact wave function, we have

$$H|\psi_{cc}\rangle = E e^Z |\psi_0\rangle, \quad (1)$$

where E is the exact energy and H is the Hamiltonian expressed with the deformed operators. The equation corresponds to a non-Hermitian-type eigenvalue problem as follows:

$$\bar{H}|\psi_0\rangle = E|\psi_0\rangle, \quad (2)$$

where the transformed Hamiltonian $\bar{H} = e^{-Z} H e^Z$ is non-Hermitian. The energy is obtained by

$$\langle \psi_0 | \bar{H} | \psi_0 \rangle = E, \quad (3)$$

where we refer to it as the CC energy hereinafter. Multiplying Eq. (2) from the left with $\langle \psi^* |$, the coefficients x are determined by the CC equation $\langle \psi^* | \bar{H} | \psi_0 \rangle = 0$, where $\langle \psi^* |$ is any state orthogonal to $\langle \psi_0 |$. Within the CCSD approximation, all operators $Z_1^{(k)}$, $Z_2^{(k)}$ and their parameters $x_1^{(k)}$, $x_2^{(k)}$ are renamed as z_i ($i = 1, \dots, n_0$) and x_i ($i = 1, \dots, n_0$), where n_0 is the number of parameters. The mean-field wave function $|\psi_0\rangle$ is simply shown by $|0\rangle$ as the 0th basis state. We evaluate the following matrix elements for an excited state $|k\rangle$ as $a_k = \langle k | H | 0 \rangle$, $b_{ki} = \langle k | [H, z_i] | 0 \rangle$, and $c_{kij} = \langle k | [[H, z_i], z_j] | 0 \rangle$ where $i, j, k = 1, \dots, n_0$. The CC equation is then rewritten as $a_k + \sum_i b_{ki} x_i + \frac{1}{2} \sum_{ij} c_{kij} x_i x_j = 0$. The parameters x can be obtained iteratively and we, thus, can evaluate the CC energy with the CCSD wave function.

Moreover, we introduce the CCD wave function as shown in Ref. [9], eliminating the one-body operator Z_1 . The e^{Z_1} operator changes the mean-field determinant $|0\rangle$ into another one $|0'\rangle = e^{Z_1} |0\rangle$. Then with this new mean-field state $|0'\rangle$, we recalculate all the matrix elements and solve the CC equation. This process is repeated iteratively. After the convergence, we obtain the optimal mean-field state $|0\rangle$ and the CCD wave function $e^{Z_2} |0\rangle$, which is the starting point of this study.

Symmetry projection of coupled-cluster wave function. In the coupled-cluster method, the energy is given by Eq. (3). This method, however, does not satisfy the Rayleigh-Ritz variational principle. Therefore, we evaluate the energy directly as

$$E = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}, \quad (4)$$

where $|\psi\rangle$ stands for the CCSD or CCD wave function. This energy gives an upper limit to the exact energy, unlike the coupled-cluster method. Hereinafter we refer to it as the RR energy to distinguish it from the CC energy.

The variational coupled-cluster methods, with which the RR energy is evaluated based on the CC wave-function by expanding e^Z , has already been tried, for instance, in Ref. [24]. In the present method, we develop another expansion.

Next, we introduce the symmetry projection operator P^L , which is a projection onto a state with a good quantum number L . For example, in nuclear structure physics, angular-momentum projection has often been utilized. For a continuous symmetry, the symmetry projection operator is generally given by

$$P^L = \frac{1}{\mathcal{N}} \int d\mu \mathcal{W}(L, \mu) R(\mu), \quad (5)$$

where \mathcal{N} is a normalization, \mathcal{W} is a weight function, and $R(\mu)$ is given by $e^{i\hat{O}\cdot\mu}$ where \hat{O} is generally defined in terms of spherical operators, not by deformed ones. The integration is carried out over the parametrization μ of the continuous group. We will show an example below in Sec. III. If $|\psi\rangle$ is a superposition of several states with different L , the projection

operator P^L extracts $|\psi, L\rangle$ with a definite quantum number L as $|\psi, L\rangle = P^L |\psi\rangle = \frac{1}{\mathcal{N}} \int d\mu \mathcal{W}(L, \mu) R(\mu) |\psi\rangle$.

