Euclidean formulation of relativistic quantum mechanics of *N* particles

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A Euclidean formulation of relativistic quantum mechanics for systems of a finite number of degrees of freedom is discussed. Relativistic treatments of quantum theory are needed to study hadronic systems at subhadronic distance scales. While direct interaction approaches to relativistic quantum mechanics have proved to be useful, they have two disadvantages. One is that cluster properties are difficult to realize for systems of more than two particles. The second is that the relation to quantum field theories is indirect. Euclidean formulations of relativistic quantum mechanics provide an alternative representation that does not have these difficulties. More surprising, the theory can be formulated entirely in the Euclidean representation without the need for analytic continuation. In this work a Euclidean representation of a relativistic N-particle system is discussed. Kernels for systems of N free particles of any spin are given and shown to be reflection positive. Explicit formulas for generators of the Poincaré group for any spin are constructed and shown to be self-adjoint on the Euclidean representation of the Hilbert space. The structure of correlations that preserve both the Euclidean covariance and reflection positivity is discussed.

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I. INTRODUCTION

Relativistic quantum mechanical models of systems with a finite number of degrees of freedom are useful for modeling strongly interacting systems because they can be solved numerically with controlled errors and can be applied consistently in both the laboratory frame and center of momentum frame. This paper discusses a Euclidean covariant representation of relativistic quantum mechanics for systems with a finite number of degrees of freedom. The Euclidean representation overcomes some of the difficulties with the direct construction of Poincaré generators on a multiparticle Hilbert space. The two key challenges of the direct construction are constructing generators satisfying cluster properties in all inertial coordinate systems for systems of more than three particles and the absence of a direct relation between the model interactions and an underlying quantum field theory.

The motivation for exploring the Euclidean formulation is that it provides a representation of a relativistic quantum theory that has a direct connection to quantum field theory and easily satisfies cluster properties. While the same is formally true of Minkowski representations of quantum field theory, most nonperturbative computations are based on relations among time-ordered vacuum expectation values of fields, while the Hilbert space structure, which is associated with the Wightman functions, is simply assumed. When truncations are involved it is not automatic that solutions of Schwinger-Dyson equations with phenomenological input are consistent with the probabilistic interpretation of quantum theory. Both Wightman functions and time-ordered Green's functions are related to Euclidean Green's functions by different analytic continuations. The Euclidean Green's functions satisfy Euclidean versions of the Schwinger-Dyson equations and at the same time are directly related to the Hilbert space structure of the field theory. An appealing feature of the Euclidean axioms is that the locality axiom is logically independent of the other axioms, so it can be relaxed (which is necessary for models of a finite number of degrees of freedom) without violating relativistic invariance, the spectral condition, cluster properties, and the Hilbert space representation of the theory. A second appealing feature is that an analytic continuation is not necessary to compute the Hilbert space inner product of the physical quantum theory. While the formulation of the dynamics discussed in this work is still phenomenological, the phenomenological kernels are in principle models of the exact Euclidean Green's functions of the field theory, so they can be constrained by field theory based phenomenology.

In a quantum theory relativistic invariance means that quantum observables, which are probabilities, expectation values and ensemble averages, have the same value for equivalent experiments that are performed in different inertial coordinate systems. This means that experiments performed in an isolated system cannot be used to distinguish inertial coordinate systems. In special relativity different inertial coordinate systems are related by the subgroup of Poincaré group connected to the identity. In 1939, Wigner [1] showed that a

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necessary and sufficient condition for a quantum system to be relativistically invariant is that vectors representing equivalent quantum states in different inertial coordinate systems are related by a unitary ray representation of this subgroup on the Hilbert space of the quantum theory. Bargmann [2] showed that this can be replaced by a single-valued representation of $SL(2, \mathbb{C})$, which is the covering group of the Lorentz group.

Relativistically invariant quantum theories are needed to study physics on distance scales that are small enough to be sensitive to the internal structure of a nucleon. This is because to get wavelengths short enough to resolve the internal structure of a nucleon it is necessary to transfer momentum to the nucleon that is comparable to or larger than its mass scale.

The direct approach for modeling relativistic systems is to construct explicit expressions for the Poincaré generators of the interacting system on a many-particle Hilbert space. Formally the Hilbert space is a direct sum of tensor products of irreducible representation spaces of the Poincaré group, representing the particle content of the system. Phenomenological interactions are added to the noninteracting Poincaré generators in a manner that preserves the commutation relations, cluster properties and the spectral condition. This is referred to as the direct interaction representation. This representation shares many of the computational advantages of nonrelativistic quantum mechanics. One problem is that the interactions are generally phenomenological and representation dependent, which makes them difficult to constrain by a more fundamental theory. In addition, satisfying cluster properties in all inertial coordinate systems puts strong constraints on the structure of the interactions. Satisfying these constraints presents computational challenges that have not been realized in applications [3–5].

Strong interactions are studied using lattice methods, which break relativistic invariance, Schwinger-Dyson equations, which are infinite systems of nonlinear equations for Euclidean Green's functions, and relativistic quantum mechanical models, which are more phenomenological and not directly related to an underlying quantum field theory. Both lattice calculations and Schwinger-Dyson calculations are normally formulated in a Euclidean representation. The purpose of this work is to formulate a class of relativistic quantum mechanical models that have many of the properties of direct interaction relativistic quantum models but have a more direct connection to lattice and Schwinger-Dyson methods. The Euclidean formulation facilitates the relation to these other methods. The formulation of relativistic quantum mechanics that will be discussed in this paper is motivated by the Euclidean reconstruction theorem of axiomatic quantum field theory.

Euclidean formulations of quantum field theory were first advocated by Schwinger [6,7] who used the spectral condition in time-ordered Green's functions to establish the existence of an analytic continuation to imaginary times. Independently, axiomatic treatments of quantum field theory [8,9] led to an understanding of the analytic properties of vacuum expectation values of products of fields, based on the spectral condition, Lorentz covariance and locality. The Euclidean approach to quantum field theory was advocated by K. Symanzik [10,11], and developed by Nelson [12]. Osterwalder and Schrader [13,14] identified properties of Euclidean covariant distributions that are sufficient to reconstruct a relativistic quantum field theory. Two observations that are implicit in the work of Osterwalder and Schrader are (1) that an explicit analytic continuation is not necessary to construct a relativistic quantum theory, and (2) the reconstruction of a relativistic quantum theory is not limited to local field theories. The Euclidean formulation of relativistic quantum mechanics presented in this work is motivated by these two observations.

An attractive feature of the Euclidean approach is that both the time-ordered Green's functions and Wightman functions can be constructed from the Euclidean Green's functions using different analytic continuations. This means the Euclidean Green's functions satisfy Euclidean Schwinger-Dyson equations while at the same time they can be used to construct the Hilbert space inner product of the underlying quantum theory. This provides a means to constrain the Hilbert space formulations of the theory from a Lagrangian based dynamics. While this formulation is intended to be phenomenological, it is designed so there is a formal relation to an underlying quantum field theory.

Aspects of this program have been discussed elsewhere [15–18]. The purpose of this paper is derive explicit expressions for the Poincaré generators with spin and discuss the structure of Euclidean covariant reflection positive distributions with spin.

This paper is organized as follows. Notation is introduced in Sec. II. The relation between the complex Lorentz group and the complex four-dimensional orthogonal group is discussed. This is central to the relation between the Euclidean and Lorentz covariant representations of the theory. Section III discusses positive mass irreducible representations of the Poincaré group. These are used to construct equivalent Lorentz and Euclidean covariant representations for massive particles in Sec. IV. The relation of Euclidean covariance to Lorentz covariance is discussed in Sec. V. Section VI contains the explicit formulas for the Poincaré generators with spin. They are shown to satisfy the Poincaré commutation relations and be Hermitian on the Euclidean representation of the Hilbert space. The generalization to systems of free particles is discussed in Sec. VII. The inclusion of dynamics and the structure of dynamical reflection positive Euclidean covariant kernels is examined in Sec. VIII. Section IX shows the self-adjointness of the Hamiltonian and boost generators in the Euclidean representation. Section X contains a summary and concluding remarks. The Appendix discusses the space-time representations of reflection positive Euclidean covariant kernels with different spins.

II. BACKGROUND

The Poincaré group is the group of space-time transformations that relate different inertial reference frames in the theory of special relativity. It is the symmetry group that preserves the proper time, τ_{ab}, or proper distance, d_{ab}, between any two events with space-time coordinates x_a^μ and x_b^μ:

$$-\tau_{ab}^2 = d_{ab}^2 = \eta_{\mu\nu}(x_a - x_b)^\mu (x_a - x_b)^\nu, \tag{1}$$

where $\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = 1$, $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$ is the Minkowski metric tensor. Repeated indices are assumed to be summed. The most general point transformation, $x'^\mu = f^\mu(x)$, satisfying Eq. (1) has the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{2}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation satisfying

$$\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \eta_{\alpha\beta} \Lambda^\beta{}_\nu$$

or in matrix form

$$\eta = \Lambda^t \eta \Lambda. \tag{3}$$

The full Poincaré group contains discrete transformations that are not associated with special relativity. Equation (3) implies that

$$\det(\Lambda)^2 = 1 \quad \text{and} \quad \left(\Lambda^0_0\right)^2 = 1 + \sum_i \left(\Lambda^0_i\right)^2. \tag{4}$$

This means that the Lorentz group can be decomposed into four topologically disconnected components

(1) $\det(\Lambda) = 1$, $(\Lambda^0_0) \geqslant 1$, includes the identity
(2) $\det(\Lambda) = -1$, $(\Lambda^0_0) \geqslant 1$, includes space reflection
(3) $\det(\Lambda) = -1$, $(\Lambda^0_0) \leqslant -1$, includes time reversal
(4) $\det(\Lambda) = 1$, $(\Lambda^0_0) \leqslant -1$, includes space-time reversal.

Since the discrete symmetries of space reflection and time reversal are not symmetries of the weak interaction, the symmetry group associated with special relativity is normally considered to be the subgroup of the Poincaré group that is continuously connected to the identity.

The relation between the Lorentz group and the four-dimensional orthogonal group is central to the development of the Euclidean formulation. The relation is illustrated by representing Minkowski, x^μ, and Euclidean, x_e^μ, four vectors as 2×2 matrices:

$$X_m = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(X\sigma_\mu), \tag{5}$$

$$X_e = x_e^\mu \sigma_{e\mu} = \begin{pmatrix} ix_e^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & ix_e^0 - x^3 \end{pmatrix} \qquad x_e^\mu = \frac{1}{2}\mathrm{Tr}(X_e \sigma_{e\mu}^\dagger), \tag{6}$$

where $\sigma_i = \sigma_{ei}$ are the Pauli matrices, σ_0 is the identity and $\sigma_{e0} = i\sigma_0$. The determinants of these matrices are related to the Minkowski and Euclidean line elements, respectively:

$$\det(X_m) = (x^0)^2 - \mathbf{x}\cdot\mathbf{x} \qquad \det(X_e) = -\left[\left(x_e^0\right)^2 + \mathbf{x}\cdot\mathbf{x}\right]. \tag{7}$$

X_m is Hermitian for real four vectors. The linear transformations that preserve the determinant and hermiticity of X_m have the form

$$X_m \to X_m' = \pm A X_m A^\dagger, \quad \det(A) = 1. \tag{8}$$

The $(-)$ sign represents a space-time reflection, which is not considered part of the symmetry group of special relativity. The group of complex 2×2 matrices with $\det(A) = 1$ is $SL(2,\mathbb{C})$. Similarly linear transformations corresponding to real four-dimensional orthogonal transformations in the 2×2 matrix representation have the general form

$$X_e \to X_e' = \pm A X_e C^t, \quad A, C \in SU(2). \tag{9}$$

Transformations of the form

$$X_e \to X_e' = A X_e C^t, \quad X_m \to X_m' = A X_m C^t, \tag{10}$$

with both A and C in $SL(2,\mathbb{C})$ preserve the Minkowski and Euclidean line elements, respectively, however they do not preserve the reality of the four vectors,

$$x'^\mu = \tfrac{1}{2}\mathrm{Tr}(X'\sigma_\mu), \quad x_e'^\mu = \tfrac{1}{2}\mathrm{Tr}(X_e'\sigma_{e\mu}^\dagger). \tag{11}$$

They represent complex Lorentz or orthogonal transformations. The corresponding complex Lorentz and orthogonal transformations are

$$\begin{aligned} \Lambda(A,C)^\mu{}_\nu &= \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu A \sigma_\nu C^t), \\ O(A,C)^\mu{}_\nu &= \tfrac{1}{2}\mathrm{Tr}(\sigma_{e\mu}^\dagger A \sigma_{e\nu} C^t). \end{aligned} \tag{12}$$

This shows that the covering group of both the complex Lorentz and complex orthogonal group is $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. These are double covers because $A, C \to -A, -C$ result in the same transformation. For $C = A^*$, Eq. (12) relates the real Lorentz group to a subgroup of the complex orthogonal group; similarly for A and C unitary Eq. (12) relates the real orthogonal group to a subgroup of the complex Poincaré group. The relation that will be exploited in this work is that Euclidean rotations that involve a space and the Euclidean time coordinate can be identified with Lorentz boosts with complex rapidity.

For the full Poincaré group it is necessary to include translations. Euclidean time translations by τ are identified with Minkowski time translations with $t = -i\tau$.

III. UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP

Any unitary representation of the Poincaré group satisfying the spectral condition ($m > 0$) can be decomposed into a direct integral of positive mass irreducible representations. The starting point of this work is to construct positive-mass irreducible unitary representations of the Poincaré group and use them to construct the corresponding Euclidean representations. Reflection positive kernels for each irreducible representation result from this construction. Since many-particle Hilbert spaces are tensor products of single-particle spaces and dynamical unitary representations of the Poincaré group can be decomposed into direct integrals of irreducible representation spaces, this construction provides a framework for constructing reflection positive kernels for different physical systems.

The 2×2 matrix representation of four vectors is used in this section. Poincaré group elements are replaced by (A, Y), where A is a $SL(2,\mathbb{C})$ matrix and Y is a 2×2 Hermitian matrix representing a translation. In the 2×2 representation

Poincaré transformations continuously connected to the identity have the form

$$X' = AXA^\dagger + Y, \tag{13}$$

where the group multiplication law is

$$(A_2, Y_2)(A_1, Y_1) = (A_2A_1, A_2Y_1A_2^\dagger + Y_2). \tag{14}$$

Four vector representations of these equations are

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + y^\mu, \tag{15}$$

$$\left(\Lambda^\mu_{12\,\nu}, y^\mu_{12}\right) = \left(\Lambda^\mu_{2\,\alpha}\Lambda^\alpha_{1\,\nu}, \Lambda^\mu_{2\,\alpha}y_1^\alpha + y_2^\mu\right), \tag{16}$$

where the four-vector representations are related to the 2×2 representations by

$$y^\mu := \tfrac{1}{2}\text{Tr}(\sigma_\mu Y) \quad \Lambda^\mu{}_\nu := \tfrac{1}{2}\text{Tr}(\sigma_\mu A\sigma_\nu A^\dagger). \tag{17}$$

SL(2, $\mathbb{C}$) is a six-parameter group. It has six independent one-parameter subgroups,

$$A_{r\hat{\theta}}(\theta) = e^{\frac{i\theta}{2}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\theta}}} = A_r(\theta), \quad A_{b\hat{\rho}}(\rho) = e^{\frac{\rho}{2}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\rho}}} = A_b(\rho), \tag{18}$$

corresponding to rotations about three different axes and rotationless Lorentz boosts in three different directions. In these expressions $\hat{\boldsymbol{\theta}}$ represents the axis and θ represents angle of a rotation while $\hat{\boldsymbol{\rho}}$ represents the direction of a rotationless boost and ρ represents the rapidity of a rotationless boost. The polar decomposition theorem expresses a general SL(2, $\mathbb{C}$) matrix A as a product of a (generalized Melosh) rotation ($R_m = (R_m^\dagger)^{-1}$ unitary) followed by rotationless (canonical) boost ($B_c = B_c^\dagger$ positive Hermitian):

$$A = B_c R_m, \tag{19}$$

where

$$B_c := (AA^\dagger)^{1/2} = B_c(\boldsymbol{\rho}), \quad R_m := (AA^\dagger)^{-1/2}A = R_m(\boldsymbol{\theta}). \tag{20}$$

