Each single-energy, single-channel partial-wave analysis is inherently model-dependent

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Švarc *et al.* [Phys. Rev. C **97**, 054611 (2018)] have shown that without fixing the reaction-amplitude phase, different partial waves at neighboring energies in single-energy, single-channel partial wave analysis reproduce experimental data identically, but are discontinuous and disconnected. To obtain the continuous solution, the phase has to be fixed to some continuous value. In the same reference it has also been shown that the change of the angular part of the reaction-amplitude phase mixes partial waves, so the pole structure of any single-energy single-channel partial wave analysis depends on the chosen phase. As in any single-channel analysis the overall reaction-amplitude phase cannot be determined because of continuum ambiguity, it is in principle free and has to be taken from some coupled-channel model. Because of the difference in the angular part of the phase, choosing different phases results in the change of the pole content of the obtained solution. Therefore, single-energy single-channel partial wave analysis is inherently model dependent, and the number of poles it contains strongly depends on the choice of the phase. In that reference these facts have been illustrated on the pseudoscalar meson production toy model. This truth is now for the full set of measured observables demonstrated on the realistic model of η photoproduction presented in Švarc, Wunderlich, and Tiator [Phys. Rev. C **102**, 064609 (2020)].

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I. INTRODUCTION

In Ref. [1] a single-channel, single-energy partial wave analysis (SC-SE-PWA) procedure[1] has been formulated. It combines amplitude and partial wave analyses into one logical sequence, and directly from the data generates a set of continuous partial waves with minimally model dependent input (AA/PWA). It has been demonstrated that by controlling the reaction-amplitude phase, and freely varying the reaction-amplitude partial waves, one obtains a continuous solution with far better agreement with the used database than the original energy dependent (ED) model. Two solutions have been generated, Sol 1 when the phase is close, but not identical to the phase of a particular coupled-channel, energy-dependent model (Ref. [4] in our case), and the second one in which the phase has been freely smoothed to a similar but simpler analytic function (Sol 2). It has been shown that the quality of both fits is fairly similar, but the obtained higher multipoles are rather different (E_0^+ is always dominantly big and very stable in η-photoproduction). Also, a promise had been given to analyze their analytic structure (pole content) using the Lauren+Pietarinen (L+P) technique [5]. In this paper I fulfill the given promise for Sol 1. However, in doing so, some unexpected effects revealed themselves. The pole content of Sol 1 turned out to be unclear. In looking for the explanation why this is so, I discovered the illuminating truth of how the precision in the phase determination decides the analytic structure of the obtained solution. The main aim of this article is to show that the analytic structure of the AA/PWA model of Ref. [1] is correct, that the change of analytic structure (pole content) of the obtained solution depends on the changes of the phase, and quantify when the notable deviation in the pole content starts. This will decide how well one can fit the given database at the same time maintaining the proper analytic structure (pole content) determined by the constraining ED model. Further improvements in the fit without controlled phase change will be possible only by spoiling good pole content, and introduce ghost poles originating in the change of the angular part of the phase. Therefore I show that the only possible improvement in the present SC-SE-PWA is model dependent, and it is accomplished by improving the constraining phase of the coupled-channel ED model. Free change of phase is not allowed, and it directly leads to ghost poles. As a direct corollary of this conclusion the analysis of Sol 2 turned out to be unnecessary.[2]

To understand that, one has to discuss the problem of analytic structure of obtained solutions in the wider context of angular dependent phase rotations.

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[1]Strictly speaking, PWA is a nontrivial mathematical problem of looking for a solution to the potentially ill-posed problem from Hadamard and Tikhonov [2,3] of determining reaction amplitude via fitting scattering data when number of equations may be less than number of unknown quantities.

[2]Sol 2 was obtained with free smoothing of the reaction amplitude phase not controlled by other channels, so the appearance of ghost poles which spoil the analytic structure is unavoidable.

In Ref. [6] it has been shown that without fixing the reaction-amplitude phase, different partial waves at neighboring energies in single-energy single-channel partial wave analysis reproduce experimental data identically, but are discontinuous and disconnected. So, to obtain the continuous solution, the phase has to be fixed to some continuous value. In the same reference it has also been shown that the change of angular part of reaction-amplitude phase mixes partial waves so the pole structure of any single-energy single-channel partial wave analysis depends on the chosen phase. As the reaction-amplitude phase because of continuum ambiguity cannot be determined in any single-channel analysis, it has to be taken from some coupled-channel model. Choosing different phases results in the change of the pole content of the obtained solution. Therefore, SC-SE-PWA must depend on the reaction-amplitude phase, and it has to be taken from some model. Hence, SC-SE-PWA must be inherently model dependent. All these statements were demonstrated on the toy model of pseudoscalar meson production.

In this paper this fact is quantitatively demonstrated on the real data in the single-channel single-energy model AA/PWA of Ref. [1] which combines amplitude and partial wave analysis in one self-consistent, two-step model for the the world collection of data in η-photoproduction. I first perform the amplitude analysis (AA) of the world collection of data fixing the amplitude phase to the theoretical ED BG2014-2 phase, varying only absolute values as the free parameters. In the second step I perform a constrained partial wave analysis (TPWA) where multipoles for the $L = 0$ to $L = 5$ are free parameters, but the fit was constrained by requiring that the final reaction amplitudes do not differ much from the AA amplitudes of Step 1. In this way I achieve a continuity of otherwise discontinuous multipoles. Let me immediately stress that the phase of the final solution depends strongly on the amount of penalization. The higher the penalization is, the closer the phase of the final solution is to the chosen theoretical ED phase. However, I have discovered that the quality of the agreement of the final fit with the data (χ^2) notably depends on the amount of penalization. If the penalization is only moderate one obtains notably better agreement with the data than in the case when penalization is strong. However, the price paid is that for the better χ^2 with the milder penalization the departure of the final phase from the theoretical BG2014-2 phase is bigger. The next step was finding poles of the obtained solution. As one is dealing with SC-SE-PWA, the preferable option is to use the L+P formalism. As one starts from the BG2014-2 ED solution, I have first extracted poles from ED multipoles using the L+P formalism. I have shown that the extracted poles, qualitative and quantitative, perfectly correspond to published values obtained by analytic continuation of theoretical coupled-channel amplitudes into the complex energy plane. Then, using the same L+P model (same number of Pietarinen coefficients, same number of variable parameters, etc). I have searched for the poles of Sol 1, whose phase is much closer to the ED phase than for Sol 2. Poles turned out to be much less stable, and notably different from the poles extracted from the ED solution. I attributed this instability to spoiling the phase with respect to the input ED phase. To test this hypothesis I have repeated the AA/PWA model with much stronger penalty where the size of penalization is determined by the penalty function coefficient λ_{penalty}—see Eq. (2) of Ref. [1] (I have increased the penalty function coefficient from 10 in Ref. [1] to 500 for this publication), and obtained a new solution Sol 1/21 which is a compromise between a good fit and correct analytic structure. As expected, the overall agreement with the data was slightly spoiled, but the analytic structure of the new solution was notably improved. And this was what I wanted to prove.

