Extraction of elastic scattering cross-section ratio $R_{e^+e^-}$ from *ep* elastic scattering experimental data

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In this work, we present a new prediction of the positron-proton to electron-proton elastic scattering cross sections ratio $R_{e^+e^-}$. While many phenomenological studies extracted the ratio $R_{e^+e^-}$ using proposed, model-independent parametrizations of the hard two-photon exchange (TPE) corrections to electron-proton (ep) elastic scattering cross section σ_R, we do not assume prior knowledge of the functional form of the hard TPE corrections to σ_R, and use combined unpolarized, and polarized ep elastic scattering experimental data to extract the TPE contributions. We provide a simple parametrization of the TPE corrections to σ_R, along with the uncertainties associated with the model dependence of the extractions, and use our TPE parametrization to predict the ratio $R_{e^+e^-}$. We compare our prediction of $R_{e^+e^-}$ to several previous phenomenological extractions, TPE-hadronic calculations, and world's data with emphasis mainly on the kinematics range of the recent precise direct measurements from the CLAS, VEPP-3, and OLYMPUS experiments.

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I. INTRODUCTION

The discrepancy on the proton's form factors (FFs) ratio $\mu_p R_p = \mu_p G_E(Q^2)/G_M(Q^2)$ as extracted using the Rosenbluth separation technique [1], and the polarization transfer (PT) or recoil polarization technique [2–4], was attributed to a missing higher order radiative corrections to σ_R, and in particular the inclusion of hard two-photon-exchange (TPE) correction diagrams [20–24]. The two techniques yield a significantly different value for $\mu_p R_p$ for $Q^2 > 1.0\,(\mathrm{GeV}/c)^2$, where they almost differ by a factor of three at high Q^2 [5–19]. While the Rosenbluth separation method predicts scaling of the ratio $\mu_p R_p \approx 1$, the PT method predicts a linearly decreasing $\mu_p R_p$ with increasing Q^2, and then flattening out for $Q^2 > 5.0\,(\mathrm{GeV}/c)^2$.

To account for TPE correction to σ_R, the real function $F(\varepsilon, Q^2)$, which represents the contributions coming from the interference of the one-photon-exchange (OPE) and TPE amplitudes, is added to the Born reduced cross section σ_{Born}:

$$\sigma_R(\varepsilon, Q^2) = [G_M(Q^2)]^2 + \frac{\varepsilon}{\tau}[G_E(Q^2)]^2 + F(\varepsilon, Q^2), \quad (1)$$

where $G_E(Q^2)$ and $G_M(Q^2)$ are the true electric and magnetic FFs of the proton, respectively, Q^2 is the four-momentum transferred squared of the virtual photon, with longitudinal polarization parameter ε, and $\tau = Q^2/4M_p^2$ is a kinematics factor, with M_p being the mass of the proton.

The impact of TPE effects on electron-proton (ep) scattering observables was studied in great details theoretically [20,23–69], phenomenologically [17,70–97], and experimentally [11–13,98]. See Refs. [23,24,91,99] for detailed reviews. Experimentally, some studies focused on measuring and/or constraining the TPE contributions to σ_R, and the ratio $\mu_p R_p$ [12,13,16,17]. Other studies examined the ε dependence and nonlinearity of σ_R [70–72], and the ε dependence of the ratio $\mu_p R_p$ [11,98] to observe any possible deviation from the OPE prediction. The measured $\mu_p R_p$ ratio showed no ε dependence consistent with OPE expectation. Phenomenologically, some studies focused on extracting the TPE contributions by using combined elastic ep cross-section and polarization measurements [17,20,72,75,78–80,82,87,89–91,93]. Other studies attempted to extract the TPE amplitudes by imposing some constraints [74,76,83,84,86,91]. However, the most direct technique used to measure TPE effect is by measuring the positron-proton to electron-proton cross sections ratio $R_{e^+e^-}(\varepsilon, Q^2)$ [100–103] as the real function $F(\varepsilon, Q^2)$ changes sign depending on the charge of the lepton (electron or positron) involved. The reduced cross section σ_R, in the Born approximation, is the measured elastic ep cross section corrected for radiative corrections such as photon radiation from the charged particle $\delta^{\pm}$, such that $\sigma_R = \sigma_{\mathrm{elastic}}(1 + \delta^{\pm})$, where the $+(-)$ sign is for positron(electron). The $\delta^{\pm}$ correction includes vertex-type corrections or charge-even terms δ_{even}, and charge-odd terms δ_{odd}, which change sign depending on the sign of the lepton involved. The δ_{odd} correction is also broken down into hard-TPE ($\delta_{2\gamma}$), and soft-TPE (δ_{soft}) contributions, such that $\delta^{\pm} = [\delta_{2\gamma} + \delta_{\mathrm{soft}} + \delta_{\mathrm{even}}]$. The measured ratio $R^{\mathrm{meas}}_{e^+e^-}(\varepsilon, Q^2)$ is now defined as

