Differential analysis of incompressibility in neutron-rich nuclei

Bao-An Li [1,*] and Wen-Jie Xie [2]

[1] *Department of Physics and Astronomy, Texas A&M University-Commerce, Texas 75429-3011, USA*
[2] *Department of Physics, Yuncheng University, Yuncheng 044000, China*

(Received 21 February 2021; revised 27 July 2021; accepted 1 September 2021; published 9 September 2021)

Both the incompressibility K_A of a finite nucleus of mass A and that (K_∞) of infinite nuclear matter are fundamentally important for many critical issues in nuclear physics and astrophysics. While some consensus has been reached about K_∞, accurate theoretical predictions and experimental extractions of K_τ characterizing the isospin dependence of K_A have been very difficult. We propose a differential approach to extract K_τ and K_∞ independently from the K_A data of any two nuclei in a given isotope chain. Applying this method to the K_A data from isoscalar giant monopole resonances (ISGMR) in even-even Pb, Sn, Cd, and Ca isotopes taken by Garg *et al.* at the Research Center for Nuclear Physics (RCNP), Osaka University, Japan, we find that the ^{106}Cd - ^{116}Cd and ^{112}Sn - ^{124}Sn pairs having the largest differences in isospin asymmetries in their respective isotope chains measured so far provide consistently the most accurate up-to-date K_τ value of $K_\tau = -616 \pm 59$ MeV and $K_\tau = -623 \pm 86$ MeV, respectively, largely independent of the remaining uncertainties of the surface and Coulomb terms in expanding K_A, while the K_∞ values extracted from different isotopes chains are all well within the current uncertainty range of the community consensus for K_∞. Moreover, the size and origin of the "soft Sn puzzle" is studied with respect to the "stiff Pb phenomenon." It is found that the latter is favored due to a much larger (by $\approx$380 MeV) K_τ for Pb isotopes than for Sn isotopes, while K_∞ from analyzing the K_A data of Sn isotopes is only about 5 MeV less than that from analyzing the Pb data.

DOI: 10.1103/PhysRevC.104.034610

I. INTRODUCTION

Because of its fundamental importance in nuclear physics and broad impact on astrophysics, the incompressibility K_∞ of infinite nuclear matter has been a long-standing and major scientific goal of many experimental and theoretical studies. Since the pioneering work of Blaizot who determined $K_\infty = (210 \pm 30)$ MeV from analyzing the experimental data on giant monopole resonance (GMR) energies in ^{40}Ca, ^{90}Zr, and ^{208}Pb [1], extensive theoretical studies and systematic experiments on the incompressibility K_A of finite nuclei extracted from GMR energies over the last four decades [1–6] have led to the community consensus that K_∞ is in the range of 220 to 260 MeV [3,6,7] or around 235 ± 30 MeV [8,9]. Thanks to the new advancement in experiments, especially at rare-isotope-beam facilities, GMR energies of neutron-rich nuclei along long isotope chains have recently become possible, facilitating more accurate and extensive explorations of the isospin dependence of K_A.

The incompressibility K_A of finite nuclei is usually parametrized in the form of a leptodermous expansion in powers of $A^{-1/3}$ in typical macroscopic models as [1]

$$K_A \approx K_\infty(1 + cA^{-1/3}) + K_\tau \delta^2 + K_{\rm Cou} Z^2 A^{-4/3} \quad (1)$$

for a nucleus of mass number A, charge number Z, and isospin asymmetry $\delta = \frac{N-Z}{A}$, with $c \approx -1.2 \pm 0.12$ [10] and $K_{\rm Cou} \approx -5.2 \pm 0.7$ MeV [11] being the surface and Coulomb parameters, respectively. The K_τ characterizing the isospin dependence of K_A has been the main focus of many recent experimental and theoretical investigations. By moving the Coulomb term to the left side of the above equation, *for all practical purposes* [3] for extracting K_τ from the experimental K_A data [12–17], $K_A - K_{\rm Cou} Z^2 A^{-4/3}$ was fit with a quadratic function of the form $\mathbf{a} + K_\tau \delta^2$ assuming $\mathbf{a} = K_\infty(1 + cA^{-1/3})$ is a constant. This approach resulted in an "experimental" value of $K_\tau = -550 \pm 100$ MeV from the K_A data of Sn isotopes and $K_\tau = -555 \pm 75$ MeV from Ca isotopes. The mass dependence of $\mathbf{a}$ and the known correlation between K_∞ and K_τ neglected in the above approach were found to affect significantly the extracted K_τ values [5,18]. For example, using the same c and $K_{\rm Cou}$ parameters but preserving the mass dependence of $\mathbf{a}$ and considering the correlation between K_∞ and K_τ in the error minimization of a multivariate χ^2 fit, $K_\tau = -595 \pm 177$ MeV, $K_\infty = 209 \pm 6$ MeV from Sn isotopes, and $K_\tau = -463 \pm 405$ MeV, $K_\infty = 211 \pm 11$ MeV from Cd isotopes were found [5]. As stressed already [3,5,18,19], the state of affairs in understanding and extracting K_τ has been very unsatisfactory for a long time.

While it is well known that K_∞ is a fundamental quantify critical for solving many interesting issues in both nuclear physics and astrophysics, to the best of our knowledge, the impact of K_τ on astrophysical observables, e.g., the radii of neutron stars, is only indirect through the shared underlying isovector interactions. Nevertheless, an accurate value of K_τ is useful for predicting the incompressibilities and thus the collective excitations of heavy neutron-rich nuclei that have not been measured or cannot be measured directly because

*Bao-An.Li@Tamuc.edu

of their instabilities. It is thus imperative to find more robust methods to extract accurately both K_∞ and especially K_τ from the K_A data. Such methods are also expected to play important roles in analyzing the coming new data from measuring the K_A of exotic, more-neutron-rich nuclei in long isotopic chains at advanced radioactive-beam facilities.