II. FORMALISM

In this paper the improved formalism of AA/PWA formulated in Ref. [1] is used. As PWA is without constraining conditions inherently discontinuous, inspired by the idea of fixed-t analyticity of Ref. [7] a two-step procedure which in the second step constrains the PWA with reaction amplitudes obtained in the AA from the previous step has been formulated. As the phase because of continuum ambiguity cannot be determined out of the data in any single-channel analysis, one had to take the phase from well-known coupled-channel theoretical model, and the BG2014-2 [4] model was chosen. In the AA of the original publication only four out of eight available η-photoproduction observables were fitted with absolute values as free parameters, and these were phase independent observables $d\sigma/d\Omega$, Σ, T, and P. This actually was the exact single energy amplitude reconstruction (four observables for four absolute values). In this publication I have extended the number of observables to all eight available observables, so this was not an exact amplitude reconstruction, but the actual fit. The phase is, as before, fixed to the BG2014-2 phase. In the ideal case of an infinitely precise complete experiment, fixing the phase is enough to obtain unique amplitudes, continuous in both energy and angle. However, as the existing database is far from a complete and self-consistent data set, I enforced the continuity by penalizing the fit with the obtained result at neighboring energies.[3] In this way the set of continuous reaction amplitudes was obtained, and all deviations from the perfectly smooth analytic function were due only to experimental errors. The next step was to obtain corresponding multipoles. In the ideal case of a complete set of exact observables available at sufficiently dense energies and angles, multipoles extraction is trivial, and boils down to integral over Legendre polynomials of a proper combination of amplitudes. However, due to the incompleteness of the available database this is not possible. So, I have applied a penalty function technique where the PWA of the data is penalized with the result of the AA step. This is the step where the departure in phase of my final result from the initially used ED phase occurs. As it turns out, the agreement of the obtained result with the data in the AA step is not perfect, and better agreement with the data can be obtained in constrained PWA. The analysis of the penalty function of Step 2 shows that the improvement in χ^2 is achieved mainly because of a deviation of the phase of

[3]It is possible as this is a single-energy analysis, meaning that the minimization is performed independently at each chosen energy.

the final solution from the input phase which is fixed in the AA step.

For the convenience of the reader I repeat the essential equations governing the two-step AA/PWA method from Ref. [1]. The most standard, classic approach was to penalize partial waves by requiring that fitted partial waves reproduce the observable $\mathcal{O}$ and are at the same time close to some partial waves taken from the theoretical model

$$\chi^2(W) = \sum_{i=1}^{N_{\rm data}} w^i \left[\mathcal{O}_i^{\rm exp}(W,\Theta_i) - \mathcal{O}_i^{th}(\mathcal{M}^{\rm fit}(W,\Theta_i))\right]^2 + \lambda_{\rm pen} \sum_{i=1}^{N_{\rm data}} [\mathcal{M}^{\rm fit}(W,\Theta_i) - \mathcal{M}^{th}(W,\Theta_i)]^2, \quad (1)$$

where

$$\mathcal{M} \stackrel{\rm def}{=} \{\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_j\},$$

w_i is the statistical weight, and j is the number of partial waves (multipoles). Here, $\mathcal{M}^{\rm fit}$ are fitting parameters and $\mathcal{M}^{th}$ are continuous functions taken from a particular theoretical model. Instead, the possibility to make the penalization function independent of a particular model was first formulated in Karlsruhe-Helsinki πN elastic PWA by Höhler in the mid 1980s [7]. Partial waves which are inherently model dependent are replaced with the penalization function which was constructed from reaction amplitudes which can be, in principle, directly linked to experimental data without any model in the amplitude reconstruction procedure. So, Eq. (1) was changed to

$$\chi^2(W) = \sum_{i=1}^{N_{\rm data}} w^i \left[\mathcal{O}_i^{\rm exp}(W,\Theta_i) - \mathcal{O}_i^{th}(\mathcal{M}^{\rm fit}(W,\Theta_i))\right]^2 + \mathcal{P},$$

$$\mathcal{P} = \lambda_{\rm pen} \sum_{i=1}^{N_{\rm data}} \sum_{k=1}^{N_{\rm amp}} |\mathcal{A}_k(\mathcal{M}^{\rm fit}(W,\Theta_i)) - \mathcal{A}_k(\mathcal{M}^{\rm pen}(W,\Theta_i))|^2, \quad (2)$$

where $\mathcal{A}_k$ is the generic name for any of reaction amplitudes (invariant, helicity, transversity). However, one is now facing two challenges: to get reaction amplitudes which fit the data, and also to make them continuous. In the Karlsruhe-Helsinki case it was accomplished by implementing fixed-t analyticity and fitting the database for fixed t. So, the first step of the KH fixed-t approach was to create the database $\mathcal{O}(W)|_{t=\rm fixed}$ using the measured base $\mathcal{O}(\cos\theta)|_{W=\rm fixed}$, and then to fit them with manifestly analytic representation of reaction amplitudes for a fixed t. Manifest analyticity was implemented by using the Pietarinen decomposition of reaction amplitudes. Then the second step was to perform a penalized PWA defined by Eq. (2) in the fixed-W channel where the penalizing factor $\mathcal{A}_k(\mathcal{M}^{\rm pen}(W,\Theta_i))$ was obtained in the first step in a fixed-t channel. In that way a stabilized SE PWA was performed. This approach was revived very recently for SE PWA of η and π^0 photoproduction by the Mainz-Tuzla-Zagreb collaboration, and analyzed in details in Refs. [8,9]. The alternative is proposed.

One also uses Eq. (2), but the penalizing factor $\mathcal{A}_k(\mathcal{M}^{\rm pen}(W,\Theta_i))$ is generated by the amplitude analysis in the same, fixed-W representation, and not in the fixed-t one. This simplifies the procedure significantly, and avoids quite some theoretical assumptions on the behavior in the fixed-t representation.

Similar to Refs. [7,9] a two-step process was also used:

Step 1: Amplitude analysis of experimental data in fixed-W system to generate penalizing factor $\mathcal{A}_k(\mathcal{M}^{\rm pen}(W,\Theta_i))$.

Step 2: Penalized PWA using Eq. (2) with the penalization factor from *Step 1*.

However, in this paper I improve the procedure of Ref. [1]. In that reference for *Step 1* (AA) only the four phase independent observables $d\sigma/d\Omega$, Σ, T, and P were fitted, obtained for absolute values as exact amplitude reconstruction,[4] and here I extend the process to all eight available observables. The phase is in both cases fixed to the theoretical BG ED phase. In this way one gets the best set of transversity amplitudes which maximally reproduce all available observables for the given phase. However, this is where the model dependence starts. The transversity amplitude phase of constraining amplitudes is fixed, so agreement of the fit for the four phase dependent observables E, F, G, and H can be improved only through the absolute value but they are also phase dependent, so the flexibility of the fit is limited. Possibility of further improvement for these observables can only be achieved in *Step 2*, constrained PWA. It is clear that the amount of departure from the BG ED phase depends on the level of penalization. Smaller penalization means a bigger departure from the BG phase, the χ^2 for the phase dependent data improves, but partial waves get less continuous and mixed. As it will be shown later in this paper, this is clearly felt by the L+P pole extraction method. For the strong penalization the analytic structure of multipoles (pole content) strongly resembles the analytic structure corresponding to the BG ED model (and PDG therewith), for the weaker penalization the position of poles gets much less precise.