$$R^{\mathrm{meas}}_{e^+e^-}(\varepsilon, Q^2) = \frac{\sigma(e^+p \to e^+p)}{\sigma(e^-p \to e^-p)} = \frac{1 + \delta_{\mathrm{even}} - \delta_{\mathrm{soft}} - \delta_{2\gamma}}{1 + \delta_{\mathrm{even}} + \delta_{\mathrm{soft}} + \delta_{2\gamma}}, \quad (2)$$

and after correcting $R^{\mathrm{meas}}_{e^+e^-}(\varepsilon, Q^2)$ for both δ_{even} and δ_{soft}, the ratio is now written as

$$R_{e^+e^-}(\varepsilon, Q^2) = \frac{1 - \delta_{2\gamma}}{1 + \delta_{2\gamma}} \approx 1 - 2\delta_{2\gamma}, \quad (3)$$

where $\delta_{2\gamma} = F(\varepsilon, Q^2)/\sigma_{\rm Born}$ is the fractional hard-TPE correction to σ_R, and any deviation of $R_{e^+e^-}(\varepsilon, Q^2)$ from unity is a clear signature of hard-TPE effect.

II. LINEAR PHENOMENOLOGICAL TWO-PHOTON-EXCHANGE CONTRIBUTION

In this section, we summarize several previous phenomenological TPE studies relevant to this work, which assumed linear TPE corrections to σ_R, and were used to extract the ratio $R_{e^+e^-}$. See Refs. [78,79,83,84,86,91] for detailed review of other TPE phenomenological studies.

Based on the framework of Ref. [20], Borisyuk and Kobushkin [73] introduced the linear combination $\mathcal{G}_E = (\tilde{F}_1 - \tau\tilde{F}_2 + \frac{\nu}{4M_p^2}\tilde{F}_3)$ and $\mathcal{G}_M = (\tilde{F}_1 + \tilde{F}_2 + \varepsilon\frac{\nu}{4M_p^2}\tilde{F}_3)$ with $\tilde{F}_1$, $\tilde{F}_2$, and $\tilde{F}_3$ corresponding to $\tilde{G}_E$, $\tilde{G}_M$, and $\tilde{F}_3$, respectively. The reduced cross section σ_R was expressed in terms of the TPE amplitudes $\mathcal{G}_E$ and $\mathcal{G}_M$ after dropping terms of order α^2 as

$$\sigma_R = \tau G_M^2 + \varepsilon G_E^2 + 2\varepsilon G_E \delta\mathcal{G}_E + 2\tau G_M \delta\mathcal{G}_M. \tag{4}$$

Because $G_M > G_E$ by a factor of μ_p, the term $2\varepsilon G_E\delta\mathcal{G}_E \ll 2\tau G_M\delta\mathcal{G}_M$, and was dropped in Eq. (4). The TPE amplitude $\mathcal{G}_M$ was parametrized as $\delta\mathcal{G}_M = [a(Q^2) + \varepsilon b(Q^2)]G_M(Q^2)$, where $a(Q^2)$ and $b(Q^2)$ are functions of Q^2, and σ_R was expressed as

$$\sigma_R = \tau G_M^2 + \varepsilon\big[G_E^2 + 2\tau b(Q^2)G_M^2\big]. \tag{5}$$

In their later analysis [74], Borisyuk and Kobushkin applied the Regge Limit such that $\delta\mathcal{G}_M \to 0$, as $\varepsilon \to 1$ yielding to

$$\delta\mathcal{G}_M/G_M = a(Q^2)(1-\varepsilon) \tag{6}$$

and

$$\sigma_R = G_M^2\big[\tau + \varepsilon R_p^2 + 2\tau a(Q^2)(1-\varepsilon)\big] \tag{7}$$

with $F(\varepsilon, Q^2) = 2\tau a(Q^2)G_M^2(1-\varepsilon)$.

Based on the work of Ref. [74], Eq. (7), Qattan, Alsaad, and Arrington [79], referred to as "QAA1" throughout the text, fitted world's data on σ_R for $Q^2 \geqslant 0.39$ (GeV$/c)^2$ to the form

$$\sigma_R = G_M^2\Big(1 + \frac{\varepsilon}{\tau}R_p^2\Big) + 2a(Q^2)G_M^2(1-\varepsilon) \tag{8}$$

with R_p constrained to its value as given by Gayou [7], and G_M and the TPE coefficient $a(Q^2)$ being the parameters of the fit. The TPE coefficient $a(Q^2)$ was best parametrized as

$$a(Q^2) = [-0.0191\sqrt{Q^2} \pm 0.0014\sqrt{Q^2} \pm 0.003]. \tag{9}$$

In their later analysis, and following the same procedure of Ref. [79], Qattan, Arrington, and Alsaad [82], referred to as "QAA2" throughout the text, included the data sets used in Ref. [79], and additional low-Q^2 data from Refs. [17,104]. They used their improved parametrization of the ratio $\mu_p R_p$ given by