In some earlier studies, see, e.g., Ref. [3] for a recent review, the leptodermous expansion of Eq. (1) was used to extract its coefficients by performing χ^2 fittings to the experimental K_A data. We regard this approach as the integral approach in our following discussions. It has been shown in numerous works, see, e.g., Ref. [20], that such approach is not very accurate. It was concluded that these leptodermous coefficients are not well constrained by the experimental data. Later, some sort of community "consensus" was reached that the model to analyze K_∞ must contain microscopic effects, reproduce the GMR, and also other observables before one extrapolates to the infinite system. In reality, in analyzing the GMR data from RCNP, for instance, the consensus approach was used in extracting only K_∞ and indeed much interesting physics was obtained. However, the "experimental" K_τ value was always extracted from the same K_A data by using Eq. (1) [12–17] because K_τ is only defined through this equation for the incompressibility of finite nuclei. Thus, regardless of whatever criticisms people may have for using Eq. (1) to extract K_∞, the same consensus approach does not apply to the extraction of K_τ which can only be extracted by using Eq. (1). Moreover, given the still very large dependencies on both the many-body theories and interactions used in the consensus approach in studying K_∞ for infinite nuclear matter, some of the same techniques cannot be used in calculating K_τ for finite nuclei, and the relevant isovector interactions are much less known than the isoscalar interactions.

Indeed, shell and pairing effects are not considered in Eq. (1). These effects may play some roles in extracting the incompressibility from ISGMR data [21], but their effects are still much smaller than the current uncertainty range of the fiducial value of K_∞, not to mention the huge uncertainty of K_τ discussed above when these effects are neglected. We notice that it was already pointed out that shell effects are not important for ISGMR [15] because giant resonances are basically collective phenomena. Obviously, more research is necessary to quantify more precisely the shell and pairing effects on ISGMR.

The well-known problems mentioned above about the reliability of the leptodermous coefficients extracted from χ^2 fitting do not necessarily mean that Eq. (1) itself is wrong or inaccurate, they may indicate instead that the χ^2 fitting approach is not appropriate for extracting the K_∞ and K_τ values. Thus, they do not prevent people from using the same Eq. (1) in better or more appropriate ways to extract accurately K_∞ and K_τ from the same K_A data.

In this work, we propose a differential approach to extract the exact values of K_∞ and K_τ independently from the K_A data of two nuclei in any isotopic chain. The nucleus-nucleus pair having the largest difference in their isospin asymmetries is found to give the most accurate K_τ and K_∞ values simultaneously. Effects of varying the c and K_{Cou} parameters by $\pm 20\%$ around their known most probable values on extracting both K_τ and K_∞ are also examined. While the variations of c and K_{Cou} lead the extracted K_∞ values to vary within its current consensus range, they have almost no effect on extracting K_τ, indicating the robustness of the differential approach. We found that both the mean value and uncertainty we extracted for K_∞ are compatible with those from using state-of-the-art microscopic theories in the consensus approach, while the accuracy of the extracted K_τ in our approach is much higher than what is available in the literature. Finally, it has been known for about a decade that the K_A values extracted experimentally from Sn isotopes are apparently smaller compared with predictions of nonrelativistic mean-field or relativistic mean-field + random-phase approximation (RPA) calculations that can successfully describe the ISGMR data of Pb isotopes [13]. However, the origin of this so-called "soft Sn puzzle" [3,22] or "stiff Pb phenomenon" [21] is still unclear. We investigate whether the differential analysis can shed new light on this issue.

The rest of the paper is organized as follows: In the next section, we present details of the proposed differential analyses. In Sec. III, we perform a differential analysis for the K_A data [12–17] from isoscalar giant monopole resonances in even-even Pb, Sn, Cd, and Ca isotopes taken by Garg *et al.* at RCNP. In Sec. IV, we study the effects of the remaining uncertainties of the surface and Coulomb parameters on extracting the K_∞ and K_τ values. Finally, we summarize our work and draw conclusions.

II. THE DIFFERENTIAL APPROACH

Applying Eq. (1) to any two isospin asymmetric ($\delta \neq 0$) nuclei of mass and charge (A_1, Z_1) and (A_2, Z_2) separately, K_τ and K_∞ can be expressed exactly as

$$K_\tau = \left[\frac{K_{A_1}}{S_1} - \frac{K_{A_2}}{S_2} - K_{\text{Cou}}\left(\frac{Z_1^2 A_1^{-4/3}}{S_1} - \frac{Z_2^2 A_2^{-4/3}}{S_2}\right)\right] \Big/ \left(\frac{\delta_1^2}{S_1} - \frac{\delta_2^2}{S_2}\right), \tag{2}$$

$$K_\infty = \left[\frac{K_{A_1}}{\delta_1^2} - \frac{K_{A_2}}{\delta_2^2} - K_{\text{Cou}}\left(\frac{Z_1^2 A_1^{-4/3}}{\delta_1^2} - \frac{Z_2^2 A_2^{-4/3}}{\delta_2^2}\right)\right] \Big/ \left(\frac{S_1}{\delta_1^2} - \frac{S_2}{\delta_2^2}\right), \tag{3}$$

