Exact expressions for the number of levels in single-j orbits for three, four, and five fermions

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We propose closed-form expressions of the distributions of magnetic quantum number M and total angular momentum J for three and four fermions in single-j orbits. The latter formulas consist of polynomials with coefficients satisfying congruence properties. Such results, derived using doubly recursive relations over j and the number of fermions, enable us to deduce explicit expressions for the total number of levels in the case of three-, four-, and five-fermion systems. We present applications of these formulas, such as sum rules for six-j and nine-j symbols, obtained from the connection with fractional-parentage coefficients, an alternative proof of the Ginocchio-Haxton relation, or cancellation properties of the number of levels with a given angular momentum.

I. INTRODUCTION

Determining the allowed total angular momenta J to which the individual half-integer spins j of N identical particles may couple is of primary importance in nuclear physics. Some values of J are forbidden by the Pauli exclusion principle; others occur more than once. Although that problem was investigated by many authors over the years, and despite the variety of approaches (number theory, recurrence relations, generating functions, etc.), exact analytical expressions for the number of states $P(M)$ with a given projection M on the quantization axis, the number of levels $Q(J)$ with spin J, or the total number of levels Q_{tot} in a configuration are not known, except in very simple cases.

Zhao and Arima have shown that there are simple structures in $Q(J)$ for j^3 or j^4, and found empirical formulas [1]. In 2005, the same authors [2] showed that $Q(J)$ could be enumerated by the reduction from SU($N+1$) to SO(3) and obtained analytical expressions of $Q(J)$ for four particles. The same year, Talmi derived a recursion formula for $Q(J)$ [3]. The latter quantity for j^N is expressed in terms of $Q(J)$ for $(j-1)^N$, $(j-1)^{N-1}$, and $(j-1)^{N-2}$. In the same work, Talmi also proved some interesting results found empirically by Zhao and Arima [1]. Zhang *et al.* extended Talmi's recursion relation to boson systems and proved empirical formulas for five bosons. They also obtained the number of states with given spin for three and four bosons by using sum rules of six-j and nine-j symbols [4,5]. Five years later, Jiang *et al.* derived the analytical formulas for $Q(J)$ for three fermions

in a single-j shell and three bosons with spin ℓ, by using a reduction rule from the U(4) to the O(3) group chain, U(4) $\supset$ Sp(4) $\supset$ O(3) [6], for $\tilde{N}$ virtual bosons which follow the U(4) symmetry (i.e., spin 3/2) [7]. One has $\tilde{N} = 2j - 2$ for fermions and $\tilde{N} = 2\ell$ for bosons. The authors were able to obtain analytical formulas of three bosons and fermions in a unified form and on a unified footing. Let us consider a system of N identical fermions in a single j (which is half-integer) shell of degeneracy $g = 2j + 1$, m_i being the angular momentum projection of electron state i ($m_1 = -j, m_2 = -j + 1, m_3 = -j + 2, \ldots, m_{g-1} = j - 1, m_g = j$). The maximum total angular momentum is

$$J_{\max} = (2j + 1 - N)N/2 \qquad (1.1)$$

and the minimum angular momentum $J_{\min}$ is 0 if N is even and 1/2 if N is odd. The distribution $P(M)$ represents the number of N-fermion states having the total projection (or magnetic quantum number) M. The number $Q(J)$ of levels with angular momentum J in a configuration can be obtained from the distribution $P(M)$ of the M values by means of the relations [8]

$$Q(J) = P(J) - P(J+1) \quad \text{if } J \leqslant J_{\max} - 1, \quad (1.2a)$$
$$Q(J_{\max}) = P(J_{\max}). \qquad (1.2b)$$

In the following we use the notation $P(M; j, N)$ instead of $P(M)$ every time it is necessary to specify the angular momentum of the shell and the number of fermions.

The fundamental relation used in the present paper to get the number of states $P(M; j, N)$ of N fermions with spin j and total magnetic quantum number M has been derived by Talmi

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(Eq. (1) in Ref. [3]):

$$P(M; j, N) = P(M; j - 1, N) + P(M - j; j - 1, N - 1)$$
$$+ P(M+j; j-1, N-1) + P(M; j-1, N-2). \quad (1.3)$$

A short alternative derivation is presented in Appendix A. From the above relation (1.2a), one also gets easily the total number of levels:

$$\sum_{J=J_{\min}}^{J_{\max}} Q(J; j, N) = P(J_{\min}; j, N), \quad (1.4)$$

where $J_{\min} = 0$ (resp. 1/2) for N even (resp. odd). A simple expression for the total number of levels for j^3 was found using coefficients of fractional parentage [9]. In the case of four fermions, no explicit formula could be obtained with the latter technique, only a triple summation involving nine-j coefficients, or equivalently products of two six-j symbols multiplied by Dunlap-Judd coefficients [10].

In the present work, using the recurrence relation (1.3), we derive explicit expressions for $P(M; j, 3)$, $Q(J; j, 3)$ (Sec. II), $P(M; j, 4)$, and $Q(J; j, 4)$ (Sec. III), as well as for the total number of J levels in the case of five fermions (Sec. IV). This leads us to deduce exact formulas for $Q_{\text{tot}}(j^3)$ (i.e., an alternative derivation much simpler than the one previously published and relying on the use of fractional parentage coefficients [9]), for $Q_{\text{tot}}(j^4)$ and for $Q_{\text{tot}}(j^5)$. To our knowledge, no expressions of the two latter formulas were published elsewhere. The algebraic forms of $Q(J; j, 3)$ and $Q(J; j, 4)$ are also likely to yield to sum rules for six-j symbols (Sec. V). We also provide some additional results, such as an alternative derivation of the Ginocchio-Haxton relation (Sec. V), cancellation properties, and particular values of the number of levels with a given angular momentum (Sec. VI).

II. THREE-FERMION SYSTEMS

A. Total number of levels

The total number of levels will be derived from Eq. (1.4). For three particles, the relation (1.3) is written as

$$P\left(\frac{1}{2}; j, 3\right) = P\left(\frac{1}{2}; j - 1; 3\right) + P\left(\frac{1}{2} - j; j - 1, 2\right)$$
$$+ P\left(\frac{1}{2} + j; j - 1, 2\right) + P\left(\frac{1}{2}; j - 1, 1\right). \quad (2.1)$$

This provides us with a recurrence relation on j for $P(1/2; j, 3)$, which is initialized by the value $P(1/2; 3/2, 3)$. Using the relation easily obtained by considering the coupling of two momenta,

$$P(M; j, 2) = \left\lfloor \frac{2j + 1 - |M|}{2} \right\rfloor, \quad (2.2)$$

where $\lfloor x \rfloor$ is the integer part of x, we get immediately, for j half-integer,

$$P\left(\frac{1}{2} - j; j - 1, 2\right) = \left\lfloor \frac{j}{2} - \frac{1}{4} \right\rfloor, \quad (2.3a)$$

$$P\left(\frac{1}{2} + j; j - 1, 2\right) = \left\lfloor \frac{j}{2} - \frac{3}{4} \right\rfloor, \quad (2.3b)$$

and a rapid inspection of the cases $j = 2n + 1/2$, $j = 2n + 3/2$ shows that, since $P(1/2; j - 1, 1) = 1$ for $j \geqslant 3/2$, one has

$$P\left(\frac{1}{2}; j, 3\right) = P\left(\frac{1}{2}; j - 1, 3\right) + j - \frac{1}{2} \quad (2.4)$$

for $j \geqslant 3/2$. Since the coupling of three angular momenta $j = 1/2$ is not possible (Pauli exclusion principle), we have $P(1/2; 1/2, 3) = 0$ and therefore

$$Q_{\text{tot}}(j^3) = P\left(\frac{1}{2}; j, 3\right) = \sum_{i=1/2}^{j} (i - 1/2)$$

$$= \sum_{t=0}^{j-1/2} t = \frac{1}{2}\left(j^2 - \frac{1}{4}\right) \quad (2.5)$$

in agreement with the formula (36) of Ref. [9].

B. Determination of the M distribution for three fermions

1. Case M greater than j

We first determine $P(j + q; j, 3)$ with q positive integer ($q = 1, 2, \ldots, J_{\max} - j$). Using Talmi's formula and the explicit value (2.2) one gets, after p iterations,

$$P(j + q; j, 3) = P(j + q; j - 1, 3) + P(q; j - 1, 2) \quad (2.6a)$$

$$= P(j + q; j - 1, 3) + \left\lfloor j - \frac{q+1}{2} \right\rfloor \quad (2.6b)$$

$$= P(j + q; j - 2, 3) + \left\lfloor j - \frac{q+1}{2} - \frac{3}{2} \right\rfloor$$
$$+ \left\lfloor j - \frac{q+1}{2} \right\rfloor \quad (2.6c)$$

$$\vdots$$

$$= P(j + q; j - p, 3) + \sum_{t=0}^{p-1} \left\lfloor j - \frac{q+1}{2} - \frac{3t}{2} \right\rfloor, \quad (2.6d)$$

where we have used the property $P(2j + q - t; j - t, 2) = 0$ and $P(j + q - t; j - t, 1) = 0$, valid for $q > 0$ and $0 \leqslant t \leqslant p - 1$. We choose p such that $P(j + q; j - p, 3)$ vanishes while $P(j + q; j - p + 1, 3)$ does not. This yields the conditions

$$1 \leqslant j - \frac{q}{2} - \frac{3p}{2} + 1, \quad j - \frac{q}{2} - \frac{3p}{2} - \frac{1}{2} < 1, \quad (2.7)$$

which amount to

$$p = \left\lfloor \frac{2j - q}{3} \right\rfloor. \quad (2.8)$$

Since q can be even or odd, for j half-integer, $j - q/2$ is either integer or half-integer. When evaluating $\lfloor(2j - q)/3\rfloor$ six cases must be considered. One obtains for the value of the maximum index p

$$p = \begin{cases} 2n & \text{if } j - \frac{q}{2} = 3n, \, 3n + 1/2, \text{ or } 3n + 1 \\ 2n + 1 & \text{if } j - \frac{q}{2} = 3n + 3/2, \, 3n+2, \text{ or } 3n + 5/2. \end{cases} \quad (2.9)$$

TABLE I. Various cases for the computation of $P(j + q; j, 3)$.

$j - q/2$	$3n$	$3n + 1/2$	$3n + 1$	$3n + 3/2$	$3n + 2$	$3n + 5/2$
First term	$\lfloor 1 \rfloor$	$\lfloor 3/2 \rfloor$	$\lfloor 2 \rfloor$	$\lfloor 1 \rfloor$	$\lfloor 3/2 \rfloor$	$\lfloor 2 \rfloor$
Last term	$\lfloor 3n - 1/2 \rfloor$	$\lfloor 3n \rfloor$	$\lfloor 3n + 1/2 \rfloor$	$\lfloor 3n + 1 \rfloor$	$\lfloor 3n + 3/2 \rfloor$	$\lfloor 3n + 2 \rfloor$
Sum	$3n^2$	$n(3n + 1)$	$n(3n + 2)$	$3n(n + 1) + 1$	$(n + 1)(3n + 1)$	$(n + 1)(3n + 2)$

In the computation of the sum (2.6d) with that value of p, we note that $P(j + q; j - p, 3)$ vanishes because of the conditions (2.7). We distinguish six cases, according to the maximum index (2.9). For instance if $j - q/2 = 3n$ the sum is, after reordering odd and even t values,

$$P(j + q; j, 3) = \lfloor 1 \rfloor + \lfloor 5/2 \rfloor + \lfloor 4 \rfloor + \cdots + \lfloor 3n - 2 \rfloor + \lfloor 3n - 1/2 \rfloor \tag{2.10a}$$

$$= 1 + 4 + \cdots + (3n - 2) + 2 + 5 + \cdots + (3n - 1) \tag{2.10b}$$

$$= \sum_{t=1}^{n}(3t - 2) + \sum_{t=1}^{n}(3t - 1) = 3n^2 = \frac{1}{3}\left(j - \frac{q}{2}\right)^2. \tag{2.10c}$$

The six cases are summed up in Table I. Expressing n versus $j - q/2$, we obtain the desired formula

$$P(j + q; j, 3) = \frac{1}{3}\left(j - \frac{q}{2}\right)^2 + \alpha(2j - q) \tag{2.11a}$$

with $\alpha(2j - q) = \left(0, -\frac{1}{12}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{3}, -\frac{1}{12}\right)$ if $2j - q \bmod 6 = (0, 1, 2, 3, 4, 5)$ respectively. $\tag{2.11b}$