The RR energy E_L with the projected coupled-cluster wave function is given by

$$E_L = \frac{\langle \psi | H P^L | \psi \rangle}{\langle \psi | P^L | \psi \rangle}, \quad (6)$$

where we refer to it as projected energy. If we follow the terminology of the projection method, e.g., variation after projection (VAP) or projection after variation (PAV), we call this procedure projection after coupled cluster (PACC). Note that this evaluation is usually carried out through spherical operators, although the above equation is expressed in the deformed basis. To evaluate the projected energy, we need the matrix elements with the projected wave function as

$$\langle \psi | \begin{bmatrix} 1 \\ H \end{bmatrix} P^L | \psi \rangle = \frac{1}{\mathcal{N}} \int d\mu \mathcal{W}(L, \mu) \langle \psi | \begin{bmatrix} 1 \\ H \end{bmatrix} | \psi(\mu) \rangle, \quad (7)$$

where the rotated state $|\psi(\mu)\rangle$ is defined by

$$|\psi(\mu)\rangle = e^{i\hat{O}\cdot\mu} |\psi\rangle. \quad (8)$$

This is the standard way for projection calculations. In realistic applications with huge Hilbert spaces, this projection is generally not feasible because $|\psi\rangle$ includes e^{Z_2} in the CCD wave function, and evaluation of deformed operators through spherical operators additionally increases computational efforts.

Monte Carlo procedure. To carry out the symmetry projection, we introduce the Monte Carlo procedure. Hereinafter we choose the label s to specify each basis of the spherical representation. Similarly, we choose the label d to specify each basis in the deformed mean-field representation. The coupled-cluster wave function is given as $|\psi\rangle$ with the deformed operators.

Inserting the unity operator $\sum_s |s\rangle \langle s| = 1$, the projected energy in Eq. (6) can be rewritten as

$$E_L = \sum_s \rho_L(s) \mathcal{E}_L(s). \quad (9)$$

For the spherical basis s , we define the projected local energy $\mathcal{E}_L(s)$ as

$$\mathcal{E}_L(s) = \sum_{s'} h_{s,s'} \frac{\langle s' | P^L | \psi \rangle}{\langle s | P^L | \psi \rangle}, \quad (10)$$

where $h_{s,s'}$ is the Hamiltonian matrix element in the spherical representation and which is generally very sparse. We can also define the projected density $\rho_L(s)$ as

$$\rho_L(s) = \frac{|\langle s | P^L | \psi \rangle|^2}{\sum_s |\langle s | P^L | \psi \rangle|^2}, \quad (11)$$

where $\rho_L(s) \geq 0$ and $\sum_s \rho_L(s) = 1$. This property allows us to stochastically generate the distribution of s according to Eq. (11) by the Markov chain Monte Carlo (MCMC) method. Note that this technique has been presented in cases with pair condensates [25–28]. Therefore, by applying the Monte Carlo

sampling, the symmetry projected energy can be estimated as

$$E_L \sim \frac{1}{N_0} \sum_i \mathcal{E}_L(s_i), \quad (12)$$

where N_0 is a number of Monte Carlo samples and we refer to it as the MC energy hereinafter.

Next, we investigate the projected overlap $\langle s|P^L|\phi\rangle$ in detail. By inserting $\sum_d |d\rangle\langle d| = 1$, the projected overlap is rewritten as

$$\langle s|P^L|\psi\rangle = \sum_d \langle s|P^L|d\rangle\langle d|\psi\rangle, \quad (13)$$

where there is another possibility for introducing the truncation instead of the expansion of e^Z .