where $S_i = 1 + cA_i^{-1/3}$ for nucleus i with $i = 1$ or 2. One can understand intuitively the physical meanings of the above expressions by using the mathematical definitions of K_τ and K_∞ based on Eq. (1). Namely, neglecting the Coulomb correction, $K_\tau \equiv (\partial K_A/\partial \delta^2)_S \approx \Delta(K_A/S)/\Delta(\delta^2/S) = (\frac{K_{A_1}}{S_1} - \frac{K_{A_2}}{S_2})/(\frac{\delta_1^2}{S_1} - \frac{\delta_2^2}{S_2})$ gives the leading term of K_τ in Eq. (2). It is simply the rate of change of K_A with respect to δ^2 evaluated by using the ratio of their finite changes. Similarly, $K_\infty \equiv (\partial K_A/\partial S)_\delta \approx \Delta(K_A/\delta^2)/\Delta(S/\delta^2) = (\frac{K_{A_1}}{\delta_1^2} - \frac{K_{A_2}}{\delta_2^2})/(\frac{S_1}{\delta_1^2} - \frac{S_2}{\delta_2^2})$ gives the leading term of K_∞ in Eq. (3).

We notice that, while K_τ and K_∞ are determined independently by the K_A data themselves of any two nuclei used, they satisfy the constraint given by Eq. (1). Therefore, there is an intrinsic correlation between K_τ and K_∞ when they are varied using the K_A data of many different nucleus-nucleus pairs in a given isotopic chain, as we shall demonstrate.

The corresponding uncertainties of K_τ and K_∞ can be calculated exactly according to the rules of error propagation using the experimental errors of K_A data, i.e., $\sigma_{K_{A_1}}$ and $\sigma_{K_{A_2}}$ in the nucleus-1 and nucleus-2 considered. Nevertheless, to see analytically what nucleus-nucleus pairs may give the most accurate K_τ and K_∞ values, we notice that, for heavy nuclei in the same isotope chain, $S_1 \approx S_2 \approx 1$, the error bars are reduced to

$$\sigma_{K_\tau} \approx \sqrt{\sigma^2_{K_{A_1}} + \sigma^2_{K_{A_2}}}\Big/\left|\delta_1^2 - \delta_2^2\right|, \tag{4}$$

$$\sigma_{K_\infty} \approx \sqrt{\left(\delta_2^2\sigma_{K_{A_1}}\right)^2 + \left(\delta_1^2\sigma_{K_{A_2}}\right)^2}\Big/\left|\delta_1^2 - \delta_2^2\right|. \tag{5}$$

They both are *inversely* proportional to $|\delta_1^2 - \delta_2^2|$, thus nuclear pairs having the largest difference in their isospin asymmetries will give the most accurate K_τ and K_∞ values simultaneously. Moreover, because of the weighting of σ_{K_A} by $\delta^2 \ll 1$ in evaluating σ_{K_∞}, K_∞ can be more precisely evaluated than K_τ, explaining the larger errors of the extracted K_τ values.

While in principle the above formalisms can be applied to any two nuclei, we shall restrict their applications to nuclei in the same isotopic chain. This will reduce not only effects of systematic experimental errors as what is being used is the *difference* in K_A scaled by either the surface factor S or isospin asymmetry δ of the two nuclei in the same isotopic chain, but also effects of the higher-order terms neglected in expanding the K_A in Eq. (1). This is also one of the reasons why the differential approach can more precisely extract both K_τ and K_∞ compared with the typical integral approaches normally used in the literature.

In cases where one of the nuclei is isospin symmetric, say $\delta_1 = 0$, its K_A alone can be used to evaluate K_∞ according to $K_\infty = K_{A_1}/S_1 - K_{\rm Cou}Z_1^2A_1^{-4/3}/S_1$ while K_τ can be evaluated from Eq. (2) by choosing nucleus-2 as neutron rich as possible to get the most accurate result, indicating the importance of using exotic heavy isotopes. As noticed already in the literature, see, e.g., Ref. [19], the leptodermous expansion in Eq. (1) itself may not be a good approximation for light nuclei, the differential approach is thus expected to work better for more heavy nuclei.

III. DIFFERENTIAL ANALYSES OF K_A DATA FROM EXPERIMENTS AT THE RESEARCH CENTER FOR NUCLEAR PHYSICS

Shown in Fig. 1 are the results of our differential analyses of the K_A data in ^{204,206,208}Pb, ^{112,114,116,118,120,122,124}Sn, ^{106,110,112,114,116}Cd, and ^{40,42,44,48}Ca from the GMR experiments at RCNP [12–17] using $c = -1.2$ and $K_{\rm Cou} = -5.2$ MeV. The extracted K_τ and K_∞ values are shown as functions of the difference $(\delta_2 - \delta_1)$ in isospin asymmetries of the two nuclei involved in each isotope chain. Except for the Pb isotopes, we took the K_A data directly from the experimental publications, as listed in Table I. They derived the K_A values by using the moment ratios for the ISGMR energies $E_{\rm ISGMR}$ and the experimental charge radii $(\langle r^2\rangle)^{1/2}$ from Ref. [23] according to the relation

$$K_{\rm A} = \left(\frac{E_{\rm ISGMR}}{\hbar c}\right)^2 Mc^2\langle r^2\rangle, \tag{6}$$

where M is the average nucleon mass. They did not publish the K_A values for the three Pb isotopes. We derived their K_A values using the published ISGMR energies (from $\sqrt{m_1/m_{-1}}$) [15] and their charge radii from Ref. [23]. More quantitatively, we found that K_A is 136.93 ± 1.99 MeV, 137.44 ± 1.99 MeV, and 136.44 ± 1.99 MeV, respectively, for ^{204}Pb, ^{206}Pb, and ^{208}Pb.