For instance one can check for $q = 1$

$$P(j + 1; j, 3) = \begin{cases} \frac{1}{3}\left(j - \frac{1}{2}\right)^2 & \text{if } j - 1/2 = 3n, \\ \frac{1}{3}\left(j - \frac{3}{2}\right)\left(j + \frac{1}{2}\right) = \frac{1}{3}\left(j - \frac{1}{2}\right)^2 - \frac{1}{3} & \text{if } j - 1/2 = 3n + 1 \text{ or } j - 1/2 = 3n + 2. \end{cases} \tag{2.12}$$

The formula (2.11a) does not assume that j is half-integer. Instead of (2.12), we would have, for integer j,

$$P(j + 1; j, 3) = \begin{cases} \frac{1}{3}\left(j - \frac{1}{2}\right)^2 - \frac{1}{12} = \frac{1}{3}(j - 1)j & \text{if } j = 3n + 1 \text{ or if } j = 3n, \\ \frac{1}{3}\left(j - \frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{3}(j^2 - j + 1) & \text{if } j = 3n + 2. \end{cases} \tag{2.13}$$

2. Case M less than or equal to j

In this section one assumes j half-integer such that $j \geqslant 3/2$. From the basic relation (1.3), one writes

$$P(j - q; j, 3) = P(j - q; j - 1, 3) + S(j, q) \tag{2.14a}$$

$$\text{where } S(j, q) = P(2j - q; j - 1, 2) + P(q; j - 1, 2) + P(j - q; j - 1, 1). \tag{2.14b}$$

The quantity $S(j, q)$ is easily transformed using the value (2.2) and the fact that $P(j - q; j - 1, 1) = 1$ if $q > 0$. Using this definition one easily checks that the terms in $S(j, 0)$ take the values $0, j - 1/2, 0$ respectively, so that $S(j, 0) = j - 1/2$. If $q > 0$ the identity (2.2) provides the result

$$S(j, q) = \lfloor (q - 1)/2 \rfloor + \lfloor j - (q + 1)/2 \rfloor + 1 \tag{2.15}$$

and considering the cases q even or odd one easily verifies that, for $j \geqslant 3/2$,

$$S(j, q) = j - \frac{1}{2}, \tag{2.16}$$

which is also valid if $q = 0$. The formula (2.14) leads to a recurrence relation,

$$P(j - q; j, 3) = P(j - q; j - 1, 3) + j - \frac{1}{2} \tag{2.17a}$$

$$= P(j - q; j - 2, 3) + j - \frac{3}{2} + j - \frac{1}{2} \tag{2.17b}$$

$$\vdots$$

$$= P(j - q; j - q - 1, 3) + \sum_{s=0}^{q}\left(j - \frac{1}{2} - s\right) \tag{2.17c}$$

$$= P(j - q; j - q - 1, 3) + (q + 1)\left(j - \frac{q + 1}{2}\right). \tag{2.17d}$$

The initial value $P(j - q; j - q - 1, 3)$ is derived from the expression (2.12). One finds for $0 \leqslant q \leqslant j - 1/2$

$$P(j - q; j, 3) = (q + 1)\left(j - \frac{q+1}{2}\right) + \begin{cases} \frac{1}{3}\left(j - q - \frac{3}{2}\right)^2 & \text{if } j - q = 3n + \frac{3}{2}, \\ \frac{1}{3}\left(j - q - \frac{5}{2}\right)\left(j - q - \frac{1}{2}\right) & \text{if } j - q = 3n + \frac{1}{2} \text{ or } 3n + \frac{5}{2}, \end{cases} \tag{2.18}$$

or after simplification

$$P(j - q; j, 3) = \frac{1}{3}\left(j + \frac{q}{2}\right)^2 - \frac{q^2}{4} + \beta(j - q - 1/2) \tag{2.19a}$$

$$\text{with } \beta(n) = \left(-\frac{1}{12}, \frac{1}{4}, -\frac{1}{12}\right) \text{ if } n \bmod 3 = (0, 1, 2) \text{ respectively.} \tag{2.19b}$$

For instance, one obtains in the $q = 0$ case

$$P(j; j, 3) = \frac{j^2}{3} \begin{cases} +\frac{1}{4} & \text{if } j = 3n + 3/2, \\ -\frac{1}{12} = \frac{1}{3}\left(j^2 - \frac{1}{4}\right) & \text{if } j = 3n + 1/2 \text{ or } 3n + 5/2. \end{cases} \tag{2.20}$$

Such formulas can be generalized for j integer but the resulting expressions will be different. The formula (2.18) was established for $q \geqslant 0$. One can check that it remains true for $q = -1$. Assuming (2.18) is valid for $q = -1$, we get a piecewise expression which is identical to (2.12). It is worth mentioning that the relation (2.18) applies in particular for $j - q = 1/2, 3/2, \ldots, n + 1/2$. A series of examples is provided in Appendix B. Finally if $j - q = n + 1/2$ with $0 \leqslant n \leqslant j - 1/2$ one has

$$P(n + 1/2; j, 3) = P(n + 1/2; n - 1/2, 3) + \frac{1}{2}\left[j^2 - \left(n - \frac{1}{2}\right)^2\right]. \tag{2.21}$$

3. General case

The formulas (2.11a), (2.19a) can be gathered in a single equation, valid for any integer q. Using the Heaviside function $H(q) = 1$ if $q \geqslant 0$, 0 otherwise, one has

$$P(j - q; j, 3) = \frac{1}{3}\left(j + \frac{q}{2}\right)^2 - H(q)\frac{q^2}{4} + H(q)\beta(j - q - 1/2)$$
$$+ [1 - H(q)]\alpha(2j + q). \tag{2.22}$$

Considering the various values of $2j + q \bmod 6$ and $q \bmod 2$, one can then easily check that $\gamma = \alpha(2j + q) - \beta(j - q - 1/2)$ is indeed a function of q and is equal to $-(q \bmod 2)/4$. The above equation transforms into

$$P(j - q; j, 3) = \frac{1}{3}\left(j + \frac{q}{2}\right)^2 + \alpha(2j + q)$$
$$- H(q)\left[\frac{q^2}{4} + \gamma(q)\right] \tag{2.23a}$$

$$\text{with } \gamma(q) = \left(0, -\frac{1}{4}\right) \text{for } q \bmod 2 = (0, 1) \tag{2.23b}$$

if $-2j + 3 \leqslant q \leqslant j - 1/2$, and α defined above (2.11a).

C. Distribution of the total angular momentum

Using the fundamental relation (1.2a), the expression (2.11a) allows us to derive the distribution of the total momentum J. The evaluation of $P(j + q; j, 3) - P(j + q + 1; j, 3)$

provides

$$Q(j + q; j, 3) = \frac{2j - q}{6} + q_{3p}$$

$$\text{with } q_{3p} = \left(0, -\frac{1}{6}, -\frac{1}{3}, \frac{1}{2}, -\frac{2}{3}, \frac{1}{6}\right) \text{ for } 2j - q \bmod 6$$

$$= (0, 1, 2, 3, 4, 5) \text{ respectively.} \tag{2.24}$$

For instance one has $Q(j; j, 3) = j/3 + (-1/6, 1/2, 1/6)$ if $j - 1/2 \bmod 3 = 0, 1, 2$ respectively, i.e., $Q(j;, j, 3) = \lfloor(2j + 1)/6\rfloor$. One also verifies that $Q(j + 1; j, 3) = (2j - 1)/6 + (0, -1/3, -2/3)$ for $j - 1/2 \bmod 3 = 0, 1, 2$ respectively, i.e., $Q(j + 1, j, 3) = \lfloor(2j - 1)/6\rfloor$. Similarly, from (2.18), the evaluation of $P(j - q; j, 3) - P(j - q + 1; j, 3)$ provides the following expression:

$$Q(j - q; j, 3) = \frac{j - q}{3} + q_{3m}$$

$$\text{with } q_{3m} = \left(-\frac{1}{6}, \frac{1}{2}, \frac{1}{6}\right)$$

$$\text{for } j - q \bmod 3 = (1/2, 3/2, 5/2) \text{ respectively.} \tag{2.25}$$

The expression (2.18) for $P(j - q; j, 3)$ remains valid for $q = -1$, therefore the above expression applies if $q = 0$. One verifies easily that (2.24), (2.25) are both correct for $q = 0$. One may also use the general expression (2.23a). When computing the difference $P(j - q; j, 3) - P(j - q + 1; j, 3)$ some attention must be paid to the case $q = 0$ for which $H(q) = 1 \neq H(q - 1) = 0$. However, the term in the factor of $H(q - 1)$ is $(q - 1)^2/4 + \gamma(q - 1) = 0$ for $q = 0$. The evaluation of $Q(j - q; j, 3)$ is then straightforward, defining

$\bar{\alpha}(n) = \alpha(n) - \alpha(n-1)$, $\bar{\gamma}(n) = \gamma(n) - \gamma(n-1)$. One gets

$$Q(j-q; j, 3) = \frac{1}{6}\left(2j + q - \frac{1}{2}\right) + \bar{\alpha}(2j + q)$$
$$-H(q)\left[\frac{q}{2} - \frac{1}{4} + \bar{\gamma}(q)\right] \qquad (2.26a)$$

with $\bar{\alpha}(n) = \left(\frac{1}{12}, -\frac{1}{12}, -\frac{1}{4}, \frac{7}{12}, -\frac{7}{12}, \frac{1}{4}\right)$

for $n \bmod 6 = (0, 1, 2, 3, 4, 5)$ respectively, $\qquad (2.26b)$

$$\bar{\gamma}(q) = \left(\frac{1}{4}, -\frac{1}{4}\right) \text{ for } q \bmod 2 = (0, 1) \text{ respectively,}$$
$$(2.26c)$$

with the conditions $3 - 2j \leqslant q \leqslant j - 1/2$, since one must have $1/2 \leqslant j - q \leqslant 3j - 3$.

III. FOUR-FERMION SYSTEMS

A. Determination of $P(M; j, 4)$ if $M \geqslant 2j$

We first derive the expressions for $P(2j + p; j, 4)$, which are easier to obtain than the expressions for $P(2j - p; j, 4)$. One has for any natural integer p

$$P(2j + p; j, 4) = P(2j + p; j - 1, 4) + P(j + p; j - 1, 3)$$
$$+ P(3j+p; j - 1, 3) + P(2j + p; j - 1, 2)$$
$$(3.1a)$$
$$= P(2j + p; j - 1, 4) + P(j + p; j - 1, 3)$$
$$(3.1b)$$
$$= P(2j + p; j - 2, 4) + P(j + p; j-1, 3)$$
$$+ P(j + p + 1; j - 2, 3) \qquad (3.1c)$$
$$= \sum_{s=1}^{\sigma} P(j + p + s - 1; j - s, 3), \qquad (3.1d)$$

where we used the properties $3j + p > J_{\max}(j - 1, 3)$ and $2j + p > J_{\max}(j - 1, 2)$. The upper bound σ in (3.1d) is chosen so that $P(j + p + s - 1; j - s, 3)$ is nonzero if $s = \sigma$ and zero if $s = \sigma + 1$, implying that $P(2j + p; j - \sigma, 4) = 0$. Explicitly

$$j + p + \sigma - 1 \leqslant J_{\max}(j - \sigma, 3) = 3j - 3\sigma - 3$$
and $\quad j + p + (\sigma + 1) - 2 > J_{\max}(j - (\sigma + 1), 3)$
$$= 3j - 3(\sigma + 1) - 3, \qquad (3.2a)$$
$$\sigma = \left\lfloor \frac{2j - p - 2}{4} \right\rfloor. \qquad (3.2b)$$

In order that the above formulas be meaningful one must have $\sigma \geqslant 1$ or

$2j - p \geqslant 6 \quad$ from which $2j + p \leqslant 4j - 6 = J_{\max}(j, 4)$.
$$(3.3)$$

The sum (3.1d) will be calculated with formulas (2.11a). This lead us to define $d(s) = j' - q'/2$ with $j' = j - s$, $q' = p + 2s - 1$, or

$$d(s) = j - \frac{p}{2} - 2s + \frac{1}{2}. \qquad (3.4)$$

The following analysis will be done according to the value of $d(1) = j - p/2 - 3/2$. From (3.2b)

$$\sigma = \left\lfloor \frac{1}{2}d(1) + \frac{1}{4} \right\rfloor = \left\lfloor \frac{1}{2}\left(j - \frac{p}{2} - \frac{3}{2}\right) + \frac{1}{4} \right\rfloor. \qquad (3.5)$$

