

Ξ -nucleus potential for Ξ^- quasifree production in the ${}^9\text{Be}(K^-, K^+)$ reaction

Toru Harada^{1,2,*} and Yoshiharu Hirabayashi³

¹Research Center for Physics and Mathematics, Osaka Electro-Communication University, Neyagawa, Osaka 572-8530, Japan

²J-PARC Branch, KEK Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), 203-1, Shirakata, Tokai, Ibaraki 319-1106, Japan

³Information Initiative Center, Hokkaido University, Sapporo 060-0811, Japan



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We study phenomenologically a Ξ^- production spectrum of the ${}^9\text{Be}(K^-, K^+)$ reaction at 1.8 GeV/ c within the distorted-wave impulse approximation using the optimal Fermi-averaged $K^-p \rightarrow K^+\Xi^-$ amplitude. We attempt to clarify properties of a Ξ -nucleus potential for Ξ^- - ${}^8\text{Li}$, comparing the calculated spectrum with the data of the BNL-E906 experiment. The results show a weak attraction in the Ξ -nucleus potential for Ξ^- - ${}^8\text{Li}$, which can sufficiently explain the data in the Ξ^- quasifree region. The strength of $V_0^\Xi = -17 \pm 6$ MeV is favored within the Woods-Saxon potential, accompanied by the reasonable absorption of $W_0^\Xi = -5$ MeV for $\Xi^-p \rightarrow \Xi^0n$ and $\Xi^-p \rightarrow \Lambda\Lambda$ transitions in nuclear medium. It is difficult to determine the value of W_0^Ξ from the data due to the insufficient resolution of 14.7 MeV FWHM. The energy dependence of the Fermi-averaged $K^-p \rightarrow K^+\Xi^-$ amplitude is also confirmed by this analysis, and its importance in the nuclear (K^-, K^+) reaction is emphasized.

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I. INTRODUCTION

Recently, Nakazawa *et al.* [1] reported the first evidence of a bound state of the Ξ^- - ${}^{14}\text{N}$ system that was identified by the ‘‘KISO’’ event in the KEK-E373 experiment. This result supports that the Ξ -nucleus potential has a weak attraction of $V_\Xi \simeq -14$ MeV in the Woods-Saxon (WS) potential, as suggested by previous analyses [2–4]. However, there still remains an uncertainty about the nature of the $S = -2$ dynamics caused by ΞN interaction and ΞN - $\Lambda\Lambda$ coupling in nuclei due to the limit to the available data. More experimental information is needed for the understanding of Ξ hypernuclei. Recently, Nagae *et al.* [5] have performed an accurate observation of the Ξ^- production spectrum in double-charge exchange reactions (K^-, K^+) on ${}^{12}\text{C}$ targets at 1.8 GeV/ c in the J-PARC E05 experiment, and their analysis is now ongoing. The double-charge exchange reactions such as (K^-, K^+) on nuclear targets produce neutron-rich Ξ hypernuclei, e.g., the neutron excess of $(N - Z)/(N + Z) = 0.25$ for a Ξ^- - ${}^8\text{Li}$ system, which is populated on ${}^9\text{Be}$. The behavior of Ξ^- in the neutron-excess environment is strongly connected with the nature of neutron stars [6] in which the baryon fraction is found to depend on properties of hypernuclear potentials [7].

Kohno [8] examined theoretically Ξ^- production spectra for the quasifree (QF) interaction region in the (K^-, K^+) reactions on ${}^9\text{Be}$ and ${}^{12}\text{C}$ targets in the semiclassical distorted-wave method, using the Ξ -nucleus potential derived from the next-to-leading order (NLO) in chiral effective field theory. However, it has shown that the calculated Ξ^- QF spectrum

on ${}^9\text{Be}$ seems to be insufficient to reproduce the experimental data, so that quantitative information on the Ξ -nucleus potential for Ξ^- - ${}^8\text{Li}$ (${}^9_\Xi\text{He}$) may be unreliable.

In this paper, we investigate phenomenologically the Ξ^- QF spectrum produced via the ${}^9\text{Be}(K^-, K^+)$ reaction at 1.8 GeV/ c in order to extract valuable information on the Ξ -nucleus (optical) potential for the Ξ^- - ${}^8\text{Li}$ system from the data of the BNL-E906 experiment [8,9]. We attempt to clarify properties of the Ξ -nucleus potential for Ξ^- - ${}^8\text{Li}$ and to understand a mechanism of the Ξ^- QF spectrum in comparison with the data [9]. Thus we demonstrate the calculated Ξ^- QF spectrum in the ${}^9\text{Be}(K^-, K^+)$ reaction within the distorted-wave impulse approximation (DWIA), taking into account the energy dependence of the $K^-p \rightarrow K^+\Xi^-$ amplitude in the optimal Fermi-averaging procedure [10,11].

II. CALCULATIONS

A. Distorted-wave impulse approximation

Let us consider production of Ξ hypernuclear states in the nuclear (K^-, K^+) reaction. According to the Green’s function method [12] in the DWIA, an inclusive K^+ double-differential laboratory cross section of the Ξ^- production on a nuclear target with a spin J_A (its z component M_A) [13–15] is given by

$$\frac{d^2\sigma}{d\Omega dE} = \frac{1}{[J_A]} \sum_{M_A} S(E), \quad (1)$$

with $[J_A] = 2J_A + 1$. The strength function $S(E)$ is written as

$$S(E) = -\frac{1}{\pi} \text{Im} \sum_{\alpha\alpha'} \int dr dr' F_\Xi^{\alpha\dagger}(\mathbf{r}) G_\Xi^{\alpha\alpha'}(E; \mathbf{r}, \mathbf{r}') F_\Xi^{\alpha'}(\mathbf{r}'), \quad (2)$$