A unitary representation of inhomogeneous SL(2, $\mathbb{C}$) is a set of unitary operators $U(A, Y)$, labeled by elements of SL(2, $\mathbb{C}$) satisfying

$$U(A_2, Y_2)U(A_1, Y_1) = U(A_2A_1, A_2Y_1A_2^\dagger + Y_2), \tag{21}$$

$$U(I, 0) = I, \tag{22}$$

$$U^\dagger(A, Y) = U^{-1}(A, Y) = U(A^{-1}, -A^{-1}Y(A^\dagger)^{-1}). \tag{23}$$

The Poincaré group is a 10 parameter group. Infinitesimal generators of $U(A, Y)$ are the 10 self-adjoint operators defined by

$$H = i\frac{d}{dy^0}U(I, y^0\sigma_0)_{|_{y_0=0}}, \tag{24}$$

$$P^j = -i\frac{d}{dy^j}U(I, y^j\sigma_j)_{|_{y_j=0}}, \tag{25}$$

$$J^j = -i\frac{d}{d\theta}U\left(e^{i\frac{\theta}{2}\sigma_j}, 0\right)_{|_{\theta=0}}, \tag{26}$$

$$K^j = -i\frac{d}{d\rho}U\left(e^{\frac{\rho}{2}\sigma_j}, 0\right)_{|_{\rho=0}}, \tag{27}$$

where there is no sum in Eq. (25) over the repeated j, and $j \in \{1, 2, 3\}$ in Eqs. (25)–(27). The group representation property Eq. (21) implies that these generators satisfy the Poincaré commutation relations

$$[J^i, J^j] = i\epsilon_{ijk}J^k, \quad [J^i, P^j] = i\epsilon_{ijk}P^k,$$

$$[J^i, K^j] = i\epsilon_{ijk}K^k, \tag{28}$$

$$[K^i, K^j] = -i\epsilon_{ijk}J^k, \quad [J^i, H] = 0, \quad [P^i, H] = 0, \tag{29}$$

$$[K^j, H] = iP^j, \quad [K^i, P^j] = i\delta_{ij}H. \tag{30}$$

These operators are components of a four-vector, P^μ, and an antisymmetric tensor operator, $J^{\mu\nu}$,

$$P^\mu = (H, \mathbf{P}), \quad J^{\mu\nu} = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J_1 & 0 \end{pmatrix}. \tag{31}$$

There are two independent polynomial invariants,

$$M^2 = (P^0)^2 - \mathbf{P}^2 = -P^\mu P_\mu \tag{32}$$

and

$$W^2 = W^\mu W_\mu \quad W^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}, \tag{33}$$

where W^μ is the Pauli-Lubanski vector. When $M \neq 0$ the square of the spin is defined by

$$S^2 = W^2/M^2. \tag{34}$$

A spin vector $\mathbf{s}$ can be defined by an *operator* rotationless (canonical) boost that transforms the angular momentum tensor to the rest frame:

$$s^i = \frac{1}{2}\sum_{j,k}\epsilon_{ijk}\Lambda_c^{-1}(P)^j{}_\mu\Lambda_c^{-1}(P)^k{}_\nu J^{\mu\nu}, \tag{35}$$

where

$$\Lambda_c(P)^\mu{}_\nu = \begin{pmatrix} V^0 & \mathbf{V} \\ \mathbf{V} & I + \frac{\mathbf{V}\otimes\mathbf{V}}{1+V^0} \end{pmatrix},$$

$$V^\mu = P^\mu/M = \frac{1}{2}\text{Tr}[B_c(\rho)\sigma_\mu B_c(\rho)], \tag{36}$$

and P^μ, M, and ρ are considered operators related by

$$\mathbf{V} = \mathbf{P}/M = \hat{\boldsymbol{\rho}}\sinh(\rho). \tag{37}$$

The spin vector defined with the rotationless boost is called the canonical spin; other types of spin vectors (helicity, light-front spin) are related to the canonical spin by momentum-dependent rotations. For the purpose of this work it is sufficient to consider the canonical spin. The canonical spin can also be expressed in terms of the Pauli-Lubanski vector:

$$\begin{pmatrix} 0 \\ \mathbf{s}_c \end{pmatrix} = \frac{1}{M}\Lambda_c^{-1}(P)^\mu{}_\nu W^\mu, \tag{38}$$

where again $\Lambda_c^{-1}(P)^\mu{}_\nu$ is a matrix of operators. The components of the spin satisfy SU(2) commutation relations:

$$[s_i, s_j] = i\epsilon_{ijk}s^k. \tag{39}$$

With these definitions, for $M > 0$, $M^2, s^2, \mathbf{P}, s_z$ are a maximal set of commuting self-adjoint functions of the Poincaré generators. The spectrum of each component of $\mathbf{P}$ is the real line since each component of $\mathbf{P}$ can be boosted to any value. Similarly the spectrum of spins are restricted to be integral or half integral as a consequence of the SU(2) commutation relations Eq. (39). In a general system these commuting observables are not complete; they can be supplemented by additional Poincaré-invariant degeneracy quantum numbers, which will be denoted by d. A basis for the Hilbert space are the simultaneous eigenstates of $M, S^2, d, \mathbf{P}, s_z$:

$$\{|(m,s,d)\mathbf{p},\mu\rangle\}. \tag{40}$$

Because these vectors are constructed out of eigenvalues of functions of P^μ and $J^{\mu\nu}$, which have well-defined Poincaré transformation properties, the Poincaré transformation properties of these basis states follow from the definitions

$$U(A,y)|(m,s,d)p,\mu\rangle = e^{i\Lambda p\cdot y}|(m,s,d)\Lambda p,\nu\rangle D^s_{\nu\mu}[R_{cw}(\Lambda,p)]\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}, \tag{41}$$

where $R_{cw}(\Lambda,p) := B_c^{-1}(\Lambda p)\Lambda B_c(p)$ is the canonical-spin Wigner rotation, $B_c(p) = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}}$, where $\boldsymbol{\rho}$ is the rapidity of a particle of mass m and momentum $\mathbf{p}$, and $\omega_m(p) := \sqrt{m^2+\mathbf{p}^2}$ is the energy of the particle. The square-root factors ensure that $U(\Lambda,a)$ is unitary for states Eq. (40) with the normalization

$$\langle(m',s',d')p',\mu'|(m,s,d)p,\mu\rangle = \delta_{m'm}\delta_{s's}\delta(\mathbf{p}'-\mathbf{p})\delta_{\mu'\mu}\delta_{d',d}. \tag{42}$$

The Wigner D-function is the finite-dimensional unitary representation of the rotation group in the $|s,\mu\rangle$ basis [19]:

$$D^s_{\mu,\mu'}[R] = \langle s,\mu|U(R)|s,\mu'\rangle = \sum_{k=0}^{s+\mu}\frac{\sqrt{(s+\mu)!(s+\mu')!(s-\mu)!(s-\mu')!}}{k!(s+\mu'-k)!(s+\mu-k)!(k-\mu-\mu')!} \times R^k_{++}R^{s+\mu'-k}_{+-}R^{s+\mu-k}_{-+}R^{k-\mu-\mu'}_{--},$$

where

$$R = \begin{pmatrix} R_{++} & R_{+-}\\ R_{-+} & R_{--}\end{pmatrix} = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \sigma_0\cos\left(\frac{\theta}{2}\right) + i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin\left(\frac{\theta}{2}\right) \tag{43}$$

is a SU(2) matrix. Because $D^s_{\mu\nu}[R]$ is a degree $2s$ polynomial in the matrix elements of R, and $R = e^{i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}}$ is an entire function of the angles, $\boldsymbol{\theta}$, it follows that $D^s_{\mu,\mu'}[e^{i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{2}}]$ is an entire function of all three components of $\boldsymbol{\theta}$. This means that the group representation property

$$\sum_{\mu''} D^s_{\mu,\mu''}[R_2]D^s_{\mu'',\mu'}[R_1] - D^s_{\mu,\mu'}[R_2R_1] = 0, \tag{44}$$

and the formulas for adding angular momenta

$$D^s_{\mu,\mu'}[R] - \sum_{s_1s_2\mu_1\mu_2\mu_1'\mu_2'}\langle s,\mu|s_1,\mu_1,s_2,\mu_2\rangle \times D^{s_1}_{\mu_1,\mu_1'}[R]D^{s_2}_{\mu_2,\mu_2'}[R]\langle s_1,\mu_1',s_2',\mu_2'|s,\mu'\rangle = 0 \tag{45}$$

and

$$D^{s_1}_{\mu_1,\mu_1'}[R]D^{s_2}_{\mu_2,\mu_2'}[R] - \sum_{s\mu\mu'}\langle s_1,\mu_1,s_2,\mu_2|s,\mu\rangle \times D^s_{\mu,\mu'}[R]\langle s,\mu'|s_1,\mu_1',s_2,\mu_2'\rangle = 0, \tag{46}$$

which hold for real angles, can be analytically continued to complex angles. This means that Eqs. (44)–(46) also hold when the SU(2) matrices R are replaced by SL(2, $\mathbb{C}$) matrices A. In these expressions, $\langle s,\mu|s_1,\mu_1,s_2,\mu_2\rangle$, are SU(2) Clebsch-Gordan coefficients. While the analytic continuation preserves the group representation Eq. (44) and angular momentum addition Eqs. (45) and (46) properties, it does not preserve unitarity.

IV. EUCLIDEAN REPRESENTATIONS

In this section the Poincaré irreducible basis states Eq. (40) are used to construct equivalent Euclidean representations of the irreducible representations of the Poincaré group.

The starting point is the irreducible representations of the Poincaré group constructed in Sec. III. The basis vectors and action of $U(\Lambda,a)$ on the basis vectors are given by Eqs. (40) and (41).

Because $R^{-1} = R^\dagger$ for $R \in$ SU(2) the $SU(2)$ representation of the Wigner rotation, $R_{wc}(\Lambda,p)$, can be expressed in two equivalent ways:

$$R_{wc}(\Lambda,p) = B_c^{-1}(\Lambda p)AB_c(p) = B_c^\dagger(\Lambda p)(A^\dagger)^{-1}B_c^{\dagger-1}(p). \tag{47}$$

The SL(2, $\mathbb{C}$) group representation property Eq. (44) implies that the unitary representation $D^s_{\nu\mu}[R_{wc}(\Lambda,p)]$ of the Wigner rotation $R_{wc}(\Lambda,p)$ can be factored in two different ways:

$$D^s_{\nu\mu}[R_{wc}(\Lambda,p)] = \sum_{\alpha\beta} D^s_{\nu\alpha}\big[B_c^{-1}(\Lambda p)\big]D^s_{\alpha\beta}[A]D^s_{\beta\mu}[B_c(p)] \tag{48}$$

or

$$D^s_{\nu\mu}[R_{wc}(\Lambda,p)] = \sum_{\alpha\beta} D^s_{\nu\alpha}[B_c^\dagger(\Lambda p)]D^s_{\alpha\beta}[(A^\dagger)^{-1}]D^s_{\beta\mu}[(B_c^\dagger)^{-1}(p)]. \tag{49}$$

These relations and the group representation properties, Eqs. (44)–(46), can be used to express Eq. (41) in terms of new Lorentz covariant basis states:

$$U(A,y)\underbrace{\sum_\alpha |(m,s)p,\alpha\rangle D^s_{\alpha\mu}\big[B_c^{-1}(p)\big]\sqrt{\omega_m(p)}}_{|(m,s)p,\mu\rangle_{\rm cov}} = e^{i\Lambda p\cdot y}\sum_\beta\underbrace{\sum_\alpha |(m,s)\Lambda p,\alpha\rangle D^s_{\alpha\beta}\big[B_c^{-1}(\Lambda p)\big]\sqrt{\omega_m(\Lambda p)}}_{|(m,s)\Lambda p,\beta\rangle_{\rm cov}}D^s_{\beta\mu}[A] \tag{50}$$

or

$$U(A,y)\underbrace{\sum_\alpha |(m,s)p,\alpha\rangle D^s_{\alpha\mu}[B^\dagger_c(p)]\sqrt{\omega_m(p)}}_{|(m,s)p,\mu\rangle_{\text{cov}*}} = e^{i\Lambda p\cdot y}\sum_\beta \underbrace{\sum_\alpha |(m,s)\Lambda p,\alpha\rangle D^s_{\alpha\beta}[B^\dagger_c(\Lambda p)]\sqrt{\omega_m(\Lambda p)}}_{|(m,s)\Lambda p,\beta\rangle_{\text{cov}*}} D^s_{\beta\mu}[(A^\dagger)^{-1}]. \quad (51)$$

The degeneracy quantum numbers are suppressed in these equations. These expressions replace the states Eq. (40) that transform covariantly with respect to the Poincaré group with states that transform covariantly with respect to SL(2, $\mathbb{C}$):

$$U(A,y)|(m,s)p,\mu\rangle_{\text{cov}} = e^{i\Lambda p\cdot y}\sum_\nu |(m,s)\Lambda p,\nu\rangle_{\text{cov}} D^s_{\nu\mu}[A] \quad (52)$$

$$U(A,y)|(m,s)p,\mu\rangle_{\text{cov}*} = e^{i\Lambda p\cdot y}\sum_\nu |(m,s)\Lambda p,\nu\rangle_{\text{cov}*} D^s_{\nu\mu}[(A^\dagger)^{-1}]. \quad (53)$$

These will be referred to as Lorentz covariant representations while the representations (41) will be referred to as Poincaré covariant representations. The transformations relating the Lorentz and Poincaré covariant representations are invertible,

$$|(m,s)p,\mu\rangle = \sum_\nu |(m,s)p,\nu\rangle_{\text{cov}} \frac{1}{\sqrt{\omega_m(p)}} D^s_{\nu\mu}[B_c(p)], \quad (54)$$

$$|(m,s)p,\mu\rangle = \sum_\nu |(m,s)p,\nu\rangle_{\text{cov}*} \frac{1}{\sqrt{\omega_m(p)}} D^s_{\nu\mu}[(B^\dagger_c)^{-1}(p)], \quad (55)$$

however, there are two distinct Lorentz covariant representations, because while $R = (R^\dagger)^{-1}$ for $R \in \text{SU}(2)$, the corresponding representations in SL(2, $\mathbb{C}$) are inequivalent. These two representations are called right- and left-handed representations for reasons that will become apparent.

In the Lorentz covariant representations, Eqs. (50) and (51), this equivalence can be used to show that the equivalent Hilbert space inner product of two SL(2, $\mathbb{C}$) covariant wave functions has a nontrivial kernel

$$\langle\psi|\phi\rangle = \sum_\mu \int \langle\psi|(m,s)p,\mu\rangle d\mathbf{p}\langle(m,s)p,\mu|\phi\rangle$$
$$= \int \sum_{\mu\nu}\langle\psi|(m,s)p,\mu\rangle_{\text{cov}} D^s_{\mu\nu}[p\cdot\sigma]2\delta(p^2+m^2) \times\theta(p^0)d^4p_{\text{cov}}\langle(m,s)p,\nu|\phi\rangle \quad (56)$$

$$\langle\psi|\phi\rangle = \int \sum_\mu \langle\psi|(m,s)p,\mu\rangle d\mathbf{p}\langle(m,s)p,\mu|\phi\rangle$$
$$= \int \sum_{\mu\nu}\langle\psi|(m,s)p,\mu\rangle_{\text{cov}*} D^s_{\mu\nu}[\Pi p\cdot\sigma]2\delta(p^2+m^2) \times\theta(p^0)d^4p_{\text{cov}*}\langle(m,s)p,\nu|\phi\rangle, \quad (57)$$

where $B_c(p)B^\dagger_c(p) = B_c(p)^2 = \sigma\cdot p$ and $B^{-1}_c(p)(B^\dagger_c)^{-1}(p) = B^{-2}_c(p) = (\Pi p)\cdot\sigma$, was used in these equations. Π is the space reflection operator and $p\cdot\sigma = \omega_m(p)\sigma_0 + \mathbf{p}\cdot\boldsymbol{\sigma}$. These equations explain why Eqs. (56) and (57) are called right- and left-handed representations. These kernels are, up to normalization and change of representation, spin-s two-point Wightman functions (see Eqs. (1.55–1.57) of Ref. [8]).

While both the left- and right-handed representations are each related to the original Poincaré covariant representation, the kernels of the Lorentz covariant representations of the Hilbert space inner product do not commute with space reflection. Instead the right-handed (left-handed) kernel gets mapped into the left-handed (right-handed) kernel under space reflection.