$$\mu_p R_p = \frac{1}{1 + 0.1430Q^2 - 0.0086Q^4 + 0.0072Q^6} \tag{10}$$

with an absolute uncertainty in the fit $\delta^2_{R_p}(Q^2) = \mu_p^{-2}[(0.006)^2 + (0.015\ln(1+{\rm Q}^2))^2]$. The extracted TPE parameter $a(Q^2)$ was best parametrized as $a(Q^2) = [0.016 - 0.030\sqrt{Q^2}]$ with Q^2 in (GeV$/c)^2$. The ratio $R_{e^+e^-}$ was extracted using the form

$$R_{e^+e^-}(\varepsilon, Q^2) = \frac{1-\delta_{2\gamma}}{1+\delta_{2\gamma}} \approx 1 - \frac{4a(Q^2)(1-\varepsilon)}{\big(1 + \frac{\varepsilon}{\tau}R_p^2\big)}. \tag{11}$$

III. EXTRACTION OF THE HARD TPE CONTRIBUTIONS

In this section, we discuss the procedure used to extract the TPE contributions to σ_R, which we use later to predict the ratio $R_{e^+e^-}$. As was discussed in Sec. II, the ratio $R_{e^+e^-} \approx (1 - 2\delta_{2\gamma})$ is usually extracted using a proposed and known phenomenological functional form of the TPE real function $F(\varepsilon, Q^2)$. However, in this work, we do not assume prior knowledge of the function $F(\varepsilon, Q^2)$, and use combined unpolarized and polarized ep elastic scattering experimental data to extract the TPE contributions to σ_R, and use the results to predict the ratio $R_{e^+e^-}$. The procedure, together with the constraints and assumptions used is outlined below where we assume:

(1) The TPE correction is responsible mainly for the discrepancy between the cross section and recoil polarization data measurements; (2) no TPE contributions to the recoil polarization measurements; (3) the reduced cross section σ_R remains linear in ε after the inclusion of TPE corrections; and (4) no TPE contribution to σ_R at $\varepsilon = 1$ or $F(\varepsilon = 1, Q^2) = 0$ (Regge limit). That way, the polarization data yield the true FFs ratio of the proton G_E/G_M, and the extrapolation of σ_R to $\varepsilon = 1$ yields a linear combination of the true FFs $G_{(E,M)}$.

We solve for $R_{e^+e^-}$ by eliminating $F(\varepsilon, Q^2)$ using Eqs. (1) and (3). The ratio $R_{e^+e^-}$ is then expressed in terms of σ_R and $\sigma_{\rm Born}$ as

$$R_{e^+e^-} \approx \left[3 - 2\frac{\sigma_R(\varepsilon, Q^2)}{\sigma_{\rm Born}(\varepsilon, Q^2)}\right], \tag{12}$$

and for a fixed Q^2 value, we calculate $R_{e^+e^-}$ using σ_R measurements, along with their quoted uncertainties, at each ε point, and $\sigma_{\rm Born} = G_M^2[1 + (\varepsilon/\tau)R_p^2]$, where R_p is the recoil-polarization ratio constrained to its value, along with its associated uncertainty, as given by Eq. (10). For G_M extraction, we use $\sigma_R(\varepsilon = 1, Q^2) = [G_M^2 + G_E^2/\tau]$, and because of the experimentally observed linearity of σ_R with ε, we fit the experimental σ_R linearly to ε using the form $\sigma_R = [c_1(Q^2) + c_2(Q^2)\varepsilon]$, where $c_{(1,2)}(Q^2)$ are the parameters of the fit, and functions of Q^2 only. Equating the two expressions for $\sigma_R(\varepsilon = 1, Q^2)$, and solving for $G_M^2(Q^2)$ yields

$$G_M^2(Q^2) = \frac{[c_1(Q^2) + c_2(Q^2)]}{1 + \big(R_p^2/\tau\big)}, \tag{13}$$

where we constrain R_p to its value, along with its associated uncertainty, as given by Eq. (10). The error on $R_{e^+e^-}$ is calculated by propagating the errors on σ_R, G_M, and R_p using Eq. (12). At a fixed Q^2 point, we fit the extracted ratio linearly to the form $R_{e^+e^-}(\varepsilon, Q^2) \approx [1 - \alpha_1(Q^2)(1-\varepsilon)]$, and extract the TPE contribution $\alpha_1(Q^2)$, which is the parameter of fit, and represents the slope. Fitting the extracted $R_{e^+e^-}$ linearly to ε suggests that both σ_R and $\delta_{2\gamma}$ are no longer linear functions