*harada@osakac.ac.jp

where $G_{\Xi}^{\alpha\alpha'}$ is a complete Green's function for a Ξ hypernuclear system, F_{Ξ}^{α} is a Ξ production amplitude defined by

$$F_{\Xi}^{\alpha} = \beta^{\frac{1}{2}} \bar{f}_{K^-p \rightarrow K^+\Xi^-} \chi_{p_{K^+}}^{(-)*} \chi_{p_{K^-}}^{(+)} \langle \alpha | \hat{\psi}_p | \Psi_A \rangle, \quad (3)$$

and α (α') denotes the complete set of eigenstates for the system. The kinematical factor β denotes the translation from a two-body K^-p laboratory system to a K^- -nucleus laboratory system. $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ is a Fermi-averaged amplitude for the $K^-p \rightarrow K^+\Xi^-$ reaction in nuclear medium [11,14,15]. $\langle \alpha | \hat{\psi}_p | \Psi_A \rangle$ is a hole-state wave function for a struck proton in the target. $\chi_{p_{K^+}}^{(-)}$ and $\chi_{p_{K^-}}^{(+)}$ are distorted waves for outgoing K^+ and incoming K^- mesons, respectively. The laboratory energy and momentum transfers are $\omega = E_{K^-} - E_{K^+}$ and $\mathbf{q} = \mathbf{p}_{K^-} - \mathbf{p}_{K^+}$, respectively; E_{K^+} and \mathbf{p}_{K^+} (E_{K^-} and \mathbf{p}_{K^-}) denote an energy and a momentum of the outgoing K^+ (incoming K^-), respectively.

Due to a high momentum transfer $q \simeq 390\text{--}600$ MeV/ c in the nuclear (K^-, K^+) reaction for K^+ forward-direction angles of $\theta_{\text{lab}} = 1.5^\circ\text{--}8.5^\circ$ at $p_{K^-} = 1.8$ GeV/ c , we simplify the computational procedure for $\chi_{p_{K^+}}^{(-)}$ and $\chi_{p_{K^-}}^{(+)}$, using the eikonal approximation [15]. To reduce ambiguities in the distorted waves, we adopt the same parameters used in calculations for the Λ and Σ^- QF spectra in nuclear (π^\pm, K^+) and (K^-, π^\pm) reactions [11,16,17]. Here we used the total cross sections of $\sigma_{K^-} = 28.9$ mb for the K^-N scattering and $\sigma_{K^+} = 19.4$ mb for the K^+N scattering, and $\alpha_{K^-} = \alpha_{K^+} = 0$, as the distortion parameters. We also take into account the recoil effects, which are very important to estimate the hypernuclear production cross section for a light nuclear system [18], leading to an effective momentum transfer having $q_{\text{eff}} \simeq (1 - 1/A)q \simeq 0.80q$ for $A = 9$.

Recently, the authors [10] have found the strong energy dependence of the $K^-p \rightarrow K^+\Xi^-$ reaction in the nuclear medium, together with the angular dependence for θ_{lab} . Therefore, we emphasize that such behavior of $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ plays a significant role in explaining the shape of the spectrum in the nuclear (K^-, K^+) reaction [10] as well as those in the nuclear (π^\pm, K^+) reactions [11,16,17]. Because $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ modifies the spectral shape including the Ξ^- QF region widely, one must extract carefully information concerning the Ξ^- -nucleus potential from the data.

B. Wave functions

For the ${}^9\text{Be}$ target, the single-particle (s.p.) description of protons is assumed for simplicity. We simulate the calculated results of the s.p. energies of the nucleons and the root-mean-square (rms) radius of $\langle r_N^2 \rangle^{1/2}$ for their wave functions in the linear combination of atomic orbits approximation [19], which well describes the ground state of ${}^9\text{Be}(3/2_{\text{g.s.}}^-; T = 1/2)$ as $\alpha + \alpha + n$ clusters. Thus we compute the s.p. wave functions for the protons in $0p$ and $0s$, using the WS potential with $R = r_0 A^{1/3}$ and $a = 0.67$ fm and omitting the spin-orbit potential; the strength parameter of the potential is adjusted to be $V_0^N = -58.0$ MeV, together with the size parameter of $R = 1.60 A^{1/3} = 3.33$ fm which may be rather large due to the structure of $\alpha + \alpha + n$. Here we obtain the s.p. energies of -22 MeV for $0p_{1/2}$ and -35 MeV for $0s_{1/2}$, which are

consistent with the data of the proton separation energies in ${}^9\text{Be}(p, 2p)$ reactions indicating widths of 8 MeV for $0p$ and 13 MeV for $0s$ [20,21]. The charge radius of ${}^9\text{Be}(3/2_{\text{g.s.}}^-)$ is estimated to be 2.53 fm, which is in good agreement with the data of 2.519 ± 0.012 fm in electron elastic scatterings on ${}^9\text{Be}$ [22]. Note that we must tune in the energies of the s.p. states for the protons as well as the matter rms radius of $\langle r_N^2 \rangle^{1/2}$ for their wave functions, leading to the fact that the shape of the calculated QF spectrum in the (K^-, K^+) reaction sufficiently explains the data.

To calculate the Ξ^- QF spectrum in the nuclear (K^-, K^+) spectrum within the DWIA, we use the Green's function method [12], which is one of the most powerful treatments in the calculation of a spectrum describing not only bound states but also continuum states with an absorptive potential for spreading components. Because non-spin-flip processes seem to dominate in the $K^-p \rightarrow K^+\Xi^-$ reaction at 1.8 GeV/ c [23], hypernuclear configurations of $[J_C^\pi \otimes j_B^\pi]_{J_B^\pi}$, with $J_B^\pi = 3/2^+, 5/2^+, 1/2^-, 3/2^-, 5/2^-, 7/2^-, 3/2^-, 5/2^-, \dots$, are populated in ${}^9_{\Xi^-}\text{He}$ with $T_B = 3/2$; we take the ${}^8\text{Li}$ core nucleus states with $J_C^\pi = 2^+, 1^+, 2^-,$ and 1^- that are given in $(3/2^- \otimes p_{3/2,1/2}^-)_{2^+,1^+}$ and $(3/2^- \otimes s_{1/2}^-)_{2^-,1^-}$ configurations formed by a proton-hole state on ${}^9\text{Be}(3/2_{\text{g.s.}}^-)$ and the Ξ^- with $j_{\Xi}^\pi = \ell_{\Xi} \otimes 1/2 = 1/2^+, 3/2^-, 1/2^-, \dots$ that are given in $\ell_{\Xi} \leq 15$ as being enough to converge in calculations for the Ξ^- spectrum. Here the components of $\Xi^0 n$ and $\Lambda\Lambda$ channels are not considered explicitly because the $\Xi^-p \rightarrow \Xi^0 n$ and $\Xi^-p \rightarrow \Lambda\Lambda$ transition processes may be described as a spreading imaginary potential in Ξ^- bound and continuum regions.