The computation of the correlation part of the overlap, $\langle d|\psi\rangle = \langle d|e^{Z_2}|0\rangle$, is feasible because Z_2 is also represented with the deformed operators and P^L does not operate on $|\psi\rangle$. The symmetry projection is easily carried out in the form $\langle s|P^L|d\rangle$, where s and d span, in principle, the whole Hilbert space, while on the other hand, for s , we use the Monte Carlo sampling so as to avoid the full use of the Hilbert space. For d , we also restrict the whole Hilbert space by introducing a truncation scheme due to the overlap $\langle d|\psi\rangle$. For a certain ϵ , we can truncate the coupled-cluster wave function as

$$|\psi\rangle \approx \sum_{|\psi(d)|^2 > \epsilon} \psi(d)|d\rangle, \quad (14)$$

where $\psi(d) = \langle d|\psi\rangle$ is an amplitude of $|\psi\rangle$ in the basis d . Under the assumption that the system has a moderately good mean field, the CCD wave function $|\psi\rangle$ can be well approximated by an expansion around the optimized deformed mean-field wave function $|0\rangle$, we can naturally expect that this truncation scheme works well, as we will see in Sec. III. Thus, we can also avoid the difficulty with the handling of the full Hilbert space for d . Note that several techniques in Refs. [27,28] can be applied to evaluate the projected overlap in Eq. (13) and we presented some discussions in the supplemental material [29].

III. BENCHMARK TEST

In this section, we want to demonstrate the feasibility of our new method by taking the three-level Lipkin model as a testing ground.

Three-level Lipkin model. First, we define the three-level Lipkin model with the levels $i = 0, 1, 2$, each having a degeneracy of N . We consider N spinless fermions. The single-particle energies ε_i are $\varepsilon_0 = 0$ and $\varepsilon_1 = \varepsilon_2 = 1$. The creation and annihilation operators of the k th particle on the i th level are c_{ik}^\dagger and c_{ik} , which we call, as before, spherical operators. The Hamiltonian is defined in this basis as $H = \varepsilon(n_1 + n_2) - \frac{v}{2}(J_{10}^2 + J_{01}^2 + J_{20}^2 + J_{02}^2)$, where $\varepsilon = 1$ and v is the strength of the two-body interaction. The J operators are defined by $J_{pq} = \sum_i c_{p,i}^\dagger c_{q,i}$. The system is specified by the dimensionless parameter $\chi = v(N - 1)/\varepsilon$. The spectra show two phases: One has a vibrational pattern around $\chi \approx 0$ and the other a rotational pattern for $\chi > 2$. These two phases continuously change from one into the other when considering a finite number N .

The HF approximation is given by the mean-field state $|\phi\rangle = a_{0,1}^\dagger \cdots a_{0,N}^\dagger |-\rangle$ with the deformed operators, $a_{i,m}^\dagger$ ($i = 0-2$, $m = 1-N$), which are canonically related to the spherical ones with coefficients D_{ij} as $a_{i,m}^\dagger = \sum_j D_{ij} c_{j,m}^\dagger$. The HF energy can be shown to be $E_{\text{HF}} = 0$ for $\chi < 1$ and $E_{\text{HF}} = \frac{N\varepsilon}{4}(2 - \chi - \frac{1}{\chi})$ for $\chi > 1$ [22].

For the degenerate two upper single-particle energies, there is a symmetry concerning the exchange of one and two levels. This can be embodied by the symmetry operator (identifiable with a rotational operator) $L = i(J_{21} - J_{12})$ which commutes with the Hamiltonian, $[H, L] = 0$. Thereby, all wave functions have the quantum number $L_0 = 0, \pm 1, \pm 2, \dots$. Except for $L_0 = 0$, the eigenstates with L_0 are doubly degenerate. With this symmetry operator, we can construct the projection operator as

$$P^{L_0} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i(L-L_0)\theta}, \quad (15)$$

which projects out the L_0 component from any wave function $|\psi\rangle$ that is, $|\psi, L_0\rangle = P^{L_0}|\psi\rangle$ where L_0 and $|\psi, L_0\rangle$ are the eigenvalue and eigenstate of L , respectively; that is, $L|\psi, L_0\rangle = L_0|\psi, L_0\rangle$. Note that L is defined by the c and c^\dagger operators and the projection should be carried out using the spherical operators.