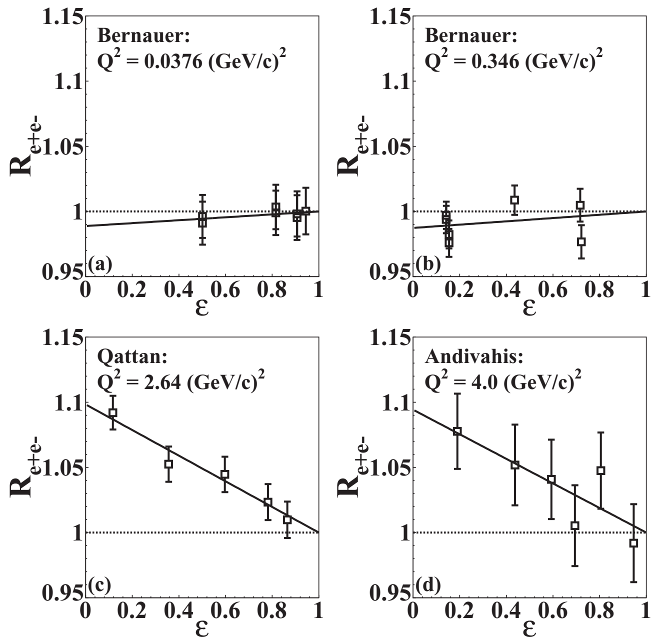

FIG. 1. The ratio $R_{e^+e^-}$ as a function of ε as extracted using Eq. (12) for a sample of low- and high-Q^2 data points from the data of Refs. [13,14,17]. Also shown is our fit (solid black line).

of ε, which is clearly inconsistent with the assumption we made above about the linearity of σ_R with ε with the inclusion of TPE corrections. However, the extracted $\alpha_1(Q^2)$ values are expected to be small, and so the deviation from linearity should not be worth much. We then attempt to parametrize the Q^2 dependence of $\alpha_1(Q^2)$, and use the parametrization to predict the ratio $R_{e^+e^-}$ as a function of ε at a fixed Q^2 value.

IV. RESULTS AND DISCUSSION

In this section, we present our results on the TPE corrections to σ_R, and use the results to predict the ratio $R_{e^+e^-}$ following the procedure outlined in Sec. III. World's data on unpolarized σ_R from Refs. [12–15,17,105–107] were used to extract the ratio $R_{e^+e^-}$ using Eq. (12) for a total of 120 Q^2 points up to $Q^2 = 5.2\ (\text{GeV}/c)^2$.

Figure 1 shows the ratio $R_{e^+e^-}$ as a function of ε extracted using Eq. (12) for a sample of low- and high-Q^2 data points. The extracted ratio $R_{e^+e^-}$ using all σ_R data sets shows linear, or nearly linear ε dependence, and so for a fixed Q^2 point, we fit $R_{e^+e^-}$ to the form $R_{e^+e^-} = [1 - \alpha_1(Q^2)(1-\varepsilon)]$ (solid black line), where the TPE coefficient $\alpha_1(Q^2)$ is the parameter of fit, and represents the slope. Note, however, that in the "QAA1" and "QAA2" extractions, the TPE contribution to the cross section is $F(\varepsilon, Q^2) = 2a(Q^2)(1-\varepsilon)G_M^2$, and so at high Q^2, the fractional slope introduced by the TPE correction is $2a(Q^2)$. In addition, $F(\varepsilon, Q^2)/G_M^2$ is linear in ε, while the $Q^2 \to 0$ limit [23,109] is more nearly linear when taken as a ratio to the reduced cross section, $[G_M^2 + (\varepsilon/\tau)G_E^2]$, and so the behavior at small Q^2 where $1/\tau$ becomes large is very different. This is seen clearly in the "QAA2" extractions when examining the ε dependence of the TPE contributions