III. Ξ^- -NUCLEUS POTENTIAL

The Ξ^- -nucleus final states are obtained by solving the Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu} \nabla^2 + U_{\Xi}(r) + U_{\text{Coul}}(r) \right] \Psi_{\Xi} = E \Psi_{\Xi}, \quad (4)$$

where μ is the Ξ^- -nucleus reduced mass, U_{Ξ} is the Ξ^- -nucleus potential, and U_{Coul} is the Coulomb potential. The Ξ^- -nucleus potential for Ξ^- - ${}^8\text{Li}$ is given by

$$\begin{aligned} U_{\Xi}(r) &= V_{\Xi}(r) + iW_{\Xi}(E, r) \\ &= [V_0^{\Xi} + iW_0^{\Xi} g(E)] f(r), \end{aligned} \quad (5)$$

with the assumption of the WS form

$$f(r) = [1 + \exp\{(r - R)/a\}]^{-1}, \quad (6)$$

where $R = r_0 A_{\text{core}}^{1/3}$ and a denote a radius and a diffuseness of the potential, respectively. V_0^{Ξ} is a strength parameter for the real part of the potential; W_0^{Ξ} is a strength parameter for the imaginary part of the potential, which denotes the Ξ^- -absorption processes including the $\Xi^-p \rightarrow \Xi^0 n$ and $\Xi^-p \rightarrow \Lambda\Lambda$ reactions. $g(E)$ is an energy-dependent function that increases linearly from 0.0 at $E = E_{\text{th}}(\Lambda)$ to 1.0 at $E = 20$ MeV with respect to the Ξ^- threshold, as often used in nuclear optical models [24], where $E_{\text{th}}(\Lambda) = -23.3$ MeV corresponds to the Λ emitted threshold.

The ground state of ${}^8\text{Li}(2_{\text{g.s.}}^+)$ is a bound state at the neutron binding energy of $B_n = 2.03$ MeV with respect to the

$n + {}^7\text{Li}_{\text{g.s.}}$ threshold [20]; the matter rms radius of $\langle r_m^2 \rangle^{1/2} = 2.39 \pm 0.06$ fm is observed experimentally. Thus the appropriate parameters of r_0 and a in Eq. (6) must be used, as we mention below.

To determine the parameters of r_0 and a for the nuclear core in the WS form, we adopt a folding-model potential obtained by convoluting the nuclear one-body density for ${}^8\text{Li}$ with a two-body Ξ^-N force. We assume the s.p. density of the spherical shell model for simplicity; the modified harmonic oscillator (MHO) model is used in a systematic description of a size and a density distribution for Li isotopes with $A = 6-9$ [25]. For ${}^8\text{Li}(2_{\text{g.s.}}^+)$, we choose carefully the MHO size parameters of $b_s = 1.42$ fm and $b_p = 1.95$ fm with center-of-mass and nucleon-size corrections, adjusting the matter rms radius of $\langle r_m^2 \rangle^{1/2} = 2.39$ fm [25]. Following the procedure in Ref. [17], we use the WS form with the parameters of r_0 and a adjusted to give a best least-squares fit to the radial shape of the form factor obtained by folding a Gaussian range of $a_{\Xi N} = 1.2$ fm into the matter MHO density distribution [17]. The parameters of the resultant WS form in Eq. (6) are $r_0 = 0.783$ fm, $a = 0.722$ fm, and $R = r_0 A_{\text{core}}^{1/3} = 1.57$ fm, which reproduce the radial shape of the form factor very well; the rms radius of the potential denotes

$$\langle r^2 \rangle_V^{1/2} = \left[\int r^2 V_{\Xi}(r) dr / \int V_{\Xi}(r) dr \right]^{1/2} = 2.81 \text{ fm.} \quad (7)$$

On the other hand, the spreading imaginary parts of W_0^{Ξ} may represent complicated continuum states of ${}^9_{\Lambda\Lambda}\text{He}^*$, ${}^9_{\Xi}\text{He}^*$, and ${}^9_{\Xi^0}\text{He}^*$. Considering the states of ${}^7\text{He}(3/2_{\text{g.s.}}^-)$ located at $E_{\text{ex}} = 0.45$ MeV above the $n + {}^6\text{He}$ threshold [20], we have the Λ emitted threshold corresponding to the $\Lambda + {}^8_{\Lambda}\text{He}$ threshold for the $\Xi^-p \rightarrow \Lambda\Lambda$ transition. The threshold-energy difference between $\Xi^- - {}^8\text{Li}$ and $\Lambda - {}^8_{\Lambda}\text{He}$ channels accounts for $\Delta M = M({}^8\text{Li}) + m_{\Xi^-} - M({}^8_{\Lambda}\text{He}) - m_{\Lambda} = 23.3$ MeV, where $M({}^8\text{Li}) = 7471.4$ MeV and $M({}^8_{\Lambda}\text{He}) = 7654.1$ MeV. For the $\Xi^-p \rightarrow \Xi^0n$ transition, the Ξ^0 emitted threshold for $\Xi^0 - {}^8\text{He}$ is located at $E_{\text{ex}} = 4.3$ MeV above the $\Xi^- - {}^8\text{Li}$ threshold. The spin-orbit potential for Ξ^- is also considered to denote the term of $V_{\text{so}}^{\Xi}(1/r)[df(r)/dr]\boldsymbol{\sigma}\cdot\mathbf{L}$, where $V_{\text{so}}^{\Xi} \simeq \frac{1}{10}V_{\text{so}}^N \simeq 2$ MeV [26]. For U_{Coul} , we use the attractive Coulomb potential with the uniform distribution of a charged sphere where $Z = 3$ for $\Xi^- - {}^8\text{Li}$.

We attempt to determine the strength parameters of V_0^{Ξ} and W_0^{Ξ} in Eq. (5) phenomenologically in comparison with the data of the ${}^9\text{Be}(K^-, K^+)$ reaction. Figure 1 shows the real and imaginary parts of the Ξ -nucleus potential for $\Xi^- - {}^8\text{Li}$, choosing the reasonable strengths of $V_0^{\Xi} = -17$ MeV and $W_0^{\Xi} = -5$ MeV, as we discuss in Sec. IV.