Symmetry projection of coupled-cluster wave function. The CCSD wave function $|\psi\rangle$ is given by $|\psi\rangle = e^Z|0\rangle$ where $|0\rangle$ is the deformed HF state, and Z is organized into a sum of one-body and two-body operators $Z = x_1 K_{10} + x_2 K_{20} + y_{11} K_{10}^2 + y_{22} K_{20}^2 + y_{12} K_{10} K_{20}$, where the K operators are defined as $K_{pq} = \sum_i a_{p,i}^\dagger a_{q,i}$. In applying the CCSD procedure described in Sec. II, we take the components $|1, 0\rangle$, $|0, 1\rangle$, $|1, 1\rangle$, $|2, 0\rangle$, and $|0, 2\rangle$ in the deformed basis, defined by $|k_1, k_2\rangle = \mathcal{N}_{k_1, k_2} (K_{20})^{k_2} (K_{10})^{k_1} |0\rangle$, with normalization factor \mathcal{N}_{k_1, k_2} . With this, the CCSD equation can be solved easily. We can also carry out the CCD calculations by changing the mean field.

Figure 1(a) shows the CC energy differences $\Delta E = E - E_{\text{exact}}$ where this is relative to the exact ground-state energy E_{exact} as a function of χ for $N = 20$. In Fig. 1(b), as a measure of improvement, we plot the ratio between ΔE and $\Delta E_{\text{HF}} = E_{\text{HF}} - E_{\text{exact}}$ which is the correlation energy. It shows that the coupled-cluster wave function contains a considerable amount of more correlations. Moreover, the CC energy of CCD is lower than that of CCSD. As the coupled-cluster method does not, however, satisfy the Rayleigh-Ritz variational principle; we calculate the RR energies in Eq. (4) with the same CCSD and CCD wave functions. CCSD and CCD give almost the same RR energies. The CCD wave function may seem not to be so good. It shows, however, a distinct aspect if we apply the symmetry projection. In Fig. 1, projected ground-state energies with CCSD and CCD relative to the exact ones ΔE and $\Delta E/\Delta E_{\text{HF}}$ are plotted. The projected energy of the ground state is vastly improved over the CC energy, especially for the CCD, where about 95% of correlation energy is taken into account. Note that VAP calculations can also be straightforwardly carried out in this simple model, and the calculated ΔE and $\Delta E/\Delta E_{\text{HF}}$ are also plotted as a reference result. The VAP correlation energy with $\approx 0.01-0.6$

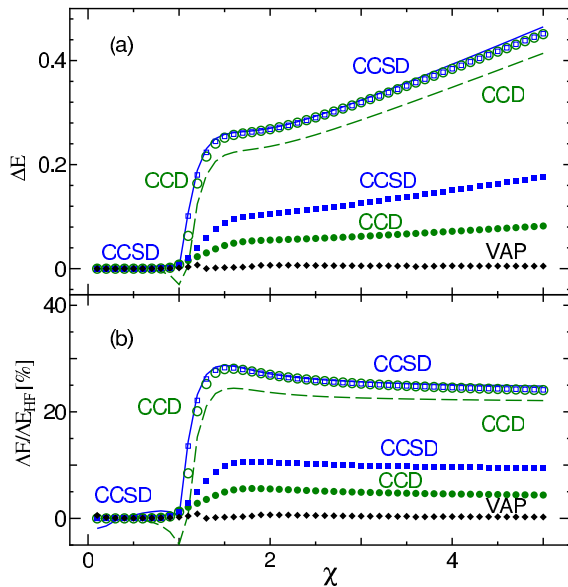


FIG. 1. Energy differences (a) ΔE and (b) ratios between ΔE and ΔE_{HF} are shown as a function of χ with $N = 20$. Blue (green dashed) line show CC energy with the CCSD (CCD) wave function. Blue open squares (green open dots) show RR energy with the CCSD (CCD) wave function. Blue filled squares (green filled dots) show projected energy with the CCSD (CCD) wave function. Black diamonds show VAP energy.

error is almost perfect. We will, however, not further dwell on the VAP procedure, since it seems to be numerically intractable for realistic cases. On the other hand, very recently the analog to the VAP calculation has been carried out for the pairing Hamiltonian in Ref. [15]. We also will come back to the VAP approach in a follow-up presentation in the near future.