More general classes of spinor representation can be constructed using tensor products

$$|(m,s;s_1,s_2)p,\mu_1,\mu_2\rangle_{cov} := \sum |(m,s)p,\nu\rangle\langle s,\nu|s_1,\nu_1,s_2,\nu_2\rangle\sqrt{\omega_m(p)} \times D^{s_1}_{\nu_1\mu_1}[B_c(p)^{-1}]D^{s_2}_{\nu_2\mu_2}[B_c(p)^\dagger] \quad (58)$$

or direct sums of right- and left-handed representations

$$\langle(m,s)p,\mu_1,\mu_2|\phi\rangle_{cov} := \frac{1}{\sqrt{2}}\sum_\nu \langle(m,s)p,\nu|\phi\rangle\sqrt{\omega_m(p)}\begin{pmatrix} D^s_{\nu\mu_1}[B_c(p)^{-1}] \\ D^s_{\nu\mu_2}[B_c(p)^\dagger]\end{pmatrix}. \quad (59)$$

Dirac spinors are direct sums of $s = 1/2$ left- and right-handed spinors while four vectors are tensor products of $s = 1/2$ left- and right-handed spinors. The discussion that follows considers the right- and left-handed representations separately. General covariant representations can be built from the right- and left-handed representations.

The motivation for considering these SL(2, $\mathbb{C}$) covariant representations is that they are directly related to the corresponding Euclidean covariant representations.

A dense set of Hilbert space vectors in the Euclidean representation are represented by Schwartz functions, $f(x_e,\mu)$ and $g(y_e,\nu)$, of Euclidean space-time variables, x_e and y_e, with positive Euclidean-time support and spins. In general, the spins are assumed to transform under finite-dimensional representations of $\text{SU}(2)\times\text{SU}(2)$.

The Euclidean time reflection operator, θ, is defined by

$$f(\theta x_e,\mu) = f\big(\theta(x^0_e,\mathbf{x}),\mu\big) := f\big[\big(-x^0_e,\mathbf{x}\big),\mu\big]. \quad (60)$$

Consider the following Euclidean covariant kernel:

$$S^s_e(x_e,\mu;y_e,\nu) := \int d^4p\frac{2}{(2\pi)^4}\frac{e^{ip_e\cdot(x_e-y_e)}}{p^2_e+m^2}D^s_{\mu\nu}(p_e\cdot\sigma_e). \quad (61)$$

This is a distribution since the integral is not convergent, however it makes perfect sense when considered as the kernel of a quadratic form.

The physical Hilbert space inner product for a particle of mass m and spin s is defined by the sesquilinear form

$$\int \sum_{\mu\nu} d^4x_e d^4y_e f^*(x_e,\mu) S_e^s(\theta x_e,\mu;y_e,\nu) g(y_e,\nu)$$
$$= \int \sum_{\mu\nu} d^4p_e f^*(\theta x_e,\mu) \frac{2}{(2\pi)^4} \frac{e^{ip_e\cdot(x_e-y_e)}}{p_e^2+m^2} \times D^s_{\mu\nu}(p_e\cdot\sigma_e) g(y_e,\nu)$$
$$= \int \sum_{\mu\nu} \psi^*_{\text{cov}}(\mathbf{p},\mu) \frac{d\mathbf{p}}{\omega_m(\mathbf{p})} D^s_{\mu\nu}(p\cdot\sigma)\phi_{\text{cov}}(\mathbf{p},\nu), \quad (62)$$

where

$$\psi^*_{\text{cov}}(\mathbf{p},\mu) := \frac{1}{(2\pi)^{3/2}} \int d^4x_e e^{i\mathbf{p}\cdot\mathbf{x}-\omega_m(\mathbf{p})x_e^0} f^*(\mathbf{x},x_e^0,\mu) \quad (63)$$

and

$$\phi_{\text{cov}}(\mathbf{p},\nu) := \frac{1}{(2\pi)^{3/2}} \int d^4x_e e^{-i\mathbf{p}\cdot\mathbf{x}-\omega_m(\mathbf{p})x_e^0} g(\mathbf{x},x_e^0,\nu). \quad (64)$$

The Euclidean time-support condition (which requires that f^* and g vanish unless $x_e^0>0$) ensures that the Laplace transforms with respect to the Euclidean times in Eqs. (63) and (64) are well defined. The resulting kernel in Eq. (62) is identical to the Lorentz covariant kernel in Eq. (56) after performing the integrals over the p_e^0. The covariant wave functions (63) and Eq. (64) are related to the Poincaré covariant wave functions by

$$\phi(\mathbf{p},\mu) = \sum_\nu D^s_{\mu\nu}[B_c(p)]\phi_{\text{cov}}(\mathbf{p},\nu)\frac{1}{\sqrt{\omega_m(\mathbf{p})}} \quad (65)$$

and

$$\psi^*(\mathbf{p},\mu) = \sum_\nu \psi^*_{\text{cov}}(\mathbf{p},\nu) D^s_{\nu\mu}(B_c(p))\frac{1}{\sqrt{\omega_m(\mathbf{p})}}. \quad (66)$$

This shows that the "Euclidean" inner product Eq. (62) can be identified with the corresponding Lorentz covariant inner product, which itself is identical to the original Poincaré covariant inner product. These steps illustrate how the correct Minkowski inner product is obtained from the Euclidean expression without analytic continuation.

This means that

$$S_r^s(x_e,\mu;y_e,\nu) := \int \frac{2d^4p_e}{(2\pi)^4}\frac{e^{ip_e\cdot(x_e-y_e)}}{p_e^2+m^2} D^s_{\mu\nu}(p_e\cdot\sigma_e) \quad (67)$$

is a Euclidean covariant reflection positive kernel for right-handed representations of mass m and spin s, respectively. The corresponding kernel for left-handed representations is

$$S_l^s(x_e,\mu;y_e,\nu) := \int \frac{2d^4p_e}{(2\pi)^4}\frac{e^{ip_e\cdot(x_e-y_e)}}{p_e^2+m^2} D^s_{\mu\nu}((\Pi p_e)\cdot\sigma_e). \quad (68)$$

Space reflection interchanges right- and left-handed representations. The space reflection operator does not commute with the Euclidean covariant kernel. This implies that space reflected states will not transform correctly under Lorentz transformations in these Lorentz covariant representations. Kernels for systems that allow a linear representation of space reflection can be constructed by taking direct sums or tensor products of right- and left-handed kernels [see Eqs. (58) and (59)].

The kernels Eqs. (67) and (68) can be evaluated analytically using the methods in Ref. [20]. The results are

$$S_r^s(z_e,\mu,\nu) := \frac{2}{(2\pi)^4}\int \frac{d^4p}{p_e^2+m^2} D^s_{\mu\nu}(p_e\sigma_e)e^{ip_e\cdot z_e}$$
$$= D^s_{\mu\nu}(-i\nabla_{ze}\cdot\sigma_e)\frac{2m^2}{(2\pi)^2}\frac{K_1\left(m\sqrt{z_0^2+\mathbf{z}^2}\right)}{m\sqrt{z_0^2+\mathbf{z}^2}}, \quad (69)$$

$$S_l^s(z_e,\mu,\nu) := \frac{2}{(2\pi)^2}\int \frac{d^4p}{p_e^2+m^2} D^s_{\mu\nu}(\Pi p_e\sigma_e)e^{ip_e\cdot z_e}$$
$$= D^s_{\mu\nu}(-i\Pi\nabla_{ze}\cdot\sigma_e)\frac{2m^2}{(2\pi)^2}\frac{K_1\left(m\sqrt{z_0^2+\mathbf{z}^2}\right)}{m\sqrt{z_0^2+\mathbf{z}^2}}, \quad (70)$$

where $z_e = x_e - y_e$ and $K_1(x)$ is a modified Bessel function. Note that $\frac{K_1(\eta)}{\eta}$ behaves like $1/\eta^2$ near the origin. Since $D^s_{\mu\nu}(-i\nabla_{ze}\cdot\sigma_e)$ is a degree $2s$ polynomial in $-i\nabla_{ze}$, these kernels have power law singularities at the origin, but fall off exponentially for large values of z_e^2. The restriction of the support of the vectors to positive Euclidean time ensures that $z_e^2>0$, so the singularities at $z_e=0$ never cause a problem. These kernels are reflection positive on this space. This is because $D^s_{\mu\nu}(p\cdot\sigma)$ factors into a product of a matrix and its adjoint:

$$D^s_{\mu\nu}(p\cdot\sigma) = \sum_\alpha D^s_{\mu\alpha}[B_c(p)]D^s_{\alpha\nu}[B_c(p)]^\dagger. \quad (71)$$

For any given spin the derivatives, $D^s_{\mu\nu}(-i\nabla_{ze}\cdot\sigma_e)$, in Eqs. (69) and (70) acting on $K_1(m\sqrt{z_e^2})$ can be expressed in terms of higher-order modified Bessel functions. See the Appendix.

The Euclidean inner product in right- and left-handed representations can be expressed directly in the x representation:

$$\langle f|g\rangle = \sum_{\mu\nu}\int f^*(x,\mu)D^s_{\mu\nu}(-i\nabla_x\cdot\sigma_e) \times \frac{2m^2}{(2\pi)^2}\frac{K_1(m\sqrt{(\theta x-y)_e^2}}{m\sqrt{(\theta x-y)_e^2}} g(y,\nu)d^4xd^4y, \quad (72)$$

$$\langle f|g\rangle = \sum_{\mu\nu}\int f^*(x,\mu)D^s_{\mu\nu}(-i\Pi\nabla_x\cdot\sigma_e) \times \frac{2m^2}{(2\pi)^2}\frac{K_1(m\sqrt{(\theta x-y)_e^2}}{m\sqrt{(\theta x-y)_e^2}} g(y,\nu)d^4xd^4y. \quad (73)$$

The construction in this section demonstrated the equivalence of the Poincaré covariant, Lorentz covariant and Euclidean covariant Hilbert space inner products for massive particles with any spin. Analytic continuation is not used to compute the physical inner product in the Euclidean representation. In addition, the Euclidean inner product, with the Euclidean time reflection on the final state, and the projection on the space of functions with positive time support was shown to be nonnegative, which demonstrates that these Euclidean kernels are reflection positive for any spin.

V. RELATIVISTIC INVARIANCE-PARTICLES

The formulation of relativistic covariance in the Euclidean representation is a consequence of the relation between the four-dimensional Euclidean group and the associated complex subgroup of the Lorentz group discussed in Sec. II.

This relation is used to relate the infinitesimal generators of Euclidean transformations to the corresponding Poincaré generators and then show that the resulting Poincaré generators are self-adjoint on the physical Hilbert space. This is not a new result, but it is desirable to construct explicit representations for the Poincaré generators for any spin to understand the relativistic transformation properties of particles with different spins or projections of multiparticle states on irreducible subspaces.

The starting point is to consider the 2×2 matrix representations of Minkowski and Euclidean four vectors:

$$p \cdot \sigma := \begin{pmatrix} p^0 + p^2 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix},$$

$$p_e \cdot \sigma_e := \begin{pmatrix} ip_e^0 + p_e^2 & p_e^1 - ip_e^2 \\ p_e^1 + ip_e^2 & ip_e^0 - p_e^3 \end{pmatrix}. \tag{74}$$

The $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ transformation properties of these matrices (denoted by P) are

$$P \to P' = APC^t. \tag{75}$$

The associated complex 4×4 Lorentz and four-dimensional orthogonal transformation matrices are

$$\Lambda(A,C)^\mu{}_\nu = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu A \sigma_\nu C^t),$$

$$\mathcal{O}(A,B)^\mu{}_\nu = \tfrac{1}{2}\mathrm{Tr}(\sigma_{e\mu}^\dagger A \sigma_{e\nu} C^t). \tag{76}$$

For ordinary rotations $A = C^* = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}}$. For rotations about the $\hat{\mathbf{z}}$ axis

$$\mathcal{O}(A,A^*)(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\lambda) & \sin(\lambda) & 0 \\ 0 & -\sin(\lambda) & \cos(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{77}$$

These transformations commute with the Euclidean time reflection operator:

$$\theta \mathcal{O}(A,A^*)(\lambda)\theta = \mathcal{O}(A,A^*)(\lambda). \tag{78}$$

For real rotations in Euclidean space-time planes, $A = C^t = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$. For the case of the x_e^0-$\hat{\mathbf{z}}$ plane,

$$\mathcal{O}(A,A^t)(\lambda)x = \begin{pmatrix} \cos(\lambda) & 0 & 0 & \sin(\lambda) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin(\lambda) & 0 & 0 & \cos(\lambda) \end{pmatrix}, \tag{79}$$

$$\theta \mathcal{O}^t(A,A^t)(\lambda)\theta = \mathcal{O}(A,A^t)(\lambda). \tag{80}$$

While ordinary three-dimensional rotations are the same for $p \cdot \sigma$ or $p_e \cdot \sigma_e$, real rotations in Euclidean space-time planes become rotationless Lorentz boosts with imaginary rapidity when applied to the Minkowski P.

These identifications imply the following algebraic relations between the infinitesimal generators of the four-dimensional orthogonal group and the Lorentz group:

$$\mathbf{P}_m = \mathbf{P}_e, \quad J_m^{ij} = J_e^{ij}, \tag{81}$$

$$H_m = iH_e, \quad K_m^i = -iJ_e^{0i}. \tag{82}$$

Because of the factor of i, if the Euclidean generators are self-adjoint operators on a representation of the Hilbert space, then the Poincaré generators Eqs. (81) and (82) cannot be self-adjoint on that representation of the Hilbert space.

In the spinless case ($s = 0$) the identifications Eqs. (76)–(80) result in the following expressions for the infinitesimal generators of the Poincaré group on the Euclidean representation of the Hilbert space with the Euclidean time reflection:

$$H_m\Psi(x_e) = \frac{\partial}{\partial x_e^0}\Psi(x_e), \quad \mathbf{P}_m\Psi(x_e) = -i\frac{\partial}{\partial \mathbf{x_e}}\Psi(x_e), \tag{83}$$

$$\mathbf{J}_m\Psi(x_e) = -i\mathbf{x} \times \boldsymbol{\nabla}_x\Psi(x_e),$$

$$K_m^j\Psi(x_e) = \left(x^j\frac{\partial}{\partial x_e^0} - x_e^0\frac{\partial}{\partial x^j}\right)\Psi(x_e). \tag{84}$$

It is straightforward to demonstrate that these operators satisfy the Poincaré commutations relations Eqs. (28)–(30). For example,

$$\left[K_m^i, H_m\right] = \left[x^i\frac{\partial}{\partial x_e^0} - x_e^0\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_e^0}\right] = i\left(-i\frac{\partial}{\partial x^i}\right) = iP_m^i, \tag{85}$$

which agrees with Eq. (30). The other commutators can be checked similarly.

The Euclidean time reflection of the final state makes both the Hamiltonian H_m and the boost generators $\mathbf{K}_m$ formally Hermitian with respect to the scalar product Eq. (62). One potential concern is that even an infinitesimal rotation in a Euclidean space-time plane can map functions with positive Euclidean time support to functions that violate the support condition. This maps Hilbert space vectors out of the Hilbert space. The resolution of this problem will be discussed in Sec. IX. The subscript m will be suppressed in what follows.