To describe the procedure used to get $P(2j + p; j, 4)$ let us consider the case $d(1) = j - p/2 - 3/2 = 3n$, where n is an integer. One has then $\sigma = \lfloor 3n/2 + 1/4 \rfloor$ so that one must split the cases n even and odd. If $n = 2\nu$ with ν integer, then $\sigma = 3\nu$. Writing $\theta(d)$ for the number on the right of the bracket in (2.11a), we have

$$P(2j + p; j, 4) = \sum_{s=1}^{\sigma} \frac{1}{3}\left(j - \frac{p}{2} - 2s + \frac{1}{2}\right)^2 + \sum_{s=1}^{\sigma} \theta(d(s)). \qquad (3.6a)$$

The quantity $d(s) = j - p/2 - 2s + 1/2$ is equal to $j - p/2 - 3/2 = 3n = 6\nu, 3n - 2, 3n - 4, \ldots, 2$ for $s = 1, \ldots, \sigma$. Since $\sigma = 3\nu$, there are ν elements in the sum such that $d \bmod 3 = 0$, and as many such that $d \bmod 3 = 1$ and $d \bmod 3 = 2$. The sum of $\theta(d)$ is, according to (2.11a), $(0 - 1/3 - 1/3)\nu = -2\nu/3$. The final result is

$$P(2j + p; j, 4) = \sum_{t=1}^{\sigma} \frac{4}{3}t^2 - \frac{2}{3}\nu \qquad (3.6b)$$
$$= \frac{2}{9}(3\nu)(3\nu + 1)(6\nu + 1) - \frac{2}{3}\nu$$
$$= 6\nu^2(2\nu + 1) \qquad (3.6c)$$
$$= \frac{1}{18}\left(j - \frac{p}{2} - \frac{3}{2}\right)^2\left(j - \frac{p}{2} + \frac{3}{2}\right) \text{ in the case } j$$
$$- \frac{p}{2} - \frac{3}{2} = 6\nu. \qquad (3.6d)$$

The procedure must be repeated in the cases $d(1) = 3n, 3n + 1/2, 3n + 1, 3n + 3/2, 3n + 2, 3n + 5/2$, with $n = 2\nu$, $n = 2\nu + 1$, where ν is integer. From the expression (2.11a), one notes that the sought number $P(M; j, 4)$ is a sum of $P(j + t; j, 3)$ that can be written as

$$P(2j + p; j, 4) = \frac{1}{3}\sum_{s=1}^{\sigma}[d(s)]^2 - \sum_{s=1}^{\sigma} \theta(d(s)) \qquad (3.7a)$$
$$= \frac{4}{3}\sum_{s=1}^{\sigma}\left(\frac{j}{2} - \frac{p-1}{4} - s\right)^2 - \sum_{s=1}^{\sigma} \theta(d(s)) \qquad (3.7b)$$
$$= \frac{4}{3}\sum_{t=0}^{\sigma-1}(t + b)^2 - \sum_{s=1}^{\sigma} \theta(d(s)) \qquad (3.7c)$$

where $b = j/2 - (p - 1)/4 - \sigma$ is the smallest value of the quantity $d(s)/2$ in the sum (3.7a). Therefore the computation of $P(2j + p; j, 4)$ amounts to obtaining the sum of the squares of numbers in arithmetical progression, which is easy to evaluate. This sum must be corrected by the term $\sum_s \theta(d(s))$.

TABLE II. Various cases for the computation of $P(2j + p; j, 4)$ in the case $p = 2q$ even, i.e., $j - p/2 + 1/2$ integer. The number of states $P(2j + p; j, 4)$ is given by $\frac{4}{3}\sum_{t=0}^{\sigma-1}(t + b)^2 + S$ with $S = \sum_s \theta(d(s))$. The number of terms in the sum defining $P(2j + p; j, 4)$ is $\sigma = \lfloor d(1)/2 + 1/4 \rfloor$. One has $e_2 = (2\nu + 1)(6\nu^2 + 6\nu + 1)$, $e_5 = (\nu + 1)(12\nu^2 + 24\nu + 11)$ for $j - p/2 - 3/2 = 6\nu + 2$, $6\nu + 5$ respectively.

$j - \frac{p}{2} - \frac{3}{2}$	6ν	$6\nu + 1$	$6\nu + 2$	$6\nu + 3$	$6\nu + 4$	$6\nu + 5$
σ	3ν	3ν	$3\nu + 1$	$3\nu + 1$	$3\nu + 2$	$3\nu + 2$
b	1	$3/2$	1	$3/2$	1	$3/2$
$\sum_s \theta(d(s))$	$-2\nu/3$	$-2\nu/3$	$-(2\nu + 1)/3$	$-2\nu/3$	$-2(\nu + 1)/3$	$-(2\nu + 1)/3$
$P(2j + p; j, 4)$	$6\nu^2(2\nu + 1)$	$3\nu(2\nu + 1)^2$	e_2	$3(\nu + 1)(2\nu + 1)^2$	$6(\nu + 1)^2(2\nu + 1)$	e_5

The parameters $\sigma, b, \sum_s \theta(d(s))$ corresponding to each case are described in the Tables II and III for p even and odd respectively. The last line of these tables provides the number of states as given by (3.7c). The ν-dependent values can be expressed back versus the physical quantities j, p. With the additional definition

$$X = j - \frac{p}{2} - \frac{1}{2} \tag{3.8}$$

the expressions for $P(2j + p; j, 4)$ will be even simpler. In the case $p = 2q$ even, from Table II results, a detailed inspection proves that the expression of $P(2j + 2q; j, 4)$ versus j, q is identical for each pair of adjacent columns. Namely, columns 6ν (resp. $6\nu + 1$, $6\nu + 2$) and $6\nu + 3$ (resp. $6\nu + 4$, $6\nu + 5$) provide the same result, so that the $P(2j + 2q; j, 4)$ value does not depend on $j - q - 1/2 \bmod 6$ but on $j - q - 1/2 \bmod 3$. Expressing ν versus j, q one has, using the definition (3.8),

$$P(2j + 2q; j, 4) = \begin{cases} \frac{1}{18}\left(j - q - \frac{3}{2}\right)^2\left(j - q + \frac{3}{2}\right) = \frac{X^3}{18} - \frac{X}{6} + \frac{1}{9} & \text{if } j - q + \frac{3}{2} = 3n, \\ \frac{1}{18}\left(j - q - \frac{5}{2}\right)\left(j - q + \frac{1}{2}\right)^2 = \frac{X^3}{18} - \frac{X}{6} - \frac{1}{9} & \text{if } j - q + \frac{3}{2} = 3n + 1, \\ \frac{1}{18}\left(j - q - \frac{1}{2}\right)\left[\left(j - q - \frac{1}{2}\right)^2 - 3\right] = \frac{X^3}{18} - \frac{X}{6} & \text{if } j - q + \frac{3}{2} = 3n + 2. \end{cases} \tag{3.9}$$

In the case of odd p one must also express $P(2j + 2q + 1; j, 4)$ versus j, q or more precisely versus $X = j - q - 1$. The six cases considered in Table III provide as many different expressions. As seen on Eq. (3.9), the final expressions are simpler as functions of X. One has

$$P(2j + 2q + 1; j, 4) = \frac{X^3}{18} - \frac{X}{24} + \psi\left(j - q - \frac{1}{2}\right),$$

where $\psi\left(j - q - \frac{1}{2}\right) = \left(-\frac{1}{72}, \frac{1}{72}, -\frac{1}{8}, \frac{17}{72}, -\frac{17}{72}, \frac{1}{8}\right)$ if $j - q - \frac{1}{2} \bmod 6 = (0, 1, 2, 3, 4, 5)$ respectively. $\tag{3.10}$

B. Determination of $P(M; j, 4)$ if $M < 2j$

From Talmi's equation one has, assuming q positive integer,

$$P(2j - q; j, 4) = P(2j - q; j - 1, 4) + P(3j - q; j - 1, 3) + P(j - q; j - 1, 3) + P(2j - q; j - 1, 2), \tag{3.11}$$

which suggests implementing a recurrence on q. Indeed the elements $P(3j - q; j - 1, 3)$, $P(j - q; j - 1, 3)$, $P(2j - q; j - 1, 4)$ are known. In addition

$$P(2j - q, j - 1, 4) = P(2(j - 1) - (q - 2), j - 1; 4) \tag{3.12}$$

TABLE III. Various cases for the computation of $P(2j + p; j, 4)$ in the case $p = 2q + 1$ odd, i.e., $j - p/2 + 1/2$ half-integer. One has $o_1 = 12\nu^3 + 15\nu^2 + 6\nu + 1$ for $j - p/2 - 3/2 = 6\nu + 3/2$ and $o_2 = 12\nu^3 + 21\nu^2 + 12\nu + 2$ for $j - p/2 - 3/2 = 6\nu + 5/2$. See Table II for details.

$j - \frac{p}{2} - \frac{3}{2}$	$6\nu + 1/2$	$6\nu + 3/2$	$6\nu + 5/2$	$6\nu + 7/2$	$6\nu + 9/2$	$6\nu + 11/2$
σ	3ν	$3\nu + 1$	$3\nu + 1$	$3\nu + 2$	$3\nu + 2$	$3\nu + 3$
b	$5/4$	$3/4$	$5/4$	$3/4$	$5/4$	$3/4$
$\sum_s \theta(d(s))$	$\nu/12$	$(\nu + 3)/12$	$(\nu - 1)/12$	$(\nu + 2)/12$	$(\nu + 2)/12$	$(\nu + 1)/12$
$P(2j + p; j, 4)$	$\nu(12\nu^2 + 9\nu + 2)$	o_1	o_2	$(\nu + 1)(12\nu^2 + 15\nu + 5)$	$3(\nu + 1)^2(4\nu + 3)$	$3(\nu + 1)^2(4\nu + 5)$

shows that the expression for $P(2j + 1, j; 4)$ [resp. $P(2j, j; 4)$] obtained above—using (3.10), (3.9) respectively—allows us to get $P(2j - 1; j, 4)$ [resp. $P(2j - 2; j, 4)$]. This leads us to split the discussion according to the parity of q. We first define

$$F(j, q) = P(3j - q; j - 1, 3) + P(2j - q; j - 1, 2). \tag{3.13}$$

Using the expression (2.11a), it is easy to prove that, if $j \geqslant q + 1/2$,

$$F(j, 2q) = \begin{cases} q^2/3 & \text{if } q \bmod 3 = 0, \\ (q^2 - 1)/3 & \text{if } q \bmod 3 = 1, 2, \end{cases} \tag{3.14}$$

and, assuming again $j \geqslant q + 1/2$, that

$$F(j, 2q + 1) = \begin{cases} q(q + 1)/3 & \text{if } q \bmod 3 = 0, 2, \\ (q^2 + q + 1)/3 & \text{if } q \bmod 3 = 1. \end{cases} \tag{3.15}$$

1. Computation of $P(2j - 2p; j, 4)$

We first consider the case where q is even. The above formula for $F(j, 2p)$ provides us with the expression for $P(2j - 2; j, 4)$. Using (2.18) for $P_3 = P(j - q, j - 1, 3)$ and (3.9) for $P_1 = P(2j - 2; j - 1, 4)$ we get $P(2j - 2; j, 4) = P_1 + P_3$. Writing $x = j - 3/2$, one considers three cases according to $j \bmod 3$.