IV. RESULTS

A. χ^2 fitting

Tamagawa *et al.* (BNL-E906 Collaboration) reported the experimental data of the Ξ^- QF spectra for the K^+ forward-direction angles of $\theta_{\text{lab}} = 1.5^\circ - 8.5^\circ$ in the ${}^9\text{Be}(K^-, K^+)$ reactions at the incident K^- momentum of $p_{K^-} = 1.8$ GeV/c [9]. The average cross section $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ in the laboratory frame

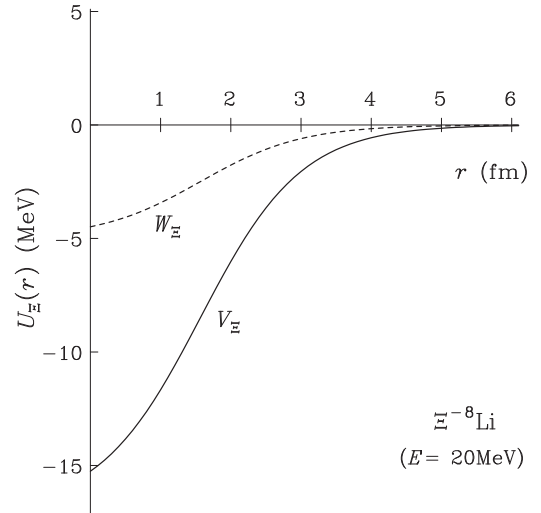


FIG. 1. Real and imaginary parts of the Ξ -nucleus potential U_{Ξ} for $\Xi^- - {}^8\text{Li}$ at the energy $E = 20$ MeV, as a function of the distance between Ξ^- and the ${}^8\text{Li}$ nucleus. Solid and dashed curves denote the calculated values for $V_0^{\Xi} = -17$ MeV and for $W_0^{\Xi} = -5$ MeV in the WS form, respectively, using $R = r_0 A_{\text{core}}^{1/3} = 1.57$ fm, where $r_0 = 0.783$ fm and $a = 0.722$ fm.

was obtained by

$$\bar{\sigma}_{1.5^\circ-8.5^\circ} \equiv \int_{\theta_{\text{lab}}=1.5^\circ}^{\theta_{\text{lab}}=8.5^\circ} \left(\frac{d^2\sigma}{dp_{K^+} d\Omega_{K^+}} \right) d\Omega / \int_{\theta_{\text{lab}}=1.5^\circ}^{\theta_{\text{lab}}=8.5^\circ} d\Omega, \quad (8)$$

with the detector resolution of 14.7 MeV full width at half maximum (FWHM) [9]. The strength parameters of V_0^{Ξ} and W_0^{Ξ} in Eq. (5) should be adjusted appropriately to reproduce the data of $\bar{\sigma}_{1.5^\circ-8.5^\circ}$.

We consider the Ξ^- QF spectrum for $\Xi^- - {}^8\text{Li}$ hypernuclear states with J_B^{Ξ} and $T_B = 3/2$, using the Green's function method [12], in order to be compared with the data of the ${}^9\text{Be}(K^-, K^+)$ reaction at the BNL-E906 experiment [9]. Calculating the spectra for $\theta_{\text{lab}} = 1.5^\circ - 8.5^\circ$, we estimate the average cross section for the corresponding $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ in Eq. (8). To make a fit to the spectral shape of the data, we introduce a renormalization factor of f_s into the absolute value of the calculated spectrum because the eikonal distortion and the amplitude of $f_{K^-p \rightarrow K^+\Xi^-}$ would have some ambiguities [10,15]. The detector resolution of 14.7 MeV FWHM is also taken into account. We obtain the values of χ^2 for fits to the data points of $N = 17$ in $p_{K^+} = 1.07-1.39$ GeV/c, varying the strengths of (V_0^{Ξ}, W_0^{Ξ}) and f_s ; we assume the value of $0.018 \mu\text{b}/(\text{sr MeV } c^{-1})$ as a constant background. Thus we estimate the average cross section in Eq. (8), calculating the spectra for $\theta_{\text{lab}} = 1.5^\circ - 8.5^\circ$ in the parameter region of $V_0^{\Xi} = (-36) - (+18)$ MeV by a 6-MeV energy step and $W_0^{\Xi} = (-10) - 0$ MeV by a 2-MeV energy step. The 1-MeV energy step is taken in the estimation near the χ_{min}^2 point.

Figure 2 displays the contour plots of χ^2 -value distributions for $\bar{\sigma}_{1.5^\circ-8.5^\circ}$. The minimum value of χ^2 is found to be $\chi_{\text{min}}^2 = 15.2$ at $V_0^{\Xi} = -17$ MeV, $W_0^{\Xi} = -5$ MeV, and $f_s = 0.940$, leading to beltlike regions of $\Delta\chi^2 = 2.30, 4.61,$ and 9.21 , which correspond to 68%, 90%, and 99% confidence

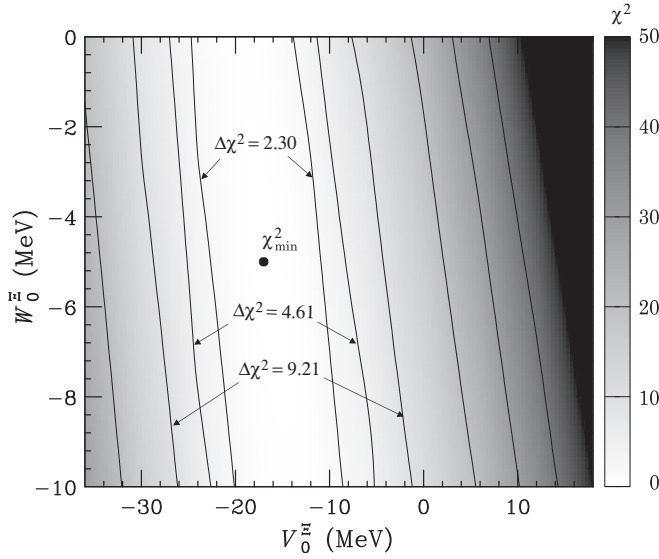


FIG. 2. Contour plots of the χ^2 -value distribution in the $\{V_0^\Xi, W_0^\Xi\}$ plane from fitting to the average cross section of $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ in the ${}^9\text{Be}(K^-, K^+)$ reaction at $p_{K^-} = 1.8$ GeV/c. A solid circle denotes the minimum position of $\chi_{\min}^2 = 15.2$ at $(V_0^\Xi, W_0^\Xi) = (-17 \text{ MeV}, -5 \text{ MeV})$ with $f_s = 0.940$. The solid lines labeled by $\Delta\chi^2 = 2.30, 4.61,$ and 9.21 correspond to 68%, 90%, and 99% confidence levels for two parameters, respectively.