Several interesting observations can be made: (1) The uncertainties of both K_τ and K_∞ generally decrease while their mean values remain approximately constant with increasing $(\delta_2 - \delta_1)$ for each isotope chain. (2) The ^{106}Cd - ^{116}Cd and ^{112}Sn - ^{124}Sn pairs give the most accurate and consistent values of $K_\tau = -616 \pm 59$ MeV and $K_\tau = -623 \pm 86$ MeV, respectively. (3) The K_τ values from analyzing the relatively light ^{40,42,44,48}Ca isotopes have larger error bars and scatter around broadly at small isospin separations. However, they seem to converge at large isospin separations and become generally consistent with the means from analyzing the Sn and Cd isotopes within error bars. We notice that, among all data available from the RCNP experiments, the $^{40\text{–}48}$Ca pair has the highest isospin separation $(\delta_2 - \delta_1) = 0.167$. This pair gives $K_\tau = -756 \pm 149$ MeV. As mentioned earlier, the K_A expansion of Eq. (1) is not expected to work well for light nuclei. The scattering of the K_τ values from analyzing the Ca data may thus indicate that our differential approach based on Eq. (1) has reached the limit of its validity. (4) The ^{106}Cd - ^{116}Cd and ^{112}Sn - ^{124}Sn pairs also give the most accurate K_∞ values of $K_\infty = 213 \pm 2$ MeV and $K_\infty = 220 \pm 3$ MeV, respectively. (5) The extracted K_∞ shows the well-known isotope dependence found earlier when the Eq. (1) was used previously in χ^2 fittings of K_A data [1,3–6,19] although our differential approach does not use any fitting at all. Nevertheless, the variation of the K_∞ from Cd to Ca isotopes is well within the uncertainty range of the current consensus value for K_∞.

While the three Pb isotopes pairs have very small isospin separations of 0.00765, 0.00781, and 0.0155, respectively, the results from the differential analyses of their K_A values set a useful reference for comparisons and favor a stiff Pb phenomenon [21] instead of the so-called soft Sn puzzle existing in the literature. The Pb data give an average value of $K_\infty = 223.1 \pm 39.5$ MeV and $K_\tau = -245 \pm 753$ MeV. Their means are indicated by the horizontal magenta bars for comparisons. Two important indications are worth emphasizing. First, it was not known before what causes the soft Sn puzzle. Our analysis here indicates that the mean value of K_∞ from Sn isotopes (218 MeV) is only 5 MeV below that from Pb