(i) If $j \bmod 3 = 1/2$, $P_1 = x^3/18 - x/6 - 1/9$, $P_3 = 2(j - 2) + \frac{1}{3}(j - 7/2)^2$, so that

$$P(2j - 2; j, 4) = \frac{1}{144}(8j^3 + 12j^2 - 16j + 5) = \frac{(j + 1/2)^3}{18} - \frac{j + 1/2}{6} + \frac{1}{9}. \tag{3.16a}$$

(ii) If $j \bmod 3 = 3/2$, $P_1 = x^3/18 - x/6$, $P_3 = 2(j - 2) + \frac{1}{3}(j - 9/2)(j - 5/2)$, whence

$$P(2j - 2; j, 4) = \frac{1}{144}(8j^3 + 12j^2 - 16j - 27) = \frac{(j + 1/2)^3}{18} - \frac{j + 1/2}{6} - \frac{1}{9}. \tag{3.16b}$$

(iii) If $j \bmod 3 = 5/2$, $P_1 = x^3/18 - x/6 + 1/9$, $P_3 = 2(j - 2) + \frac{1}{3}(j - 9/2)(j - 5/2)$, from which

$$P(2j - 2; j, 4) = \frac{1}{144}(8j^3 + 12j^2 - 16j - 11) = \frac{(j + 1/2)^3}{18} - \frac{j + 1/2}{6}. \tag{3.16c}$$

A series of similar computations for greater values of q has been performed and leads us to propose the formula

$$P(2j - 2q; j, 4) = \frac{(j + q - 1/2)^3}{18} - \frac{(j + q - 1/2)}{6} - f(q) + \theta(j + q), \tag{3.17}$$

which we will prove by recurrence on q. The initial computations show that $f(0) = 0$, $f(1) = 0$, and the general expression for $f(q)$ will be obtained below. The initial value $q = 0$ (3.16) requires that

$$\theta(j + q) = \begin{cases} -1/9 & \text{if } j + q + 1/2 \bmod 3 = 0, \\ 0 & \text{if } j + q + 1/2 \bmod 3 = 1 \\ +1/9 & \text{if } j + q + 1/2 \bmod 3 = 2. \end{cases} \tag{3.18}$$

Let us assume the recurrence (3.17) true up to $q = p$ (e.g., $p = 0$ or 1), and prove it for $q = p + 1$. With definition (3.14),

$$F(j, 2p + 2) = \tfrac{1}{3}(p + 1)^2 + \varepsilon_{p+1}, \quad \text{with } \varepsilon_{p+1} = -\tfrac{1}{3} \text{ if } p + 1 \bmod 3 = 1 \text{ or } 2, \text{ otherwise } 0, \tag{3.19}$$

we get from the fundamental relation (1.3)

$$P(2j - 2p - 2; j, 4) = P(2(j - 1) - 2p; j - 1, 4) + P(j - 2p - 2; j - 1, 3) + F(j, 2p + 2). \tag{3.20}$$

The recurrence hypothesis applies to the first term of (3.20):

$$P(2(j - 1) - 2p; j, 4) = \frac{(j + p - 3/2)^3}{18} - \frac{(j + p - 3/2)}{6} - f(p) + \theta(j - 1, p). \tag{3.21}$$

The second term of (3.20) is obtained from (2.18):

$$P(j - 2p - 2; j - 1, 3) = \frac{1}{3}(j + p - 1/2)^2 - (p + 1/2)^2 - \frac{1}{12} + \tau(j + p) \tag{3.22a}$$

$$\text{with } \tau(j + p) = \frac{1}{3} \text{ if } j - 2p - \frac{1}{2} \bmod 3 = j + p - \frac{1}{2} \bmod 3 = 0, \tag{3.22b}$$

$$\text{and } \tau(j + p) = 0 \text{ if } j - 2p - \frac{1}{2} \bmod 3 = j + p - \frac{1}{2} \bmod 3 = 1 \text{ or } 2. \tag{3.22c}$$

In order to verify the recurrence for $q = 2p + 2$, according to Eq. (3.20) one must verify for every j

$$\Delta - f(p+1) + \theta(j+p+1) = -f(p) + \theta(j-1+p) + \varepsilon_{p+1} + \tau(j+p), \tag{3.23}$$

where Δ contains the terms function of j, p except $f(p)$ and the quantities defined modulo 3:

$$\Delta = \frac{(j+p+1/2)^3}{18} - \frac{(j+p+1/2)}{6} - \frac{(j+p-3/2)^3}{18} + \frac{(j+p-3/2)}{6}$$
$$- \frac{1}{3}(j+p-1/2)^2 + (p+1/2)^2 + \frac{1}{12} - \frac{(p+1)^2}{3}. \tag{3.24}$$

After some basic algebraic manipulations one obtains

$$\Delta = \frac{2}{3}p^2 + \frac{p}{3} - \frac{2}{9}. \tag{3.25}$$

In addition, one may verify that $\delta = \theta(j+p+1) - \theta(j-1+p) - \tau(j+p)$ does not depend on j. Indeed

$$\delta = \begin{Bmatrix} 1/9 \\ -1/9 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -1/9 \\ 0 \\ 1/9 \end{Bmatrix} - \begin{Bmatrix} 1/3 \\ 0 \\ 0 \end{Bmatrix} \quad \text{if } j+p - \frac{1}{2} \mod 3 = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} \tag{3.26}$$

which leads to $\delta = -1/9$ in all cases. Equation (3.23) may be rewritten, using ε_{p+1} as given by (3.19):

$$f(p+1) - f(p) = \Delta + \delta - \varepsilon_{p+1} \tag{3.27a}$$

$$= \begin{cases} 2p^2/3 + p/3 - 1/3 + 1/3 & \text{if } p \bmod 3 = 0 \text{ or } 1, \\ 2p^2/3 + p/3 - 1/3 & \text{if } p \bmod 3 = 2, \end{cases} \tag{3.27b}$$

which is

$$f(p+1) - f(p) = \begin{cases} \frac{1}{3}p(2p+1) & \text{if } p \bmod 3 = 0 \text{ or } 1, \\ \frac{1}{3}(p+1)(2p-1) = \frac{1}{3}p(2p+1) - \frac{1}{3} & \text{if } p \bmod 3 = 2. \end{cases} \tag{3.28}$$

Since $f(p+1) - f(p)$ is indeed independent of j, the recurrence assumption (3.17) is verified. The proof is completed by the determination of $f(p)$. Applying Eq. (3.28) for p, $p+1$, $p+2$, we get, whatever $p \bmod 3$,

$$f(p+3) - f(p) = \frac{p}{3}(2p+1) + \frac{(p+1)}{3}(2p+3) + \frac{(p+2)}{3}(2p+5) - \frac{1}{3} = 2p^2 + 5p + 4. \tag{3.29}$$

From the known initial values $f(0) = f(1) = 0$ using (3.28) one gets $f(2) = 1$, and more generally $f(p)$:

$$f(3n + p_0) - f(p_0) = \sum_{t=0}^{n-1} [2(3t + p_0)^2 + 5(3t + p_0) + 4]; \tag{3.30}$$

and considering $p_0 = 0, 1, 2$ separately we obtain

$$f(p) = f_0(p) = \frac{m}{2}(12m^2 - 3m - 1) = \frac{2}{9}p^3 - \frac{p^2}{6} - \frac{p}{6} \qquad \text{if} \quad p = 3m, \tag{3.31a}$$

$$f(p) = f_1(p) = \frac{m}{2}(12m^2 + 9m + 1) = \frac{2}{9}p^3 - \frac{p^2}{6} - \frac{p}{6} + \frac{1}{9} \qquad \text{if} \quad p = 3m + 1, \tag{3.31b}$$

$$f(p) = f_2(p) = \frac{(m+1)}{2}(12m^2 + 9m + 2) = \frac{2}{9}p^3 - \frac{p^2}{6} - \frac{p}{6} + \frac{2}{9} \quad \text{if} \quad p = 3m + 2. \tag{3.31c}$$

A further generalization consists in verifying that the expression (3.17) may be applied *even for q negative* provided one cancels the $f(q)$ term. Indeed, comparing this expression to the known values (3.9), one notes that

$$P(2j - 2q; j, 4) = \frac{(j+q-1/2)^3}{18} - \frac{(j+q-1/2)}{6} + \theta(j+p) \quad \text{if } q < 0 \tag{3.32}$$

from which one gets, whatever the sign of the integer q,

$$P(2j - 2q; j, 4) = \frac{(j+q-1/2)^3}{18} - \frac{(j+q-1/2)}{6} - H(q)f(q) + \theta(j+q), \tag{3.33}$$

$H(q)$ being the Heaviside function, $H(q) = 1$ if $q \geqslant 0$, 0 otherwise.

2. Computation of $P(2j - 2p - 1; j, 4)$

As a first example, the computation of $P(2j - 1; j, 4)$ is detailed in Appendix C. In order to discover the general formula, we also got expressions for $P(2j - 3; j, 4)$ and $P(2j - 5; j, 4)$. An analysis on $P(M; j, 4)$ with $2j - M$ odd similar to the case $2j - M$ even leads us to propose the relation

$$P(2j - 2q - 1; j, 4) = \frac{(j+q)^3}{18} - \frac{(j+q)}{24} - g(q) + \phi(j+q), \tag{3.34}$$

which will be demonstrated by recurrence. The direct computation in the first two cases show that $g(0) = 0$ and $g(1) = 0$. From the analysis of Appendix C one imposes that, if $q \geqslant 0$,

$$\phi(j+q) = \left(\frac{1}{72}, -\frac{1}{8}, \frac{17}{72}, -\frac{17}{72}, \frac{1}{8}, -\frac{1}{72}\right) \text{ if } j + q - \frac{1}{2} \text{ mod } 6 = (0, 1, 2, 3, 4, 5) \text{ respectively.} \tag{3.35}$$

Assuming that (3.34) is true up to $q = p$, we now try to prove it for $q = p + 1$. Using the value of (3.15),

$$F(j, 2p + 3) = \frac{1}{3}(p + 1)(p + 2) + \mu_{p+1}, \text{ where } \mu_{p+1} = \frac{1}{3} \text{ if } (p + 1) \text{ mod } 3 = 1, \text{ otherwise } \mu_{p+1} = 0, \tag{3.36}$$

the fundamental relation (1.3) may be written

$$P(2j - 2p - 3; j, 4) = P(2(j - 1) - 2p - 1; j - 1, 4) + P(j - 2p - 3; j - 1, 3) + F(j, 2p + 3). \tag{3.37}$$

The recurrence hypothesis is again applied to the first term at the second member of (3.37):

$$P(2(j - 1) - 2p - 1; j, 4) = \frac{(j + p - 1)^3}{18} - \frac{(j + p - 1)}{24} - g(p) + \phi(j - 1, p). \tag{3.38}$$

The second term at the second member of (3.37) is given by (2.19a)

$$P(j - 2p - 3; j - 1, 3) = \frac{1}{3}(j - 1 + p + 1)^2 - (p + 1)^2 - \frac{1}{12} + \upsilon(j + p) \tag{3.39a}$$

$$\text{with } \upsilon(j + p) = \frac{1}{3} \text{ if } j - 2p - \frac{1}{2} \text{ mod } 3 = j + p - \frac{1}{2} \text{ mod } 3 = 1, \tag{3.39b}$$

$$\text{and } \upsilon(j + p) = 0 \text{ if } j - 2p - \frac{1}{2} \text{ mod } 3 = j + p - \frac{1}{2} \text{ mod } 3 = 0 \text{ or } 2. \tag{3.39c}$$

In order to verify the recurrence for $q = 2p + 3$, from (3.37) one must have for every j

$$\Delta' - g(p + 1) + \phi(j + p + 1) = -g(p) + \phi(j - 1 + p) + \mu_{p+1} + \upsilon(j + p), \tag{3.40}$$

where Δ' contains the terms function of j, p except $g(p)$ and the terms defined modulo 3 or modulo 6

$$\Delta' = \frac{(j + p + 1)^3}{18} - \frac{(j + p + 1)}{24} - \frac{(j + p - 1)^3}{18} + \frac{(j + p - 1)}{24} - \frac{1}{3}(j + p)^2 + (p + 1)^2 + \frac{1}{12} - \frac{1}{3}(p + 1)(p + 2). \tag{3.41}$$

After some algebra one gets

$$\Delta' = \frac{2}{3}p^2 + p + \frac{4}{9}. \tag{3.42}$$

In addition one can check that $\delta' = \phi(j + p + 1) - \phi(j - 1 + p) - \upsilon(j + p)$ is independent of j. Indeed

$$\delta' = \begin{cases} -1/8 \\ 17/72 \\ -17/72 \\ 1/8 \\ -1/72 \\ 1/72 \end{cases} - \begin{cases} -1/72 \\ 1/72 \\ -1/8 \\ 17/72 \\ -17/72 \\ 1/8 \end{cases} - \begin{cases} 0 \\ 1/3 \\ 0 \\ 0 \\ 1/3 \\ 0 \end{cases} \text{ if } j + p - \frac{1}{2} \text{ mod } 3 = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{cases} \tag{3.43}$$

therefore $\delta' = -1/9$ in all cases. Equation (3.40) may be rewritten as

$$g(p + 1) - g(p) = \Delta' + \delta' - \mu_{p+1} = \frac{2}{3}p^2 + p + \frac{1}{3} - \mu_{p+1}, \tag{3.44}$$

which is, using μ_{p+1} as given by (3.36),

$$g(p + 1) - g(p) = \begin{cases} \frac{1}{3}p(2p + 3) = \frac{1}{3}(p + 1)(2p + 1) - \frac{1}{3} & \text{if } p \text{ mod } 3 = 0, \\ \frac{1}{3}(p + 1)(2p + 1) & \text{if } p \text{ mod } 3 = 1 \text{ or } 2. \end{cases} \tag{3.45}$$