To show the hermiticity of the rotationless boost generators Eq. (84) note that rotational invariance of the Euclidean Green's function in Euclidean space-time planes means that the Euclidean rotation generators commute with the Euclidean Green's function:

$$\left(-ix^i\frac{\partial}{\partial x_e^0} + ix_e^0\frac{\partial}{\partial x^i}\right)S_e^0(x-y)$$

$$= S_e^0(x-y)\left(-iy^i\frac{\partial}{\partial y_e^0} + iy_e^0\frac{\partial}{\partial y^i}\right). \tag{86}$$

Multiplying both sides by i gives

$$\left(x^i\frac{\partial}{\partial x_e^0} - x_e^0\frac{\partial}{\partial x^i}\right)S_e^0(x-y) = S_e^0(x-y)\left(y^i\frac{\partial}{\partial y_e^0} - y_e^0\frac{\partial}{\partial y^i}\right). \tag{87}$$

Next, consider the inner product

$$\langle f|K^i|g\rangle = \int d^4x d^4y f^*(\mathbf{x}, -x_e^0) S_e^0(x-y)\left(y^i \frac{\partial}{\partial y_e^0} - y_e^0 \frac{\partial}{\partial y^i}\right) g(\mathbf{y}, y_e^0). \tag{88}$$

Using Eq. (87) in Eq. (88) gives

$$= \int d^4x d^4y f^*(\mathbf{x}, -x_e^0)\left(x^i \frac{\partial}{\partial x_e^0} - x_e^0 \frac{\partial}{\partial x^i}\right) S_e^0(x-y) g(\mathbf{y}, y_e^0). \tag{89}$$

Integrating by parts again gives

$$= -\int d^4x d^4y \left(x^i \frac{\partial}{\partial x_e^0} + x_e^0 \frac{\partial}{\partial x^i}\right)(\theta f)^*(\mathbf{x}, x_e^0) S_e^0(x-y) g(\mathbf{y}, y_e^0). \tag{90}$$

Finally, factoring the Euclidean time reversal out of f gives

$$-\left(x^i \frac{\partial}{\partial x_e^0} + x_e^0 \frac{\partial}{\partial x^i}\right)\theta f^*(\mathbf{x}, x_e^0) = \theta\left[\left(x^i \frac{\partial}{\partial x_e^0} - x_e^0 \frac{\partial}{\partial x^i}\right) f^*(\mathbf{x}, x_e^0)\right], \tag{91}$$

which when used in Eq. (90) gives

$$\begin{aligned}\langle f|K^i|g\rangle &= \int d^4x d^4y f^*(\mathbf{x}, -x_e^0) S_e^0(x-y)\left(y^i \frac{\partial}{\partial y_e^0} - y_e^0 \frac{\partial}{\partial y^i}\right) g(\mathbf{y}, y_e^0)\\ &= \int d^4x d^4y \theta\left(\left(x^i \frac{\partial}{\partial x_e^0} - x_e^0 \frac{\partial}{\partial x^i}\right) f(\mathbf{x}, x_e^0)\right)^* S_e^0(x-y) g(\mathbf{y}, y_e^0) = \langle K^i f|g\rangle.\end{aligned} \tag{92}$$

This shows that K^i is a Hermitian operator on this representation of the Hilbert space.

The other nontrivial operator is the Hamiltonian Eq. (83). In this case,

$$\begin{aligned}\langle f|H|g\rangle &= \int d^4x d^4y f^*(\mathbf{x}, -x_e^0) S_e^0(x-y) \frac{\partial}{\partial y_e^0} g(\mathbf{y}, y_e^0)\\ &= -\int d^4x d^4y f^*(\mathbf{x}, -x_e^0) \frac{\partial}{\partial y_e^0} S_e^0(x-y) g(\mathbf{y}, y_e^0) = \int d^4x d^4y f^*(\mathbf{x}, -x_e^0) \frac{\partial}{\partial x_e^0} S_e^0(x-y) g(\mathbf{y}, y_e^0)\\ &= -\int d^4x d^4y \frac{\partial}{\partial x_e^0} f^*(\mathbf{x}, -x_e^0) S_e^0(x-y) g(\mathbf{y}, y_e^0) = \int d^4x d^4y \frac{\partial f^*}{\partial x^0}(\mathbf{x}, -x_e^0) S_e^0(x-y) g(\mathbf{y}, y_e^0) = \langle Hf|g\rangle.\end{aligned} \tag{93}$$

Euclidean time reversal does not change the linear or angular momentum operators. These methods can be used to demonstrate that all of the $s = 0$ generators Eqs. (83) and (84) are Hermitian in the Euclidean representation of the Hilbert space and satisfy the Poincaré Lie algebra.

VI. SPIN

For application in hadronic physics or relativistic many-body physics it is necessary to consider representations of the Poincaré Lie algebra with higher spins. In this section explicit formulas for generators for particles with arbitrary spin are derived, generalizing the method used in Sec. V for scalar particles. While these results are not new, explicit formulas are needed for applications.

In the original Poincaré covariant theory the spin is associated with the observable that is the $\hat{\mathbf{z}}$-component of the spin that would be measured in the particle's rest frame if it was transformed to the rest frame with a rotationless Lorentz transformation. The spin in the covariant wave function is related to this spin by multiplying by one of the SL$(2, \mathbb{C})$ matrices, $D^s_{\mu\nu}[B_c(p)^{-1}]$ or $D^s_{\mu\nu}[B_c(p)^{\dagger}]$. These transformations lead to distinct right- or left-handed spinors. In discussing spin it is important to understand that the Poincaré covariant spinors and the Lorentz covariant spinors are related, but have different transformation properties. Representations of the Poincaré generators for right- and left-handed covariant spins must be considered separately. In addition, for each type of covariant spinor there are invariant linear functionals that define dual spinors. The dual spinors are spinor analogs of covariant and contravariant vectors. In conventional treatments [8,21,22] the right-handed spinors are denoted by ξ^a, left-handed spinors are denoted by $\xi^{\dot{a}}$ and their duals are denoted by ξ_a and $\xi_{\dot{a}}$, respectively. In this section we consider each of these four cases.

The first step is to determine the Euclidean covariance properties of the Euclidean kernels for right and left handed covariant spinors and their duals. Euclidean four vectors can be represented by any of the four matrices:

$$\begin{aligned}p_e \cdot \sigma_e &= p_e^\mu \sigma_{e\mu}, \quad p_e \cdot (\sigma_2 \sigma_e \sigma_2) = p_e^\mu \sigma_2 \sigma_{e\mu} \sigma_2,\\ p_e \cdot \sigma_e^t &= p_e^\mu \sigma_{e\mu}^t, \quad p_e \cdot \left(\sigma_2 \sigma_e^t \sigma_2\right) = p_e^\mu \sigma_2 \sigma_{e\mu}^t \sigma_2.\end{aligned} \tag{94}$$

The determinant of each of these matrices is $(-)$ the square of the Euclidean length of p_e, which is preserved under linear transformations of the form

$$P' = APC^t, \tag{95}$$

where P represents any of the matrices in Eq. (94), and $A, C \in SL(2, \mathbb{C})$. Real four-dimensional orthogonal transformations are obtained by restricting A and C to be elements of SU(2).

The 4×4 orthogonal matrix $\mathbb{O}(A, C)^\mu{}_\nu$ is related to the pair (A, C) by

$$\mathbb{O}(A, C)^\mu{}_\nu := \frac{1}{2} Tr(\sigma_{e\mu}^\dagger A \sigma_{e\nu} C^t). \tag{96}$$

It follows that

$$A p_e^\mu \sigma_{e\mu} C^t = \sigma_{e\mu} \mathbb{O}(A, C)^\mu{}_\nu p_e^\nu = \sigma_{e\mu} (\mathbb{O}(A, C) p_e)^\mu. \tag{97}$$

Multiplying Eq. (97) by σ_2 on both sides using $\sigma_2 A \sigma_2 = A^*$ for $A \in SU(2)$ gives

$$A^*(p_e \cdot (\sigma_2 \sigma_e \sigma_2)) C^\dagger = (\mathbb{O}(A, C) p)_e \cdot (\sigma_2 \sigma_e \sigma_2). \tag{98}$$

Taking transposes of the 2×2 matrices Eqs. (97) and (98) gives

$$C\big(p_e \cdot \sigma_e^t\big) A^t = \sigma_e^t \cdot (\mathbb{O}(A, C) p_e) \tag{99}$$

and

$$C^*\big(p_e \cdot \big(\sigma_2 \sigma_e^t \sigma_2\big)\big) A^\dagger = (\sigma_2 \sigma_e^t \sigma_2) \cdot (\mathbb{O}(A, C) p_e). \tag{100}$$

In all four of these expressions A, C and the orthogonal matrix $\mathbb{O}(A, C)$ are unchanged. All four of the matrices Eq. (94) become positive when p_e is replaced by the on-shell Minkowski four-momentum, $p_m^\mu = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$, and σ_e^μ is replaced by σ^μ.

These identities will be used to derive the covariance properties of each type of Euclidean kernel.

The matrices Eq. (94) appear in the Euclidean covariant kernels for the right- and left-handed representations and their duals. The spin s Euclidean covariant inner product kernels for each type of covariant spinor are

$$S_e^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}[p_e \cdot \sigma_e]}{p_e^2 + m^2} e^{ip_e x_e} d^4 p_e, \tag{101}$$

$$S_{ed}^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}[p_e \cdot (\sigma_2 \sigma_e \sigma_2)]}{p_e^2 + m^2} e^{ip_e x_e} d^4 p_e, \tag{102}$$

$$S_{e*}^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}\big[p_e \cdot \sigma_e^t\big]}{p_e^2 + m^2} e^{ip_e x_e} d^4 p_e, \tag{103}$$

$$S_{ed*}^s(x_e; \mu, \nu) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}\big[p_e \cdot \big(\sigma_2 \sigma_e^t \sigma_2\big)\big]}{p_e^2 + m^2} e^{ip_e x_e} d^4 p_e. \tag{104}$$

It is possible to construct more general classes of kernels using products or direct sums of left- and right-handed representations, for example,

$$S_{e:e*}^{s\dot{s}}(x_e; \mu, \dot{\mu}, \nu, \dot{\nu}) = \frac{2}{(2\pi)^4} \int \frac{D^s_{\mu\nu}[p_e \cdot (\sigma_2 \sigma_e \sigma_2)] D^{\dot{s}}_{\dot{\mu}\dot{\nu}}\big[p_e \cdot \sigma_e^t\big]}{p_e^2 + m^2} e^{ip_e x_e} d^4 p_e. \tag{105}$$

The physical Hilbert space inner product associated with each of these kernels is

$$\langle \psi_e | \phi_e \rangle = \int \sum_{\mu\nu} \psi_e^*(\theta x, \mu) S_e^s(x_e - y_e; \mu, \nu) \phi_e(y, \nu) d^4x d^4y, \tag{106}$$

$$\langle \psi_{ed} | \phi_{ed} \rangle = \int \sum_{\mu\nu} \psi_{ed}^*(\theta x, \mu) S_{ed}^s(x_e - y_e; \mu, \nu) \phi_{ed}(y, \nu) d^4x d^4y, \tag{107}$$

$$\langle \psi_{e*} | \phi_{e*} \rangle = \int \sum_{\mu\nu} \psi_{e*}^*(\theta x, \mu) S_{e*}^s(x_e - y_e; \mu, \nu) \phi_{e*}(y, \nu) d^4x d^4y, \tag{108}$$

$$\langle \psi_{ed*} | \phi_{ed*} \rangle = \int \sum_{\mu\nu} \psi_{ed*}^*(\theta x, \mu) S_{ed*}^s(x_e - y_e; \mu, \nu) \phi_{ed*}(y, \nu) d^4x d^4y. \tag{109}$$

For wave functions with positive Euclidean time support, the p_e^0 integral can be evaluated by the residue theorem, closing the contour in the upper-half plane. This replaces p_e^0 by $i\omega_m(\mathbf{p})$. The kernels become the two-point Minkowski Wightman functions [8] for mass m spin s irreducible representations of the Lorentz group. Equations (106) and (107) are dual representations of the right-handed kernel, while Eqs. (108) and (109) are dual representations of the left-handed kernel. σ_2 behaves like a metric tensor for the Lorentz covariant spinors, relating the representations Eqs. (106) and (107) or Eqs. (108) and (109). Contraction of the two types of right- or left-handed spinors are Lorentz invariant. The results of performing the p_e^0 integral for each type of kernel are

$$\langle \psi_e | \phi_e \rangle = \int \sum_{\mu\nu} f_m^*(\mathbf{p}, \mu) \frac{d\mathbf{p} D^s_{\mu\nu}[p_m \cdot \sigma]}{\omega_m(\mathbf{p})} g_m(\mathbf{p}, \nu), \tag{110}$$

$$\langle \psi_{ed} | \phi_{ed} \rangle = \int \sum_{\mu\nu} f_m^*(\mathbf{p}, \mu) \frac{d\mathbf{p} D^s_{\mu\nu}[p_m \cdot (\sigma_2 \sigma \sigma_2)]}{\omega_m(\mathbf{p})} g_m(\mathbf{p}, \nu), \tag{111}$$

$$\langle\psi_{e*}|\phi_{e*}\rangle = \int \sum_{\mu\nu} f_m^*(\mathbf{p},\mu)\frac{d\mathbf{p}D^s_{\mu\nu}[p_m\cdot\sigma^*]}{\omega_m(\mathbf{p})}g_m(\mathbf{p},\nu), \tag{112}$$

$$\langle\psi_{ed*}|\phi_{ed*}\rangle = \int \sum_{\mu\nu} f_m^*(\mathbf{p},\mu)\frac{d\mathbf{p}D^s_{\mu\nu}[p_m\cdot(\sigma_2\sigma^*\sigma_2)]}{\omega_m(\mathbf{p})}g_m(\mathbf{p},\nu), \tag{113}$$

where

$$f_m^*(\mathbf{p},\mu) := \int \frac{d^4x}{(2\pi)^{3/2}}\psi^*(x,\mu)e^{i\mathbf{p}\cdot\mathbf{x}-\omega_m(\mathbf{p})x^0}, \tag{114}$$

$$g_m(\mathbf{p},\nu) := \int \frac{d^4y}{(2\pi)^{3/2}}\phi(y,\nu)e^{-i\mathbf{p}\cdot\mathbf{y}-\omega_m(\mathbf{p})y^0}, \tag{115}$$

for each type of spinor wave function.

Each of the spin matrices, $D^s_{\mu\nu}[p_m\cdot\sigma]$, $D^s_{\mu\nu}[p_m\cdot(\sigma_2\sigma\sigma_2)]$, $D^s_{\mu\nu}[p_m\cdot\sigma^*]$, and $D^s_{\mu\nu}[p_m\cdot(\sigma_2\sigma^*\sigma_2)]$ are positive Hermitian matrices, so the Euclidean Green's functions [Eqs. (101)–(104)] are all reflection positive.

The spinor transformation properties [Eqs. (97)–(100)] of the right- and left-handed spinors and their duals are used construct the spinor parts of the Poincaré generators in the Euclidean representation:

$$\int \sum_{\mu\nu}\psi_e^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[(\mathbb{O}p)\cdot\sigma_e]\phi_e(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\mu\nu}\psi_e^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[p\cdot(A\sigma_eC^t)]\phi_e(y,\nu)d^4xd^4yd^4p, \tag{116}$$

$$\int \sum_{\mu\nu}\psi_{ed}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[(\mathbb{O}p)\cdot(\sigma_2\sigma_e\sigma_2)]\phi_{ed}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\mu\nu}\psi_{ed}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}[p\cdot(A^*\sigma_2\sigma_e\sigma_2C^\dagger)]\phi_{ed}(y,\nu)d^4xd^4yd^4p, \tag{117}$$

$$\int \sum_{\mu\nu}\psi_{e*}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}\big[(\mathbb{O}p)\cdot\sigma_e^t\big]\phi_{e*}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\mu\nu}\psi_{e*}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}\big[p\cdot\big(C\sigma_e^tA^t\big)\big]\phi_{e*}(y,\nu)d^4xd^4yd^4p, \tag{118}$$

$$\int \sum_{\mu\nu}\psi_{ed*}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}\big[(\mathbb{O}p)\cdot\big(\sigma_2\sigma_e^t\sigma_2\big)\big]\phi_{ed*}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\mu\nu}\psi_{ed*}^*(\theta x,\mu)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}D^s_{\mu\nu}\big[p\cdot\big(C^*\sigma_2\sigma_e^t\sigma_2A^\dagger\big)\big]\phi_{ed*}(y,\nu)d^4xd^4yd^4p. \tag{119}$$

The next step is to move the transformations in the kernels to the wave functions. The Euclidean invariance of the measures and scalar products, the group representation properties of the Wigner functions, and re-definitions of the wave functions can be used to show that Eqs. (116)–(119) are equivalent to

$$\int \sum_{\alpha\mu\nu}\big[D^s_{\mu\alpha}[A^\dagger]^{-1}\psi_e(\theta\mathbb{O}^t\theta x,\alpha)\big]^*\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2}D^s_{\mu\nu}[p\cdot\sigma_e]\phi_e(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\alpha\mu\nu}\psi_e^*(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2}D^j_{\mu\alpha}[p\cdot\sigma_e]D^s_{\alpha\nu}[C^t]\phi_e(\mathbb{O}y,\nu)d^4xd^4yd^4p, \tag{120}$$

$$\int \sum_{\alpha\mu\nu}\big[D^s_{\mu\alpha}[A^t]^{-1}\psi_{ed}(\theta\mathbb{O}^t\theta x,\alpha)\big]^*\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2}D^s_{\mu\nu}[p\cdot(\sigma_2\sigma_e\sigma_2)]\phi_{ed}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\alpha\mu\nu}\psi_{ed}^*(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2}D^j_{\mu\alpha}[p\cdot(\sigma_2\sigma_e\sigma_2)]D^s_{\alpha\nu}[C^\dagger]\phi_{ed}(\mathbb{O}y,\nu)d^4xd^4yd^4p, \tag{121}$$

$$\int \sum_{\alpha\mu\nu} \left[D^s_{\mu\alpha}[C^\dagger]^{-1}\psi_{e*}(\theta\mathbb{O}^t\theta x,\alpha)\right]^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}\left[p\cdot\sigma_e^t\right]\phi_{e*}(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum_{\alpha\mu\nu} \psi^*_{e*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^j_{\mu\alpha}\left[p\cdot\sigma_e^t\right]D^s_{\alpha\nu}[A^t]\tilde{\phi}_{e*}(\mathbb{O}y,\nu)d^4xd^4yd^4p, \tag{122}$$

$$\int \sum_{\alpha\mu\nu} \left[D^s_{\mu\alpha}[C^t]^{-1}\psi_{ed*}(\theta\mathbb{O}^t\theta x,\alpha)\right]^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}\left[p\cdot\left(\sigma_2\sigma_e^t\sigma_2\right)\right]\phi_{ed*}(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum_{\alpha\mu\nu} \psi^*_{ed*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}\left[p\cdot\left(\sigma_2\sigma_e^t\sigma_2\right)\right]D^s_{\alpha\nu}[A^\dagger]\phi_{ed*}(\mathbb{O}y,\nu)d^4xd^4yd^4p. \tag{123}$$

For ordinary rotations, as well as rotations in space Euclidean time planes, the SU(2) matrices A and C are related.