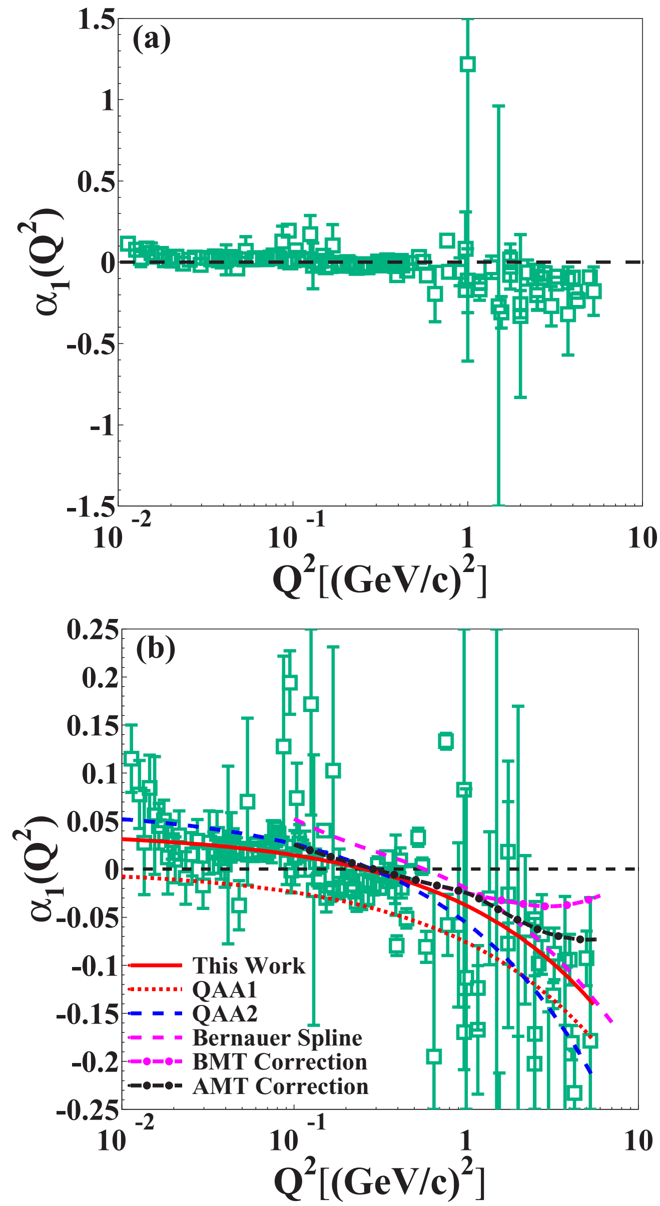

FIG. 2. The TPE coefficient $\alpha_1(Q^2)$ as obtained for all data sets from Refs. [12–15,17,105–107] (open dark-green squares). The bottom plot is on a smaller vertical scale for clarity. Also shown are the simple parametrization of our results (solid red line), and curves representing $\alpha_1(Q^2)$ as determined using TPE-hadronic calculations from Ref. [26] "BMT Correction" (dashed-dotted magenta line), Ref. [108] "AMT Correction" (dashed-dotted black line), and previous analyzes "QAA1" [78] (dashed red line), "QAA2" [82] (long-dashed blue line), and "Bernauer Spline" [17] (large-dashed magenta line). Note that we do not show extractions of $\alpha_1(Q^2)$ for the TPE-hadronic calculations and "Bernauer Spline" at very low Q^2, as the ε dependence is quite different in our parametrization when the cross sections are dominated by the electric FF G_E.

at very low Q^2 in Fig. 3. Moreover, the "QAA1" extraction used cross-section data which were limited to $Q^2 \geqslant 0.39$, and a parametrization of R_p from Ref. [7], which was not well constrained at low Q^2. In the "QAA2" extraction, however,