Since $g(p+1) - g(p)$ is indeed independent of j the recurrence relation (3.34) is proved and $g(p)$ may be computed. One may use (3.45) in order to get $g(1) = 0$, $g(2) = 2$ from $g(0) = 0$. One has then, whatever p mod 3,

$$g(p+3) - g(p) = \frac{(p+1)}{3}(2p+1) + \frac{(p+2)}{3}(2p+3) + \frac{(p+3)}{3}(2p+5) - \frac{1}{3} = 2p^2 + 7p + 7. \quad (3.46)$$

From the values of $g(p)$ for $p_0 = 0, 1, 2$ one obtains the general expression

$$g(3n + p_0) - g(p_0) = \sum_{t=0}^{n-1} [2(3t + p_0)^2 + 7(3t + p_0) + 7] \quad (3.47)$$

and, splitting cases $p_0 = 0, 1, 2$,

$$g(p) = g_0(p) = \frac{n}{2}(12n^2 + 3n - 1) = \frac{2}{9}p^3 + \frac{p^2}{6} - \frac{p}{6} \qquad \text{if} \quad p = 3n, \quad (3.48a)$$

$$g(p) = g_1(p) = \frac{n}{2}(12n^2 + 15n + 5) = \frac{2}{9}p^3 + \frac{p^2}{6} - \frac{p}{6} - \frac{2}{9} \qquad \text{if} \quad p = 3n + 1, \quad (3.48b)$$

$$g(p) = g_2(p) = \frac{(n+1)}{2}(12n^2 + 15n + 4) = \frac{2}{9}p^3 + \frac{p^2}{6} - \frac{p}{6} - \frac{1}{9} \qquad \text{if} \quad p = 3n + 2. \quad (3.48c)$$

As for even p, comparing expressions (3.10) and (3.34) for q negative or positive, one may write the general relation

$$P(2j - 2q - 1; j, 4) = \frac{(j+q)^3}{18} - \frac{(j+q)}{24} - H(q)g(q) + \phi(j+q), \quad (3.49)$$

where $H(q)$ is the Heaviside function, and $\phi(j + q)$ is given by (3.35).

3. General expression for the distribution of the magnetic quantum number

The formulas (3.17), (3.34) for $P(2j - n; j, 4)$ in the cases n even and odd can even be gathered in a single expression. One notices that the first term can be simply written as $[j + (n-1)/2]^3/18$, while the second is $-[j + (n-1)/2]/6 + \pi(n)[j + (n-1)/2]/8$, where $\pi(n)$ is 0 if n is even, 1 if n is odd. The third term of the quoted formulas may also be unified, noting that from the values (3.31a), (3.48) one has

$$g_2(x - 1/2) = f_1(x) - \frac{1}{8}, \quad (3.50)$$

which allows us to write $f(n/2)$ and $g[(n-1)/2]$ with a single formula, namely $f(n/2) = f_1(n/2) + \xi(n)$ with

$\xi(n) = (-1/9, 0, 1/9)$ if $n/2$ mod $3 = 0, 1, 2$ respectively, and $g((n-1)/2) = f_1(n/2) + \xi(n)$ with $\xi(n) = (-1/8 + 1/9, -1/8 - 1/9, -1/8)$ for $(n-1)/2$ mod $3 = 0, 1, 2$ respectively. Finally the term $\theta(j + n/2)$ and $\phi(j + (n-1)/2)$ in these formulas can be collected in a single expression, if one considers $2j + n$ mod 12 value. If n is even, from the expression (3.18) one may write this term as $\theta(j + n/2) = (0, 1/9, -1/9, 0, 1/9, -1/9)$ for $2j + n - 1$ mod $12 = (0, 2, 4, 6, 8, 10)$ respectively. If n is odd, from (3.35) this term is $\phi(j + q) = \phi(j + (n-1)/2) = (1/72, -1/8, 17/72, -17/72, 1/8, -1/72)$ for $2j + n - 1$ mod $12 = (1, 3, 5, 7, 9, 11)$ respectively. One obtains the single formula

$$P(2j - n; j, 4) = \frac{1}{18}\left(j + \frac{n-1}{2}\right)^3 - \left(\frac{1}{6} - \frac{\pi(n)}{8}\right)\left(j + \frac{n-1}{2}\right) - H(n)\left[f_1\left(\frac{n}{2}\right) + \xi(n)\right] + \omega(2j + n - 1) \quad (3.51a)$$

with $\pi(n) = n$ mod 2, $\quad \xi(n) = \left(-\frac{1}{9}, -\frac{1}{72}, 0, -\frac{17}{72}, \frac{1}{9}, -\frac{1}{8}\right)$ if n mod $6 = (0, 1, 2, 3, 4, 5)$ respectively, $\quad (3.51b)$

$$f_1(n) = \frac{2}{9}n^3 - \frac{n^2}{6} - \frac{n}{6} + \frac{1}{9}, \quad (3.51c)$$

$$\omega(2j + n - 1) = \left(0, \frac{1}{72}, \frac{1}{9}, -\frac{1}{8}, -\frac{1}{9}, \frac{17}{72}, 0, -\frac{17}{72}, \frac{1}{9}, \frac{1}{8}, -\frac{1}{9}, -\frac{1}{72}\right)$$

if $2j + n - 1$ mod $12 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$ respectively. $\quad (3.51d)$

In this formula $2j - n$ must be non-negative, n may be negative. Explicitly, n must be such that $-2j + 6 \leqslant n \leqslant 2j$.

Though this paper is not devoted to deriving approximations, one will observe that for $n \geqslant 0$ one has $|\omega(2j + n - 1) - \xi(n)| \leqslant 17/36$, and for $n < 0$ one has $|\omega(2j + n - 1)$
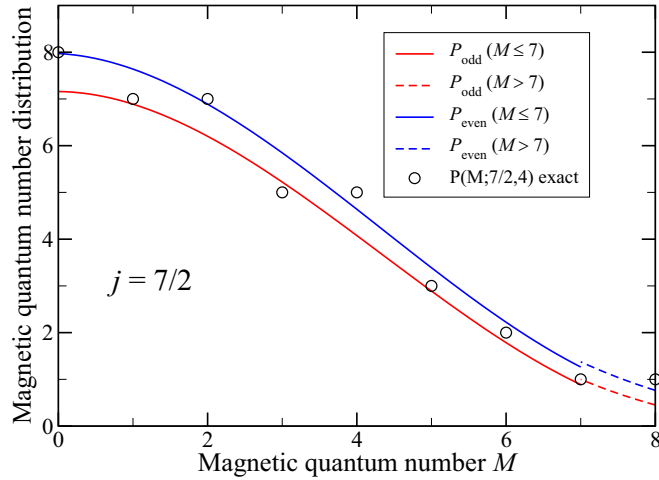
FIG. 1. Magnetic quantum number distribution $P(M; j, 4)$ for a 4-fermion system with spin $j = 7/2$. The red (resp. blue) curve is the approximation (3.52) with $\pi(n) = 0$ (resp. 1). The black circles are the exact values.

$1)| \leqslant 17/72$, so that the approximation

$$P(2j - n; j, 4) \simeq P_{\text{app}}(2j - n; j, 4)$$

$$= \frac{1}{18}\left(j + \frac{n-1}{2}\right)^3 - \left(\frac{1}{6} - \frac{\pi(n)}{8}\right)$$

$$\times \left(j + \frac{n-1}{2}\right) - H(n)f_1\left(\frac{n}{2}\right) \quad (3.52)$$

results in an absolute error below $1/2$. The relative error will be small if conditions $[j + (n-1)/2] \gg 1, n \gg 1$ are met. For really large j, even the $\pi(n)$ dependent term may be omitted, but the resulting approximation is not as good. This is illustrated by Figs. 1 and 2 for $j = 7/2$ and $15/2$ respectively. As can be seen in the approximate form above, both approximations $\pi(n) = 0$ and 1 exhibit a discontinuity of $1/9$ at $n = 0$ or $M = 2j$ since $f_1(0) = 1/9$. Though the above approximation is rough for $j = 7/2$ it proves to be fair for
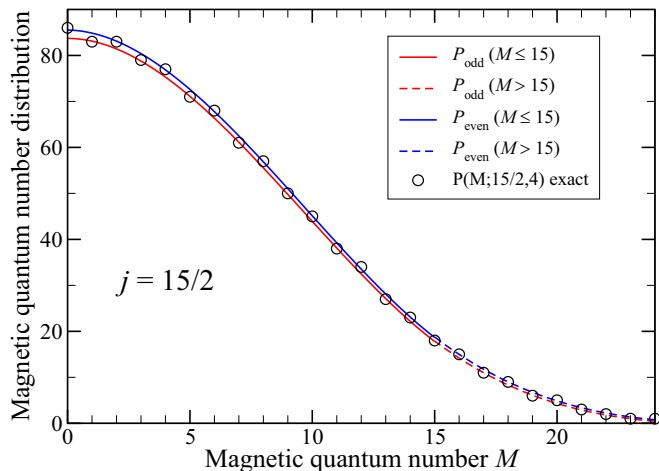

FIG. 2. Magnetic quantum number distribution $P(M; j, 4)$ for a four-fermion system with spin $j = 15/2$. See Fig. 1 for details.

higher j, and correctly reproduces the even-odd staggering, previously noticed in the atomic physics context [11,12].

C. Total number of levels

A direct application of the above derived expression for $P(M; j, 4)$ is the determination of the total number of levels. From the relation (1.2a), one verifies that the total number of levels for four fermions of spin j is given by $P(0; j, 4)$, which is easily obtained with (3.34). Writing $q = j - 1/2$ in this equation, one gets

$$P(0; j, 4) = \frac{(2j - 1/2)^3}{18} - \frac{(2j - 1/2)}{24}$$

$$- g(j - 1/2) + \phi(2j - 1/2). \quad (3.53)$$

One has to consider three cases according to $j - 1/2 \bmod 3$. If $j - 1/2 = 3n$, the first equation in the group (3.48) applies, and one has $\phi(2j - 1/2) = 1/72$. If $j - 1/2 = 3n + 1$, the second equation in the group (3.48) applies, and $\phi(2j - 1/2) = 17/72$. If $j - 1/2 = 3n + 1$, the third equation (3.48) is relevant, and $\phi(2j - 1/2) = 1/8$. One obtains the general formula

$$P(0; j, 4) = \frac{2}{9}j^3 - \frac{j^2}{6}$$

$$+ \frac{j}{6}\begin{cases} -5/72 & \text{if } j - 1/2 \bmod 3 = 0, \\ +3/8 & \text{if } j - 1/2 \bmod 3 = 1, \\ +11/72 & \text{if } j - 1/2 \bmod 3 = 2. \end{cases}$$

$$(3.54)$$

D. Distribution of the total angular momentum

Once again, the fundamental relation (1.2a), together with the expression (3.51) of the M distribution for a four-fermion system, allow us to derive the distribution of the total momentum J. One must evaluate $Q(2j - n; j, 4) = P(2j - n; j, 4) - P(2j - (n-1); j, 4)$ which we will write as $Q_1 + Q_2 + Q_3$. The quantity Q_1 consists in the contribution of first two terms of (3.51), which is easily obtained noticing that $\pi(n-1) = 1 - \pi(n)$,

$$Q_1 = \frac{X^3}{18} - \frac{X}{6} - \frac{(X - 1/2)^3}{18} + \frac{X - 1/2}{6} - \frac{j + n/2 - 1}{8}$$

$$+ \frac{\pi(n)}{8}(2j + n - 3/2) \quad \text{with } X = j + \frac{n-1}{2}$$

$$(3.55)$$

$$= \frac{1}{12}\left(j + \frac{n-3}{2}\right)^2 - \frac{7}{72} + \frac{\pi(n)}{8}(2j + n - 3/2). \quad (3.56)$$

The quantity Q_2 is the difference of terms involving the Heaviside factors $H(n)$ and $H(n-1)$. These factors are equal except in the case $n = 0$, which requires more attention: one must note that the factor of $H(n-1)$ for $n = 0$ is $f_1(-1/2) + \xi(-1)$, which is zero according to the values (3.51b), (3.51c).