To derive expressions for the generators for each type of spinor, check the hermiticity and verify the commutation relations the first step is to replace A and C with the pairs of SU(2) matrices representing one-parameter groups for both ordinary rotations about a fixed axis and rotations in a Euclidean space-time plane.

For ordinary rotations about the $\hat{\mathbf{n}}$ axis, the one-parameter group is

$$A(\lambda) = C^*(\lambda) = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} \tag{124}$$

and $[\theta\mathbb{O}^t(\lambda)\theta] = \mathbb{O}^t(\lambda)$, while for rotations in Euclidean $\hat{\mathbf{n}}$-x^0 space-time planes the one-parameter group is

$$A(\lambda) = C^t(\lambda) = e^{i\frac{\lambda}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} \tag{125}$$

and $[\theta\mathbb{O}^t(\lambda)\theta] = \mathbb{O}(\lambda)$. The 4×4 orthogonal transformations, $\mathbb{O}(\lambda)$ associated with each type of transformation are shown explicitly for rotations about the $\hat{\mathbf{z}}$ axis and for rotations in the $\hat{\mathbf{z}}$-x^0 plane: For rotations about the $\hat{\mathbf{z}}$ axis,

$$\mathbb{O}(A,A^*)(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos(\lambda) & \sin(\lambda) & 0\\ 0 & -\sin(\lambda) & \cos(\lambda) & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{126}$$

and

$$\theta\mathbb{O}(A,A^*)(\lambda)\theta = \mathbb{O}(A,A^*)(\lambda). \tag{127}$$

For rotations in the $\hat{\mathbf{z}}$-x^0 plane,

$$\mathbb{O}(A,A^t)(\lambda) = \begin{pmatrix} \cos(\lambda) & 0 & 0 & \sin(\lambda)\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\sin(\lambda) & 0 & 0 & \cos(\lambda)\end{pmatrix} \tag{128}$$

and

$$\theta\mathbb{O}^t(A,A^t)(\lambda)\theta = \mathbb{O}(A,A^t)(\lambda). \tag{129}$$

For the case of ordinary rotations $A = C^*$ and Eqs. (120)–(123) become

$$\int \sum_{\alpha\mu\nu} \{D^s_{\mu\alpha}[A]\psi_e[\mathbb{O}^t(\lambda)x,\alpha]\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot\sigma_e]\phi_e(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum_{\alpha\mu\nu} \psi^*_e(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^j_{\mu\alpha}[p\cdot\sigma_e]D^s_{\alpha\nu}[A^\dagger]\phi_e[\mathbb{O}(\lambda)y,\nu]d^4xd^4yd^4p, \tag{130}$$

$$\int \sum_{\alpha\mu\nu} \{D^s_{\mu\alpha}[A^*]\psi_{ed}[\mathbb{O}^t(\lambda)x,\alpha]\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot\sigma_2\sigma_e\sigma_2]\phi_{ed}(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum_{\alpha\mu\nu} \psi^*_{ed}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}[p\cdot(\sigma_2\sigma_e\sigma_2)]D^s_{\alpha\nu}[A^t]\phi_{ed}[\mathbb{O}(\lambda)y,\nu]d^4xd^4yd^4p, \tag{131}$$

$$\int \sum_{\alpha\mu\nu} \left\{D^s_{\mu\alpha}[A^*]\psi_{e*}[\mathbb{O}^t(\lambda)x,\alpha]\right\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}\left[p\cdot\sigma_e^t\right]\phi_{e*}(y,\nu)d^4xd^4yd^4p$$

$$= \int \sum_{\alpha\mu\nu} \psi^*_{e*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}\left[p\cdot\sigma_e^t\right]D^s_{\alpha\nu}[A^t]\phi_{e*}(\mathbb{O}(\lambda)y,\nu)d^4xd^4yd^4p, \tag{132}$$

$$\int \sum_{\alpha\mu\nu} \{D^s_{\mu\alpha}[A]\psi_{ed*}[\mathbb{O}^t(\lambda)x,\alpha]\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}\big[p\cdot\big(\sigma_2\sigma_e^t\sigma_2\big)\big]\phi_{ed*}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\alpha\mu\nu} \psi^*_{ed*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}\big[p\cdot\big(\sigma_2\sigma_e^t\sigma_2\big)\big]D^s_{\alpha\nu}[A^\dagger]\phi_{ed*}(\mathbb{O}(\lambda)y,\nu)d^4xd^4yd^4p. \tag{133}$$

For the case of rotations in Euclidean space-time planes for $A = C^t$ Eqs. (120)–(123) become

$$\int \sum_{\alpha\mu\nu} \{D^s_{\mu\alpha}[A]\psi_{e}[\mathbb{O}(\lambda)x,\alpha]\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot\sigma_e]\phi_{e}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\alpha\mu\nu} \psi^*_{e}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}[p\cdot\sigma_e]D^s_{\alpha\nu}[A]\phi_{e}(\mathbb{O}(\lambda)y,\nu)d^4xd^4yd^4p, \tag{134}$$

$$\int \sum_{\alpha\mu\nu} \{D^s_{\mu\alpha}[A^*]\psi_{ed}[\mathbb{O}(\lambda)x,\alpha]\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}[p\cdot(\sigma_2\sigma_e\sigma_2)]\phi_{ed}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\alpha\mu\nu} \psi^*_{ed}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}[p\cdot(\sigma_2\sigma_e\sigma_2)]D^s_{\alpha\nu}[A^*]\phi_{ed}(\mathbb{O}(\lambda)y,\nu)d^4xd^4yd^4p, \tag{135}$$

$$\int \sum_{\alpha\mu\nu} \{D^s_{\mu\alpha}[A^t]\psi_{e*}[\mathbb{O}(\lambda)x,\alpha]\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}\big[p\cdot\big(\sigma_e^t\big)\big]\phi_{e*}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\alpha\mu\nu} \psi^*_{e*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}\big[p\cdot\big(\sigma_e^t\big)\big]D^s_{\alpha\nu}[A^t]\phi_{e*}(\mathbb{O}(\lambda)y,\nu)d^4xd^4yd^4p, \tag{136}$$

$$\int \sum_{\alpha\mu\nu} \{D^s_{\mu\alpha}[A^\dagger]\psi_{ed*}[\mathbb{O}(\lambda)x,\alpha]\}^* \frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\nu}\big[p\cdot\big(\sigma_2\sigma_e^t\sigma_2\big)\big]\phi_{ed*}(y,\nu)d^4xd^4yd^4p$$
$$= \int \sum_{\alpha\mu\nu} \psi^*_{ed*}(x,\mu)\frac{e^{ip\cdot(\theta x-y)}}{p^2+m^2} D^s_{\mu\alpha}\big[p\cdot\big(\sigma_2\sigma_e^t\sigma_2\big)\big]D^s_{\alpha\nu}[A^\dagger]\phi_{ed*}(\mathbb{O}(\lambda)y,\nu)d^4xd^4yd^4p. \tag{137}$$

To construct generators of ordinary rotations differentiate the right-hand side of Eqs. (130)–(133) by λ, set $\lambda = 0$, and multiply the result by i. To construct the generators of Euclidean space-time rotations differentiate the right-hand side of Eqs. (134)–(137) by λ, set $\lambda = 0$, and multiply the result by i to get expressions for the generators. To get expressions for the Lorentz boost generators multiply the Euclidean space-time rotation generators by an additional factor of $-i$. The derivatives of the Wigner functions can be computed using

$$\frac{d}{d\lambda}D^s_{\mu\nu}[A(\lambda)]_{|_{\lambda=0}} = \frac{d}{d\lambda}\langle s,\mu|e^{i\lambda\hat{\mathbf{n}}\cdot\mathbf{S}}|s,\nu\rangle_{|_{\lambda=0}} = i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle, \tag{138}$$
$$\frac{d}{d\lambda}D^s_{\mu\nu}[A(\lambda)^\dagger]_{|_{\lambda=0}} = \frac{d}{d\lambda}\langle s,\mu|e^{-i\lambda\hat{\mathbf{n}}\cdot\mathbf{S}}|s,\nu\rangle_{|_{\lambda=0}} = -i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle, \tag{139}$$
$$\frac{d}{d\lambda}D^s_{\mu\nu}[A^*(\lambda)]_{|_{\lambda=0}} = \frac{d}{d\lambda}(D^s_{\mu\nu}[A(\lambda)])^*_{|_{\lambda=0}} = -i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle^* = -i\langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle, \tag{140}$$
$$\frac{d}{d\lambda}D^s_{\mu\nu}[A^t(\lambda)]_{|_{\lambda=0}} = \frac{d}{d\lambda}(D^s_{\mu\nu}([A(\lambda)])^*)^{-1}_{|_{\lambda=0}} = i\langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle^* = i\langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle. \tag{141}$$

These can be evaluated using S_z and angular momentum raising and lowering operators. The rotation generators for each type of spinor representation can be read off of Eqs. (130)–(133):

$$\langle x,s,\nu|\mathbf{J}|\psi_e\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times\left(-i\frac{\partial}{\partial\mathbf{x}}\right) + \langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle\right)\langle x,s,\nu|\psi_e\rangle, \tag{142}$$

$$\langle x,s,\nu|\mathbf{J}|\psi_{ed}\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times\left(-i\frac{\partial}{\partial\mathbf{x}}\right) - \langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle\right)\langle x,s,\nu|\psi_{ed}\rangle, \tag{143}$$

$$\langle x,s,\nu|\mathbf{J}|\psi_{e*}\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times\left(-i\frac{\partial}{\partial\mathbf{x}}\right) - \langle s,\nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\mu\rangle\right)\langle x,s,\nu|\psi_{e*}\rangle, \tag{144}$$

$$\langle x,s,\nu|\mathbf{J}|\psi_{ed*}\rangle = \sum_\nu \left(\delta_{\mu\nu}\mathbf{x}\times\left(-i\frac{\partial}{\partial\mathbf{x}}\right) + \langle s,\mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s,\nu\rangle\right)\langle x,s,\nu|\psi_{ed*}\rangle. \tag{145}$$

The first and fourth term are representations of standard rotation generators . In the second and third terms the spin generator matrix elements are transposed and multiplied by with a (-) sign. To show that these operator satisfy SU(2) commutation relations, consider matrices satisfying SU(2) commutation relations:

$$[M_m, M_n] = i\sum_k \epsilon_{mnk}M_k. \tag{146}$$

The transposes satisfy

$$\left[M_n^t, M_m^t\right] = i\sum_k \epsilon_{mnk}M_k^t, \tag{147}$$

$$\left[\left(-M_m^t\right), \left(-M_n^t\right)\right] = i\sum_k \epsilon_{mnk}\left(-M_k^t\right), \tag{148}$$

which shows that the negative transpose of these matrices also satisfy SU(2) commutation relations. This shows that all of the spin generators satisfy SU(2) commutation relations.

Generators for rotations in Euclidean space-time planes are constructed the same way from

$$\langle x, s, \nu|J^{0\hat{n}}|\psi_e\rangle = \sum_\nu \left(i\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) - \langle s, \mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \nu\rangle\right)\langle x, s, \nu|\psi_e\rangle, \tag{149}$$

$$\langle x, s, \nu|J^{0\hat{n}}|\psi_{ed}\rangle = \sum_\nu \left(i\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) + \langle s, \nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \mu\rangle\right)\langle x, s, \nu|\psi_{ed}\rangle, \tag{150}$$

$$\langle x, s, \nu|J^{0\hat{n}}|\psi_{e*}\rangle = \sum_\nu \left(i\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) - \langle s, \nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \mu\rangle\right)\langle x, s, \nu|\psi_{e*}\rangle, \tag{151}$$

$$\langle x, s, \nu|J^{0\hat{n}}|\psi_{ed*}\rangle = \sum_\nu \left(i\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) + \langle s, \mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \nu\rangle\right)\langle x, s, \nu|\psi_{ed*}\rangle. \tag{152}$$

To construct the boost generators it is necessary to multiply these expression by an additional factor of $(-i)$

$$\langle x, s, \nu|\mathbf{K}|\psi_e\rangle = \sum_\nu \left(\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) + i\langle s, \mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \nu\rangle\right)\langle x, s, \nu|\psi_e\rangle, \tag{153}$$

$$\langle x, s, \nu|\mathbf{K}|\psi_{ed}\rangle = \sum_\nu \left(\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) - i\langle s, \nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \mu\rangle\right)\langle x, s, \nu|\psi_{ed}\rangle, \tag{154}$$

$$\langle x, s, \nu|\mathbf{K}|\psi_{e*}\rangle = \sum_\nu \left(\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) + i\langle s, \nu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \mu\rangle\right)\langle x, s, \nu|\psi_{e*}\rangle, \tag{155}$$

$$\langle x, s, \nu|\mathbf{K}|\psi_{ed*}\rangle = \sum_\nu \left(\delta_{\mu\nu}\left(\mathbf{x}\frac{\partial}{\partial x^0} - x^0\frac{\partial}{\partial \mathbf{x}}\right) - i\langle s, \mu|\hat{\mathbf{n}}\cdot\mathbf{S}|s, \nu\rangle\right)\langle x, s, \nu|\psi_{ed*}\rangle. \tag{156}$$

The continuous part of these expressions agree with Eqs. (83) and (84) for spinless operators. The relevant commutators involving the spin parts of the boost generators in each of the four representations are

$$[K_i, K_j]_{\text{spin}} = [iS_i, iS_j] = -i\sum_k \epsilon_{ijk}S_k = -i\sum_k \epsilon_{ijk}J_{k\,\text{spin}}, \tag{157}$$

$$[K_i, K_j]_{\text{spin}} = \left[-iS_i^t, -iS_j^t\right] = -i\sum_k \epsilon_{ijk}\left(-S_k^t\right) = -i\sum_k \epsilon_{ijk}J_{k\,\text{spin}}, \tag{158}$$

$$[K_i, K_j]_{\text{spin}} = \left[iS_i^t, iS_j^t\right] = -i\sum_k \epsilon_{ijk}\left(-S_k^t\right) = -i\sum_k \epsilon_{ijk}J_{k\,\text{spin}}, \tag{159}$$

$$[K_i, K_j]_{\text{spin}} = [-iS_i, -iS_j] = -i\sum_k \epsilon_{ijk}S_k = -i\sum_k \epsilon_{ijk}J_{k\,\text{spin}}, \tag{160}$$

$$[K_i, S_j]_{\text{spin}} = [iS_i, S_j] = i\sum_k \epsilon_{ijk}(iS_k) = i\sum_k \epsilon_{ijk}K_{k\,\text{spin}}, \tag{161}$$

$$[K_i, S_j]_{\text{spin}} = \left[-iS_i^t, -S_j^t\right] = i\epsilon_{ijk}\left(-iS_k^t\right) = i\sum_k \epsilon_{ijk}K_{k\,\text{spin}}, \tag{162}$$

$$[K_i, S_j]_{\text{spin}} = \left[iS_i^t, -S_j^t\right] = i\epsilon_{ijk}iS_k^t = i\sum_k \epsilon_{ijk}K_{k\,\text{spin}}, \tag{163}$$

$$[K_i, S_j]_{\text{spin}} = [-iS_i, S_j] = i\epsilon_{ijk}(-iS_k) = \sum_k \epsilon_{ijk}K_{k\,\text{spin}}, \tag{164}$$

where the spin generators in Eqs. (158), (159), (162), and (163) are (−) the transposes of the matrices satisfying SU(2) commutation relations, which were shown in Eqs. (146)–(148) to satisfy SU(2) commutation relations. It follows that Eqs. (142)–(145) and (153)–(156) for the Lorentz generators in each of the four spinor representations satisfy the Poincaré commutation relations.