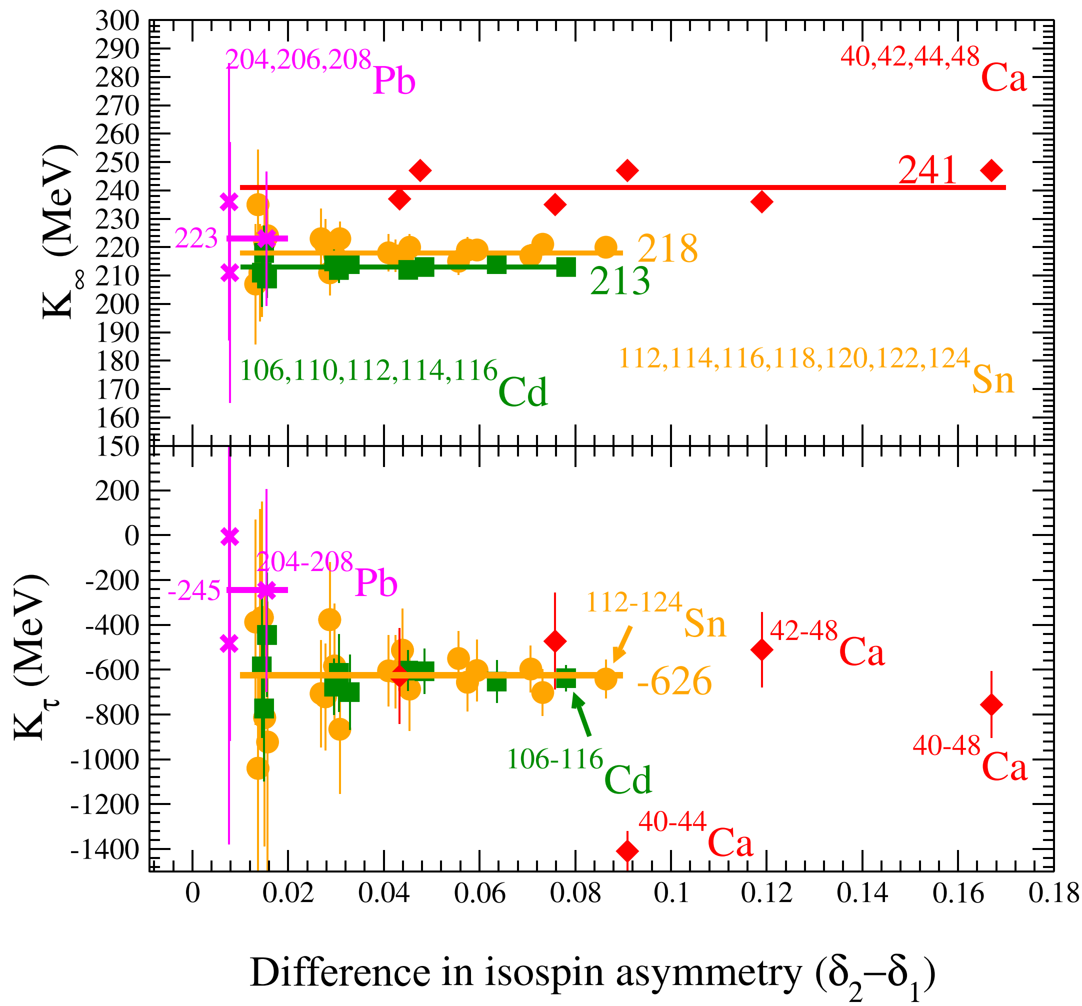

FIG. 1. K_τ (lower window) and K_∞ (upper window) from differential analyses of the incompressibilities in finite nuclei as functions of the difference $(\delta_2 - \delta_1)$ in isospin asymmetries of the isotope pairs used. The solid lines are the mean values of K_τ and K_∞ for the respective isotope chains. The arrows indicate the Cd and Sn isotope pairs giving the most accurate K_τ and K_∞ values.

isotopes (223 MeV), well within the experimental uncertainties. This finding happens to be the same as that found in a very recent Bayesian uncertainty quantification of the nuclear matter incompressibility using the original GMR data of the same sets of isotopes analyzed within the Skyrme Hartree-Fock plus RPA approach [24]. On the other hand, the mean value of K_τ from Sn isotopes (−626 MeV) is significantly below that (−245 MeV) from the Pb isotopes although the latter also has a large error bar. It indicates that the soft Sn puzzle is mainly due to the significantly smaller K_τ value for Sn isotopes or larger K_τ value for Pb isotopes. The overpredictions of the K_A values by the state-of-the-art microscopic theories are most likely due to the model ingredients controlling the K_τ instead of K_∞ values. Second, it is interesting to note that the K_τ values from Sn, Cd, and Ca isotopes all converged asymptotically at large isospin separations to relatively precise values with less than about 20% errors. However, the average K_τ value for Pb isotopes is significantly higher than these asymptotic values, although it is only slightly higher than the K_τ values for Sn and Cd isotopes at the same small isospin separations. These findings may give us some hints about whether there is a soft Sn puzzle or a stiff Pb phenomenon. Our results seem to indicate that it is probably more meaningful to speak about a stiff Pb phenomenon. To verify the latter experimentally, more K_A data for Pb isotopes are obviously necessary.

It is interesting and necessary to check in more detail the consistency between the results of our differential analyses and those from the traditional integral analyses. Shown in Fig. 2 are the correlations between K_τ and K_∞ with each point representing one nucleus-nucleus pair in the Cd or Sn isotope chain corresponding to the results shown in Fig. 1. Averaging over these results is equivalent to performing a typical integral analysis, e.g., a multivariate χ^2 fitting or Bayesian analysis. The solid lines are results of a χ^2 fit to all points in the two isotope chains, separately. The mean of K_τ is -625 ± 100 MeV for the Cd isotopes and -626 ± 188 MeV for the Sn isotopes. The corresponding mean of K_∞ is 213 ± 3 MeV for the Cd isotopes and 218 ± 6 MeV for the Sn isotopes. These mean values are in general agreement with the results of earlier χ^2 analyses [3,5] of the same K_A data using essentially identical surface and Coulomb parameters within error bars.

TABLE I. Results of analysis of incompressibility data of finite nuclei.

Nucleus	K_A (MeV)	Reference
^{40}Ca	144.46 ± 0.33	[17]
^{42}Ca	139.00 ± 1.09	[17]
^{44}Ca	137.36 ± 0.66	[17]
^{48}Ca	131.90 ± 4.13	[17]
^{106}Cd	127.84 ± 0.86	[14]
^{110}Cd	124.59 ± 0.86	[14]
^{112}Cd	123.59 ± 0.77	[14]
^{114}Cd	120.95 ± 1.24	[14]
^{116}Cd	118.96 ± 0.86	[14]
^{112}Sn	131.86 ± 1.53	[12,13]
^{114}Sn	129.45 ± 1.64	[12,13]
^{116}Sn	127.11 ± 1.53	[12,13]
^{118}Sn	126.39 ± 1.54	[12,13]
^{120}Sn	125.45 ± 1.63	[12,13]
^{122}Sn	121.33 ± 1.54	[12,13]
^{124}Sn	120.17 ± 1.62	[12,13]
^{204}Pb	136.93 ± 1.99	[15]
^{206}Pb	137.44 ± 1.99	[15]
^{208}Pb	136.44 ± 1.99	[15]

Interestingly, within the error bars of the mean values there is a clear anticorrelation between K_τ and K_∞. It can be understood easily. With the surface and Coulomb parameters fixed, for a given K_A value, K_τ and K_∞ is expected to be anticorrelated according to their relationship given in Eq. (1). Notice that the K_τ vs K_∞ correlations for the Cd and Sn isotopes are almost in parallel in the direction of K_∞ because they give approximately the same K_τ values but slightly different (about 5 MeV) K_∞ values (notice the fine K_∞ scale used).