Therefore one may write

$$Q_2 = -H(n)[f_1(n/2) + \xi(n) - f_1((n-1)/2) - \xi(n-1)]$$
(3.57a)

$$= -H(n)\left[\frac{n^2}{12} - \frac{n}{6} - \frac{1}{72} + \overline{\xi}(n)\right]$$
(3.57b)

with

$$\overline{\xi}(n) = \xi(n) - \xi(n-1)$$

$$= \left(\frac{1}{72}, \frac{7}{72}, \frac{1}{72}, -\frac{17}{72}, \frac{25}{72}, -\frac{17}{72}\right)$$

for $n \bmod 6 = 0, 1, 2, 3, 4, 5$ respectively. (3.57c)

Finally the ω-dependent term is simply

$$Q_3 = \overline{\omega}(2j + n - 1) = \omega(2j + n - 1) - \omega(2j + n - 2)$$
(3.58a)

$$= \left(\frac{1}{72}, \frac{1}{72}, \frac{7}{72}, -\frac{17}{72}, \frac{1}{72}, \frac{25}{72}, -\frac{17}{72},\right.$$

$$\left. -\frac{17}{72}, \frac{25}{72}, \frac{1}{72}, -\frac{17}{72}, \frac{7}{72}\right)$$
(3.58b)

for $2j + n - 1 \bmod 12 = 0 - 11$ respectively. The complete formula is

$$Q(2j - n; j, 4) = \frac{1}{12}\left(j + \frac{n-3}{2}\right)^2$$

$$- \frac{7}{72} + \frac{\pi(n)}{8}(2j + n - 3/2)$$

$$- H(n)\left[\frac{(n-1)^2}{12} - \frac{7}{72} + \overline{\xi}(n)\right]$$

$$+ \overline{\omega}(2j + n - 1).$$
(3.59)

Similarly to the M-distribution study, one observes that for $n \geqslant 0$ one has $|\overline{\omega}(2j + n - 1) - \overline{\xi}(n)| \leqslant 7/12$, while for $n < 0$ one has $|\overline{\omega}(2j + n - 1)| \leqslant 25/72$, so that the congruence-free approximation

$$Q(2j - n; j, 4) \simeq Q_{\mathrm{app}}(2j - n; j, 4) = \frac{1}{12}\left(j + \frac{n-3}{2}\right)^2$$

$$- \frac{7}{72} + \frac{\pi(n)}{8}(2j + n - 3/2)$$

$$- H(n)\left[\frac{(n-1)^2}{12} - \frac{7}{72}\right]$$
(3.60)

holds with an error less than unity. The approximation is tested in Figs. 3 and 4 for $j = 7/2$ and $15/2$ respectively. Since the main contribution to $Q(J; j, 4)$ scales as the squares j^2 or n^2 instead of cubes in the $P(M; j, 4)$ case, the above approximation is not as good as $P_{\mathrm{app}}(M; j, 4)$. Nevertheless the above formula is quite simple and efficient for moderate j values. As for the above $P(M; j, 4)$ analysis, one notices a significant even-odd staggering [11,12] which is correctly reproduced by the above formula. Finally one will note that the discontinuity on the approximate values at $n = 0$ is only
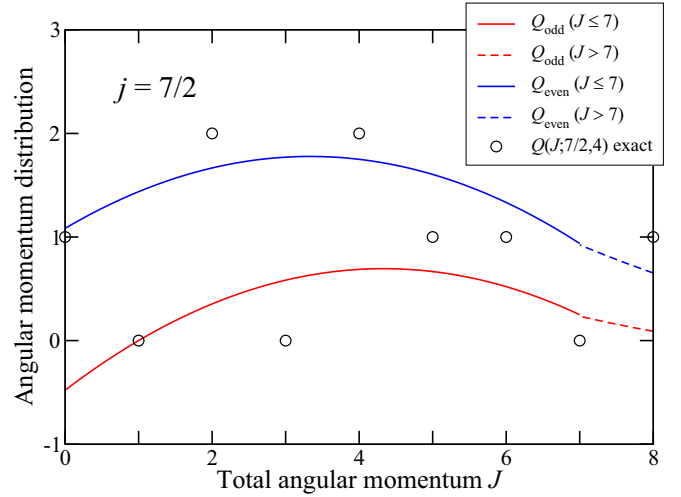

FIG. 3. Angular momentum distribution $Q(J; j, 4)$ for a four-fermion system with spin $j = 7/2$. The red (resp. blue) curve is the approximation (3.60) with $\pi(n) = 0$ (resp. 1). The black circles are the exact values.

1/72 so that the red and blue curves look almost continuous at $J = 2j$.

IV. TOTAL NUMBER OF LEVELS IN FIVE-FERMION SYSTEMS

The formula (3.51) allows us to get the total number of levels for a five-fermion system, which is equal to $P(1/2; j, 5)$. From (1.3), one may write, for s from 1 to $s = j - 3/2$,

$$P(1/2; j - s + 1, 5) = P(1/2; j - s, 5)$$

$$+ P(j - s + 3/2; j - s, 4)$$

$$+ P(j - s + 1/2; j - s, 4)$$
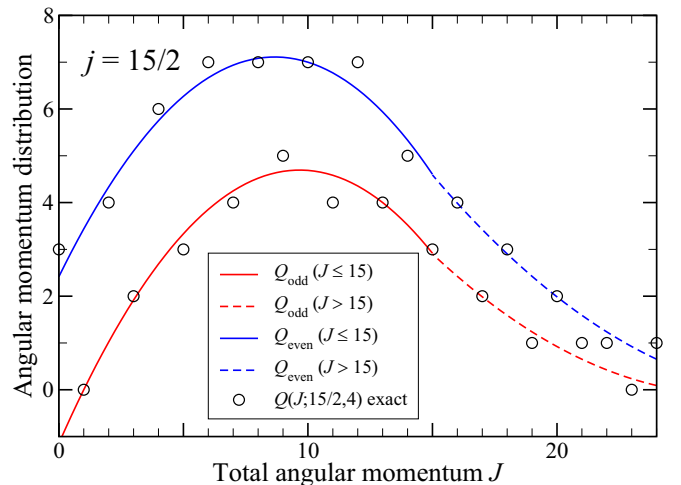
$$+ P(1/2; j - s, 3),$$
(4.1)

FIG. 4. Angular momentum distribution $Q(J; j, 4)$ for a four-fermion system with spin $j = 15/2$. See Fig. 3 for details.

which gives the total number of levels as a sum,

$$P(1/2; j, 5) = S_1 + S_2 + S_3 \tag{4.2a}$$

$$S_1 = \sum_{s=1}^{j-5/2} P(j - s + 3/2; j - s, 4) \tag{4.2b}$$

$$S_2 = \sum_{s=1}^{j-5/2} P(j - s + 1/2; j - s, 4) \tag{4.2c}$$

$$S_3 = \sum_{s=1}^{j-3/2} P(1/2; j - s, 3), \tag{4.2d}$$

$$P(1/2; j, 5) = A - \Xi + \Omega \tag{4.4a}$$

$$A = \sum_{u=0,1} \sum_{s=1}^{j-5/2} \left[\frac{(3j - 3s - u - 1/2)^3}{144} - \frac{(3j - 3s - u - 1/2)}{12} - f_1\left(\frac{j - s - u - 1/2}{2} \right) \right]$$

$$+ \sum_{u=0,1} \sum_{s=1}^{j-5/2} \left[\pi(j - s - u - 1/2) \frac{(3j - 3s - u - 1/2)}{16} \right] + S_3, \tag{4.4b}$$

$$\Xi = \sum_{u=0,1} \sum_{s=1}^{j-5/2} \xi(j - s - u - 1/2), \tag{4.4c}$$

$$\Omega = \sum_{u=0,1} \sum_{s=1}^{j-5/2} \omega(3j - 3s - u - 3/2). \tag{4.4d}$$

In order that $P(j - s + 1 \pm 1/2; j - s, 4)$ be nonzero, on must have $j \geqslant 7/2$. When evaluating the last part of the sum A, because of the factor $\pi(j - s - u - 1/2)$ one must consider separately the cases $j - 1/2$ even and odd. We define $\bar{n} = j - s - u - 1/2$. If $j - 1/2 = 2\nu$ is even, we have $\bar{n} = 2\nu - 1 - s$ (resp. $2\nu - s$) for $u = 1$ (resp. $u = 0$), and $\bar{n}$ will be odd if $s = 2t$, $1 \leqslant t \leqslant (j - 5/2)/2 = \nu - 1$ (resp. $s = 2t - 1$, $1 \leqslant t \leqslant (j - 5/2)/2 = \nu - 1$). If $j - 1/2 = 2\nu + 1$ is odd, we have $\bar{n} = 2\nu - s$ (resp. $2\nu + 1 - s$) for $u = 1, 0$ respectively, and since $\bar{n}$ must be odd, the summation index is $s = 2t - 1$, $1 \leqslant t \leqslant (j - 3/2)/2 = \nu$ (resp. $s = 2t$, $1 \leqslant t \leqslant (j - 7/2)/2 = \nu - 1$). This allows one to compute A as a sum of first, second, and third powers of terms in arithmetic progression, which is a simple operation. Namely we get

$$A = \frac{23 j^4}{288} - \frac{23 j^3}{144} + \frac{49 j^2}{576} - \frac{139 j}{576} + \alpha, \tag{4.5a}$$

$$\alpha = \frac{2063}{4608} \text{ if } j - 1/2 \text{ even}, \quad \frac{1919}{4608} \text{ if } j - 1/2 \text{ odd}. \tag{4.5b}$$

For $j \geqslant 7/2$, specifying the contributions $u = 0, 1$ the sum Ξ may be written $\Xi = \Xi_1 + \Xi_0$, $\Xi_1 = \sum_{t=1}^{j-5/2} \xi(t)$, $\Xi_0 = \sum_{t=2}^{j-3/2} \xi(t)$. The quantities Ξ_0, Ξ_1, Ξ as functions of j are easily derived from the definition (3.51b) of ξ. Since $\xi(n)$ is periodic with period 6, one will note that $\sum_{n=0}^{5} \xi(n) = -3/8$, and therefore $\Xi_1(j + 6) = \Xi_1(j) - 3/8$, $\Xi_0(j + 6) = \Xi_0(j) - 3/8$, and $\Xi(j + 6) = \Xi(j) - 3/4$.

knowing that for $s = j - 3/2$ the elements $P(j - s + 1 \pm 1/2; j - s, 4)$ vanish. The sum S_3 is easily derived from (2.5)

$$S_3 = \sum_{s=1}^{j-3/2} \frac{1}{2} \left[(j - s)^2 - \frac{1}{4} \right] = \frac{1}{48}(2j - 3)(2j - 1)(2j + 1)$$

$$= \frac{j^3}{6} - \frac{j^2}{4} - \frac{j}{24} + \frac{1}{16}. \tag{4.3}$$

Using the formula for the four-fermion distribution (3.51), the sum (4.2) may be rewritten by gathering the contributions to S_1 and S_2:

We have, from the definition (4.4c), defining a new table $T_\xi(n)$ equally periodic with period 6,

$$\Xi = \sum_{t=0}^{j-7/2} [\xi(2 + t) + \xi(1 + t)]$$

$$= -\frac{3}{4} \left\lfloor \frac{j - 7/2}{6} \right\rfloor + T_\xi(j - 7/2)$$

$$= -\frac{1}{8} \left[j - \frac{7}{2} - \left(j - \frac{7}{2} \bmod 6 \right) \right] + T_\xi(j - 7/2),$$

where $T_\xi(n) = \left(-\frac{1}{72}, -\frac{1}{4}, -\frac{3}{8}, -\frac{7}{18} - \frac{5}{8}, -\frac{3}{4} \right)$

if $n \bmod 6 = (0, 1, 2, 3, 4, 5)$ respectively. $\tag{4.6}$

The sum over $\omega(n)$ is obtained in a similar way. One has, from definition (4.4d),

$$\Omega = \sum_{s=1}^{j-5/2} [\omega(3j - 3s - 5/2) + \omega(3j - 3s - 3/2)]$$

$$= \sum_{t=0}^{j-7/2} [\omega(5 + 3t) + \omega(6 + 3t)] = \Omega_1 + \Omega_0 \tag{4.7}$$

and using the ω value (3.51d) it easy to check that $\Omega_1(j + 4) = \Omega_1(j) + 4/9$, $\Omega_0(j + 4) = \Omega_0(j)$, $\Omega(j + 4) = \Omega(j) +$