And now I am bound to say something about the importance of the reaction amplitude phase. Continuum ambiguity forbids to conclude onto the correct phase in any single channel analysis because unitarity loss to other channels starts after the first inelastic threshold opens. The only way to solve the continuum ambiguity problem is to reintroduce the unitarity introducing the coupled-channel formalism. If one picks the phase in a single-channel analysis arbitrarily by hand, one is departing from the genuine phase, the phase in which partial waves do not mix, and via the angular part of continuum ambiguity I introduce pole transfer from one partial wave into another. However, each coupled-channel model by construction results in the nonmixing pole solution. Namely, some form of interaction introducing poles is formulated, and the background contribution is added to it. Then, the data in all channels are simultaneously fitted forcing the phase to be the correct one, and the pole nonmixing situation is established. Background contributions automatically

[4]Transversity amplitude representation is used.

enforce the phase to be a nonmixing one. Needless to say is is that all coupled-channel models should end up with the same phase in the ideal case, but incompleteness of the data forbids that to happen. Therefore, phases of different models [4,10–12] are somewhat different, and one cannot avoid this. However, fixing the phase to a phase of a particular model ensures to obtain the nonmixing pole solution; departure from it automatically enforces pole mixing, so the analytic structure of such a solution is spoiled. So, one can chose a different phase, a phase coming from any model, but it has to be the proper phase originating from that model. Free, uncontrolled departure from the ED model phase is not allowed.

Of course the question is purely quantitative: How much is one allowed to depart from the "diagonal" phase in an uncontrolled way to maintain the correct analytic property. In other words, the question is how much is one allowed to reduce the importance of the penalty function and maintain the correct analyticity.

III. RESULTS AND DISCUSSION

A. Analytic structure (pole content) of Sol 1

As one is interested in analyzing the pole structure of the obtained solution Sol 1 which is the SE quantity, the most sophisticated method which enables pole extraction from SE quantities is used, and that is the Laurent+Pietarinen (L+P) formalism [5]. This, relatively recent model is based on fitting, instead of exact mathematical analytic continuation methods. The assumption is simple: one does not construct a complicated analytic function on the real axis which is the solution of the elaborated theoretical model, and continue it into the complex energy plane, but instead, using the most general principles, one constructs the simplest analytic function in the complex energy plane with a number of free parameters, and fits them to the data on the real axis up to the precision allowed by the data. The method entirely relies on analyticity. As it must be assumed that each physical process is described by an analytic function, one starts with its Laurent decomposition. It is known that by Laurent theorem[5] any analytic function is locally, within limited area of convergency, uniquely determined with its poles and cuts, and cuts are generated by channel opening branch points. For the singular part the method therefore has four parameters per pole, and for the regular part one uses the fast converging expansions over conformal variables generated by a chosen number of relevant branch points. This power series, in particular branch points, actually parametrize the most general function which can exist having exactly this branch point. As the analytic function is entirely determined by its poles and cuts, by choosing enough poles and covering all relevant branch points, the obtained solution represents the simplest analytic function which corresponds to the data on the real axis. The simplicity is enforced by choosing the lowest number of free parameters for poles and cuts which reproduce the data within given precision.

The model is a standard, coupled-multipole model of Ref. [5]. It uses three Pietarinen expansions, first and third branch points are left free. The middle branch point is fixed to the η-photoproduction threshold. The number of Pietarinen terms is limited to 3–5 per expansion, and starting number of poles is set to the accepted three-, four-star PDG resonances for the given multipole.

First one uses the described L+P model to extract pole parameters from the BG 2014-2 solution of Ref. [4] to obtain the reference points, and then use these reference values to extract the poles from Sol 1 of Ref. [1]. Extracting poles from BG 2014-2 multipoles is straightforward, results are very confident, and completely correspond to the values of BG2014-2 multipoles given in the literature [4,13]. Observe that one can compare only pole positions with the literature, residues for the η-photoproduction process are extracted in this article. The L+P results for BG 2014-2 and Sol 1 are given in Table I. It is clearly visible that some pole parameters extracted from Sol 1, especially for higher multipoles, very poorly match the corresponding values extracted from the BG 2014-2 solution, and this should not be so. The agreement between the two should be good for all poles in all multipoles. Therefore, the explanation for this discrepancy has to be found.

B. Analysis of phase cependence of Sol 1

In Table I all results obtained for poles using L+P analysis are presented.

First one observes that nice and stable poles are obtained when L+P is used on the BG2014-2 model. However, some problems which will be blown up by additional ambiguities in SE analysis could be spotted even for the ED case. First, the most stable parts of the L+P formalism are pole positions. This is not at all unexpected as this is one of the main features of the L+P formalism. The main model dependence of the L+P formalism lies in pole-background separation. This part is rather arbitrary. The background is completely unknown, and is practically fitted to the input data, so a direct consequence is that residues of the singular part must strongly depend on the way pole-background separation is performed (every change in the background part is counterbalanced by the change in residue). On the other hand, pole position (position of the singularity in the complex energy plane) is, for all possible backgrounds, always the same. So, a direct consequence is that residues are much less precisely determined in the L+P formalism, and may have some meaning only in the analysis of ED functions. There exists another reason for the uncertainty in residues in inelastic processes like η photoproduction, and this is the ambiguity in the reaction amplitude phase. As the phase can be determined only when multichannel unitarity is restored, the phase in singe-channel analyses is unknown. One also knows that the change in the angular part of the reaction amplitude phase mixes multipoles, so residues associated with a certain pole also depend on the chosen phase. So, this is another source of ambiguity for the

[5]More precisely Mittag-Leffler theorem; see discussion in Ref. [5].

TABLE I. Pole parameters for BG 2014-2, Sol 1, and Sol 1/21 extracted using L+P expansion. M_i, Γ_i, r_i, and Θ_i, $i = 1, 2$ are pole masses, widths, absolute values of the residue, and its phase, while χ^2 and χ^2_{red} are total and reduced χ-squared values (reduced χ-squared is defined as total χ-squared divided by the difference of total number of points and free fitting parameters). Particle Data Group values from Ref. [13] are for reference given in bold text.