We emphasize that the error bars of the mean values of both K_τ and K_∞ in the integral approaches, i.e., by averaging over

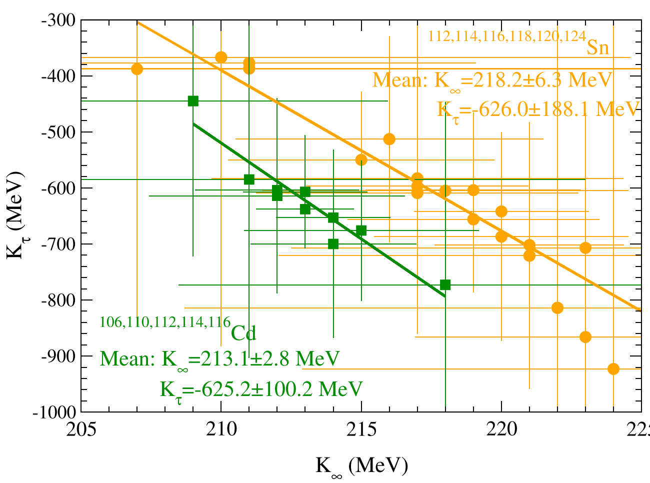

FIG. 2. The correlation between K_τ and K_∞, with each point representing one nucleus-nucleus pair in the Cd or Sn isotope chain corresponding to the results shown in Fig. 1. The solid lines are results of a χ^2 fit to all points in each isotope chain.

all isotope pairs, are all much larger than those we found in the differential analyses of ^{106}Cd - ^{116}Cd and ^{112}Sn - ^{124}Sn pairs. Besides the advantage of largely canceling the systematic errors in the differential analyses, another reason is that the $K_\tau\delta^2$ contribution to K_A is very small even for the most neutron-rich nuclei available. For instance, with $\delta = 0.2$, $K_\tau = -600$ MeV, $K_\tau\delta^2 = -24$ MeV that is still only about 10% of the acceptable K_∞ values around 240 MeV. It is actually significantly less than the current uncertainty of about 40 MeV of the consensus value for K_∞. A global χ^2 fit to the K_A data or a Bayesian analysis of all K_A data available thus cannot reliably extract the value of K_τ from its small contribution relative to K_∞ to K_A. In turn, the uncertainty of extracting the K_∞ cannot be better than $K_\tau\delta^2/K_\infty$ in the integral analyses of the K_A data. On the contrary, the differential approach decouples completely the extractions of K_τ and K_∞ for each isotope pair used. Only K_τ and K_∞ extracted independently for different isotope pairs along an isotope chain show an expected intrinsic correlation within their respective error bars.

IV. EFFECTS OF THE SURFACE AND COULOMB PARAMETERS

We have used above the most probable known values of $c = -1.2$ [10] and $K_{\rm Cou} = -5.2$ MeV [11]. It is also known that the Coulomb parameter is rather model-independent [11,25] while the calculations [1,26–30] of the surface parameter c show somewhat larger variations around $c \approx -1$. It is generally accepted that both the c and $K_{\rm Cou}$ parameters have less than about (10%–20)% uncertainties [3,5]. How do theses uncertainties affect the accuracies of extracting the K_τ and K_∞ in the differential analyses? To answer this question, we have carried out systematic calculations by varying the two parameters independently by $\pm 20\%$ around their most probable values.

As an example, shown in Fig. 3 are the variations of K_τ (lower panels) and K_∞ (upper panels) due to the variation of the surface parameter c (left panels) and Coulomb parameter $K_{\rm Cou}$ (right panels) for the Sn isotopes. Qualitatively, effects of varying the $K_{\rm Cou}$ and especially the parameter c are much smaller on K_τ than on K_∞. Quantitatively, for the ^{112}Sn - ^{124}Sn pair, changing the c parameter by 40% from -1.2×0.8 to -1.2×1.2 makes K_τ change by about 6% from -624 ± 84 to -663 ± 87 MeV, while K_∞ changes by about 13% from 205 ± 29 to 236 ± 34 MeV, respectively. On the other hand, by changing $K_{\rm Cou}$ by 40% from -5.2×0.8 to -5.2×1.2, K_τ changes by about 8% from -616 ± 87 to -669 ± 87 MeV, while K_∞ changes by about 6% from 213 ± 31 to 227 ± 31 MeV. Thus, the 6%–8% uncertainty of K_τ due to the $\pm 20\%$ uncertainty of the surface parameter is much smaller than the approximately 14% uncertainty due to the experimental errors of K_A. While the 8%–13% uncertainty of K_∞ due to the $\pm 20\%$ uncertainty in the Coulomb parameter is compatible with that due to the experimental errors of the K_A data. Thus, the remaining uncertainties of the surface and Coulomb parameters of about 10%–20% have essentially no effect on the extraction of K_τ.

The observed dependencies of K_τ and K_∞ on the variations of the surface and Coulomb parameters can be understood