In Fig. 2, the exact excitation energies for $L_0 = 1-3$ are plotted as a function of χ for $N = 20$. For $\chi \approx 0$, spectra show a vibrational feature, while for $\chi \approx 5$, the spectra show a deformed band. For $1 < \chi < 2$, a crossover between these

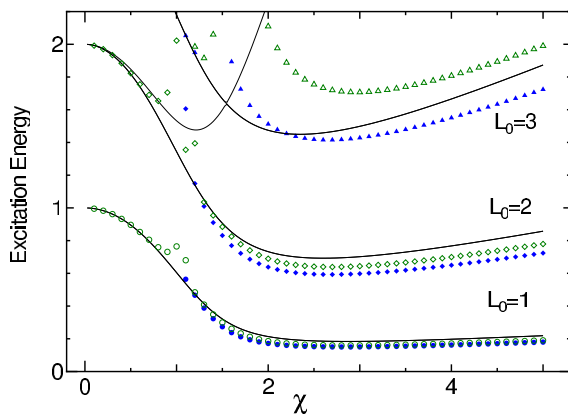


FIG. 2. Excitation energies are shown as a function of χ with $N = 20$. Black lines show exact energies ($L_0 = 1-3$). Blue filled (green open) symbols show the projected energies ($L_0 = 1-3$) with the CCSD (CCD) wave function.

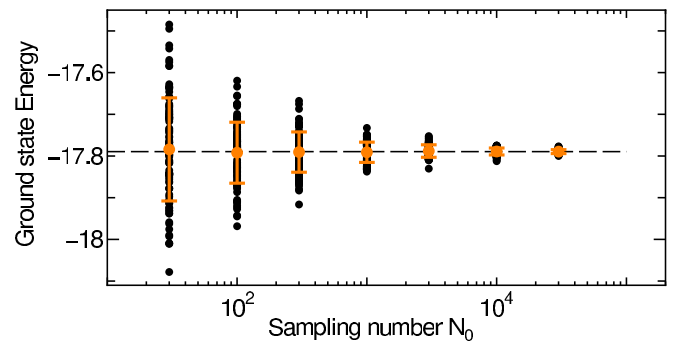


FIG. 3. MC energies with $L_0 = 0$ as a function of Monte Carlo sampling number N_0 . For each N_0 , 100 sets of MC energies with different random numbers are plotted. Orange-filled circle and error bars stand for average energy and its one-standard deviations, respectively.

two phases occurs. We plot the projected excitation energies of Eq. (6) with the CCSD and CCD wave functions, using the symmetry projection in Eq. (15) to show the results obtained with the PACC procedure. The results show reasonably good excitation energies except for the crossover region.

For $0 < \chi < 1$, CCD and CCSD wave functions are the same by construction and coupled-cluster method gives a pure spherical mean field. Thereby, in the projected calculations with CCD, we added a slight deformation to the pure spherical mean-field and succeeded in reproducing the excitation energies. Thus, the symmetry projection for the coupled-cluster wave function is quite promising. The remainder of the problem is how to calculate such a symmetry projection in realistic cases. Next, we introduce the Monte Carlo method and mixed representation for the projected overlap as in Eq. (13).

Tests of Monte Carlo procedure. The symmetry projection in Eq. (7) requires heavy computations in realistic applications, so we introduce the Monte Carlo method as described in Sec. II. The spherical basis s is sampled by the MCMC, which stochastically generates the distribution obeyed to $\rho_L(s)$ in Eq. (11). To perform the MCMC, we take the basis states specified by s as a random walker, and we move this s basis to another nearby s' basis under the control of the Metropolis-Hasting (MH) algorithm, keeping the detailed balance as in Refs. [27,28].

As a benchmark test of the MC calculation, we take the projected energy in Eq. (6) with the CCD wave function, whose value for $L_0 = 0$ is -17.789 . We investigate the same projected energy by the Monte Carlo method. As the MC calculations have statistical errors, we examine the convergence by taking several sampling numbers $N_0 = 30, 100, 300, 1000, 3000, 10\,000, 30\,000$. For each N_0 , we carry out 100 sets with different random numbers. In Fig. 3, the MC energy in Eq. (12) for each calculation is displayed. The average energy and its average variance over 100 sets are also plotted. Thus, the statistical errors of the Monte Carlo procedure are well controlled. The sampling number is, in general, relatively constant for various quantum systems with a larger Hilbert space.