	Model	M_1	Γ_1	$\lvert a_1\rvert$	Θ_1	M_2	Γ_2	$\lvert a_2\rvert$	Θ_2	χ^2	χ^2_{red}
S_{11} $1/2^-$	PDG	**1510(19)**	**130(20)**	-	-	**1655(15)**	**135(35)**	-	-		
	BG 2014-2	1498(107)	158(157)	1780(5300)	164(345)	1661(5)	85(12)	126(47)	24(19)	18	0.13
E_0^+	Sol 1	1489(66)	158(78)	2043(5054)	148(146)	1664(5)	92(9)	140(36)	37(15)	140	0.6
	Sol 1/21	1484(37)	196(189)	2926(7330)	166(172)	1662(3)	101(7)	158(26)	34(9)	97	0.43
P_{11} $1/2^+$	PDG	**1379(10)**	**175(15)**	-	-	**1700(20)**	**120(40)**	-	-		
	BG 2014-2	-	-	-	-	1698(1)	123(1)	105(2)	-90(1)	102	0.75
M_1^-	Sol 1	-	-	-	-	1730(6)	80(10)	48(12)	-22(18)	140	0.7
	Sol 1/21	-	-	-	-	1660(6)	112(13)	49(15))	-168(16)	106	0.47
	Sol 1/21	1526(25)	73(37)	19(32)	-123(110)	1681(7)	103(12)	39(10))	-124(17)	25	0.11
P_{13} $3/2^+$	PDG	-	-	-	-	**1675(15)**	**250(150)**	-	-		
	BG 2014-2	-	-	-	-	1705(7)	195(21)	$\begin{pmatrix}38(10)\\38(13)\end{pmatrix}$	$\begin{pmatrix}-133(15)\\-107(16)\end{pmatrix}$	58	0.23
$\begin{pmatrix}E_1^+\\M_1^+\end{pmatrix}$	Sol 1	-	-	-	-	1879(46)	200(68)	$\begin{pmatrix}328(359)\\260(280)\end{pmatrix}$	$\begin{pmatrix}-8(44)\\67(50)\end{pmatrix}$	310	0.6
	Sol 1/21	-	-	-	-	1714(7)	102(13)	$\begin{pmatrix}10(3)\\1(1)\end{pmatrix}$	$\begin{pmatrix}-167(16)\\20(47)\end{pmatrix}$	297	0.6
D_{13} $3/2^-$	PDG	**1510(5)**	**110(10)**	-	-	**1700(50)**	**200(100)**	-	-		
	BG 2014-2	1508(3)	106(7)	$\begin{pmatrix}52(11)\\25(6)\end{pmatrix}$	$\begin{pmatrix}122(12)\\118(13)\end{pmatrix}$	1664(76)	399(159)	$\begin{pmatrix}119(155)\\72(86)\end{pmatrix}$	$\begin{pmatrix}73(71)\\103(77)\end{pmatrix}$	1.7	0.06
$\begin{pmatrix}E_2^-\\M_2^-\end{pmatrix}$	Sol 1	1528(23)	63(37)	$\begin{pmatrix}11(22)\\2(3)\end{pmatrix}$	$\begin{pmatrix}-160(82)\\148(98)\end{pmatrix}$	1721(6)	64(13)	$\begin{pmatrix}10(3)\\4(1)\end{pmatrix}$	$\begin{pmatrix}149(19)\\-168(18)\end{pmatrix}$	368	0.8
	Sol 1/21	1525(23)	121(60)	$\begin{pmatrix}37(52)\\24(39)\end{pmatrix}$	$\begin{pmatrix}-156(91)\\158(94)\end{pmatrix}$	1664(12)	121(24)	$\begin{pmatrix}11(6)\\13(7)\end{pmatrix}$	$\begin{pmatrix}-31(33)\\46(33)\end{pmatrix}$	50	0.1
D_{15} $5/2^-$	PDG	-	-	-	-	**1660(5)**	**135(15)**	-	-		
	BG 2014-2	-	-	-	-	1673(4)	225(6)	$\begin{pmatrix}1(0.3)\\23(1)\end{pmatrix}$	$\begin{pmatrix}54(17)\\-17(6)\end{pmatrix}$	45	0.16
$\begin{pmatrix}E_2^+\\M_2^+\end{pmatrix}$	Sol 1	-	-	-	-	1784(1)	11(1)	$\begin{pmatrix}0.5(0.01)\\0.8(0.1)\end{pmatrix}$	$\begin{pmatrix}23(9)\\95(9)\end{pmatrix}$	310	0.6
	Sol 1/21	-	-	-	-	1659(10)	145(23)	$\begin{pmatrix}6(2)\\9(4)\end{pmatrix}$	$\begin{pmatrix}17(20)\\-40(22)\end{pmatrix}$	90	0.2
F_{15} $5/2^+$	PDG	-	-	-	-	**1675(10)**	**120(15)**	-	-		
	BG 2014-2	-	-	-	-	1677(1)	117(1)	$\begin{pmatrix}13(1)\\7(0.5)\end{pmatrix}$	$\begin{pmatrix}147(1)\\145(11)\end{pmatrix}$	12	0.05
$\begin{pmatrix}E_3^-\\M_3^-\end{pmatrix}$	Sol 1	-	-	-	-	1767(2)	34(4)	$\begin{pmatrix}3(0.5)\\2(0.5)\end{pmatrix}$	$\begin{pmatrix}-91(9)\\33(10)\end{pmatrix}$	690	1.5
	Sol 1/21	-	-	-	-	1690(4)	166(11)	$\begin{pmatrix}11(2)\\23(4)\end{pmatrix}$	$\begin{pmatrix}172(8)\\164(77)\end{pmatrix}$	156	0.34

residues.[6] Because of these two reasons, in the analysis of SE data, residues are something just a little bit more than pure fitting parameters. This is obvious from Table I. Even for ED input, the confidence level of residues is very low, sometimes on the level of 100% (for the S_{11} partial wave even much lower for the first resonance, but this is altogether another story[7]). However, poles are pretty precisely determined. Even for pole position determination, one can see a clear hierarchy in confidence level: pole position is always more precisely determined than its width (how deep the pole is in the complex energy plane is more unreliable). This is the direct consequence of the fact that all the fits are done with the data lying on the real

[6]Observe that this is not so in elastic processes like πN elastic scattering as unitarity gets violated only above the first inelastic threshold. So the phase is up to this energy fully determined. First uncertainties occur at higher energies. However, let us observe that the phase of inelastic processes like π photoproduction is also fairly well determined at lower energies because of Watson's theorem which connects the π-photoproduction phase with the πN elastic phase at lower energies. Uncertainties also rise with energy, but are still small in the N^* energy range.

[7]First the S_{11} pole is very near the threshold of η photoproduction which is taken as the fixed branch point in the L+P formalism, so to have some impact onto the process the residue in addition to being unprecise must also be very high. However, this is typical even for ED models [12].

axis of the complex energy plane, so it is more sensitive to the position which is more directly influencing the shape of the scattering matrix on the real axis.

Making an L+P analysis of the SE Sol 1, and even taking these facts into account, one can see that the analytic structure of Sol 1 is unclear, and clearly rather different from the the ED solution the model has started with. With the exception of the E_{0+} multipole all poles are notably shifted in energy with the results obtained for the ED solution, which is not allowed, and pole width is unreasonably narrow. This is also not admissible, so there must be some effect which spoils the analytic structure of Sol 1.

From my earlier research [6] I know that the natural candidate for such an effect is the change of the reaction amplitude phase. As the essential part of the proposed AA/PWA model is the penalty factor $\mathcal{P}$, and the possible change of phase is introduced via this factor, I shall put this factor under a magnifying glass. This factor introduces two phenomena, ensures the continuity in partial waves, and ensures the proximity of phase to the known input phase of the ED model which, I repeat, is a matter of my choice. In Ref. [1] we have opted for the moderate constraint, and used $\lambda_{\text{pen}} = 10$. To fully test the consequence of such a choice, I have, in this article, generated a new solution with a much stronger constraint using $\lambda_{\text{pen}} = 500$ called Sol 1/21. The outcome is shown in Figs. 1 and 2. To quantify the amount of the likeness of the new solution phase with the original phase, I also show the penalty function $\mathcal{P}$ for both solutions in Fig. 3.