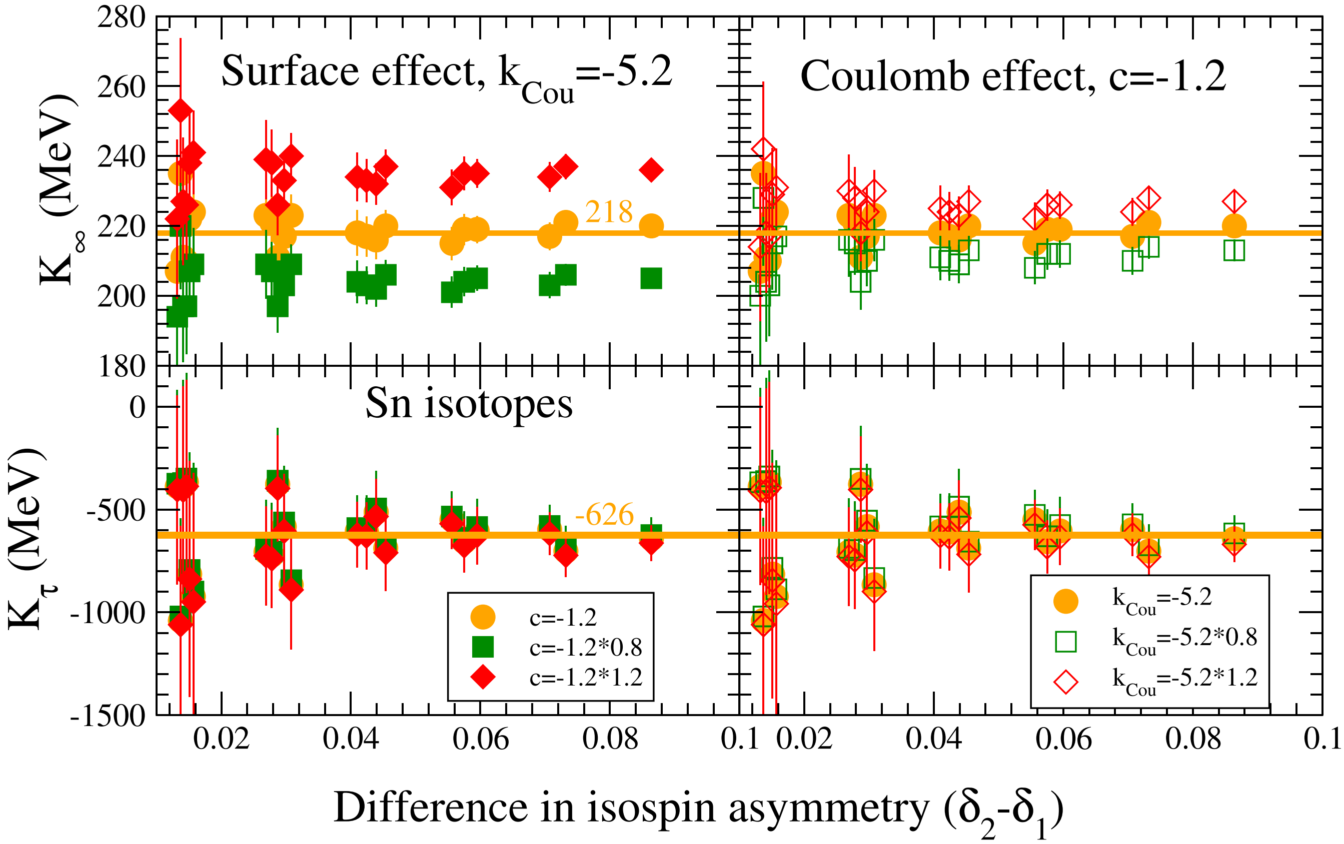

FIG. 3. Variations of K_τ (lower panels) and K_∞ (upper panels) due to the variations of the surface parameter c (left panels) and Coulomb parameter $K_{\rm Cou}$ (right panels), respectively, for the Sn isotopes.

analytically by further examining the expressions of K_τ and K_∞ in Eqs. (2) and (3), respectively. First, we examine effects of the parameter c. The c-dependent part of K_τ is

$$K_\tau \propto \frac{\left(1+cA_2^{-1/3}\right)K_{A_1}-\left(1+cA_1^{-1/3}\right)K_{A_2}}{\left(1+cA_2^{-1/3}\right)\delta_1^2-\left(1+cA_1^{-1/3}\right)\delta_2^2}. \quad (7)$$

Because the parameter c appears in all terms, its effect largely cancels out. In particular, for heavy nuclei $c/A^{1/3}\approx 0$, K_τ then becomes independent of c, i.e.,

$$K_\tau \rightarrow \frac{K_{A_1}-K_{A_2}}{\delta_1^2-\delta_2^2}. \quad (8)$$

The c-dependent part of K_∞ is

$$K_\infty \propto \frac{\delta_2^2 K_{A_1}-\delta_1^2 K_{A_2}}{c\cdot\left(\delta_2^2A_1^{-1/3}-\delta_1^2A_2^{-1/3}\right)+\delta_2^2-\delta_1^2}. \quad (9)$$

The parameter c only appears in the first term of the denominator. Since both terms in the denominator are very small, a very small change in the parameter c can thus lead to a large change in the value of K_∞. This also implies that the surface properties of different nuclei may affect significantly the extraction of K_∞ from the K_A data, as already noticed in the χ^2 analyses in Ref. [5].

Similar analyses can be done to understand effects of the Coulomb parameter $K_{\rm Cou}$. More specifically,

$$K_\tau \propto -K_{\rm Cou}Z^2\frac{\left(1+cA_2^{-1/3}\right)A_1^{-4/3}-\left(1+cA_1^{-1/3}\right)A_2^{-4/3}}{\left(1+cA_2^{-1/3}\right)\delta_1^2-\left(1+cA_1^{-1/3}\right)\delta_2^2}. \quad (10)$$

Again, the parameter c has little effect because it appears in all terms. Considering $cA^{-1/3}\approx 0$ for heavy nuclei, the above expression reduces to

$$K_\tau \rightarrow -K_{\rm Cou}Z^2\times\frac{A_1^{-4/3}-A_2^{-4/3}}{\delta_1^2-\delta_2^2}. \quad (11)$$

As heavy nuclei are more neutron rich in a given chain of isotopes, i.e., for $A_2>A_1$, $\delta_2>\delta_1$, the fraction in the above equation is always negative. Thus, one obtains $K_\tau \propto K_{\rm Cou}$. While $K_{\rm Cou}$ itself is negative, thus a larger negative $K_{\rm Cou}$ decreases the value of K_τ, as seen in our numerical calculations. Given that the overall contribution of the Coulomb term to K_τ is small, its variation causes little change in the final K_τ value, while for the Coulomb effect on K_∞, a similar analysis leads to

$$K_\infty \propto -K_{\rm Cou}\frac{Z^2}{A_1^{4/3}A_2^{4/3}}\frac{\delta_2^2A_2^{4/3}-\delta_1^2A_1^{4/3}}{\delta_2^2-\delta_1^2}. \quad (12)$$

Since the last fraction is always positive, $K_\tau \propto -K_{\rm Cou}$. Therefore, a larger negative $K_{\rm Cou}$ increases the value of K_∞. Moreover, the above analysis clearly explains why the Coulomb parameter has opposite effects on extracting the K_τ and K_∞ values.