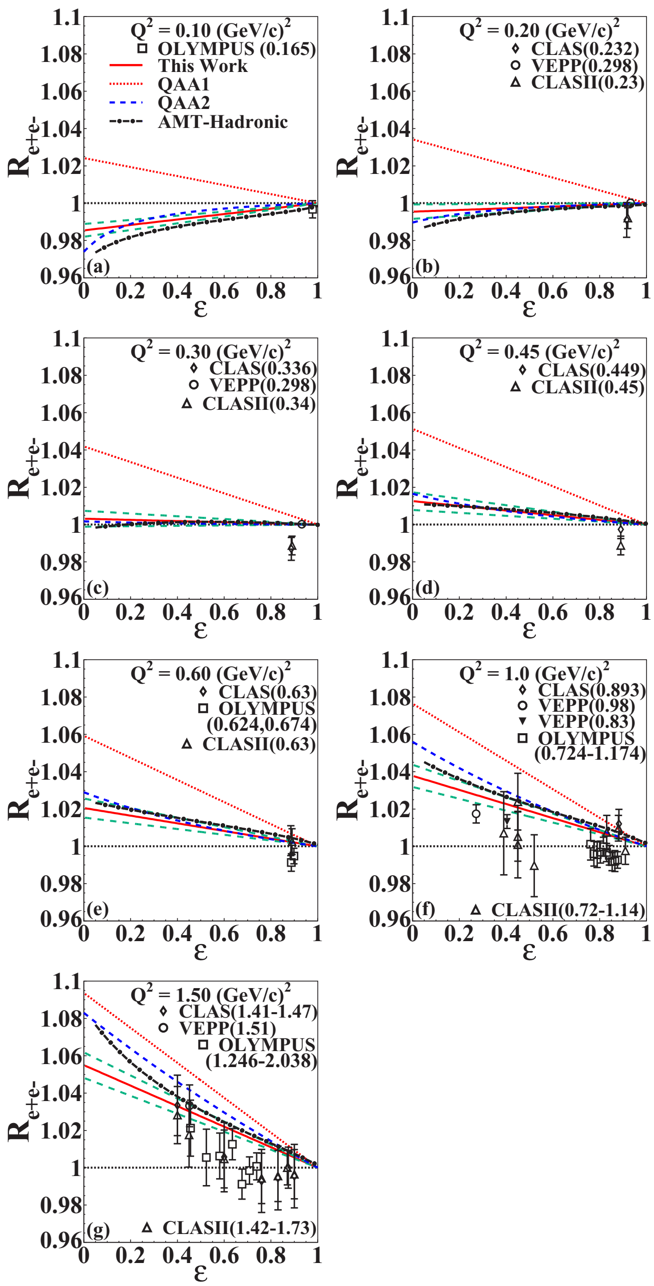

FIG. 3. The ratio $R_{e^+e^-}$ as a function of ε as extracted from this work (solid red line), along with its error band (long-dashed dark-green line), at the Q^2 values listed in the figure. Also shown are previous TPE hadronic calculations "AMT-Hadronic" [108] (dashed-dotted black line), and several previous phenomenological extractions: "QAA1" [78] (dashed red line), and "QAA2" [82] (long-dashed blue line). The data points are direct measurements of $R_{e^+e^-}$ from Refs. [100–103]. For the world's data, the measurement and the Q^2 value(s) are given in (GeV/c)^2.

they included the Mainz data and low Q^2 polarization data, which provided meaningful uncertainties for $a(Q^2)$ below $Q^2 = 1.0\,(\mathrm{GeV/c})^2$. Note that both extractions, and similar to most similar phenomenological analyses, assumed that the TPE contributions are significant for the cross-section measurements, but negligible for polarization data. On the other hand, analysis based on TPE-hadronic calculations "BMT-Correction" [26], and "AMT-Correction" [108], which adds an additional phenomenological TPE contribution at higher Q^2, as well as partonic calculations [36] suggest that the TPE corrections are at the few-percentage-point level for both observables.

The TPE coefficient $\alpha_1(Q^2)$ extracted from this work as a function of Q^2 for all data sets is shown in Fig. 2. We also show parametrizations of $\alpha_1(Q^2)$ from TPE hadronic calculations of Ref. [26] "BMT-Correction" and Ref. [108] "AMT-Correction", which adds and additional phenomenological TPE contribution at higher Q^2, and from previous phenomenological extractions "QAA1", "QAA2", and Ref. [17] "Bernauer Spline". In comparison to our ratio functional form $R_{e^+e^-} = [1 - \alpha_1(Q^2)(1-\varepsilon)]$, the "QAA1" and "QAA2" extractions, which determine $R_{e^+e^-}$ based on Eq. (11), suggest that $\alpha_1(Q^2) = 4a(Q^2)$, where we use the $a(Q^2)$ parametrization from each analysis. For the "BMT-Correction", "AMT-Correction", and "Bernauer Spline", we also use $\alpha_1(Q^2) = 4a(Q^2)$. To calculate $a(Q^2)$, and in comparison to Eq. (11), we first correct $\delta_{2\gamma}$ for each calculation using $\delta_{\mathrm{Corr.}} = [1 + (\varepsilon/\tau)R_p^2]\delta_{2\gamma}$ with R_p given by Eq. (10), and then fit to the form $\delta_{\mathrm{Corr.}} = 2a(Q^2)(1-\varepsilon)$ with $a(Q^2)$ being the parameter of the fit, limiting our extraction of $a(Q^2)$ to $Q^2 \geqslant 0.10\,(\mathrm{GeV}/c)^2$, as the ε dependence in the "BMT-Correction", "AMT-Correction", and "Bernauer Spline" extractions at very low Q^2 value is quite different from the one assumed in this work, and that of the "QAA1" and "QAA2" extractions when the cross sections are dominated by the charge FF. For the "Bernauer Spline" extraction, an additional low Q^2 Feshbach-Coulomb correction $\delta_{\mathrm{Feshbach}}$ was also added to $\delta_{2\gamma}$. Note that for the "AMT-Correction" calculations, the ratio $R_{e^+e^-}$, shown in Fig. 3, was calculated using their published $\delta_{2\gamma}$ as $R_{e^+e^-} = (1-\delta_{2\gamma})/(1+\delta_{2\gamma})$.

The TPE coefficient $\alpha_1(Q^2)$ extracted from this work is on the few percent level, and for most data sets, increases in magnitude with increasing Q^2, and shows a change in sign at low Q^2, as seen in previous low-Q^2 TPE calculations [26,40,95,96]. However, the behavior seen in the "QAA1" and "QAA2" extraction is driven mainly by the fitting function used for $a(Q^2)$. Similar extraction using neural networks [49] included a discussion of the procedure used to minimize and evaluate the model dependence, but did not include the Mainz low-Q^2 data as in this present work, and that of Ref. [82].

