Energy-weighted sum rule for Gamow-Teller giant resonances in high-spin isomeric states of $N = Z$ nuclei

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The sum-rule approach and the shell model are used to estimate the energies of Gamow-Teller (GT) giant resonances in high-spin isomeric (HSI) states in $N = Z$ nuclei. The newly derived energy-weighted sum rules show that the GT resonance energies of HSI states are related to the $\sigma\sigma\tau\tau$ and $\sigma\sigma$ interactions. Comparing the results of the sum-rule approach with those of large-scale shell-model calculations for the ^{52}Fe(12^+) and ^{94}Ag(21^+) $N = Z$ high-spin isomers, the deduced κ_σ, which is the strength of the $\sigma\sigma$ residual interaction in a simple residual-interaction model, is 2.5 MeV.

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I. INTRODUCTION

A giant resonance (GR) is a collective oscillation mode of an atomic nucleus and also a feature of quantum many-body systems [1]. The Gamow-Teller (GT) giant resonance (GTR) is the oscillation in the spin and isospin degrees of freedom [2–8]. Its transition operator $\vec{\sigma} t_\pm$ is the simplest operator combined from the spin $\vec{\sigma}$ and isospin t operators. Because it does not include space coordinate operators, GT transitions are more sensitive to the spin-isospin properties of nuclear matter than the nuclear geometrical properties. Migdal [9] predicted so-called pion condensation, which is a candidate for phase transitions in nuclear matter such as the interior of a neutron star. To calibrate the interaction causing pion condensation, many experimental studies have been performed for stable nuclei with neutron excess [7,8]. Recently, such experimental studies have been extended to the region of neutron-rich unstable nuclei [10]. Moreover, due to the similarity of the GT transition operator to weak interactions, studies on GTR are crucial to improve nuclear models used to predict the rates of weak processes in astrophysics [11] as well as in experiments such as double-β-decay searches [12].

Theoretically, in addition to the various microscopic calculations developed to date [13–16], sum-rule approaches have been employed to analyze the observed spectra [6,17–22]. Roughly speaking, the non-energy-weighted sum rule (NEWSR) provides a criterion for the collectivity of the observed resonances, while the energy-weighted sum rule (EWSR) provides a measure of the interaction strength driving the oscillation.

On the other hand, the GTR has yet to be observed in $N = Z$ nuclei. This is because, in $N = Z$ nuclei, the initial state lacks neutron excess as a medium to propagate the oscillation in the particle-hole (ph) channel. However, we recently showed that GT transitions from high spin isomeric (HSI) states in $N = Z$ nuclei can create a strong collectivity [23] because HSI states can be regarded as a mixture of two Fermi liquids filled with nucleons with opposite spin directions and the spin excess (i.e., the excess of one fluid to the other) can serve as a medium to propagate the ph-channel oscillation. We confirmed this picture using the NEWSR and the shell model for two $N = Z$ HSI states in ^{52}Fe and ^{94}Ag with spins and parities of 12^+ and 21^+, respectively. These nuclei are strongly deformed and spin-aligned along the direction of the total spin.

In this work, we extend our approach to the EWSR and evaluate the strength of the spin-isospin residual interactions behind the GTR. We start from a simple Bohr-Mottelson Hamiltonian [24] according to the method in Ref. [17]. This Hamiltonian contains three parameters κ_σ, κ_τ, and $\kappa_{\sigma\tau}$, which represent the strengths of the residual interactions. κ_σ, κ_τ, and $\kappa_{\sigma\tau}$ depend on whether the interacting nucleons are coupled through spin ($\sigma\sigma$), isospin ($\tau\tau$), or both ($\sigma\sigma\tau\tau$), respectively. These parameters are useful for investigating the properties of the short-range spin- and isospin-dependent parts of nuclear interactions in nuclear medium. In addition, they are crucial for describing nuclear weak responses such as neutrino mean-free path [25] as well as the onset of the pion condensation in the interiors of neutron stars [9,26,27].

Compared with the $\sigma\sigma\tau\tau$ and $\tau\tau$ parts, the $\sigma\sigma$ part is poorly understood but is considered to be weak compared with the $\sigma\sigma\tau\tau$ part. An empirical evidence is 1^+ isospin doublets in ^{12}C: The $T = 1$ 1^+ state has an excitation energy of 15.1 MeV, which is 2.4 MeV higher than the $T = 0$ 1^+ state at 12.7 MeV. Also in heavier nuclei, the weakness of the $\sigma\sigma$ part, which is close to zero, was shown by analyzing $M1$ excitations [28]. On the other hand, microscopic models whose parameters are adjusted to describe nuclear ground-state properties often fail to evaluate these spin-dependent interactions [29]. This means that, in nuclear ground states, the spin-spin correlation is hidden and difficult to investigate. Therefore, studies on the excited states such as GTR should be crucial to elucidate the effect of the $\sigma\sigma$ part.