First from Figs. 1 and 2 one sees that new multipoles of Sol 1/21 (red symbols) are notably different from the old ones of Sol 1 (black squares), and that they are much smoother. From Fig. 3 one sees that the normalized penalty function of Sol 1/21 (penalty function divided by the penalty coefficient λ_{pen}), as expected, is much lower that the normalized penalty function of Sol 1, and uniform over energies. Let me stress that it should not vanish, as transversity amplitudes generated in *Step 1* are in principle discontinuous both in energy and angle, and the continuity of the solution is imposed by the very weak condition of constraining the fit to the neighboring energy solution. For Sol 1, a deviation of the penalty function from the constraining value is also notably rising with energy. From this figure alone it is not clear whether this increase is due to a change of phase or to the change of absolute value as the penalty function $\mathcal{P}$ is defined on full amplitudes. Therefore, I, in Fig. 4 on the same scale, show only the difference of absolute values of Sol 1 and Sol 1/21. One sees that these quantities are very stable in energy, both of the same order of magnitude, and much lower then the penalty function itself. This shows that an increase of penalty function of Sol 1 is solely due to the change of phase. In addition, let me illustrate another claim made before: strengthening the constraint (imposing the phase closer to the ED phase) spoils the agreement with the data as the phase of the ED solution is not ideal for phase dependent observables. Therefore, in Fig. 5 I show χ^2/N_{data} (χ^{**2} per data point) for both solutions. The χ^2/N_{data} for Sol 21/1 is still good, but the χ^2/N_{data} for Sol 1 is notably better. Light grey lines and red dots in Fig. 5 deserve some more detailed explanation which is taken from Ref. [1].

Two sets of interpolated data are generated:

Set 1 : All σ_0 data are used, and all spin data are interpolated. So the whole minimization is performed on the set which consists of σ_0 data + observables interpolated at energies and angles where σ_0 is measured. *InterpolationOrder=3* level interpolation code from Mathematica has been used. These data are marked in light grey. Observe that all data are very dense, but in practice the only factually measured data are σ_0 values, all other data are obtained by interpolation from measured values. This set of data is somewhat model dependent, and serves only as an indication. This set will be used in *Step 1*.

Set 2 : One uses only part of σ_0 data at energies where at least one additional spin observable is exactly measured. This set is not so dense in energy, but model dependence is reduced. The results corresponding to this set are denoted with red discrete symbols.

Let me summarize the results of L+P analysis shown in Table I. As said before, poles of the energy dependent solution BG2014-2 correspond to the values published by the Bonn-Gatchina group [4] as shown in Ref. [13]. Extraction of the first pole in the S wave is unreliable as the pole lies in the vicinity of the η-photoproduction threshold. This means that it lies on a different Riemann sheet, so the residue value has to be big to have any influence upon the data. However, the L+P fit unquestionably needs this pole, but the residues are big, and an unreliable meaning that one can generate similar solutions with a wide choice of different backgrounds. This confirms a general problem of residue in the L+P formalism. As pole-background separation in Laurent expansion is not theoretically well defined, one relies on a fit, so unquestionably one can have different backgrounds which differently combine with the residue of the singular part giving identical results. This is the general feature of all fits. However, the agreement with known values is good.

Next I summarize the analysis of SE multipoles obtained with the AA/PWA method. First let me stress that the poles of SE solutions should be close to BG2014-2 values, but definitely should not be identical to them. Namely, ED BG2014-2 is a coupled-channel model, so it is expected that the pole parameters will be formed as the overall agreement of the fit to all channels. SE partial waves are on the other hand a single-channel quantity, and will be ideally matched only to this particular channel. It is theoretically known that these poles should be identical in all channels, but in practice experimental error introduces unwanted uncertainty, especially when all channels are treated simultaneously. Therefore, poles of ED models are expected to differ from poles of SE PWA. One can only expect that our results are within the confidence level of PDG. In this table one systematically sees that Sol 1 poles are less certain, sometimes with unrealistic errors, and unrealistic residues (P_{13} 3/2+ partial wave), and the obtained χ^2 is poor. On the other hand, Sol 1/21, the solution with the phase much closer to the ED phase, has much more reliable poles. First one finds that the number of poles needed to obtain a good L+P fit, is identical to the number of 4* resonances

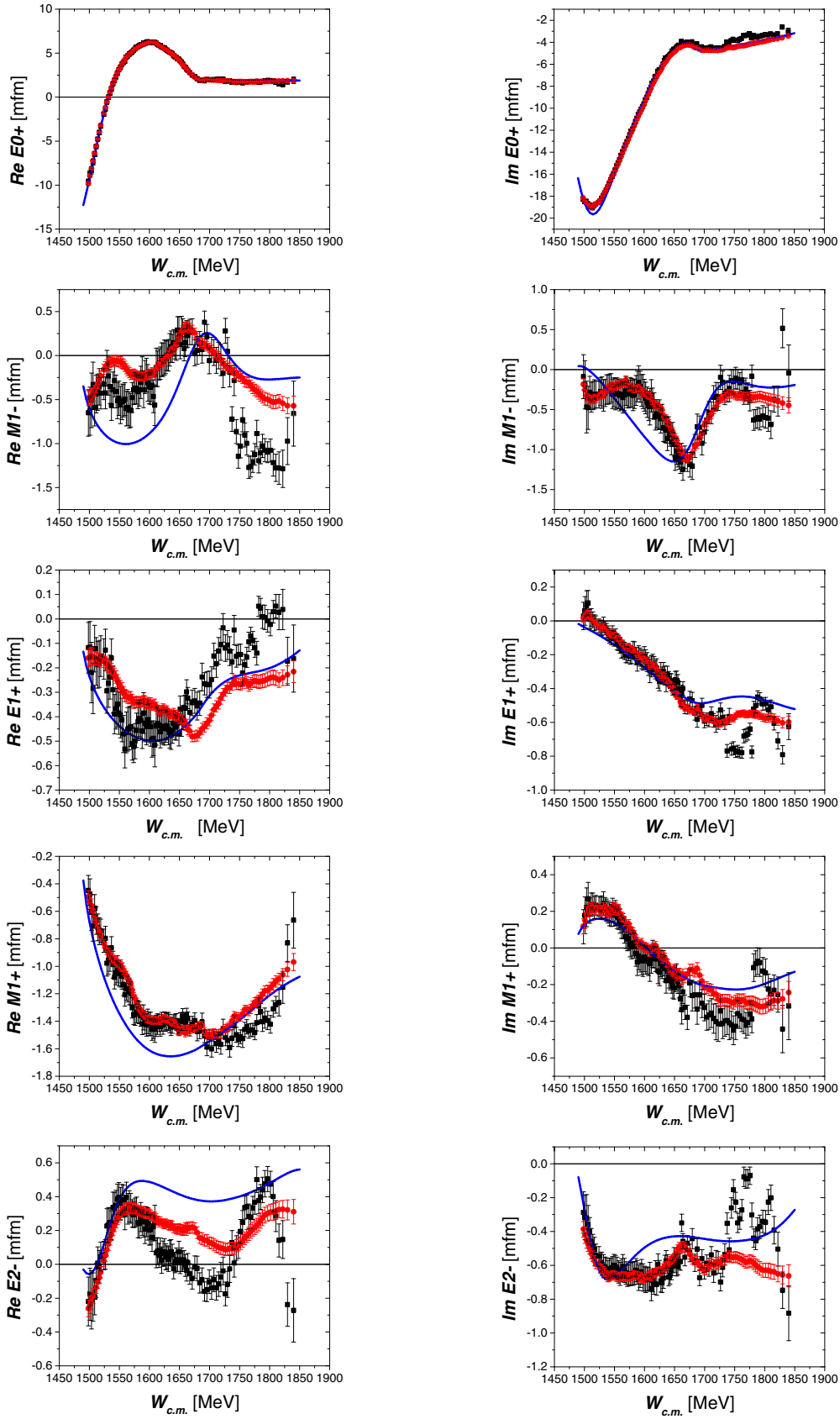

FIG. 1. Comparison of lowest multipoles for Sol 1 [1] (black squares), Sol 1/21 (red full circles) this article, and BG 2014-2 ED model [4] (full blue line).