levels for two parameters, respectively, where $\Delta\chi^2 \equiv \chi^2 - \chi_{\min}^2$. We find that the value of χ^2 is almost insensitive to W_0^Ξ . This fact implies that the parameter of W_0^Ξ cannot be determined from the BNL-E906 data due to the insufficient resolution of 14.7 MeV FWHM. Nevertheless, we recognize that the calculated spectrum for $V_0^\Xi \simeq -17$ MeV seems to be in good agreement with the data when $W_0^\Xi \simeq -5$ MeV; it gives the minimum value of $\chi^2/N = 15.2/17 = 0.89$ and the standard deviation of $\sigma \simeq 6$ MeV. In Table I, we list the reduced χ^2 values of χ^2/N in calculations when $V_0^\Xi = -30, -24, -18, -12, -6, 0,$ and $+12$ MeV, and $W_0^\Xi = -10, -5,$ and 0 MeV, comparing the calculated spectra with the data. Note that the absolute values of the calculated cross section can explain the magnitude of the data, as seen by $f_s \simeq 0.9-1.0$.

Figure 3 shows the absolute values of the calculated spectrum for $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ in the best-fit calculation and compares them with the data of the BNL-E906 experiment at $p_{K^+} = 1.07-1.39$ GeV/c. We recognize that an attraction in the Ξ^- - ${}^8\text{Li}$ potential is needed to reproduce the data. The contribution of p -hole configurations is larger than that of s -hole configurations in the Ξ^- QF region of $p_{K^+} = 1.2-1.4$ GeV/c, whereas the former is similar to the latter in the region of $p_{K^+} < 1.2$ GeV/c where the recoil momentum grows into $q > 540$ MeV/c. Consequently, we confirm that the Ξ potential for Ξ^- - ${}^8\text{Li}$ has a weak attraction in the real part of the WS potential with $r_0 = 0.738$ fm and $a = 0.722$ fm;

$$V_0^\Xi = -17 \pm 6 \text{ MeV} \quad \text{for } W_0^\Xi = -5 \text{ MeV}. \quad (9)$$

This potential provides the ability to explain the ${}^9\text{Be}(K^-, K^+)$ data at the BNL-E906 experiment. Several authors [2,3] have

TABLE I. The χ^2 -fitting for various strength parameters, V_0^Ξ and W_0^Ξ , in the WS potential with $r_0 = 0.738$ fm and $a = 0.722$ fm for Ξ^- - ${}^8\text{Li}$. The value of χ^2/N and the renormalization factor f_s are obtained by comparing the calculated spectrum with the $N = 17$ data points of the average cross sections of $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ for $p_{K^+} = 1.07-1.39$ GeV/c. The data were taken from Ref. [9].

V_0^Ξ (MeV)	W_0^Ξ (MeV)	$N = 17$ data points	
		χ^2/N	f_s
+12	0	69.8/17	0.988
0	0	37.6/17	0.964
-6	0	26.4/17	0.951
-12	0	18.9/17	0.939
-18	0	15.6/17	0.927
-24	0	16.8/17	0.914
-30	0	22.8/17	0.902
+12	-5	58.3/17	0.999
0	-5	30.7/17	0.975
-6	-5	21.8/17	0.962
-12	-5	16.5/17	0.950
-17	-5	15.2/17	0.940
-18	-5	15.3/17	0.938
-24	-5	18.4/17	0.925
-30	-5	26.1/17	0.913
+12	-10	49.0/17	1.010
0	-10	25.7/17	0.985
-6	-10	18.9/17	0.973
-12	-10	15.6/17	0.961
-18	-10	16.3/17	0.948
-24	-10	21.1/17	0.936
-30	-10	30.4/17	0.923

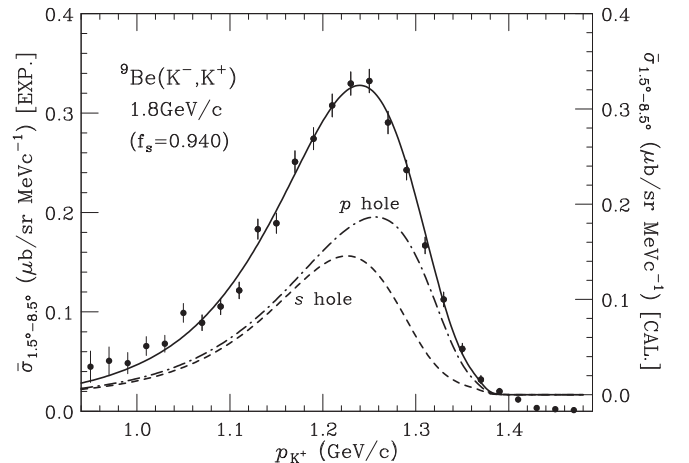


FIG. 3. Calculated spectrum for $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ in the WS potential with $V_0^\Xi = -17$ MeV, $W_0^\Xi = -5$ MeV, $r_0 = 0.738$ fm, and $a = 0.722$ fm, together with the data of the ${}^9\text{Be}(K^-, K^+)$ reaction at $p_{K^-} = 1.8$ GeV/c [9]. The calculated spectrum is normalized by $f_s = 0.940$ for fits to the data. Solid, dashed, and dot-dashed curves denote the contributions of total, s -hole, and p -hole configurations, respectively. The calculated values are folded with a detector resolution of 14.7 MeV FWHM.

TABLE II. Binding energies B_{Ξ^-} and widths Γ_{Ξ^-} of the Ξ^- -nucleus (nl) bound states for Ξ^- - ${}^8\text{Li}$ (${}^9_{\Xi^-}\text{He}$). The strengths of $V_0^{\Xi} = -17$ MeV and $W_0^{\Xi} = -2.5$ (-1.5) MeV are used in the WS potential for the Ξ^- bound region. These values are estimated in combination with the Ξ -nucleus potential $U_{\Xi} = V_{\Xi} + iW_{\Xi}$ and the Coulomb potential U_{Coul} .