In the present work, and that of "QAA2" [82], the inclusion of the low-Q^2 data from Mainz [17], and the use of a well-constrained R_p parametrization, Eq. (10), along with its associated uncertainty, provide meaningful uncertainties for $\alpha_1(Q^2)$ below $Q^2 = 1\,(\mathrm{GeV}/c)^2$. However, the uncertainties we quote on the extracted $\alpha_1(Q^2)$ are rather an underestimate of the true uncertainties, as the uncertainties quoted on the Mainz cross sections do not include correlated systematic uncertainties, which are considered to be a significant contribution to the total uncertainty in their final FFs parametrization. In our analysis, we do not attempt to account for any residual uncertainty in the normalization of the different data subsets used in the analysis, but rather use the total uncertainties on the experimental σ_R as published. To parametrize the Q^2 dependence of $\alpha_1(Q^2)$, several functional forms were tried. The best fit was achieved when we used the simple two-parameter function used in the "QAA2" analysis [82], yielding $\alpha_1(Q^2) = [(0.039 \pm 0.002) + (-0.076 \pm 0.004)\sqrt{Q^2}]$ with Q^2 is in $(\mathrm{GeV}/c)^2$, and error matrix elements of $\sigma_1^2 = 5.261 \times 10^{-6}$, $\sigma_2^2 = 1.494 \times 10^{-5}$, and $\sigma_{12}^2 = \sigma_{21}^2 = 7.671 \times 10^{-6}$. However, we do not quote a reduced χ^2 value on our $\alpha_1(Q^2)$ fit as this would not be meaningful given that we do not have a complete evaluation of the uncertainties on $\alpha_1(Q^2)$ from the Mainz data. Our $\alpha_1(Q^2)$ parametrization is in generally good quantitative agreement with TPE hadronic calculations, and the previous "QAA2" and "Bernauer-Spline" phenomenological extractions shown in Fig. 2(b), although our fit is clearly driven by the tension between the different data sets used in the analysis, as well as the relatively larger uncertainties on $\alpha_1(Q^2)$ at large-Q^2 values, rather than a limitation of the fit function used.

Figure 3 shows the ratio $R_{e^+e^-}$ as a function of ε extracted from this work, using our new parametrization of the TPE coefficient $\alpha_1(Q^2)$ "This Work", along with the error bands on our extraction, shown as long-dashed dark-green lines, as computed using the covariance matrix of the $\alpha_1(Q^2)$ fit. In addition, we compare our results to TPE hadronic calculations "AMT-Hadronic" [108], several previous phenomenological extractions: "QAA1" [79], and "QAA2" [82], and world's data with emphasis mainly on the kinematics range of the recent precise direct measurements from the CLAS collaboration [100,103], VEPP-3 collaboration [101], and OLYMPUS collaboration [102]. Note that for the experimental data, the measurement and the Q^2 value(s) are given in $(\mathrm{GeV}/c)^2$.

The ratio $R_{e^+e^-}$ as extracted from this work is below unity, and behaves linearly with ε at low Q^2. The ratio increases, changes sign (above unity), and continues to behave linearly with ε as Q^2 increases. However, note that such linearity is already built in our parametrization of $R_{e^+e^-}$, and so it is not a natural outcome and consequence of the analysis. Our extractions are in generally good quantitative agreement with the "QAA2" and "AMT-Hadronic" calculations, although the "QAA2" extractions exhibit some nonlinearity at low Q^2 and low ε, and with relatively larger slope at high Q^2 values. The nonlinearity observed in the "QAA2" analysis at low Q^2 and low ε is a natural consequence of the fit function used, Eq. (8), where $F(\varepsilon, Q^2)/G_M^2$ is linear in ε, but the cross section is dominated by G_E^2 at very low Q^2, except for $\varepsilon \to 0$, strongly suppressing TPE as a fractional contribution as one moves away from $\varepsilon = 0$. Note that at very low Q^2, the TPE coefficient $a(Q^2)$ increases in value, enhancing the nonlinearity in the ratio $R_{e^+e^-}$. This issue can be fixed by modifying the functional form used where one can use a linear function in ε times the full reduced cross section [17]. Note that for the kinematics range of the new direct measurements considered in this work, $0.10 \leqslant Q^2 \leqslant 2.10\,(\mathrm{GeV}/c)^2$, calculating the ratio $R_{e^+e^-}$ for the "QAA1" and "QAA2" extractions using either Eq. (11), or $R_{e^+e^-} = [1 - \alpha_1(1-\varepsilon)] \approx [1 - 4a(Q^2)(1-\varepsilon)]$, yields effectively the same results, as the term $(\varepsilon/\tau)R_p^2$ decreases with increasing Q^2 value, making the ratio $R_{e^+e^-}$ more linear in ε.

The "QAA1" extraction, on the other hand, predicts a ratio that is above unity, with noticeably larger slope compared to other extractions and calculations for all Q^2 range. Again this is mainly driven by the fit function used, as the $a(Q^2)$ parametrization is limited to $Q^2 \geqslant 0.39$ $(\mathrm{GeV}/c)^2$, yielding negative value for $a(Q^2)$ and $\alpha_1(Q^2)$, and no change of sign as seen in other extractions and calculations. In general, our extractions are in good quantitative agreement with the "QAA2" extractions, and TPE hadronic calculations but not with the "QAA1" extractions which seem to yield relatively larger values for $R_{e^+e^-}$ for all Q^2 range.