Interestingly, GTR energies for HSI states are sensitive to κ_σ and $\kappa_{\sigma\tau}$. This is in contrast with the case of 0^+ $N > Z$ initial states [17], where GTR energies are sensitive to κ_τ and $\kappa_{\sigma\tau}$. Thus, we suggest that the GTR energies from HSI states can provide a new type of sensitivity on the spin-isospin interaction, especially for the $\sigma\sigma$ interaction. In addition to GT transitions, we also discuss isoscalar $M1$ transitions. Because the sum rules of $M1$ transitions are not sensitive to the spin-isospin residual interactions, they can be used as a reference to cancel other effects.

This work is stimulated by an experimental program, which aims to measure GT transitions from the ^{52}Fe(12^+) state provided as a radioactive-isotope beam [30]. To date, there have been no experimental data on GT and isoscalar $M1$ giant resonances in high-spin isomers. In this work, instead of experimental data, we compare the results of the sum-rule approach with those obtained from shell-model calculations and evaluate the theoretical value of the $\sigma\sigma$ interaction strength. We use the same shell-model calculations employed in Ref. [23] for the ^{52}Fe($I^\pi = 12^+$) and ^{94}Ag(21^+) HSI states. Although the GT transition strength distributions are already shown, the results for the isoscalar $M1$ transitions are shown in this work for the first time.

This paper is organized as follows. In Sec. II, the EWSRs and NEWSRs for GT and isoscalar $M1$ transitions from high-spin isomers are shown. In Sec. III, the relation of the resonance energies and the spin-isospin interactions is deduced. Shell-model calculations for ^{52}Fe and ^{94}Ag high spin isomers are used to estimate κ_σ in Sec IV, which is followed by a discussion and summary in Secs. V and VI, respectively.

II. SUM RULES

A. Gamow-Teller transition

According to Ref. [23], we consider the Gamow-Teller transition operators, which are given as

$$\hat{O}^{\pm}_{\nu} = \sum_{\alpha} \sigma_\nu(\alpha) t_{\pm}(\alpha), \tag{1}$$

where $\nu = \pm 1, 0$ and α is the index of nucleons. Here, the $\nu = 0$ direction of the spin operators is the direction of the total spin of the high-spin states in their intrinsic frame. The HSI states considered herein are highly deformed prolate shapes and the direction of the total spin is along the symmetry axis. The transition operator with ν changes the quantum number K (i.e., the projection of the total spin I onto the axis) by $\Delta K = \nu$.

We evaluate the energy-weighted sum rules for these operators as

$$S^{(1)}(\sigma_\nu t_\pm) = \sum_f (E_f - E_i) |\langle f|\hat{O}^\pm_\nu|i\rangle|^2, \tag{2}$$

where i and f are the initial and final states, $E_f - E_i$ is the excitation energy from i to f, and the summation runs over all the final states. Using the Hamiltonian of the system, H, these sum rules can be written as

$$S^{(1)}(\sigma_\nu t_\pm) = \langle i|(\hat{O}^\pm_\nu)^\dagger [H, \hat{O}^\pm_\nu]|i\rangle. \tag{3}$$

Hence

$$S^{(1)}(\sigma_{-1} t_\pm) + S^{(1)}(\sigma_{+1} t_\pm) = \langle i|DCT[H, \hat{O}^\pm_{\pm 1}]|i\rangle, \tag{4}$$

$$2S^{(1)}(\sigma_0 t_\pm) = \langle i|DCT[H, \hat{O}^\pm_0]|i\rangle, \tag{5}$$

$$2S^{(1)}(\boldsymbol{\sigma} t_\pm) = \sum_\nu \langle i|DCT[H, \hat{O}^\pm_\nu]|i\rangle, \tag{6}$$

where $DCT[A, B]$ is the double commutation relation,

$$DCT[A, B] = [B^\dagger, [A, B]]. \tag{7}$$

Following Ref. [17], we select a simple Hamiltonian [24] as

$$H = T + V, \tag{8}$$

where T is the kinetic energy of each nucleon and can be ignored because it commutes with the transition operators. V is taken as a separation interaction model, which is expressed by

$$V = \frac{1}{2A} \sum_{\alpha,\alpha'} \sum_\xi \kappa_\xi I_\xi(\alpha, \alpha') + \sum_\alpha \kappa_{ls} \boldsymbol{l}(\alpha) \cdot \boldsymbol{s}(\alpha), \tag{9}$$

where $I_\xi(\alpha, \alpha')$ is the two-body operators for two different nucleons (α, α') and is defined as