(nl)	$V_{\Xi} + U_{\text{Coul}} + iW_{\Xi}$			$V_{\Xi} + U_{\text{Coul}}$		V_{Ξ}		U_{Coul}	
	$-B_{\Xi^-}$ (MeV)	Γ_{Ξ^-} (MeV)	rms (fm)	$-B_{\Xi^-}$ (MeV)	rms (fm)	$-B_{\Xi^-}$ (MeV)	rms (fm)	$-B_{\Xi^-}$ (MeV)	rms (fm)
$W_0^{\Xi} =$	-2.5 MeV								
1S	-1.851	1.118	3.95	-1.897	3.96	-0.475	5.56	-0.255	14.6
2S	-0.122	1.3×10^{-2}	29.3	-0.122	29.5	—	—	-0.066	53.4
2P	-0.068	3.1×10^{-4}	43.4	-0.068	43.4	—	—	-0.067	44.0
$W_0^{\Xi} =$	-1.5 MeV								
1S	-1.880	0.669	3.95						
2S	-0.122	8.0×10^{-3}	29.4						
2P	-0.068	1.9×10^{-4}	43.4						

attempted to determine the values of V_0^{Ξ} for fits to the shape and magnitude of the Ξ^- QF spectra from the data of the ${}^{12}\text{C}(K^-, K^+)$ reaction [3]. They have suggested that the Ξ -nucleus potential has a weak attraction of $V_0^{\Xi} \simeq -14$ MeV in the WS potential. It is shown that the results of Eq. (9) in our analysis are considerably consistent with the results of the previous studies [2,3].

B. Ξ^- -nucleus bound states

In Table II, we show the numerical results of binding energies and widths of the Ξ^- -nucleus (nl) bound states for Ξ^- - ${}^8\text{Li}$, where (nl) denotes the principal and angular momentum quantum numbers for the relative motion between Ξ^- and ${}^8\text{Li}$. By solving the Schrödinger equation of Eq. (4) with the WS potential U_{Ξ} and the finite Coulomb potential U_{Coul} , we obtain a complex eigenvalue as a Gamow state,

$$E_{nl} = -B_{\Xi^-} - i\frac{\Gamma_{\Xi^-}}{2}, \quad (10)$$

where B_{Ξ^-} and Γ_{Ξ^-} denote a binding energy and a width of the bound state, respectively. When we use $V_0^{\Xi} = -17$ MeV in the WS potential, we confirm that there exists a very shallow Ξ^- (1S) bound state due to the weak attraction in the Ξ -nucleus potential even if the Coulomb potential is switched off; the binding energy accounts for $B_{\Xi^-}(1S) = 0.475$ MeV and the rms radius of $\langle r^2 \rangle^{1/2} = 5.56$ fm. When the Coulomb potential is switched on, the binding energy is significantly shifted downward in comparison with the corresponding Coulomb eigenstate, as seen in Table II. Thus this state is often regarded as a ‘‘Coulomb-assisted’’ Ξ^- -nucleus bound state; $B_{\Xi^-}(1S) = 1.897$ MeV and $\langle r^2 \rangle^{1/2} = 3.96$ fm.

A Ξ^- hyperon bound in nuclei must be absorbed by strong interaction via the $\Xi^-p \rightarrow \Lambda\Lambda$ conversion process. To estimate the width of the Ξ^- bound state, we assume the value of $W_0^{\Xi} = -2.5$ MeV, which corresponds to the strength of $W_{\Xi}(E)$ at the Ξ^- threshold ($E = 0.0$ MeV). Thus we obtain the width of $\Gamma_{\Xi^-}(1S) = 1.118$ MeV, together with $B_{\Xi^-}(1S) = 1.851$ MeV. When we use $W_0^{\Xi} = -1.5$ MeV arising from the $\Xi^-p \rightarrow \Lambda\Lambda$ conversion in the ΞN NLO potential [27,28],

we obtain $\Gamma_{\Xi^-}(1S) = 0.669$ MeV and $B_{\Xi^-}(1S) = 1.880$ MeV (see also Sec. VB).

V. DISCUSSION

A. Effects of the real part of the Ξ -nucleus potential

To see effects of the attraction in the Ξ -nucleus potential for Ξ^- - ${}^8\text{Li}$, we discuss the shapes and magnitudes of the calculated spectra. Figure 4 shows the absolute values of the calculated spectra for $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ in the Ξ^- QF region, using various strengths of V_0^{Ξ} . We find that the shape and the magnitude of the calculated spectrum are considerably sensitive to the value of V_0^{Ξ} . This confirms that the value of χ^2/N is significantly changed by V_0^{Ξ} . The peak position of the QF spectrum is scarcely shifted downward for p_{K^+} , as V_0^{Ξ} changes from -24 to $+12$ MeV, whereas the magnitude of this peak is slightly reduced by 7.4%. There is a difference between the spectra of $V_0^{\Xi} = (-24)$ - $(+12)$ MeV in the momentum

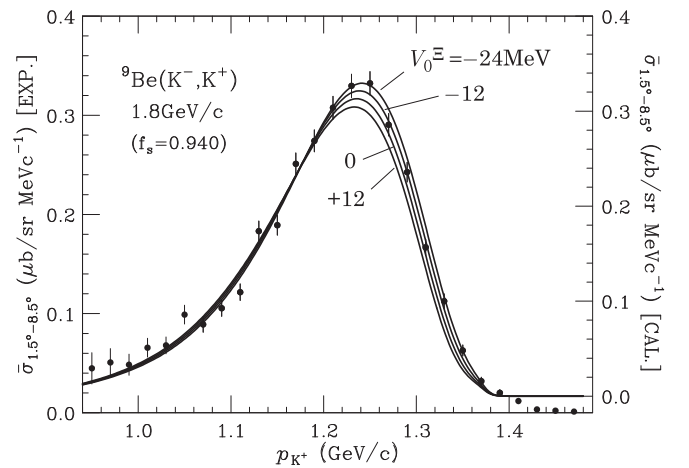


FIG. 4. Shapes and magnitudes of the calculated spectra for $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ in the ${}^9\text{Be}(K^-, K^+)$ reaction at $p_{K^-} = 1.80$ GeV/c, depending on the strengths of $V_0^{\Xi} = -24, -12, 0,$ and $+12$ MeV in the WS potential with $W_0^{\Xi} = -5$ MeV. The spectra are folded with a detector resolution of 14.7 MeV FWHM.

region of $p_{K^+} > 1.2$ GeV/c, corresponding to the region of lower energies $E < 140$ MeV. On the other hand, the shapes and magnitudes of the spectra with $V_0^{\Xi} = (-24)$ – $(+12)$ MeV become similar to each other in the region of higher energies $E > 140$ MeV ($p_{K^+} < 1.2$ GeV/c).