4/9. One obtains the last contribution

$$\Omega = \frac{4}{9} \left\lfloor \frac{j - 7/2}{4} \right\rfloor + U_\omega(j - 7/2)$$

with $U_\omega(n) = \left(\frac{17}{72}, \frac{17}{36}, \frac{11}{24}, \frac{4}{9} \right)$

if $n \bmod 4 = (0, 1, 2, 3)$ respectively. (4.8)

With $\lfloor (j - 7/2)/4 \rfloor = [j - 7/2 - (j - 7/2 \bmod 4)]/4$, one gets

$$\Omega = \frac{1}{9} \left[j - \frac{7}{2} - \left(j - \frac{7}{2} \bmod 4 \right) \right] + U_\omega(j - 7/2). \quad (4.9)$$

Collecting (4.9), (4.6), one obtains

$$\Omega - \Xi = \frac{17}{72} j - \frac{119}{144} + \mathscr{T}(j - 7/2 \bmod 6)$$
$$+ \mathscr{U}(j - 7/2 \bmod 4) \quad (4.10a)$$

with, for $n = 0, 1, 2, 3, 4, 5$, $\mathscr{T}(n) = -T_\xi(n) - n/8$ or

$$\mathscr{T}(n) = \left(\frac{1}{72}, \frac{1}{8}, \frac{1}{8}, \frac{1}{72}, \frac{1}{8}, \frac{1}{8} \right), \quad (4.10b)$$

and, for $n = 0, 1, 2, 3$, $\mathscr{U}(n) = U_\omega(n) - n/9$ or

$$\mathscr{U}(n) = \left(\frac{17}{72}, \frac{17}{36}, \frac{17}{72}, \frac{1}{9} \right). \quad (4.10c)$$

The expression for $P(1/2; j, 5) = A - \Xi + \Omega$ comes from relations (4.5a), (4.10). We get

$$P(1/2; j, 5) = \frac{23 j^4}{288} - \frac{23 j^3}{144} + \frac{49 j^2}{576} - \frac{j}{192} + p_0 \quad (4.11a)$$

with

$$p_0 = \left(-\frac{73}{512}, -\frac{737}{4608}, \frac{55}{512}, -\frac{25}{512}, -\frac{1169}{4608}, \right.$$
$$-\frac{25}{512}, \frac{55}{512}, -\frac{737}{4608}, -\frac{73}{512}, -\frac{25}{512},$$
$$\left. -\frac{17}{4608}, -\frac{25}{512} \right)$$

if $j - \frac{5}{2} \bmod 12 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$.

(4.11b)

For instance, one gets $P(1/2; 7/2, 5) = 6$. Since a $j = 7/2$ subshell has a degeneracy $g = 8$, this corresponds to a three-hole system. One expects that the total number of levels is the same for a three-fermion $j = 7/2$ shell. Using Eq. (2.5) one indeed finds $P(1/2; 7/2, 3) = 6$, in agreement with the j^5 number of levels. This is a simple consistency check of Eq. (4.11a).

V. DERIVATION OF SUM RULES FOR SIX-j AND NINE-j SYMBOLS

A. Three-fermion case: Sum rules for six-j symbols

It was shown in Ref. [9] that, for three-fermion systems,

$$Q(J, j, 3) = \frac{1}{3} \sum_{\substack{J_{\min} \leqslant J_1 \leqslant J_{\max} \\ J_1 \text{ even}}} \left[1 + 2(2J_1 + 1) \begin{Bmatrix} J_1 & j & J \\ J_1 & j & j \end{Bmatrix} \right],$$
(5.1)

where $J_{\min} = |J - j|$ and $J_{\max} = \min(2j, j + J)$. Replacing the left-hand side of Eq. (5.1) by the expressions (2.24) and (2.25) of $Q(J, j, 3)$ provides a new sum rule on six-j coefficients:

$$2 \sum_{\substack{J_{\min} \leqslant J_1 \leqslant J_{\max} \\ J_1 \text{ even}}} \left[1 + 2(2J_1 + 1) \begin{Bmatrix} J_1 & j & J \\ J_1 & j & j \end{Bmatrix} \right]$$

$$= \begin{cases} (3j - J) + \tilde{q}_{3p} \text{ with } \tilde{q}_{3p} = (0, -1, -2, 3, -4, 1) \text{ for } j \geqslant J \\ \text{and } (3j - J) \bmod 6 = (0, 1, 2, 3, 4, 5) \text{ respectively,} \\ \\ 2J + \tilde{q}_{3m} \text{ with } \tilde{q}_{3m} = (-1, 3, 1) \text{ for for } j \leqslant J \\ \text{and } J \bmod 3 = (1/2, 3/2, 5/2) \text{ respectively.} \end{cases}$$
(5.2)

To our knowledge, the above sum rule is not included in reference books such as Ref. [13], nor can it be deduced in a simple way from elementary sum rules.

B. Four-fermion case: Connection to Ginocchio-Haxton and Rosensteel-Rowe sum rules

The number of $J = 0$ states for four fermions in a single-j shell was originally solved by Ginocchio and Haxton [14–16]. They found that

$$Q(0; j, 4) = \left\lfloor \frac{2j + 3}{6} \right\rfloor. \quad (5.3)$$

Using formula (3.59) with $n = 2j$, one gets after simple operations $Q(0; j, 4) = j/3 - 1/12 + \overline{\omega}(4j - 1) - \overline{\xi}(2j)$. With the above definitions of $\overline{\xi}$ and $\overline{\omega}$, one gets $Q(0; j, 4) = (j - 1/2)/3$, $(j + 3/2)/3$, and $(j + 1/2)/3$ for $j - 1/2 \bmod 3 = 0, 1, 2$ respectively. It is then simple to verify that such expressions are identical to $\lfloor j/3 + 1/2 \rfloor$. Rosensteel and Rowe showed that the number of linear constraints and algebraic expressions for conservation of seniority can be derived with the quasispin tensor decomposition of the two-body interaction. They proposed a matrix which can project the eigenvectors to two quasispin subspaces, stated that the eigenvalues of the matrix must equal to 2 or -1, and showed that way [17] that the number of $J = 0$ states for four fermions is equal to

$$Q(0; j, 4) = \frac{1}{3} \left(\frac{2j + 1}{3} + 2 \sum_{\text{even } J_0} (2J_0 + 1) \begin{Bmatrix} j & j & J_0 \\ j & j & J_0 \end{Bmatrix} \right). \quad (5.4)$$

From Eqs. (5.3) and (5.4), Zhao pointed out that

$$\sum_{\text{even } J_0} (2J_0 + 1) \begin{Bmatrix} j & j & J_0 \\ j & j & J_0 \end{Bmatrix} \quad (5.5)$$

has a modular behavior [18] (the sum over all J_0 was calculated by Schwinger for instance [19] but none of these sums—over all values of J_0 or over even values only—are given in the handbook by Varshalovich *et al.* [13]). The values are $(-0.5, 0.5, 0)$ for j values $(1/2, 3/2, 5/2)$, and repeat after that, i.e., are the same for j values $(7/2, 9/2, 11/2)$, $(13/2, 15/2, 17/2)$, $(19/2, 21/2, 23/2)$, etc. The first three values -0.5, 0.5, 0 for $j = 1/2, 3/2$, and $5/2$ respectively were obtained by Zamick and Escuderos using recursion relations for coefficients of fractional parentage [20,21]. Noticing that the number of $J = j$ states for three fermions is equal to the number of $J = 0$ states for four fermions,

Zamick and Escuderos proposed an alternate derivation [15] of $Q(0; j, 4)$. In 2010, Qi *et al.* published an alternative proof of the Rosensteel-Rowe relation relying on a decomposition of the total angular momentum. In this work, a matrix similar to that of Ref. [1] has been constructed from the decomposition and the eigenvalue problem was explored in a general way with symmetry properties of angular-momentum coupling coefficients [22]. All those properties (Ginocchio-Haxton and Rosensteel-Rowe relations, sum rules over six-j symbols) are obtained in a straightforward way by the formulas given in the preceding sections.

C. Four-fermion case: Sum rules for nine-*j* symbols

In the same paper [9], the following expression was derived for j^4:

$$Q(J, j, 4) = \frac{1}{6} \sum_{\substack{J_1 \text{ even} \\ 0 \leqslant J_1 \leqslant 2j}} \sum_{\substack{J_2 \text{ even} \\ 0 \leqslant J_2 \leqslant 2j}} \Delta(J_1, J_2, J) \left[1 + (-1)^J \delta_{J_1, J_2} - 4(2J_1 + 1)(2J_2 + 1) \begin{Bmatrix} j & j & J_2 \\ j & j & J_1 \\ J_2 & J_1 & J \end{Bmatrix} \right], \tag{5.6}$$

where $\Delta(J_1, J_2, J) = 1$ if (J_1, J_2, J) verify the triangular conditions, 0 otherwise. Setting $J = 2j - n$ in Eq. (3.59), we get the sum rule

$$4 \sum_{\substack{J_1 \text{ even} \\ 0 \leqslant J_1 \leqslant 2j}} \sum_{\substack{J_2 \text{ even} \\ 0 \leqslant J_2 \leqslant 2j}} \Delta(J_1, J_2, J) \left[1 + (-1)^J \delta_{J_1, J_2} - 4(2J_1 + 1)(2J_2 + 1) \begin{Bmatrix} j & j & J_2 \\ j & j & J_1 \\ J_2 & J_1 & J \end{Bmatrix} \right]$$

$$= 2\left(2j - \frac{J+3}{2}\right)^2 - \frac{7}{3} - H(2j - J)\left[2(2j - J - 1)^2 - \frac{7}{3} + 24\overline{\xi}(2j - J)\right]$$

$$+ 24\overline{\omega}(4j - J - 1) \begin{cases} & \text{if } (2j - J) \text{ even,} \\ +3(4j - J - 3/2) & \text{if } (2j - J) \text{ odd.} \end{cases} \tag{5.7}$$

As implied by the triangular and parity conditions, the above relation is derived assuming that $J \leqslant 4j - 2$. For higher J, the left-hand side always vanishes while the right-hand side does vanish if $4j - 1 \leqslant J \leqslant 4j + 2$, but equals 24 if $J = 4j + 3$. The total number of levels in j^4 reads

$$Q_{\text{tot}}(j^4) = \sum_{J=0}^{2(2j-3)} Q(J, j, 4) \tag{5.8a}$$

$$= \frac{1}{72}(2j + 1)(8j^2 + 2j + 9) - \frac{2}{3} \sum_{J_1, J_2 \text{ even}} (2J_1 + 1)(2J_2 + 1) \sum_{J=|J_1 - J_2|}^{J_1 + J_2} \begin{Bmatrix} j & j & J_2 \\ j & j & J_1 \\ J_2 & J_1 & J \end{Bmatrix}, \tag{5.8b}$$

and therefore expression (3.53) enables one to write the sum rule

$$4 \sum_{J_1, J_2 \text{ even}} (2J_1 + 1)(2J_2 + 1) \sum_{J=|J_1 - J_2|}^{J_1 + J_2} \begin{Bmatrix} j & j & J_2 \\ j & j & J_1 \\ J_2 & J_1 & J \end{Bmatrix} = 2j^2 + \frac{2}{3}j \begin{cases} +\frac{7}{6} & \text{if } \left(j - \frac{1}{2}\right) \bmod 3 = 0, \\ -\frac{3}{2} & \text{if } \left(j - \frac{1}{2}\right) \bmod 3 = 1, \\ -\frac{1}{6} & \text{if } \left(j - \frac{1}{2}\right) \bmod 3 = 2. \end{cases} \tag{5.9}$$