$$\hat{I}_\xi(\alpha, \alpha') = \{\boldsymbol{\tau}(\alpha) \cdot \boldsymbol{\tau}(\alpha'),\ \boldsymbol{\sigma}(\alpha) \cdot \boldsymbol{\sigma}(\alpha'),\quad \boldsymbol{\sigma}(\alpha) \cdot \boldsymbol{\sigma}(\alpha')\boldsymbol{\tau}(\alpha) \cdot \boldsymbol{\tau}(\alpha')\}, \tag{10}$$

and ξ is τ, σ, $\sigma\tau$ labeling the isospin, spin, and spin-isospin channels, respectively. The second term in Eq. (9) is the one-body spin-orbit (ls) potential, and κ_{ls} represents its strength. Here, we introduce one-body orbital-angular-momentum $\boldsymbol{l}(\alpha)$ and spin operators $\boldsymbol{s}(\alpha)$. In the same manner as $\boldsymbol{\sigma}$, the $\boldsymbol{l}$ and $\boldsymbol{s}$ operators each have directions of $\nu = \pm 1, 0$. Consequently,

$$S^{(1)}(\sigma_{-1} t_\pm) + S^{(1)}(\sigma_{+1} t_\pm) = \frac{1}{2A} \sum_\xi \kappa_\xi \sum_{\alpha,\alpha'} \langle i|DCT[I_\xi(\alpha, \alpha'), \hat{O}^\pm_{\pm 1}]|i\rangle + \sum_\alpha \kappa_{ls} \langle i|DCT[\boldsymbol{l}(\alpha) \cdot \boldsymbol{s}(\alpha), \hat{O}^\pm_{\pm 1}]|i\rangle, \tag{11}$$

$$2S^{(1)}(\sigma_0 t_\pm) = \frac{1}{2A} \sum_\xi \kappa_\xi \sum_{\alpha,\alpha'} \langle i|DCT[I_\xi(\alpha, \alpha'), \hat{O}^\pm_0]|i\rangle + \sum_\alpha \kappa_{ls} \langle i|DCT[\boldsymbol{l}(\alpha) \cdot \boldsymbol{s}(\alpha), \hat{O}^\pm_0]|i\rangle, \tag{12}$$

$$S^{(1)}(\boldsymbol{\sigma} t_\pm) = \sum_\nu S^{(1)}(\sigma_\nu t_\pm). \tag{13}$$

For the one-body part, we find that

$$\langle i|DCT[\boldsymbol{l}(\alpha)\cdot\boldsymbol{s}(\alpha),\hat{O}^{\pm}_{\pm1}]|i\rangle = -2\langle i|l_0(\alpha)s_0(\alpha)|i\rangle, \tag{14}$$

$$\langle i|DCT[\boldsymbol{l}(\alpha)\cdot\boldsymbol{s}(\alpha),\hat{O}^{\pm}_{0}]|i\rangle = -2\langle i|\boldsymbol{l}(\alpha)\cdot\boldsymbol{s}(\alpha)-l_0(\alpha)s_0(\alpha)|i\rangle. \tag{15}$$

To reduce the two-body part to a simpler form, we use two properties of the double commutators based on symmetry. First, the two-body isospin interaction part, $DCT[I_\tau,\hat{O}^{\pm}_{\nu}]$, vanishes because the initial ground state has the isospin of zero. Second, the sum of spin, isospin, and isospin-spin double commutators is canceled out [see Eq. (B2)]. Hence, the excitation energy due to the two-body interaction is zero when the coupling constants $\kappa_\sigma=\kappa_\tau=\kappa_{\sigma\tau}$. These properties are explained in Appendix B. Thus, the two-body parts of Eqs. (11) and (12) are simplified as

$$\frac{1}{2A}(\kappa_\sigma-\kappa_{\sigma\tau})\sum_{\alpha,\alpha'}\langle i|DCT[I_\sigma(\alpha,\alpha'),\hat{O}^{\pm}_{\nu}]|i\rangle. \tag{16}$$

For $\nu=\pm1$, we find [see Eq. (A18)] that

$$\begin{aligned} &DCT[I_\sigma(\alpha,\alpha'),\hat{O}^{\pm}_{\nu}]\\ &\quad=-\{\boldsymbol{\sigma}(\alpha)\cdot\boldsymbol{\sigma}(\alpha')+\sigma_0(\alpha)\sigma_0(\alpha')\}\\ &\qquad\times\{2-\boldsymbol{\tau}(\alpha)\cdot\boldsymbol{\tau}(\alpha')+\tau_0(\alpha)\tau_0(\alpha')\}\\ &\qquad+\frac{1}{2}\{\sigma_{\pm1}(\alpha)\sigma_{\mp1}(\alpha')-\sigma_{\mp1}(\alpha)\sigma_{\pm1}(\alpha')\}\\ &\qquad\times\{\tau_{\pm}(\alpha)\tau_{\mp}(\alpha')-\tau_{\mp}(\alpha)\tau_{\pm}(\alpha')\}. \end{aligned} \tag{17}$$