V. SUMMARY AND CONCLUSIONS

In summary, we emphasize the following aspects of our work

(i) The analytical expressions for K_∞ and K_τ are derived from solving exactly two linear equations for two unknowns using the K_A data as the only input. While the approach is very simple, its physics is sound. It also has no dependence on any nuclear many-body theories nor interactions. There is absolutely no fitting procedure involved, thus it does not suffer from the well-known problems in fitting the K_A data using Eq. (1).

(ii) K_∞ and K_τ are directly extracted from the experimental K_A data of any two nuclei in a given isotope chain. Besides showing that the proposed approach gives K_∞ consistent with its fiducial value from the consensus approach, and a K_τ that is much more accurate than what is available in the literature, we also ask which isotope pairs are most useful for extracting especially K_τ at rare-isotope-beam facilities. Our answer to this question is expected to be useful for future experiments using rare isotopes to study the equation of state of neutron-rich matter.

(iii) The so-called soft Sn puzzle (when one uses the interactions that correctly reproduce the GMR strength in ^{208}Pb to calculate the GMR strengths in the Sn and Cd isotopes within the consensus approach, the experimental values are always overestimated) has been alive for over 10 years. However, people have not really understood the underlying cause of the puzzle, leading to the conclusion that it is not feasible to simultaneously reproduce both ^{208}Pb and Sn's GMR data by the same interaction [22] using the state-of-the-art theories within the consensus approach. While we did not solve this puzzle in this work, we showed that K_∞ from analyzing the GMR data of Pb, Sn, and Gd isotopes is not much different within the experimental error bars to indicate strongly a soft Sn puzzle. Quantitatively, K_∞ from Sn isotopes is only about 5 MeV smaller than that from Pb isotopes. On the other hand, K_τ from analyzing the ^{204,206,208}Pb data is significantly higher (by ≈380 MeV) than the converged asymptotic K_τ value at large isospin separations in analyzing the Sn and Gd isotopes, indicating strongly a stiff Pb phenomenon [21]. To verify this further, differential analyses of future GMR data of more Pb isotopes will be very useful. The suggestion of having more Pb data was also made for addressing the same puzzle from a different perspective in Ref. [21].

(iv) There are several caveats in our work. First, if the leptodermous expansion of Eq. (1) is perfect, one expects K_∞ and K_τ extracted from all pairs of nuclei to be identical. In reality, this is of course not the case. As mentioned earlier, we expect the differential approach to work better for heavy nuclei along the same isotope chains. Indeed, K_τ from most isotopes fall approximately on the same line at large isospin separations within still relatively large error bars, while K_∞ from different isotope chains, especially the light nuclei, scatter more broadly due to mostly the remaining (approximately ±20%) uncertainty of the surface parameter c. Second, to avoid introducing any model dependence in presenting their K_A data, the experimentalists translated their original GMR observables to the experimental K_A data by using the experimentally measured charge radii instead of the matter radii which are inherently model dependent. This probably introduced a systematic error in the experimental K_A data and its effects have not been evaluated yet. We used the experimental K_A data as in all previous analyses in the literature. Thus, all results presented here should be understood within the context and with the cautions discussed above. Nevertheless, the importance and new physics revealed in our work can be clearly seen from comparing our approach and results with the traditional ones in the literature. We also emphasize that the focus of this work is a more accurate determination of K_τ for finite nuclei, while K_∞ for infinite nuclear matter just came out naturally consistent with its fiducial value that has not changed much since 1980.

In conclusion, we proposed a differential approach to analyze the incompressibilities of neutron-rich nuclei and investigated which nuclear pairs give the most accurate results using K_A data and analytically. The nucleus-nucleus pair having the largest difference in their isospin asymmetries in a given isotope chain is found to give the most accurate values for both K_τ and K_∞ simultaneously. Applying this approach to the K_A data from RCNP, we found that the ^{106}Cd - ^{116}Cd and ^{112}Sn - ^{124}Sn pairs give consistently the most accurate up-to-date K_τ values of -616 ± 59 and -623 ± 86 MeV, respectively, largely independent of the remaining uncertainties of the surface and Coulomb parameters. These results can exclude many predictions based on various microscopic and/or phenomenological nuclear many-body theories in the literature. We also studied the stiff Pb phenomenon versus the soft Sn puzzle and found that the former is favored. Thus, compared with the integral approach widely used in the literature, the differential analysis can reveal some interesting physics underlying the incompressibilities of finite nuclei.

ACKNOWLEDGMENTS

We would like to thank Umesh Garg for many fruitful discussions. BALI is supported in part by the U.S. Department of Energy, Office of Science, under Award No. DE-SC0013702 and the CUSTIPEN (China-U.S. Theory Institute for Physics with Exotic Nuclei) under the U.S. Department of Energy Grant No. DE-SC0009971. W.-J.X. is supported in part by the Yuncheng University Research Project under Grant No. YQ-2017005 and the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi under Grant No. 2020L0550.

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