B. Validity of the imaginary part of the Ξ -nucleus potential

In Sec. IV A, we have found that the shapes and magnitudes of the calculated spectra are not so sensitive to the value of W_0^{Ξ} when we change $W_0^{\Xi} = (-10)$ – 0 MeV in the imaginary part of the Ξ -nucleus potential. This is because a mask of W_0^{Ξ} is inevitable due to the insufficient resolution of 14.7 MeV FWHM. Thus we recognize that it is difficult to determine the value of W_0^{Ξ} .

According to the procedure of Gal, Tokar, and Alexander [29], we examine theoretically an appropriate parameter for W_0^{Ξ} from a viewpoint of the first-order optical ($t\rho$) potential,

$$U_{\Xi}^{(1)}(r) = t_{\Xi^-p}\rho_p(r) + t_{\Xi^-n}\rho_n(r), \quad (11)$$

in terms of the effective two-body ΞN elastic $t_{\Xi N}$ scattering matrices in the laboratory frame, where $\rho_{p,n}(r)$ are the proton and neutron densities of the core nucleus. By the optical theorem $4\pi \text{Im}f_{\Xi N}(0) = k_{\Xi}\sigma_{\text{tot}}$ and considering collisions of zero energy Ξ with bound nucleons, we obtain the imaginary part $W_{\Xi}^{(1)}$ of the optical potential involving the $\Xi^-p \rightarrow \Xi^0n$ and $\Xi^-p \rightarrow \Lambda\Lambda$ conversions, which is given by

$$W_{\Xi}^{(1)}(r) = -\langle v_{\Xi^-p}\sigma(\Xi^-p \rightarrow \Xi^0n, \Lambda\Lambda) \rangle \rho_p(r)/2, \quad (12)$$

where v is the relative velocity of a Ξ^-p pair, and $\langle \dots \rangle$ indicates nuclear medium corrections to the free space value of $v\sigma$ arising from Fermi averaging, binding effects, and the Pauli principle, etc. The cross section is well approximated up to 300 MeV/c in the laboratory system by the form

$$v_{\Xi^-p}\sigma = (v_{\Xi^-p}\sigma)_0/(1 + \alpha v), \quad (13)$$

with the two representative parametrizations of $(v_{\Xi^-p}\sigma)_0 = 25$ mb and $\alpha = 18$ for the $\Xi^-p \rightarrow \Xi^0n$ and $\Xi^-p \rightarrow \Lambda\Lambda$ reactions, fitting to $v_{\Xi^-p}\sigma$, which are given by the ΞN NLO potential [27,28]. Taking into account the closure assumption and nuclear medium corrections [29], we obtain $\langle v_{\Xi^-p}\sigma \rangle = 7.02$ mb within the Fermi gas model. Using the relation between $\langle v\sigma \rangle$ and $\text{Im}b$, where b is the effective parameter of a complex scattering length for Ξ^-p , we roughly estimate

$$\text{Im}b = \mu \langle v\sigma \rangle / 8\pi = 0.078 \text{ fm}, \quad (14)$$

of which the value corresponds to $W_0^{\Xi} = -6.2$ MeV in the WS potential. We find that this value is similar to $W_0^{\Xi} = -5$ MeV for the minimum value of χ_{min} , as shown in Fig. 2. If we replace the momentum distributions of the Fermi gas model by those of the s.p. shell model for the finite nuclei, the results may not change. Therefore, we believe that the Ξ -nucleus potential with $V_0^{\Xi} = -17$ MeV and $W_0^{\Xi} = -5$ MeV is appropriate to the study of the Ξ^- QF spectrum in the ${}^9\text{Be}(K^-, K^+)$ reaction at $p_{K^-} = 1.8$ GeV/c.

Considering the same manner for only the $\Xi^-p \rightarrow \Lambda\Lambda$ conversion [27,28], we also obtain $(v_{\Xi^-p}\sigma)_0 = 4.5$ mb and $\alpha = 20$. Thus we estimate $\text{Im}b = 0.018$ fm, which corresponds to $W_0^{\Xi} = -1.5$ MeV. Such a small absorption of

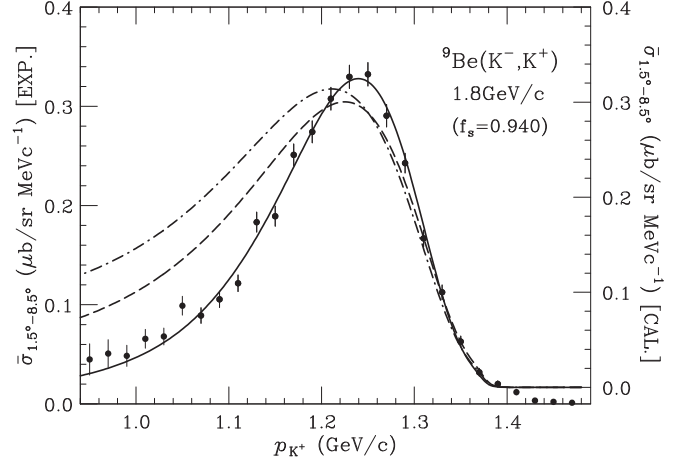


FIG. 5. Comparison of the calculated spectra for $\bar{\sigma}_{1.5^\circ-8.5^\circ}$ with the data of the ${}^9\text{Be}(K^-, K^+)$ reaction at $p_{K^-} = 1.80$ GeV/c [9], using the WS potential with $V_0^{\Xi} = -17$ MeV and $W_0^{\Xi} = -5$ MeV. Solid and dashed curves denote the spectra obtained by the optimal and standard Fermi-averaged $K^-p \rightarrow K^+\Xi^-$ amplitudes for $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$, respectively. A dot-dashed curve denotes the spectrum obtained by $\beta(d\sigma/d\Omega)_{\text{lab}}^{\text{av}} = \text{constant}$. The spectra are folded with a detector resolution of 14.7 MeV FWHM.