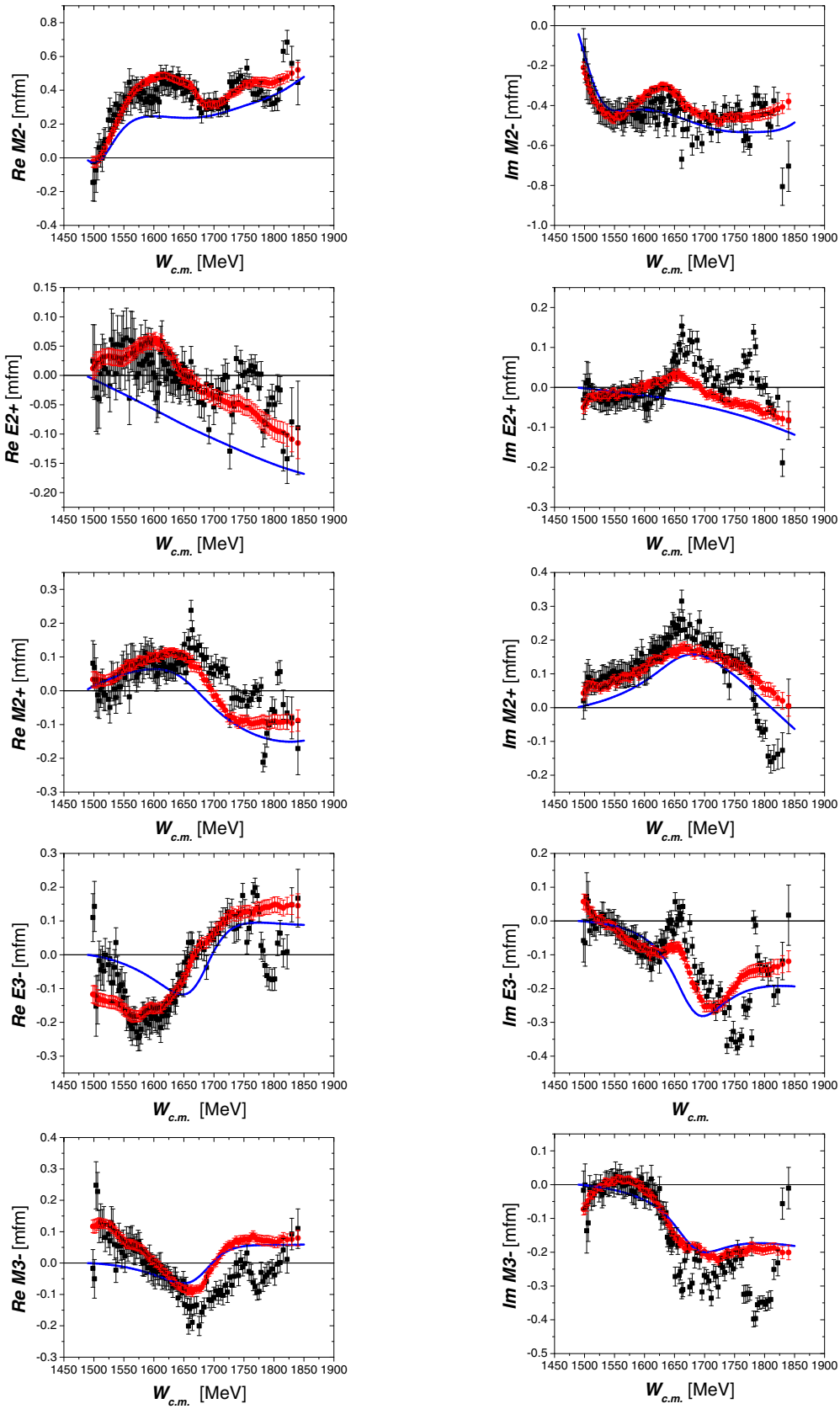

FIG. 2. Continuation of Fig. 1.

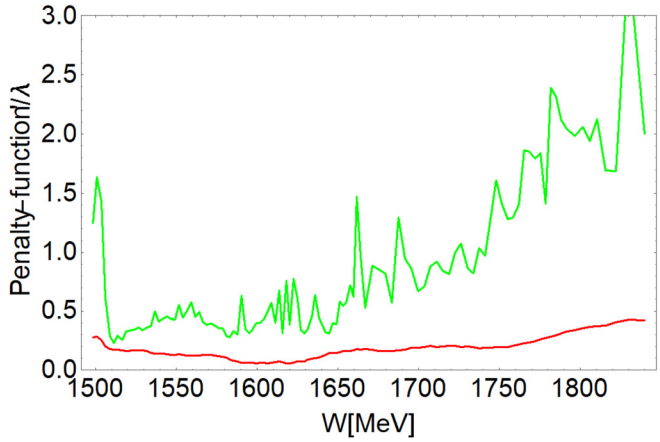

FIG. 3. Comparison of penalty function $\mathcal{P}$ for Sol 1 (green line) and Sol 1/21 (red line).

reported in PDG.[8] Pole masses are in principle within one standard deviation with PDG results, pole widths are somewhat less reliable. In addition, looking at Figs. 1 and 2 one sees that the Sol 1/21 solution is much smoother than Sol 1 as some structures especially at higher energies are smeared out, and much closer to theoretical BG2014-2 ED value. This is not at all surprising, and this solution is much more constrained, so has to be smoother and closer to the ED value.

C. New P_{11} 1/2+ state?

As can be seen from Table I, the fit with only the standard $N(1710)$ $1/2+$ state is poor for this partial wave when compared to the rest. However, if one allows for one extra pole in the fit, the result is drastically improved, and the χ^2 drops by the factor of 4 (for details see Table I). The difference in fits is shown in Fig. 6. The needed pole falls too far away to be interpreted as the known $N(1440)$ $1/2+$ state, but is pretty reliable. It has a well-defined mass of 1526 ± 25 MeV and width of 73 ± 37 MeV with a reasonable residue.