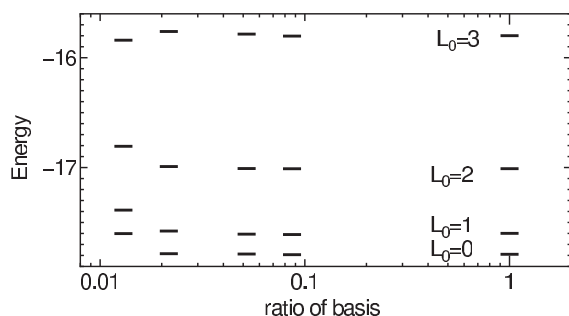


FIG. 4. MC energies with $L_0 = 0-3$ as a function of basis truncation ratio. At ratio 1, exact projected energies are also plotted. The MC statistical errors are invisible within the bars.

Finally, we discuss the remaining problem: the computation of the projected overlaps $\langle s|P^{L_0}|\psi\rangle$. For the MC sampling in Eq. (10), the spherical basis is better than the deformed one due to the advantage of the sparsity in $h_{s,s'}$. However, the projected overlap includes the exponential-type operators, of which exact projections become intractable for larger systems in general. Therefore we decompose the computation of the projected overlap as in Eq. (13), which yields the following natural truncation scheme of the deformed complete set of states. As the $|d\rangle$ states comprise the ground state $|0\rangle$ and all excited states thereof, the overlaps $\langle s|P^{L_0}|\psi\rangle$ for highly excited states $|d\rangle$ are expected to be negligible, which then can be truncated by the limiting value of $\langle d|\psi\rangle$. As the CCD wave function $|\psi\rangle$ has no dependence of θ in Eq. (15) and is expressed by the deformed operators, the numerical evaluation of the values of $\langle d|\psi\rangle$ is simple. Moreover, contributions of higher excited states $|d\rangle$ are, in general, expected to be smaller. Therefore, the truncation of d by $\langle d|\psi\rangle$ is feasible.

In Fig. 4, we present a benchmark test of such a truncation scheme for the states with $L_0 = 0-3$. By setting a given threshold for the value $\langle d|\psi\rangle$, we can truncate the summation for d in Eq. (13). The MC energies in Eq. (12) with these projected states are plotted as a function of the ratio of the truncated basis number to the one of the whole Hilbert-space dimension. The parameters for the three-level Lipkin model are $\chi = 5$ and $N = 20$. The calculation at the ratio of 0.02 shows that the truncation scheme works quite well. This ratio

is expected to be increasingly smaller for larger systems. Its investigation for various quantum systems will, however, be our task for the future.

IV. SUMMARY

This study combines the Rayleigh-Ritz variational procedure, the coupled-cluster wave function, and symmetry projection through the Monte Carlo method. We start from the mean-field wave function and extend it by applying an exponential-type correlation factor e^Z as within the coupled-cluster theory. We use the CCSD wave function, and its parameters are optimized by the coupled-cluster method as shown in Sec. II. Next, we apply the symmetry projection to this wave function and we directly evaluate the expectation value of the Hamiltonian with this projected wave function as in Eq. (6). Such an approach is, however, numerically intractable for practical applications. Therefore, we introduce the Monte Carlo method as in Eqs. (9)–(12). We also introduce an efficient truncation scheme for the deformed set of states as in Eq. (14).

To evaluate the feasibility of this new method, we employ the three-level Lipkin model, which has spherical and deformed phases. Moreover, it has an exchange symmetry under degenerate single-particle energies of the upper two levels. Therefore we can introduce the symmetry projection concerning this symmetry in Eq. (15). By numerical calculations, we found that the symmetry projected CCD wave function can give a better ground-state energy than the coupled-cluster method and gives reasonable spectra for low-lying states. We showed that the Monte Carlo method and the truncation scheme we introduced also work quite well.

This method can be applied to many quantum systems. For instance, the shell-model calculation is under investigation with promising preliminary results. The other direction of this method is to extend it to variation after projection, for which the Monte Carlo realization is also under study.

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