$W_0^{\Xi} \simeq -1$ MeV may be acceptable because the $\Xi^-p \rightarrow \Lambda\Lambda$ coupling is recently predicted to be rather small [8,30].

C. Verification of the optimal Fermi-averaged $K^-p \rightarrow K^+\Xi^-$ amplitude

In a previous paper [10], we emphasized the importance of the energy dependence of the $K^-p \rightarrow K^+\Xi^-$ amplitude of $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ arising from the optimal Fermi-averaging procedure [11] in the nuclear (K^-, K^+) reaction. We discuss the calculated Ξ^- QF spectra involving the energy dependence of $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ in comparison with the data of the ${}^9\text{Be}(K^-, K^+)$ reaction in the BNL-E906 experiment. To see the importance of the energy dependence of $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$, we also estimate the spectrum in the DWIA using the “standard” Fermi-averaged cross section $(d\sigma/d\Omega)_{\text{lab}}^{\text{av}}$ for the $K^-p \rightarrow K^+\Xi^-$ reaction, which may be given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}}^{\text{av}} = \int d\mathbf{p}_N \rho(p_N) \left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}}^{\text{free}}, \quad (15)$$

where $\rho(p_N)$ is a proton momentum distribution in the target nucleus, and $(d\sigma/d\Omega)_{\text{lab}}^{\text{free}}$ is the differential cross section for the $K^-p \rightarrow K^+\Xi^-$ reaction in free space. This spectrum is proportional to $\beta(d\sigma/d\Omega)_{\text{lab}}^{\text{av}}$, indicating the energy dependence of β , whereas the value of $(d\sigma/d\Omega)_{\text{lab}}^{\text{av}}$ at each θ_{lab} becomes constant in Eq. (15). In Table III, we show the calculated values of $(d\sigma/d\Omega)_{\text{lab}}^{\text{av}}$ and $(d\sigma/d\Omega)_{\text{lab}}^{\text{free}}$ [10]. Figure 5 displays the calculated Ξ^- QF spectra obtained by the optimal and standard Fermi-averaged $K^-p \rightarrow K^+\Xi^-$ amplitudes in the ${}^9\text{Be}(K^-, K^+)$ reaction at $p_{K^-} = 1.8$ GeV/c, together with the spectrum obtained by $\beta(d\sigma/d\Omega)_{\text{lab}}^{\text{av}} = \text{constant}$, omitting the energy dependence of β . We find that the energy dependence of $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ acts on the shape and magnitude of the QF spectrum remarkably, and it makes its width narrower.

TABLE III. Comparison of the standard Fermi-averaged differential cross sections $(d\sigma/d\Omega)_{\text{lab}}^{\text{av}}$ for the $K^-p \rightarrow K^+\Xi^-$ reaction at $p_{K^-} = 1.8$ GeV/c with the differential cross sections $(d\sigma/d\Omega)_{\text{lab}}^{\text{free}}$ for the $K^-p \rightarrow K^+\Xi^-$ reaction in free space [10]. The values are in units of mb/sr.

θ_{lab}	1°	2°	3°	4°	5°	6°	7°	8°	9°
$(d\sigma/d\Omega)_{\text{lab}}^{\text{av}}$	55.4	54.7	53.5	51.9	50.0	47.8	45.5	43.1	40.7
$(d\sigma/d\Omega)_{\text{lab}}^{\text{free}}$	67.1	65.9	64.1	61.7	58.8	55.6	52.3	49.0	45.8

If we use a constant value for $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ in our calculations, the shape and the magnitude of the calculated Ξ⁻ QF spectrum cannot explain the data qualitatively. We show clearly that the optimal Fermi averaging for the $K^-p \rightarrow K^+\Xi^-$ reaction provides a good description of the energy dependence of the Ξ⁻ QF spectrum in the nuclear (K^-, K^+) reaction [10]. Therefore, we recognize that the optimal Fermi-averaged amplitude for $\bar{f}_{K^-p \rightarrow K^+\Xi^-}$ is essential to explain the shape and the magnitude of the spectrum including the Ξ⁻ QF region with a wide energy range. Thus it is required that we extract information concerning the Ξ-nucleus potential carefully from the data of the experimental spectrum.

VI. SUMMARY AND CONCLUSION

We have studied phenomenologically the Ξ⁻ production spectrum of the ${}^9\text{Be}(K^-, K^+)$ reaction at 1.8 GeV/c within the DWIA using the optimal Fermi-averaged $K^-p \rightarrow K^+\Xi^-$ amplitude. We have attempted to clarify properties of the Ξ-nucleus potential for Ξ⁻- ${}^8\text{Li}$, comparing the calculated spectrum with the data of the BNL-E906 experiment. We have performed the χ^2 fitting to the $N = 17$ data points for $\bar{\sigma}_{1.5^\circ-8.5^\circ}$, varying the strength parameters of V_0^Ξ and W_0^Ξ in the WS potential.

In conclusion, we show the weak attraction in the Ξ-nucleus potential for Ξ⁻- ${}^8\text{Li}$, which provides the ability to explain the data for the Ξ⁻ QF region in the ${}^9\text{Be}(K^-, K^+)$ reaction at 1.8 GeV/c, consistent with analyses for previous experiments [1,3]. The attraction of $V_0^\Xi = -17 \pm 6$ MeV is favored within the WS potential, accompanied by the reasonable absorption of $W_0^\Xi = -5$ MeV for the Ξ⁻ $p \rightarrow \Xi^0 n$ and Ξ⁻ $p \rightarrow \Lambda\Lambda$ transitions in nuclear medium, although it is difficult to determine the value of W_0^Ξ from the data due to the insufficient resolution of 14.7 MeV FWHM. The importance of the energy dependence of the Fermi-averaged $K^-p \rightarrow K^+\Xi^-$ amplitude is confirmed by this analysis. The detailed analysis is also required for the J-PARC E05 experiment of the ${}^{12}\text{C}(K^-, K^+)$ reaction at 1.8 GeV/c [5]. This investigation is a subject for future research.

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