The hermiticity of these generators follows from Eqs. (130)–(133) and (134)–(137). Each of Eqs. (130)–(133) has the form

$$\langle U^{\dagger}(\lambda)\psi|\phi\rangle = \langle\psi|U(\lambda)|\phi\rangle, \tag{165}$$

so the rotation operators, which are generators of unitary one-parameter groups [23], are self-adjoint in the Hilbert spaces with inner products Eqs. (106)–(109).

For the boost generators hermiticity follows from Eqs. (134)–(137). In this case all of these equations have the form

$$\langle T(\lambda)\psi|\phi\rangle = \langle\psi|T(\lambda)|\phi\rangle. \tag{166}$$

In these cases $T(\lambda)$ is Hermitian, but the generators are constructed by multiplying the λ derivative by $1 = (i)(-i)$ rather than i, resulting in Hermitian operators.

The self-adjointness of the Hamiltonian and boost generators is discussed in Sec. IX.

In these covariant representations the spin does not enter in the Hamiltonian or the linear momentum operators. These operators all commute with the spin operators and commutators with these operators follow from the scalar case.

The main result of this section is Eqs. (149)–(156) for the Poincaré generators. These operators are formally Hermitian on the different representations of the Euclidean Hilbert space and they satisfy the Poincaré commutation relations with the translations generators Eq. (83).

The construction in this section is limited to a description of a particle of mass $m > 0$ and spin s.

VII. SYSTEMS OF FREE PARTICLES

The Hilbert space for systems of free particles is the direct sum of tensor products of single-particle Hilbert spaces.

A dense set of vectors in Euclidean Hilbert space for a system N noninteracting particles are represented by functions of the form

$$\psi(x_1, \mu_1, x_2, \mu_2 \cdots x_N, \mu_N), \tag{167}$$

which vanish unless $x_{ei}^0 > 0$. The indices μ_i are $SU(2) \times SU(2)$ spinor indices.

The Hilbert space inner product is

$$\langle\psi|\phi\rangle = \sum \int d^{4N}x d^{4N}y \psi^*(x_1, \mu_1, x_2, \mu_2 \cdots, x_N, \mu_N) \times \prod_{n=1}^{N} S(\theta x_n - y_n, \mu_n, \nu_n)\phi(y_1, \nu_1, y_2, \nu_2 \cdots y_N, \nu_N). \tag{168}$$

This is reflection positive since each of the $S(x_n - y_n, \mu_n, \nu_n)$ is reflection positive and given explicitly by Eqs. (101)–(104). This is simply an N-fold tensor product single-particle Hilbert spaces.

When the particles are identical the initial and final states can be symmetrized or antisymmetrized as appropriate. Both of these operations commute with the Euclidean time reflection and consequently preserve the reflection positivity.

The $\theta x_n^0 - y_m^0$ will always be negative for functions with support for positive Euclidean time.

Unlike the field theory case, the kernels are not assumed to be completely symmetric (antisymmetric) which leads to locality.

The Poincaré generators are sums of single-particle generators.

VIII. DYNAMICS

In Lorentz and Euclidean covariant representations of relativistic quantum mechanics the dynamics enters through a kernel. The reflection positivity constraint on Euclidean kernels is less restrictive in the particle case than it is in the local field theory case. In a local field theory there is one N-point kernel for any combination of M initial degrees of freedom and K final degrees of freedom for $N = M + K$. When locality is not required there can be different reflection positive kernels for each combination of M initial degrees of freedom and K final degrees of freedom that add up to N. This is a weaker form of reflection positivity.

For the purpose of making models the biggest challenge is to understand the structure of model reflection positive kernels or to verify that model kernels are reflection positive. While in general any positive-mass positive-energy unitary representation of the Poincaré group can be decomposed into a direct integral of irreducible representations, where the methods of Sec. IV can be applied to construct the equivalent Euclidean kernels, typical model input is normally a collection of multipoint Euclidean covariant distributions, where the reflection positivity must be established.

This section discusses the structure of reflection positive multipoint kernels. This is illustrated by considering the example of a four-point function, however the method can be applied to more general kernels. Four-point functions have a cluster decomposition as the sum of products of two-point functions and a connected four-point function. The dynamics appears in the connected part of the four-point function. Reflection positivity of two-point functions was demonstrated in Sec. IV. This is also true for products of these kernels. A sufficient condition for the dynamical four-point function to be reflection positive is that the connected part of the four-point function is reflection positive.

The method that was used to construct reflection positive two-point functions is used to examine the structure of reflection positive four-point functions. This construction is performed in two steps. Lorentz covariant kernels are defined as vacuum expectation values of formal Lorentz covariant fields and their adjoints. Complete sets of Poincaré irreducible states are inserted between the fields. These states are assumed to be positive-mass positive-energy states. These are replaced by equivalent complete sets of Lorentz covariant intermediate states. This results in a decomposition of the kernel in terms of

Lorentz covariant matrix elements of Lorentz covariant fields. The spin structure of these matrix elements follows from the covariance. Analytic properties of covariant matrix elements that are sufficient to construct an equivalent reflection positive Euclidean kernel are identified.

In Sec. IV right- and left-handed representations were treated separately. This section considers the general case of products of these representations. This is relevant for four vectors which transform as a product of $s = 1/2$ right- and left-handed representations. To distinguish the right- and left-handed degrees of freedom, left-handed spin degrees of freedom appear with a dot superscript, $\dot{s}$, $\dot{\mu}$.

The analysis begins by considering fields

$$\phi^{s\dot{s}}_{\mu\dot{\mu}}(x), \tag{169}$$

which transform covariantly under SL$(2,\mathbb{C})$:

$$U(A)\phi^{s\dot{s}}_{\mu\dot{\mu}}(0)U^{\dagger}(A) = \sum_{\nu\dot{\nu}} \phi^{s\dot{s}}_{\nu\dot{\nu}}(\Lambda x)D^{s}_{\nu\mu}[A]D^{\dot{s}}_{\dot{\nu}\dot{\mu}}[(A^{\dagger})^{-1}] \tag{170}$$

and

$$U(A)\phi^{s\dot{s}\dagger}_{\mu\dot{\mu}}(x)U^{\dagger}(A) = \sum_{\nu\dot{\nu}} \phi^{s\dot{s}\dagger}_{\nu\dot{\nu}}(\Lambda x)D^{s'}_{\nu\mu}[A^{*}]D^{\dot{s}}_{\dot{\nu}\dot{\mu}}[(A^{t})^{-1}]. \tag{171}$$

Locality is not assumed. Next consider the vacuum expectation value of the product of two such fields and their adjoints:

$$\begin{aligned}&\langle 0|\phi^{s_2\dot{s}_2\dagger}_{\mu_2\dot{\mu}_2}(x_2)\phi^{s_1\dot{s}_1\dagger}_{\mu_1\dot{\mu}_1}(x_1)\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(y_1)\phi^{s_2\dot{s}_2}_{\nu_2\dot{\nu}_2}(y_2)|0\rangle\\ &\quad := W_{2:2}(x_2,\mu_2,\dot{\mu}_2,x_1,\mu_1,\dot{\mu}_1;y_1,\nu_1,\dot{\nu}_1,y_2,\nu_2,\dot{\nu}_2).\end{aligned} \tag{172}$$

This kernel is Lorentz covariant and *manifestly positive* since it has the from

$$\langle 0|O^{\dagger}O|0\rangle. \tag{173}$$

This is referred to as a quasi-Wightman function.

The next step is to insert complete sets of Poincaré irreducible intermediate states between the fields. Vacuum intermediate states do not appear in the truncated part of the kernel. The states that appear are assumed to be positive-mass positive-energy intermediate states.

This results in a decomposition of Eq. (172) in the form

$$\begin{aligned}&W_{2:2}(x_2,\mu_2,\dot{\mu}_2,x_1,\mu_1,\dot{\mu}_1;y_1,\nu_1,\dot{\nu}_1,y_2,\nu_2,\dot{\nu}_2)\\ &\quad = \sum_{\mu_a\mu_b\mu_c}\int \langle 0|\phi^{s_2\dot{s}_2}_{\mu_2\dot{\mu}_2}(x_2)^{\dagger}|p_a,\mu_a\rangle d\mathbf{p}_a\langle p_a,\mu_a|\phi^{s_1\dot{s}_1}_{\mu_1\dot{\mu}_1}(x_1)^{\dagger}\\ &\qquad \times |p_b,\mu_b\rangle d\mathbf{p}_b\langle p_b,\mu_b|\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(y_1)|p_c,\mu_c\rangle\\ &\qquad \times d\mathbf{p}_c\langle p_c,\mu_c|\phi^{s_2\dot{s}_2}_{\nu_2\dot{\nu}_2}(y_2)|0\rangle,\end{aligned} \tag{174}$$

where invariant degeneracy quantum numbers have been suppressed.

To take advantage of the Lorentz covariance of the fields, the intermediate states are replaced by equivalent Lorentz covariant intermediate states as was done in Sec. IV. As was mentioned in Sec. IV, the Lorentz covariant states can be represented by right- or left-handed representations. In this application the Poincaré covariant states are decomposed into products of spin states which are transformed to products of right- and left-handed Lorentz covariant states. This is done by decomposing the Poincaré irreducible intermediate spins states into tensor products using SU(2) Clebsch-Gordan coefficients. In transforming to the Lorentz covariant representation one factor in the tensor product is put in a right-handed representation and the other in a left-handed representation.

The resulting mixed Lorentz covariant states are defined by

$$|(m,s)\mathbf{p},s_1,\dot{s}_2;\mu_1,\dot{\mu}_2\rangle_{\rm cov} := \sum |(m,s)\mathbf{p},\mu\rangle\sqrt{\omega_m(\mathbf{p})}\langle s,\mu|s_1,\nu_1,\dot{s}_2,\dot{\nu}_2\rangle D^{s_1}_{\nu_1\mu_1}\big[B_c^{-1}(p)\big]D^{\dot{s}_2}_{\dot{\nu}_2\dot{\mu}_2}[B_c^{\dagger}(p)]. \tag{175}$$

These states transform covariantly

$$U(A)|(m,s)\mathbf{p},s_1,\dot{s}_2;\mu_1,\dot{\mu}_2\rangle_{\rm cov} := \sum |(m,s)\Lambda p,s_1,\dot{s}_2;\nu_1,\dot{\nu}_2\rangle_{\rm cov}D^{s_1}_{\nu_1\mu_1}[A]D^{\dot{s}_2}_{\dot{\nu}_2\dot{\mu}_2}[(A^{\dagger})^{-1}]. \tag{176}$$

The identity can be expressed in terms of these states as

$$\begin{aligned}I &= \int\sum_{s\mu}|(m,s)\mathbf{p},\mu\rangle d\mathbf{p}\langle (m,s)\mathbf{p},\mu\rangle = \sum\int |(m,s)\mathbf{p},s_1,\dot{s}_2;\nu_1,\dot{\nu}_2\rangle_{\rm cov}\\ &\quad\times \frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^{s_1}_{\nu_1\mu_1}[p\cdot\sigma]D^{\dot{s}_2}_{\dot{\nu}_2\dot{\mu}_2}[(\Pi p)\cdot\sigma]_{\rm cov}\langle (m,s)\mathbf{p},s_1,\dot{s}_2;\mu_1,\dot{\mu}_2|.\end{aligned} \tag{177}$$

The choice of how to break up the intermediate Poincaré covariant states into right- and left-handed Lorentz covariant states is determined by the spin structure of the fields.

Replacing the Poincaré covariant intermediate states by the corresponding mixed Lorentz covariant intermediate states in Eq. (174) gives

$$\begin{aligned}&W_{2:2}(x_2,\mu_2,\dot{\mu}_2,x_1,\mu_1,\dot{\mu}_1;y_1,\nu_1,\dot{\nu}_1,y_2,\nu_2,\dot{\nu}_2)\\ &\quad = \sum\int \langle 0|\phi^{s_2\dot{s}_2\dagger}_{\mu_2\dot{\mu}_2}(x_2)|p_a,s_{a1},\dot{s}_{a2};\mu_{a1},\dot{\mu}_{a2}\rangle_{\rm cov}\frac{d\mathbf{p}_a}{\omega_{m_a}(\mathbf{p}_a)}D^{s_{a1}}_{\mu_{a1}\nu_{a1}}[p_a\cdot\sigma]D^{\dot{s}_{a2}}_{\dot{\mu}_{a2}\dot{\nu}_{a2}}[(\Pi p_a)\cdot\sigma]\end{aligned}$$

$$\times_{\rm cov}\langle p_a, s_{a1}, \dot{s}_{a2}; \nu_{a1}, \dot{\nu}_{a2}|\phi^{s_1\dot{s}_1\dagger}_{\mu_1\dot{\mu}_1}(x_1)|p_b, s_{b1}, \dot{s}_{b2}; \mu_{b1}, \dot{\mu}_{b2}\rangle_{\rm cov}\frac{d\mathbf{p}_b}{\omega_{m_b}(\mathbf{p}_b)}D^{s_{b1}}_{\mu_{b1}\nu_{b1}}[p_b\cdot\sigma]D^{\dot{s}_{b2}}_{\dot{\mu}_{b2}\dot{\nu}_{b2}}[(\Pi p_b)\cdot\sigma]$$

$$\times_{\rm cov}\langle p_b, s_{b1}, \dot{s}_{b2}; \nu_{b1}, \dot{\nu}_{b2}|\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(y_1)|p_c, s_{c1}, \dot{s}_{c2}; \mu_{c1}, \dot{\mu}_{c2}\rangle_{\rm cov}\frac{d\mathbf{p}_c}{\omega_{m_c}(\mathbf{p}_c)}D^{s_{c1}}_{\mu_{c1}\nu_{c1}}[p_c\cdot\sigma]D^{\dot{s}_{c2}}_{\dot{\mu}_{c2}\dot{\nu}_{c2}}[(\Pi p_c)\cdot\sigma]$$

$$\times_{\rm cov}\langle p_c, s_{c1}, \dot{s}_{c2}; \nu_{c1}, \dot{\nu}_{c2}|\phi^{s_2'\dot{s}_2}_{\nu_2'\dot{\nu}_2}(y_2)|0\rangle. \tag{178}$$

This expression contains four Lorentz covariant matrix elements of Lorentz covariant field operators.