Equation (5.8) can also be expressed using the coefficients introduced by Dunlap and Judd [10],

$$D_{J_a, J_b; k} = \frac{1}{2k + 1} \left[\frac{(2J_a - k)!(2J_b + k + 1)!}{(2J_b - k)!(2J_a + k + 1)!} \right]^{1/2}, \tag{5.10}$$

as

$$Q_{\text{tot}}(j^4) = \frac{2j + 1}{72}[2j(4j + 1) + 9]$$

$$- \frac{2}{3} \sum_{\substack{J_1, J_2 \\ J_1, J_2 \text{ even}}} (2J_1 + 1)(2J_2 + 1) \sum_{k=0}^{\min(2j, 2J_1, 2J_2)} (2k + 1)(-1)^\phi D_{J_M, J_m; k} \begin{Bmatrix} j & j & k \\ J_2 & J_2 & j \end{Bmatrix} \begin{Bmatrix} j & j & k \\ J_1 & J_1 & j \end{Bmatrix} \tag{5.11}$$

with $\phi = J_1 + J_2 + k$, $J_m = \min(J_1, J_2)$, $J_M = \max(J_1, J_2)$. Of course the sum is restricted to conditions $0 \leqslant J_1 \leqslant 2j$, $0 \leqslant J_2 \leqslant 2j$ imposed by the six-j symbol. The corresponding sum rule is therefore

$$4 \sum_{\substack{J_1, J_2 \\ J_1, J_2 \text{ even}}} (2J_1 + 1)(2J_2 + 1) \sum_{k=0}^{\min(2j, 2J_1, 2J_2)} (2k + 1)(-1)^\phi D_{J_M, J_m; k} \begin{Bmatrix} j & j & k \\ J_2 & J_2 & j \end{Bmatrix} \begin{Bmatrix} j & j & k \\ J_1 & J_1 & j \end{Bmatrix}$$

$$= 2j^2 + \frac{2}{3} j \begin{cases} +\frac{7}{6} & \text{if } \left(j - \frac{1}{2}\right) \bmod 3 = 0, \\ -\frac{3}{2} & \text{if } \left(j - \frac{1}{2}\right) \bmod 3 = 1, \\ -\frac{1}{6} & \text{if } \left(j - \frac{1}{2}\right) \bmod 3 = 2. \end{cases} \tag{5.12}$$

To our knowledge, Eqs. (5.7), (5.9), and (5.12) are not included in reference books such as Ref. [13], nor can they be deduced in a simple way from elementary sum rules.

VI. PARTICULAR VALUES OF THE NUMBER OF LEVELS WITH A GIVEN SPIN J

A property mentioned by Talmi is the vanishing of $Q(1/2; j, 3)$. It is worth mentioning that for $j = 1/2$ it is not possible to get three distinct values m_1, m_2, m_3 because $m_i = \pm 1/2$ and therefore $Q(1/2; 1/2, 3) = 0$. From the above relation (1.2a), one also gets

$$Q(J; j, N) = Q(J; j - 1, N) + Q(J - j; j - 1, N - 1)$$
$$+ Q(J + j; j - 1, N - 1)$$
$$+ Q(J; j - 1, N - 2). \tag{6.1}$$

The recurrence (6.1) reads, for $J = 1/2$, $N = 3$ and accounting for the formal symmetry property $Q(-J - 1) = -Q(J)$,

$$Q\left(\frac{1}{2}; j, 3\right) = Q\left(\frac{1}{2}; j - 1, 3\right) + Q\left(\frac{1}{2} - j; j - 1, 2\right)$$
$$+ Q\left(\frac{1}{2} + j; j - 1, 2\right) + Q\left(\frac{1}{2}; j - 1, 1\right)$$
$$\tag{6.2a}$$

$$= Q\left(\frac{1}{2}; j - 1, 3\right) - Q\left(j - \frac{3}{2}; j - 1, 2\right)$$
$$+ Q\left(j + \frac{1}{2}; j - 1, 2\right) + Q\left(\frac{1}{2}; j - 1, 1\right). \tag{6.2b}$$

Let us note first that the fourth term of that equation is zero except if $j = 3/2$. For $j = 3/2$, the second term is $-Q(0; 1/2, 2) = -1$, the third $Q(2; 1/2, 2) = 0$, and the fourth $Q(1/2; 1/2, 1) = 1$ according to the elementary properties of the coupling of angular momenta $j = 1/2$. The sum of the last three terms of (6.2b) is therefore zero. For $j = 5/2$, $-Q(1; 3/2, 2) = Q(3; 3/2, 2) = 0$ because the total momentum J must be even, and the sum of the last three terms of (6.2b) cancels as well. For $j \geqslant 7/2$, one has $j - 3/2 < j + 1/2 \leqslant J_{\max} = 2j - 3$. For $J \geqslant 0$, $Q(J; j - 1, 2) = 1$ if J

even, 0 otherwise. One checks

$$- Q\left(j - \frac{3}{2}; j - 1, 2\right) + Q\left(\frac{1}{2} + j; j - 1, 2\right)$$
$$= \begin{cases} -1 + 1 = 0 & \text{for } j = 2n - \frac{1}{2}, \\ 0 + 0 = 0 & \text{for } j = 2n + \frac{1}{2}, \end{cases} \tag{6.3}$$

and therefore for each j the summation of the last three terms of (6.2b) cancels. Such an equation implies that for j half-integer

$$Q\left(\frac{1}{2}; j, 3\right) = Q\left(\frac{1}{2}; j - 1, 3\right) = \cdots = Q\left(\frac{1}{2}; \frac{3}{2}, 3\right)$$
$$= Q\left(\frac{1}{2}; \frac{1}{2}, 3\right) = 0. \tag{6.4}$$

In addition, it is easy to show that $Q(J_{\max}(j, N); j, N) = 1$ and that $Q(J_{\max}(j, N) - 1; j, N) = 0$. Indeed, for each configuration j^N, the value $J = J_{\max}$ is realized only once. This manifests clearly if one notes that in order to get $M = J_{\max}$ there is only one solution except permutations of the m_i, which is $m_1 = j - N + 1$, $m_2 = j - N + 2, \ldots, m_N = j$, yielding $P(J_{\max}; j, N) = 1$. For $M = J_{\max} - 1$, the only possibility is to reduce m_1 by 1 with respect to the $J_{\max}$ case: $m_1 = j - N$, $m_2 = j - N + 2, \ldots, m_N = j$ and one has also $P(J_{\max} - 1; j, N) = 1$ and thus $Q(J_{\max} - 1; j, N) = 0$.

VII. CONCLUSION

Closed-form expressions for the number of levels for three, four, and five fermions in a single-j shell are obtained using recursion relations for $P(M)$, the number of states with a given magnetic quantum number M. We derive exact expressions for $P(M)$ and $Q(J)$, the number of levels with a given total angular momentum J, in the cases of j^3 and j^4. The formulas involve polynomials, the coefficients of which are defined by congruence relations. We provide supplementary results, such as proofs of empirical formulas published by several authors over the last years, cancellation properties and peculiar values of $Q(J)$, or new sum rules over six-j and nine-j symbols.

TABLE IV. Coefficient for the three-fermion distribution of the quantum magnetic number for the lowest M values. This number is given by $P(M; j, 3) = \frac{1}{2}(j^2 - c_M/4)$, assuming $j \geqslant M$.

M	1/2	3/2	5/2	7/2	9/2	11/2
c_M	1	1	9	17	25	41

APPENDIX A: RECURRENCE RELATION ON THE NUMBER OF FERMIONS FOR THE QUANTUM NUMBERS j AND $j - 1$

We have established in Appendix B of Ref. [23] the two relations [respectively (B4) and (B8)]

$$P(M; j, N) = P\left(M - \frac{N}{2}; j - \frac{1}{2}, N\right) + P\left(M - \frac{N}{2} + j + \frac{1}{2}; j - \frac{1}{2}, N - 1\right), \tag{A1a}$$

$$P(M; j, N) = P\left(M + \frac{N}{2}; j - \frac{1}{2}, N\right) + P\left(M + \frac{N}{2} - j - \frac{1}{2}; j - \frac{1}{2}, N - 1\right) \tag{A1b}$$

from the recurrences for the Gaussian binomial coefficient. The first term on the right-hand side of (A1a) can be transformed using (A1b):

$$P\left(M - \frac{N}{2}; j - \frac{1}{2}, N\right) = P(M; j - 1, N) + P(M - j; j - 1, N - 1). \tag{A2}$$

In the same way, the second term on the right-hand side of (A1a) transforms with (A1b) into

$$P\left(M - \frac{N}{2} + j + \frac{1}{2}; j - \frac{1}{2}, N - 1\right) = P\left(M - \frac{N}{2} + j + \frac{1}{2} + \frac{N-1}{2}; j - 1, N - 1\right)$$

$$+ P\left(M - \frac{N}{2} + j + \frac{1}{2} + \frac{N-1}{2} - j; j - 1, N - 2\right) \tag{A3a}$$

$$= P(M + j; j - 1, N - 1) + P(M; j - 1, N - 2), \tag{A3b}$$

and, gathering equations (A1a), (A2), (A3b), we get the basic equation (1.3) which was previously obtained by Talmi [Eq. (1) of Ref. [3]].

APPENDIX B: EXAMPLES OF $P(M; j, 3)$ VALUES FOR $M \leqslant 11/2$

The relation (2.18) may be used to get $P(j - q; j, 3)$ for $j - q = 1/2, 3/2, \ldots, n + 1/2$. Examples for the first $j - q$ values are given in Table IV, with the notation $P(M; j, 3) = \frac{1}{2}(j^2 - c_M/4)$, and assuming $M \leqslant j$. For instance $P(11/2; j, 3) = \frac{1}{2}(j^2 - 41/4)$ only if $j \geqslant 11/2$. One calculates $P(11/2; 7/2, 3) = 2$, although this formula would give 1. We obtain again from $P(1/2; j, 3)$ the total number of levels for three fermions derived above (2.5) and also obtained in Ref. [9] using fractional parentage coefficients.

APPENDIX C: DETERMINATION OF THE DISTRIBUTION $P(2j - 1; j, 4)$

The value $P(2j - 1; j, 4)$ is derived starting from Eq. (3.10), that can be rewritten as

$$P(2j + 1; j, 4) = \frac{(j-1)^3}{18} - \frac{(j-1)}{24} + \varphi, \tag{C1a}$$

$$\varphi = \left(-\frac{1}{72}, \frac{1}{72}, -\frac{1}{8}, \frac{17}{72}, -\frac{17}{72}, \frac{1}{8}\right) \text{ if } j - \frac{1}{2} \mod 6 = (0, 1, 2, 3, 4, 5) \text{ respectively.} \tag{C1b}$$

We will obtain $P(2j - 1; j, 4)$ from the fundamental equation (1.3), and the definition (3.13):

$$P(2j - 1; j, 4) = P(2j - 1; j - 1, 4) + P(j - 1; j - 1, 3) + F(j, 1). \tag{C2}$$

We note that $F(j, 1) = 0$ according to (3.15). With the notations $x = j - 2$, $P_1 = P(2j - 1; j - 1, 4)$, $P_3 = P(j - 1; j - 1, 3)$, and the value (C1) for P_1 we obtain the following results:

(1) If $j \mod 6 = 1/2$, $P_1 = x^3/18 - x/24 + 1/8$, $P_3 = (j-1)^2/3 - 1/12$, $P(2j - 1; j, 4) = j^3/18 - j/24 + 1/72$,
(2) if $j \mod 6 = 3/2$, $P_1 = x^3/18 - x/24 - 1/72$, $P_3 = (j-1)^2/3 - 1/12$, $P(2j - 1; j, 4) = j^3/18 - j/24 - 1/8$,
(3) if $j \mod 6 = 5/2$, $P_1 = x^3/18 - x/24 + 1/72$, $P_3 = (j-1)^2/3 + 1/4$, $P(2j - 1; j, 4) = j^3/18 - j/24 + 17/72$,
(4) if $j \mod 6 = 7/2$, $P_1 = x^3/18 - x/24 - 1/8$, $P_3 = (j-1)^2/3 - 1/12$, $P(2j - 1; j, 4) = j^3/18 - j/24 - 17/72$,

(5) if $j \bmod 6 = 9/2$, $P_1 = x^3/18 - x/24 + 17/72$, $P_3 = (j-1)^2/3 - 1/12$, $P(2j-1; j, 4) = j^3/18 - j/24 + 1/8$,

(6) if $j \bmod 6 = 11/2$, $P_1 = x^3/18 - x/24 - 17/72$, $P_3 = (j-1)^2/3 + 1/4$, $P(2j-1; j, 4) = j^3/18 - j/24 - 1/72$.

These expressions are needed for initializing the recurrence (3.34).

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