We now compare our extractions of $R_{e^+e^-}$, along with their associated error bands, to world's data with focus mainly on the very recent precise measurements from the CLAS, VEPP-3, and OLYMPUS collaborations at the Q^2 value listed in the figure. All three collaborations measured $R_{e^+e^-}$ for $Q^2 < 2.1$ $(\mathrm{GeV}/c)^2$, which is below where the discrepancy on $\mu_p R_p$ is significant. The CLAS and VEPP-3 collaborations measured the ratio $R_{e^+e^-}$ at Q^2 values of 0.2–1.5 $(\mathrm{GeV}/c)^2$. They provided precise measurements of $R_{e^+e^-}$ at $Q^2 \approx 1.0$ and 1.5 $(\mathrm{GeV}/c)^2$. The ratio $R_{e^+e^-}$ is larger than unity, and shows clear ε dependence, mainly at low ε points, providing an evidence for a sizable hard-TPE correction at larger Q^2 values consistent with the FFs discrepancy at Q^2 values of 1.0–1.6 $(\mathrm{GeV}/c)^2$. The data showed clear deviation, and change of sign from the exact calculations, high proton mass limit, at $Q^2 = 0$ [109], as well as the finite-Q^2 calculations for a point-proton [23]. The OLYMPUS experiment measured the ratio $R_{e^+e^-}$ at Q^2 values of 0.165–2.038 $(\mathrm{GeV}/c)^2$. The ratio is below unity at high ε, and changes sign (above unity) and increases gradually with decreasing ε reaching about 2% at $\varepsilon = 0.46$.

Our extracted TPE contributions as well as that of previous "QAA2" extractions and "AMT-Hadronic" calculations are in generally good quantitative agreement with existing precise data on $R_{e^+e^-}$, including the recent measurements which show clear ε dependence, consistent with the FF discrepancy, at Q^2 values of 1.0–1.5 $(\mathrm{GeV}/c)^2$. For the CLAS measurements in the Q^2 range of 0.85–1.0 $(\mathrm{GeV}/c)^2$, the data are generally consistent with our extractions and those of "QAA2" and "AMT-Hadronic" calculations, which are $\leqslant$2% in the measurements range, and increase at lower ε and higher Q^2. Our extractions are also in very good agreement with the VEPP-3 measurements at Q^2 values of 0.83–1.5 $(\mathrm{GeV}/c)^2$. However, the "QAA2" and "AMT-Hadronic" results are slightly larger than the VEPP-3 measurements at $Q^2 = 1.0\,(\mathrm{GeV}/c)^2$. The OLYMPUS measurements show clear enhancement of $R_{e^+e^-}$ for $\varepsilon \leqslant 0.6$, and then a dip below unity for $\varepsilon \geqslant 0.7$, which still consistent with no deviation from unity. Such suppression of $R_{e^+e^-}$ seen at large ε is not seen in other measurements. Our results and those of "QAA2" and "AMT-Hadronic" do not show a decrease in $R_{e^+e^-}$ below unity at large ε, but still in a reasonable agreement with the OLYMPUS results within the experimental uncertainties. Finally, until new data on the ratio $R_{e^+e^-}$ are accumulated for $Q^2 > 2.1$ $(\mathrm{GeV}/c)^2$ in the region where the discrepancy on $\mu_p R_p$ is significant, the assumption whether hard-TPE corrections could account for the discrepancy on $\mu_p R_p$ is still an open question.

V. CONCLUSIONS

In conclusion, we presented a new prediction of the positron-proton to electron-proton elastic scattering cross sections ratio $R_{e^+e^-}$. Contrary to several previous phenomenological studies which extracted $R_{e^+e^-}$ using proposed, model-independent parametrizations of the hard TPE corrections to σ_R, we do not assume prior knowledge of the functional form of the hard TPE corrections, and use combined unpolarized, and polarized ep elastic scattering experimental data to extract the TPE contributions to σ_R. We provided a simple parametrization of the TPE corrections to σ_R, along with the uncertainties associated with the model dependence of the extractions, and used our TPE parametrization to predict the ratio $R_{e^+e^-}$. We compared our prediction of $R_{e^+e^-}$ to several previous phenomenological extractions, TPE-hadronic calculations, and world's data with emphasis mainly on the kinematics range of the recent precise direct measurements from the CLAS, VEPP-3, and OLYMPUS experiments from Refs. [100–103]. Our extractions are in generally good quantitative agreement with previous phenomenological extractions of "QAA2" [82], TPE hadronic calculations of "AMT-Hadronic" [108], and with existing precise data on $R_{e^+e^-}$, including the recent measurements which show clear ε dependence, consistent with the FF discrepancy at Q^2 values of 1.0–1.5 $(\text{GeV}/c)^2$.

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