Translational covariance can be used to remove the space-time dependence from each of these four matrix elements:

$$\langle 0|\phi^{s_2\dot{s}_2\dagger}_{\mu_2\dot{\mu}_2}(x_2)|p_a, s_{a1}, \dot{s}_{a2}; \mu_{a1}, \dot{\mu}_{a2}\rangle_{\rm cov} = e^{ip_a\cdot x_2}\langle 0|\phi^{s_2\dot{s}_2\dagger}_{\mu_2\dot{\mu}_2}(0)|p_a, s_{a1}, \dot{s}_{a2}; \mu_{a1}, \dot{\mu}_{a2}\rangle_{\rm cov}, \tag{179}$$

$$\begin{aligned}&{}_{\rm cov}\langle p_a, s_{a1}, \dot{s}_{a2}; \nu_{a1}, \dot{\nu}_{a2}|\phi^{s_1\dot{s}_1\dagger}_{\mu_1\dot{\mu}_1}(x_1)|p_b, s_{b1}, \dot{s}_{b2}; \mu_{b1}, \dot{\mu}_{b2}\rangle_{\rm cov}\\ &\quad = e^{i(p_b-p_a)\cdot x_1}{}_{\rm cov}\langle p_a, s_{a1}, \dot{s}_{a2}; \nu_{a1}, \dot{\nu}_{a2}|\phi^{s_1\dot{s}_1}_{\mu_1\dot{\mu}_1}(0)^\dagger|p_b, s_{b1}, \dot{s}_{b2}; \mu_{b1}, \dot{\mu}_{b2}\rangle_{\rm cov},\end{aligned} \tag{180}$$

$$\begin{aligned}&{}_{\rm cov}\langle p_b, s_{b1}, \dot{s}_{b2}; \nu_{b1}, \dot{\nu}_{b2}|\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(y_1)|p_c, s_{c1}, \dot{s}_{c2}; \mu_{c1}, \dot{\mu}_{c2}\rangle_{\rm cov}\\ &\quad = e^{i(p_c-p_b)\cdot y_1}{}_{\rm cov}\langle p_b, s_{b1}, \dot{s}_{b2}; \nu_{b1}, \dot{\nu}_{b2}|\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(0)|p_c, , s_{c1}, \dot{s}_{c2}; \mu_{c1}, \dot{\mu}_{c2}\rangle_{\rm cov},\end{aligned} \tag{181}$$

$${}_{\rm cov}\langle p_c, s_{c1}, \dot{s}_{c2}; \nu_{c1}, \dot{\nu}_{c2}|\phi^{s_2\dot{s}_2}_{\nu_2\dot{\nu}_2}(y_2)|0\rangle = e^{-ip_c\cdot y_2}{}_{\rm cov}\langle p_c, s_{c1}, \dot{s}_{c2}; \nu_{c1}, \dot{\nu}_{c2}|\phi^{s_2\dot{s}_2}_{\nu_2\dot{\nu}_2}(0)|0\rangle. \tag{182}$$

The Lorentz covariance properties of these matrix elements with the space-time coordinate set to 0 are

$$\begin{aligned}&\langle 0|\phi^{s_2\dot{s}_2\dagger}_{\mu_2\dot{\mu}_2}(0)|p_a, s_{a1}, \dot{s}_{a2}; \mu_{a1}, \dot{\mu}_{a2}\rangle_{\rm cov}\\ &\quad = \sum\langle 0|\phi^{s_2\dot{s}_2\dagger}_{\nu_2\dot{\nu}_2}(0)|\Lambda p_a, s_{a1}, \dot{s}_{a2}; \nu_{a1}, \dot{\nu}_{a2}\rangle_{\rm cov}D^{s_2}_{\nu_2\mu_2}[A^*]D^{\dot{s}_2}_{\dot{\nu}_2\dot{\mu}_2}[(A^t)^{-1}]D^{s_{a1}}_{\nu_{a1}\mu_{a1}}[A]D^{\dot{s}_{a2}}_{\dot{\nu}_{a2}\dot{\mu}_{a2}}[(A^\dagger)^{-1}],\end{aligned} \tag{183}$$

$$\begin{aligned}&{}_{\rm cov}\langle p_a, s_{a1}, \dot{s}_{a2}; \mu_{a1}, \dot{\mu}_{a2}|\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(0)^\dagger|p_b, s_{b1}, \dot{s}_{b2}; \mu_{b1}, \dot{\mu}_{b2}\rangle_{\rm cov}\\ &\quad = \sum{}_{\rm cov}\langle\Lambda p_a, s_a, s_{a1}, \dot{s}_{a2}; \nu_{a1}, \dot{\nu}_{a2}|\phi^{s_1'\dot{s}_1}_{\mu_1'\dot{\mu}_1}(0)^\dagger|\Lambda p_b, s_b, s_{b1}, \dot{s}_{b2}; \nu_{b1}, \dot{\nu}_{b2}\rangle_{\rm cov}\\ &\qquad \times D^{s_{a1}}_{\nu_{a1}\mu_{a1}}[A^*]D^{\dot{s}_{a2}}_{\dot{\nu}_{a2}\dot{\mu}_{a2}}[(A^t)^{-1}]D^{s_1}_{\nu_1\mu_1}[A^*]D^{\dot{s}_1}_{\dot{\nu}_1\dot{\mu}_1}[(A^t)^{-1}]D^{s_{b1}}_{\nu_{b1}\mu_{b1}}[A]D^{\dot{s}_{b2}}_{\dot{\nu}_{b2}\dot{\mu}_{b2}}[(A^\dagger)^{-1}],\end{aligned} \tag{184}$$

$$\begin{aligned}&\sum{}_{\rm cov}\langle p_b, s_{b1}, \dot{s}_{b2}; \nu_{b1}, \dot{\nu}_{b2}|\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(0)|p_c, s_{c1}, \dot{s}_{c2}; \mu_{c1}, \dot{\mu}_{c2}\rangle_{\rm cov}\\ &\quad = {}_{\rm cov}\langle\Lambda p_b, s_{b1}, \dot{s}_{b2}; \nu_{b1}, \dot{\nu}_{b2}|\phi^{s_1\dot{s}_1}_{\nu_1\dot{\nu}_1}(0)|\Lambda p_c, s_{c1}, \dot{s}_{c2}; \nu_{c1}, \dot{\nu}_{c2}\rangle_{\rm cov}\\ &\qquad \times D^{s_{b2}}_{\nu_{b2}'\mu_{b2}}[A^*]D^{\dot{s}_{b2}}_{\dot{\nu}_{b2}\dot{\mu}_{b2}}[(A^t)^{-1}]D^{s_1}_{\nu_1\mu_1}[A]D^{\dot{s}_1}_{\dot{\nu}_1\dot{\mu}_1}[(A^\dagger)^{-1}]D^{s_{c1}}_{\nu_{c1}\mu_{c1}}[A]D^{\dot{s}_{c2}}_{\dot{\nu}_{c2}\dot{\mu}_2}[(A^\dagger)^{-1}],\end{aligned} \tag{185}$$

$$\begin{aligned}&\sum{}_{\rm cov}\langle p_c, s_{c1}, \dot{s}_{c2}; \nu_{c1}, \dot{\nu}_{b2}|\phi^{s_2\dot{s}_2}_{\nu_2\dot{\nu}_2}(0)|0\rangle\\ &\quad = {}_{\rm cov}\langle\Lambda p_c, s_{c1}, \dot{s}_{c2}; \nu_{c1}, \dot{\nu}_{b2}|\phi^{s_2\dot{s}_2}_{\nu_2\dot{\nu}_2}(0)|0\rangle D^{s_{c2}}_{\nu_{c2}\mu_{c2}}[A^*]D^{\dot{s}_{c2}}_{\dot{\nu}_{c2}\dot{\mu}_{c2}}[(A^t)^{-1}]D^{s_2}_{\nu_2\mu_2}[A]D^{\dot{s}_2}_{\dot{\nu}_2\dot{\mu}_2}[(A^\dagger)^{-1}].\end{aligned} \tag{186}$$

In Eqs. (184) and (185) SU(2) Clebsch-Gordan coefficients can be used to replace

$$D^{s_{a1}}_{\nu_{a1}\mu_{a1}}[A^*]D^{\dot{s}_{a2}}_{\dot{\nu}_{a2}\dot{\mu}_{a2}}[(A^t)^{-1}]D^{s_1}_{\nu_1\mu_1}[A^*]D^{\dot{s}_1}_{\dot{\nu}_1\dot{\mu}_1}[(A^t)^{-1}] \tag{187}$$

and

$$D^{s_{1'}}_{\nu_{1'}\mu_{1'}}[A]D^{\dot{s}_1}_{\dot{\nu}_1\dot{\mu}_1}[(A^\dagger)^{-1}]D^{s_{b1}}_{\nu_{b1}\mu_{b1}}[A]D^{\dot{s}_{b2}}_{\dot{\nu}_{b2}\dot{\mu}_{b2}}[(A^\dagger)^{-1}], \tag{188}$$

by

$$\sum\langle s_{a1}, \nu_{a1}, s_1, \nu_1|s, \nu\rangle D^s_{\nu\mu}[A^*]\langle s, \mu, |s_{a1}, \mu_{a1}, s_1, \mu_1\rangle\langle\dot{s}_{a2}, \dot{\nu}_{a2}, \dot{s}_1, \dot{\nu}_1|\dot{s}, \dot{\nu}\rangle D^s_{\dot{\nu}\dot{\mu}}[(A^t)^{-1}]\langle\dot{s}, \dot{\mu}, |\dot{s}_{a2}, \dot{\mu}_{a2}, \dot{s}_1, \dot{\mu}_1\rangle \tag{189}$$

and

$$\sum\langle s_1, \nu_1, s_{b1}, \nu_{b1}|s, \nu\rangle D^s_{\nu\mu}[A]\langle s, \mu, |s_1, \mu_1, s_{b1}, \mu_{b1}\rangle\langle\dot{s}_1, \dot{\nu}_1, \dot{s}_{b2}, \dot{\nu}_{b2}|\dot{s}, \dot{\nu}\rangle D^s_{\dot{\nu}\dot{\mu}}[(A^\dagger)^{-1}]\langle\dot{s}, \dot{\mu}, |\dot{s}_1, \dot{\mu}_1, \dot{s}_{2b}, \dot{\mu}_{2b}\rangle. \tag{190}$$

After these replacements the spin dependence of Eqs. (184) and (185) has the same structure as Eqs. (183) and (186):

$$D^{s_2}_{\nu_{2'}\mu_2}[A^*]D^{s_{a1}}_{\nu_{a1}\mu_{a1}}[A]D^{\dot{s}_2}_{\dot{\nu}_2\dot{\mu}_2}[(A^t)^{-1}]D^{\dot{s}_{a2}}_{\dot{\nu}_{a2}\dot{\mu}_{a2}}[(A^\dagger)^{-1}] \tag{191}$$

and

$$D^{s_{c1}}_{\nu_{c1}\mu_{c1}}[A^*]D^{s_2}_{\nu_2\mu_2}[A]D^{\dot{s}_{c2}}_{\dot{\nu}_{c2}\dot{\mu}_{c2}}[(A^t)^{-1}]D^{\dot{s}_2}_{\dot{\nu}_2\dot{\mu}_2}[(A^\dagger)^{-1}]. \tag{192}$$

The spins appearing in the fields are properties of the kernel. The spins appearing in the intermediate states are determined by the spin of the fields. For example the spin of the field applied to the vacuum fixes the spins of the first set of intermediate states. These states, along with the spin of the second field fix the allowed spins in the next set of intermediate, etc.

The building blocks of each of these covariant matrix elements are the four momenta and the covariant spinors, σ_μ, σ^*_μ, $\sigma_2\sigma_\mu\sigma_2$ and $\sigma_2\sigma^*_\mu\sigma_2$. Functions of these quantities that that have the transformation properties of Eqs. (183)–(186) follow from Eq. (8). They are products of the matrices

$$D^{\dot{s}}_{\dot{\mu}\dot{\nu}}[p\cdot\sigma]:\quad D^{\dot{s}}_{\dot{\mu}\dot{\nu}}[p\cdot\sigma]=\sum D^{\dot{s}}_{\dot{\alpha}\dot{\beta}}[(\Lambda p)\cdot\sigma]D^{\dot{s}}_{\dot{\alpha}\dot{\mu}}[(A^t)^{-1}]D^{\dot{s}}_{\dot{\beta}\dot{\nu}}[(A^\dagger)^{-1}] \tag{193}$$

and

$$D^{s}_{\mu\nu}[p\cdot(\sigma_2\sigma^*\sigma_2)]:\quad D^{s}_{\mu\nu}[p\cdot(\sigma_2\sigma^*\sigma_2)]=\sum D^{s}_{\alpha\beta}[(\Lambda p)\cdot(\sigma_2\sigma^*\sigma_2)]D^{s}_{\alpha\mu}[A^*]D^{s}_{\beta\nu}[A], \tag{194}$$

for any four-momentum p^μ appearing in the matrix element. The following expressions have the covariance properties of Eqs. (183)–(186) of each matrix element in Eq. (178):

$$\begin{aligned}&\langle 0|\phi^{s_2\dot{s}_2}_{\mu_2\dot{\mu}_2}(0)^\dagger|p_a,s_{a1},\dot{s}_{a2};\mu_{a1},\dot{\mu}_{a2}\rangle_{\rm cov}\\&\quad=\delta_{s_2s_{a1}}\delta_{\dot{s}_2\dot{s}_a2}F(s_2,s_{a1},\dot{s}_2,\dot{s}_{a2},p_{a2},m_a)D^{s_2}_{\mu_2\mu_{a1}}[p_a\cdot(\sigma_2\sigma^*\sigma_2)]D^{\dot{s}_2}_{\dot{\mu}_2\dot{\mu}_{a1}}[p_a\cdot\sigma],\end{aligned} \tag{195}$$

$$\begin{aligned}&{}_{\rm cov}\langle p_a,s_{a1},\dot{s}_{a2};\mu_{1a},\dot{\mu}_{a2}|\phi^{s_1'\dot{s}_1}_{\mu_1'\dot{\mu}_1}(0)^\dagger|p_b,s_b,s_{b1},\dot{s}_{b2};\mu_{b1},\dot{\mu}_{b2}\rangle_{\rm cov}\\&\quad=\sum F_{ij}(\cdots)\times\langle s_{a1},\nu_{a1},s_1,\nu_1,|s_{b1},\nu_{b1}\rangle D^{s_{b1}}_{\nu_{b1}\mu_{b1}}[p_i\cdot(\sigma_2\sigma^*\sigma_2)]\langle\dot{s}_1,\dot{\nu}_1,\dot{s}_{a2},\dot{\mu}_{a2}|\dot{s}_{b2},\dot{\nu}_{b2}\rangle D^{\dot{s}_{b2}}_{\dot{\nu}_{b2}\dot{\mu}_{b2}}[p_j\cdot\sigma],\end{aligned} \tag{196}$$

$$\begin{aligned}&{}_{\rm cov}\langle p_b,s_{b1},\dot{s}_{b2};\nu_{b1},\dot{\nu}_{b2}|\phi^{s_1\dot{s}_1}_{\nu_1'\dot{\nu}_1}(0)|p_c,s_{c1},\dot{s}_{c2};\mu_{c1},\dot{\mu}_{c2}\rangle_{\rm cov}\\&\quad=\sum F_{ij}(\cdots)\langle s_{b1},\nu_{1b}|s_1,\mu_1,s_{c1},\mu_{c1}\rangle D^{s_{b1}}_{\mu_{b1}\nu_{b1}}[p_i\cdot(\sigma_2\sigma^*\sigma_2)]D^{\dot{s}_{b2}}_{\dot{\mu}_{b2}\dot{\nu}_{b2}}[p_j\cdot\sigma]\langle\dot{s}_{21},\dot{\nu}_{b2}|\dot{s}_1,\dot{\nu}_1,\dot{s}_{c2},\dot{\mu}_{c2},\rangle,\end{aligned} \tag{197}$$

$$\begin{aligned}&{}_{\rm cov}\langle p_c,s_{c1},\dot{s}_{c2};\nu_{c1},\dot{\nu}_{c2}|\phi^{s_2\dot{s}_2}_{\nu_2\dot{\nu}_2}(0)|p_c,s_{c1},\dot{s}_{c2};\mu_{c1},\dot{\mu}_{c2}\rangle_{\rm cov}\\&\quad=F(s_1,s_{c1},\dot{s}_1,\dot{s}_{c2},p_c^2,m_c)\delta_{s_{c1}s_1}\delta_{\dot{s}_{c2}\dot{s}_1}\sum D^{\dot{s}_1}_{\dot{\mu}_{c2}\dot{\mu}_1}[p_j\cdot\sigma]D^{s_1}_{\mu_{c1}\nu_1}[p_i\cdot(\sigma_2\sigma^*\sigma_2)],\end{aligned} \tag{198}$$

where the coefficient functions, $F(\cdots)$, are scalars. In the Lorentz covariant expressions all of the energies are on shell. The Wigner functions, $D[p\cdot\sigma]$ and $D[\Pi p\cdot\sigma]$, are polynomials in the components of p. In what follows the coefficient functions $F(\cdots)$ are assumed to be analytic functions of the momenta in the upper- or lower-half energy planes.