The obvious reason to doubt this state is that it has not been seen in other processes. However, one is allowed to speculate that this is not entirely correct. In Fig. 7 the relevant partial wave for other processes is shown: πN elastic scattering from KH80 [7,14], SE part of πN photoproduction from Ref. [15] (discrete data points), and π^0 photoproduction from Ref. [9]. First it is important to stress that this hypothetic state is never seen in ED models. This is not surprising, as to get a resonance in the ED model you have to put it in by hand in some way. As this resonance was never discussed, it was never added to the ED model, so it cannot be found there. This is also not surprising as strong, direct, experimental indications were never found. However, some SE models show some problems (discontinuities, larger uncertainties) exactly on that energy. First, πN elastic scattering KH80 [7,14] is a fixed-t analysis, so by default it is an energy independent one. It is depicted in Fig. 7(a). You see that it contains two distinct structures, one identified as $N(1440)$ $1/2+$ at low energies, and the second at high energies above 2000 MeV identified as $N(2100)$ $1/2+$. However, in addition one sees two energy ranges of a notable departure from linearity: one at ≈1700 MeV (indicated with a yellow circle), but one also around 1550 MeV (indicated by the full red circle). Historically, there had been, for quite some time, a strong dispute about the effects at the energy range of 1700 MeV. GWU/SAID PWA claimed smooth behavior without any poles, while KH80 [7] and some coupled-channel analyses [14,16] identified it with the $N(1710)$ $1/2+$ resonance. The dispute was solved in favor of $N(1710)$ $1/2+$ when inelastic-channel reactions like $\pi N \to K\Lambda$ and $\pi N \to \eta N$ were included in coupled-channel formalisms [10,14]. However, the second area of nonlinearity around 1550 MeV (marked with full red circles) was never discussed and left unexplained. Another SE PWA for π photoproduction [15] depicted in Fig. 7(b) also shows continuity in the ED solution, but a definite departure from linearity in this energy range (indicated with a full red circle) is observed. The third process is a fixed-t SE π^0-photoproduction analysis depicted in Fig. 7(c). Here, the SE analysis also shows notable departures from the smooth, ED model, exactly in that range (also indicated with full red circles). Neither effects have ever been explained.

Let me conclude that analyzed few-body processes of πN elastic scattering, π, and η photoproduction show the disturbance in that energy range, and our results in η photoproduction strongly require a new resonant state in that range which couples dominantly to inelastic channels. So, it will be very interesting to see if such effect will be confirmed or refuted when other inelastic photoproduction data are also included in SE analyses. Of course, there always exists another possibility: the phase we use is not a correct η-photoproduction phase, so this resonance is the result of angular mixing of other multipoles, and has different quantum numbers. In other words, it could easily be the reflection of either $N(1535)$ $1/2-$ or $N(1520)$ $3/2-$ states due to incorrect phase and angular mixing.

[8]With the exception of P_{11} $1/2-$ partial wave what will be discussed later.

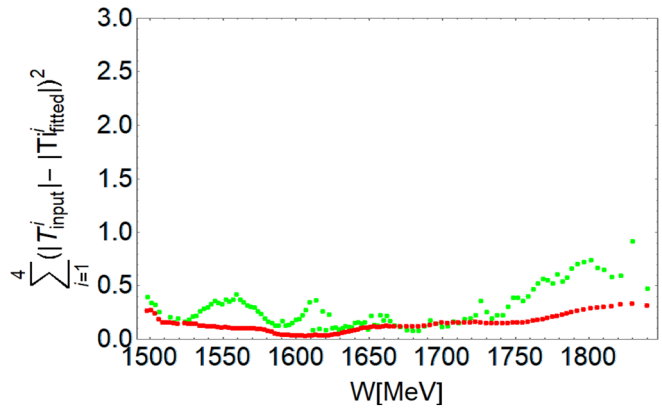

FIG. 4. Comparison of absolute values of penalty function $\mathcal{P}$ for Sol (green line) 1 and Sol 1/21 (red line).

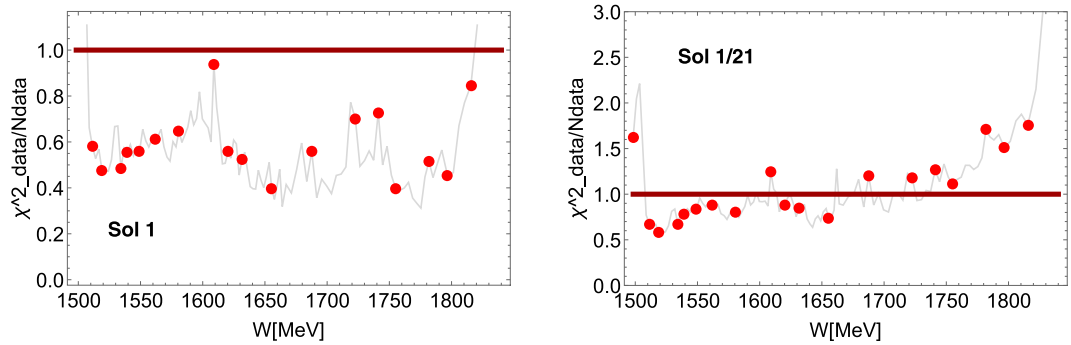

FIG. 5. Comparison χ^2/N_{data} for Sol 1 and Sol 1/21. The meaning of the grey lines and red dots is taken from Ref. [1], and is given in the text.

IV. CONCLUSIONS

Analytic structure of Sol 1 of Ref. [1] is unclear in spite of the fact that it fits the data almost perfectly. Following the theoretical arguments of Ref. [6] one attributes these effects to the small, but still uncontrolled phase change of reaction amplitudes. To prove this hypothesis I constrained the phase much stronger to the theoretical ED phase of Ref. [4], and generated a new solution Sol 1/21. As expected, the agreement of the fit with the data was somewhat spoiled (χ^2 was worse but still acceptable), but L+P analysis showed that the analytic structure (pole content) of the solution is much improved. Poles are clearly determined, and all of them are within one standard deviation consistent with the most recent compilation in PDG. The only exception is the P_{11} $1/2+$ partial wave where the need for an additional analytic structure in the solution was strongly needed. In spite of that, it could be interpreted as some kind of uncontrolled threshold behavior, I speculate that it might be the sign of a new, low lying P_{11} $1/2+$ state ($M = 1526 \pm 25$ MeV and $\Gamma = 73 \pm 37$ MeV). This state was never seen before in any SE PWA, but unclear and completely uninterpreted discontinuities in that energy range in πN elastic and π photoproduction were reported [7,9,14,15]. The unification of all troubles with accepting a new state is offered as the simplest explanation. Of course, another valid interpretation is that it is the result of admixture of either $N(1535)1/2-$ or $N(1520)3/2-$ states due to

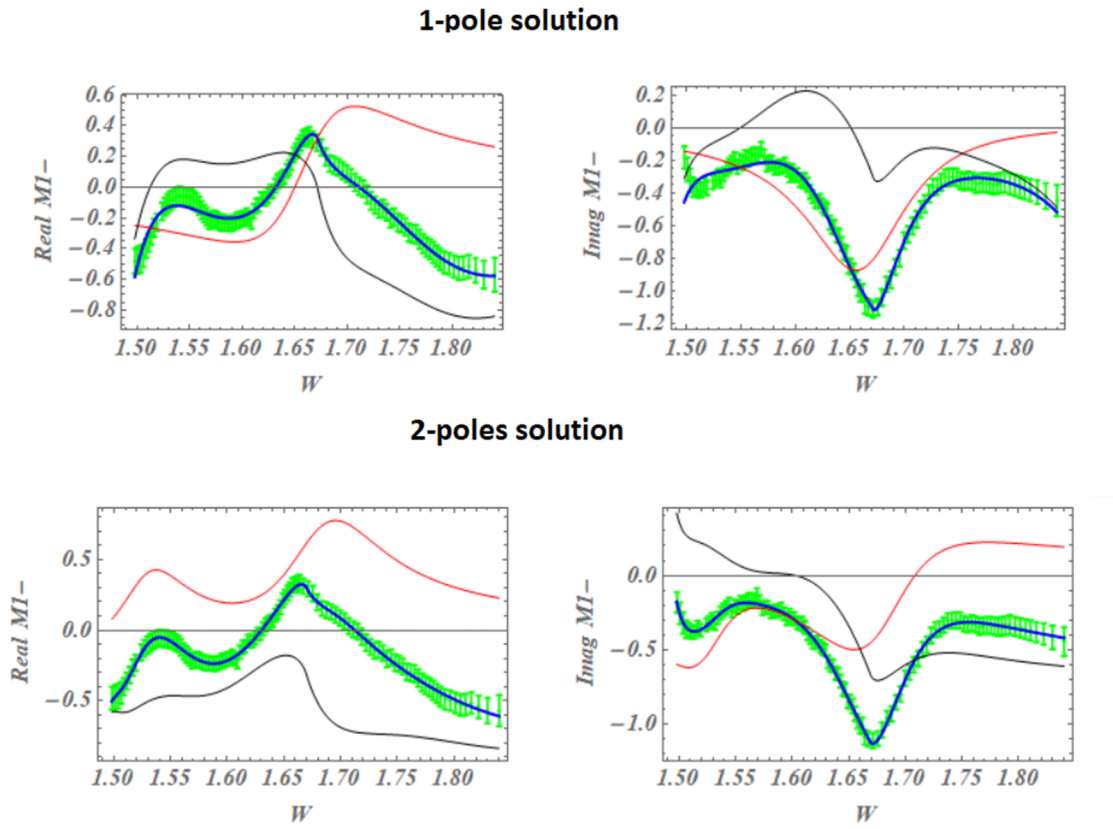

FIG. 6. One-pole and two-pole L+P fits of M_1^- multipole. Blue lines represent the L+P fits, while red and black lines represent resonant and background contributions, respectively.

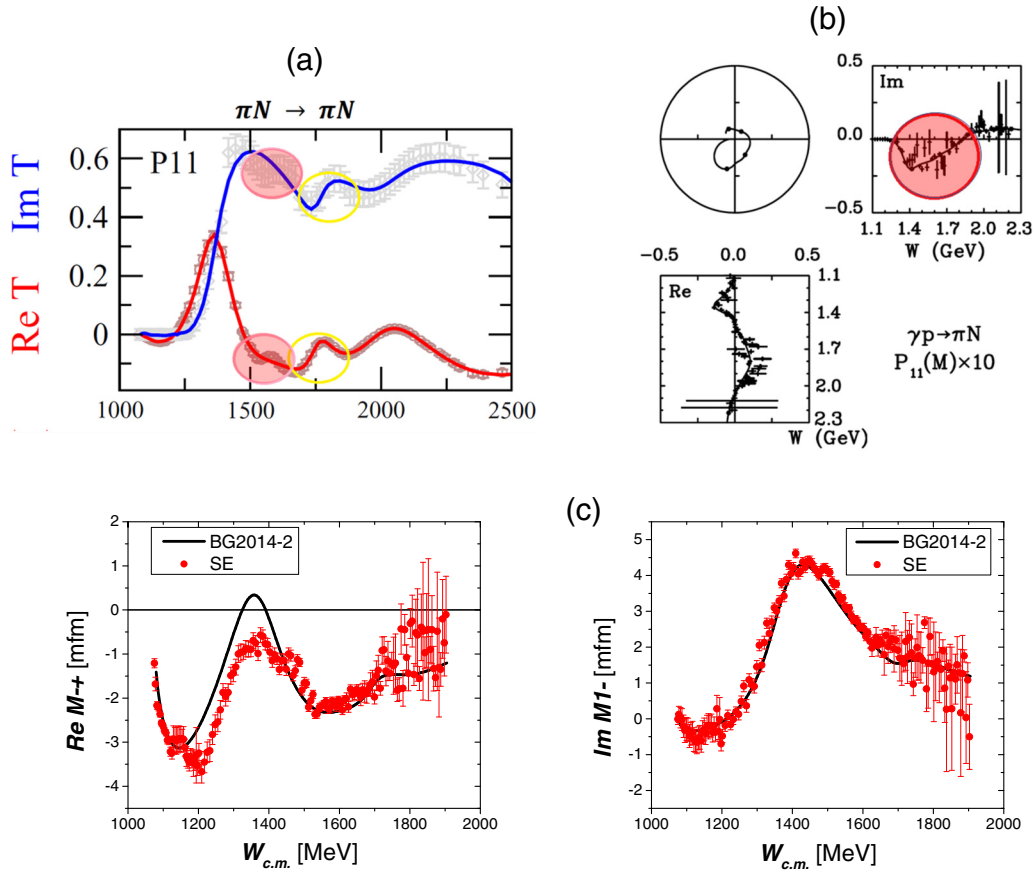

FIG. 7. Signs for the possibility of a new pole in P_{11} $1/2+$ partial wave in other processes. Filled, red, transparent circles indicate areas where obtained partial waves/multipoles show sudden, and until now, unexplained erratic, structure-like behavior. Yellow circle in a is explained in the text.

angular mixing because of the incorrectness of the used η-photoproduction phase is also possible. In any case, I need an extra pole in this multipole, either it is a genuine or a ghost one. Residues, as discussed in the text, are much less confident, and must be correlated with the phase used in the model.

This confirms that the AA/PWA model of Ref. [1] in addition to reproducing the data almost perfectly has a correct analytic structure, so it makes it a reasonable tool to be applied for a SE PWA in other processes. In the presently analyzed process of η photoproduction, strong S-wave dominance pushes the rest of the multipoles strongly down, so imprecision of currently available database strongly influences their shape. One expects that a much more stable situation in the case of $K\Lambda$ photoproduction is that the next target process will improve the situation and refine the insight into the preset problems.

It has been demonstrated that the good analytic structure of the model fails if the input reaction-amplitude phase is violated in order to improve the agreement of the result with the data which is not ideal if the input phase is strongly enforced. The needed improvement in χ^2 in AA/PWA can be accomplished only by changing the phase; and this results in the destruction of clear analytic structure. This indicates that the presently used BG2014-2 phase is good, but it is still not perfect. There is definitely quite some room for further improvements, but strictly within the framework of the coupled-channel formalism to avoid continuum ambiguity effects.

Even a small change of the angular part of the reaction amplitude phase can dramatically change the analytic structure (pole content) of obtained partial waves, so any change of the phase must be done in the controlled way, strongly correlated with other channels. As in any inelastic, single-channel model, the continuum ambiguity reaction-amplitude phase cannot be determined from the first principles, it has to be taken from somewhere, usually a coupled-channel model. Free change of the phase in any SE PWA is not allowed. So, each SE PWA must be model dependent from the first principles.

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