The integrals appearing in Eq. (178) have the form

$$\int\frac{e^{ip_a\cdot(x_2-x_1)+ip_b\cdot(x_1-y_1)+ip_c\cdot(y_1-y_2)}d\mathbf{p}_a d\mathbf{p}_b d\mathbf{p}_c}{\omega_{m_a}(\mathbf{p}_a)\omega_{m_b}(\mathbf{p}_b)\omega_{m_c}(\mathbf{p}_c)}\cdots, \tag{199}$$

with

$$p_a^0=\omega_{m_a}(\mathbf{p}_a),\quad p_b^0=\omega_{m_b}(\mathbf{p}_b),\quad p_c^0=\omega_{m_c}(\mathbf{p}_c). \tag{200}$$

If $x^0_{2e}>x^0_{1e}>0$ and $y^0_{2e}>y^0_{1e}>0$, then the following Euclidean integral over the p^0_e's below can be evaluated:

$$\begin{aligned}&\int\frac{e^{ip_{ae}\cdot(\theta x_{2e}-\theta x_{1e})+ip_{be}\cdot(\theta x_{1e}-y_{1e})+ip_{ce}\cdot(y_{1e}-y_{2e})}8d^4p_{ae}d^4p_{be}d^4p_{ce}}{(2\pi)^3\left(p^2_{ae}+m^2_a\right)\left(p^2_{be}+m^2_b\right)\left(p^2_{ce}+m^2_c\right)}\cdots\\&\quad=\int\frac{e^{i\mathbf{p}_a\cdot(\mathbf{x}_2-\mathbf{x}_1)+i\mathbf{p}_b\cdot(\mathbf{x}_1-\mathbf{y}_1)+i\mathbf{p}_c\cdot(\mathbf{y}_1-\mathbf{y}_2)}d\mathbf{p}_a d\mathbf{p}_b d\mathbf{p}_c}{\omega_{m_a}(\mathbf{p}_a)\omega_{m_b}(\mathbf{p}_b)\omega_{m_c}(\mathbf{p}_c)}e^{-\omega_{m_a}(\mathbf{p}_a)(x^0_{2e}-x^0_{1e})-\omega_{m_b}(\mathbf{p}_b)(x^0_{1e}+y^0_{1e})-\omega_{m_c}(\mathbf{p}_c)(y^0_{2e}-y^0_{1e})}\cdots,\end{aligned} \tag{201}$$

with

$$p^0_{ea}=-i\omega_{m_a}(\mathbf{p}_a)=-ip^0_a,\quad p^0_{eb}=-i\omega_{m_b}(\mathbf{p}_b)=-ip^0_b,\quad p^0_{ec}=-i\omega_{m_b}(\mathbf{p}_c)=-ip^0_c. \tag{202}$$

Except for the Euclidean and Minkowski time components, which can be absorbed in the test functions, both integrals Eqs. (199) and (201) are identical. The $\cdots$ that appear in Eq. (199) are only functions of three momenta (including the on-shell energies). The covariance condition means that the on shell-four momenta only appear in Lorentz invariant inner products or in the combinations $p\cdot\sigma$ or $p\cdot(\sigma_2\sigma^*\sigma_2)$ that appear in the Wigner functions. If the σ's and the on-shell Minkowski momenta are replaced by the Euclidean 4 momenta and σ_e's, then all of the Euclidean quantities become the corresponding Minkowski quantities with the replacements Eq. (202). This replacement will be made in the residue of the pole term in Eq. (201) provided the $\cdots$ terms contain no additional p^0_e singularities in the lower-half p^0_e plane. This is not a problem for the Wigner functions since they are polynomials in $p_e\cdot\sigma_e$ or $p\cdot(\sigma_2\sigma^*\sigma_2)$. It does require that when the replacements $p^0 \to ip_e^0$ Eq. (202) are made in the invariant functions F, the resulting functions must be analytic in the lower-half Euclidean energy plane. If this is true, then it follows from the positivity of Eq. (173) that the resulting Euclidean kernel will be reflection positive.

This general structure is not surprising. It illustrates how reflection positivity in the Euclidean representation is related to positivity and the spectral condition in the Lorentz covariant representation. This is of limited value, since for models it requires building in the observed mass spectrum. On the other hand it shows that the spin structures do not introduce new singularities.

It is also important to note that unlike the case of irreducible representations where the test functions of one variable are required to have support for positive Euclidean time, for the case of general multipoint functions the test functions in the physical Hilbert space must have support for

$$0 < x_{1e}^0 < x_{2e}^0 \cdots x_{Ne}^0 \tag{203}$$

[see below Eq. (200)]. In the field theory case the ordering of the Euclidean time supports fixes the ordering of the fields in the corresponding Wightman functions.

A standard method to construct the Euclidean covariant four-point functions is to solve the Bethe-Salpeter integral equation, which has the form

$$S_4 = S_0 + S_0 K S_4, \tag{204}$$

where K is the Euclidean Bethe-Salpeter kernel. This integral equation can be iterated to get

$$S_4 = S_0 + S_0 K S_0 + S_0 K S_0 K S_0 + \cdots . \tag{205}$$

The corresponding series for the connected part of the four point function is

$$S_4^c = S_0 K S_0 + S_0 K S_0 K S_0 + \cdots . \tag{206}$$

This series can be formally expressed as

$$S_4^c = S_0 T S_0 \quad T = K + K S_0 K + K S_0 K S_0 K + \cdots . \tag{207}$$

Each term in the series for K has the same Euclidean covariance property. If the series converges, then the sum will not generate any new singularities that are not already in K. This suggest that if $S_0 K S_0$ is reflection positive and the series converges, then $S_0 T S_0$ will be reflection positive. When the series does not converge, the solution of the Bethe-Salpeter equation could result in new singularities that violate the spectral condition, which would necessarily also lead to a violation of reflection positivity, since it implies the spectral condition.

These observations suggest that suitable Bethe-Salpeter kernels K should have the property that $S_0 K S_0$ is reflection positive, however this condition alone is not sufficient to ensure reflection positivity of the resulting connected four point function. The condition that $S_0 K S_0$ is reflection positive should be good starting point for constructing model dynamical four point function.

For dynamical models, while the dynamics appears in the kernel, the expression for the Poincaré generators are sums of the generators in Sec. (IV) for each degree of freedom.

IX. SELF ADJOINTNESS

While the self-adjointness of the generators of ordinary rotations follows from the unitarity of the one-parameter group of rotations on the Hilbert spaces [Eqs. (106)–(109)], this argument does not apply to either the Hamiltonian or the boost generators. In both cases the operators were derived from the corresponding Euclidean generators by multiplication by an imaginary constant. The Euclidean generators and corresponding Lorentz generators act on different Hilbert space representations. The problem is that the corresponding finite Euclidean transformations can map functions with positive time support to functions that violate this condition.

For the Hamiltonian this can be treated by only considering translations in the positive Euclidean time direction. These translations map functions with positive Euclidean time support into functions with positive Euclidean time support. Reflection positivity can be used to show that translations in the positive Euclidean time direction define a contractive Hermitian semigroup on the Hilbert space with the scalar product Eqs. (106)–(109). The argument [24] uses the Schwartz inequality on both the physical and Euclidean Hilbert spaces. One application of the Schwartz inequality on the physical Hilbert space gives

$$\begin{aligned}\| |e^{-Hx^0}|\phi\rangle\| &= \langle e^{-Hx^0}\phi|e^{-Hx^0}|\phi\rangle^{1/2} = \langle\phi|e^{-H2x^0}|\phi\rangle^{1/2}\\ &\leqslant \| |e^{-H2x^0}|\phi\rangle\|^{1/2} \| |\phi\rangle\|^{1/2}.\end{aligned} \tag{208}$$

Repeating these steps n-times gives

$$\| |e^{-Hx^0}|\phi\rangle\| \leqslant \| |e^{-H2^n x^0}|\phi\rangle\|^{1/2^n} \| |\phi\rangle\|^{1-1/2^n}. \tag{209}$$

The quantity

$$\| |e^{-H2^n x^0}|\phi\rangle\| \leqslant \|\theta U_e(2^n x^0)|\phi\rangle\|_e < \| |\phi\rangle\|_e < \infty \tag{210}$$

is bounded by the Euclidean norm, $\|\cdot\|_e$, since $U_e(2^n x^0)$ is unitary and $\|\theta\|_e = 1$ on that Hilbert space. Since this is finite and independent of n, taking the limit as $n \to \infty$ gives

$$\| |e^{-Hx^0}|\phi\rangle\| \leqslant \| |\phi\rangle\|. \tag{211}$$

It follows that positive Euclidean time translations define a contractive Hermitian semigroup on the Hilbert spaces [Eqs. (106)–(109)]. The generator is a positive self-adjoint operator [23,25].

Boosts present additional complications. Even an infinitesimal rotation in a Euclidean space-time plane will map a general function with positive Euclidean time support to one that violates this condition. The self-adjointness of the boost generator cannot be demonstrated by showing that it defines a unitary one-parameter group or contractive semigroup, however it turns out that rotations in Euclidean space-time planes, which are interpreted as boosts with complex rapidity, define local symmetric semigroups [26–28] on the Hilbert spaces Eqs. (106)–(109). These have self-adjoint generators, which are exactly the boost generators.

The conditions for a local symmetric semigroup [26] are

1. For each $\theta \in [0, \theta_0]$, there is a linear subset $\mathcal{D}_\theta$ such that $\mathcal{D}_{\theta_1} \supset \mathcal{D}_{\theta_2}$ if $\theta_1 < \theta_2$, and $\cup_{0<\theta<\theta_0}\mathcal{D}_{\theta_2}$ is dense.
2. For each $\theta \in [0, \theta_0]$, $E(\theta)$ is a linear operator on the Hilbert space with domain $\mathcal{D}_\theta$.

3. $E(0) = I$, $E(\theta_1) : \mathcal{D}_{\theta_2} \to \mathcal{D}_{\theta_2 - \theta_1}$, and $E(\theta_1)E(\theta_2) = E(\theta_1 + \theta_2)$ on $\mathcal{D}_{\theta_1+\theta_2}$ for $\theta_1, \theta_2, \theta_1 + \theta_2 \in [0, \theta_0]$.
4. $E(\theta)$ is Hermitian for $\theta \in [0, \theta_0]$.
5. $E(\theta)$ is weakly continuous on $[0, \theta_0]$.

When these conditions are satisfied there is a unique self-adjoint operator K such that $\mathcal{D}_\theta \subset \mathcal{D}_{e^{-K\theta}}$ and $E(\theta)$ is the restriction of $e^{-K\theta}$ to $\mathcal{D}_\theta$.

In this case, $E(\theta)$ represents Euclidean space-time rotations considered as operators on the Hilbert space Eqs. (106)–(109) restricted to domains that will be described below.

The domains are Schwartz functions with space Euclidean time support in the wedge shaped region defined by

$$\mathbf{x} \cdot \hat{\mathbf{n}} - \frac{x_e^0}{\epsilon} + \epsilon < 0, \tag{212}$$

$$\mathbf{x} \cdot \hat{\mathbf{n}} + \frac{x_e^0}{\epsilon} - \epsilon > 0. \tag{213}$$

The wedge shaped region becomes the positive Euclidean time half plane in the limit that $\epsilon \to 0$. Schwartz functions with support on this half plane are dense. In addition, if this domain is rotated by an angle less than $\theta_\epsilon := \pm \tan^{-1}(\epsilon)$, it will still be contained in the positive Euclidean time half plane. Schwartz functions with support in these wedge shaped regions can be constructed from Schwartz functions that have support for positive Euclidean time by multiplying the function by $g(x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon)$, where

$$g(x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon)) = h\left(\frac{x_e^0}{\epsilon} - \epsilon + \mathbf{x} \cdot \hat{\mathbf{n}}\right) h\left(\frac{x_e^0}{\epsilon} - \epsilon - \mathbf{x} \cdot \hat{\mathbf{n}}\right) \tag{214}$$

and

$$h(\lambda) = \begin{cases} e^{-\frac{1}{(\lambda)^2}} & \lambda > 0 \\ 0 & \lambda \leqslant 0 \end{cases} \tag{215}$$

is a smoothed Heaviside function. $g[x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon)]$ is a Schwartz function with support in the wedge shaped region [Eqs. (212)–(213)] that approaches 1 as $\epsilon(\theta)$ approaches 0.

The domain $\mathcal{D}_\theta$ is taken as the space of Schwartz functions with positive time support multiplied by the function $g(x^0, \mathbf{x} \cdot \hat{\mathbf{n}}, \epsilon)$, where $\theta = \theta_\epsilon$. The Euclidean space-time rotations restricted to these domains have all of the properties of local symmetric semigroup. It follows that the boost generators $\mathbf{K}$ are self-adjoint on the physical Hilbert space.

X. SUMMARY AND CONCLUSION

Relativistic formulations of quantum mechanics are useful for understanding the short-distance properties of strongly interacting systems. The advantage is that they can be solved using the same Hilbert space methods that are used in nonrelativistic quantum theories. The challenges are formulating the models so isolated subsystems are separately Poincaré invariant (cluster properties) and relating the phenomenological interactions to QCD. A Euclidean approach provides a one way of addressing these challenges, while creating a different set of challenges. The Euclidean formulation of relativistic quantum mechanics is motivated by the axioms of Euclidean quantum field theory. The axiom that leads to microscopic locality is logically independent of the other axioms. This investigation of Euclidean formulations of relativistic quantum theories of particles is motivated by the possibility of being able to satisfy all of the axioms of relativistic quantum field theory without having to require locality. While microscopic locality is desirable, it is the source of most of the difficulties of quantum field theory, and is difficult to test experimentally. In the Euclidean formulation cluster properties can be easily satisfied, there is a natural relation to quantum field theories, and the formalism is still a theory of linear operators acting on a Hilbert space.

An appealing feature of the Euclidean representation is that the physical Hilbert space and the infinitesimal generators of the Poincaré group can be constructed without any need for an analytic continuation to Minkowski space.

The new property of this representation is that the Hilbert space inner product has a nontrivial kernel that is not manifestly symmetric. The requirement that Hilbert space inner product has a positive norm is called reflection positivity, which constrains the form of the kernel. While being able to use standard Hilbert space methods in the Euclidean representation has some advantages, these methods get modified in unfamiliar ways when the inner product has a nontrivial kernel. Among the unfamiliar properties are that self-adjoint operators have unfamiliar forms, distributions, like delta functions, can become normalizable vectors, and the Poincaré generators have a form that does not depend on the interactions. In addition, a deviation from the Euclidean formulation of field theory is that a single N-point kernel can be replaced by $N - 1$, $K + M = N$ point kernels.

The primary purpose of this paper is to provide explicit expressions for the Poincaré generators for particles of any spin in the Euclidean representation. While these formulas follow from the definitions in a straightforward manner, explicit formulas are needed for applications, especially for hadronic and nuclear physics applications that can involve particles with high spins. Since the forms of these operators are unfamiliar, the commutation relations and self-adjointness of each one is demonstrated explicitly.

Generators and Euclidean kernels were derived by starting with positive mass irreducible representations of the Poincaré group, constructing equivalent Lorentz covariant representations, and using these to construct Euclidean covariant representations. This automatically results in reflection positive irreducible representations. These results are general since any unitary representation of the Poincaré group can be decomposed into a direct integral of positive-mass positive-energy irreducible representations. While this also applies to systems of particles, for systems it is useful to replace the irreducible representation by products of single particle irreducible representations. Interactions require introducing correlations in the N free particle kernel. While the correlations preserve the covariance properties, the requirement that they preserve reflection positivity is not automatic. The formulas for the Poincaré generators remain unchanged.

The structure of reflection positive kernels with arbitrary spin was investigated. On one hand the spin structures that result from covariance do not impact the reflection positivity, however the coefficient functions must be analytic in the lower-half Euclidean energy planes. Ideally one would like to be able find sufficient conditions on the input to Schwinger Dyson equations so the solution generates reflection positive kernels. Even for the simplest case of the Bethe-Salpeter equation, the solution can introduce singularities that violate reflection positivity.

In the Euclidean representation the dynamics appears in the Hilbert space kernel. While in principle Hilbert space methods can be used in calculations, because of the kernel, applications favor different methods of computation. These have been discussed elsewhere, [15–18]. In addition, the triviality of the Poincaré generators puts the burden of constructing dynamical models on the structure of reflection positive kernels.

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APPENDIX

The kernels in the Euclidean space-time representation for higher spin are given by Eq. (70). This requires the computation of a finite number of derivatives applied to

$$\frac{K_1\left(m\sqrt{z_0^2+\mathbf{z}^2}\right)}{m\sqrt{z_0^2+\mathbf{z}^2}}. \tag{A1}$$

These derivatives can be expressed in terms of higher-order modified Bessel functions using

$$\begin{aligned}\frac{d^n}{dx^n}\frac{K_1(x)}{x} &= \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{d^m}{dx^m}K_1(x)(-)^{n-m}\frac{(n-m)!}{x^{n-m-1}} = \sum_{m=0}^{n}\frac{n!(-)^{n-m}}{m!x^{n-m+1}}\frac{d^m}{dx^m}K_1(x)\\ &= \frac{d^m}{dx^m}K_1(x) = \frac{1}{2^m}(-)^{1-m}\left[K_{1-m}(x)+\frac{m!}{1!(m-1)!}K_{1-m-2}(x)\frac{m!}{2!(m-2)!}K_{1-m-4}(x)+\cdots+K_{1+m}(x)\right],\end{aligned} \tag{A2}$$

where

$$K_n(x) = K_{-n}(x) \tag{A3}$$

and

$$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x}K_n(x). \tag{A4}$$

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