This equation also holds for the case of $\alpha=\alpha'$, and the isospin-dependent terms vanish for the $T=0$ initial state. Consequently, the two-body part is

$$4\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\langle i|\big(\boldsymbol{S}\cdot\boldsymbol{S}+S_0^2\big)|i\rangle, \tag{18}$$

where $\boldsymbol{S}$ is the sum of the spin operators for all the nucleons

$$\boldsymbol{S}=\sum_\alpha \boldsymbol{s}(\alpha)=\sum_\alpha \boldsymbol{\sigma}(\alpha)/2, \tag{19}$$

and S_0 is the projection onto the z direction. As a result,

$$\begin{aligned} S^{(1)}(\sigma_{-1}t_\pm)+S^{(1)}(\sigma_{+1}t_\pm) &=\langle i|DCT\big[H,\hat{O}^{\pm}_{\pm1}\big]|i\rangle\\ &=4\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\langle i|\boldsymbol{S}\cdot\boldsymbol{S}+S_0^2|i\rangle\\ &\quad-2\kappa_{l\sigma}\langle i|\sum_\alpha l_0(\alpha)s_0(\alpha)|i\rangle. \end{aligned} \tag{20}$$

Similarly, for $\nu=0$,

$$\begin{aligned} S^{(1)}(\sigma_0t_\pm) &=4\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\langle i|\boldsymbol{S}\cdot\boldsymbol{S}-S_0^2|i\rangle\\ &\quad-2\kappa_{l\sigma}\langle i|\sum_\alpha\{\boldsymbol{l}(\alpha)\cdot\boldsymbol{s}(\alpha)-l_0(\alpha)s_0(\alpha)\}|i\rangle. \end{aligned} \tag{21}$$

We also find that

$$\begin{aligned} S^{(1)}(\boldsymbol{\sigma}t_\pm)&=\sum_\nu S^{(1)}(\sigma_\nu t_\pm)8\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\langle i|\boldsymbol{S}\cdot\boldsymbol{S}|i\rangle\\ &\quad-2\kappa_{l\sigma}\langle i|\sum_\alpha \boldsymbol{l}(\alpha)\cdot\boldsymbol{s}(\alpha)|i\rangle. \end{aligned} \tag{22}$$

Now we evaluate the NEWSR, which is defined as

$$S^{(0)}(\sigma_\nu t_\pm)=\sum_f |\langle f|\hat{O}^{\pm}_{\nu}|i\rangle|^2. \tag{23}$$

According to Ref. [23], in the $\nu=\pm1$ channels, there is a relation that

$$S^{(0)}(\sigma_{-1}t_\pm)-S^{(0)}(\sigma_{+1}t_\pm)=2\langle i|S_0|i\rangle. \tag{24}$$

This relation corresponds to the so-called Ikeda's sum rule for GT transitions from $N>Z$ nuclei [6].

It is interesting to represent NEWSRs using spin operators, similar to the handling in Ref. [31]:

$$S^{(0)}(\sigma_{-1}t_\pm)+S^{(0)}(\sigma_{+1}t_\pm)=2\langle i|(\boldsymbol{S}^n-\boldsymbol{S}^p)^2-\big(S_0^n-S_0^p\big)^2|i\rangle, \tag{25}$$

$$S^{(0)}(\boldsymbol{\sigma}t_\pm)=2\langle i|(\boldsymbol{S}^n-\boldsymbol{S}^p)^2|i\rangle, \tag{26}$$

$$S^{(0)}(\sigma_0t_\pm)=2\langle i|\big(S_0^n-S_0^p\big)^2|i\rangle. \tag{27}$$

Here $\boldsymbol{S}^n$ and $\boldsymbol{S}^p$ are the total spin operators for neutrons and protons, respectively. In Eq. (19), the nucleon index α runs over only either neutrons or protons. Thus, NEWSRs reflect the spin and isospin structures of the initial states. Details of the derivation of these three equations are described in Appendix C.

B. Isoscalar $M1$ transition

For isoscalar $M1$ transitions, we consider the transition operator

$$\hat{O}_\nu=\sum_\alpha \sigma_\nu(\alpha). \tag{28}$$

The $M1$ transition operator commutes with the two-body operators of I_ξ. Therefore,

$$S^{(1)}(\sigma_{-1})+S^{(1)}(\sigma_{+1})=-4\kappa_{ls}\langle i|\sum_\alpha l_0(\alpha)s_0(\alpha)|i\rangle, \tag{29}$$

$$S^{(1)}(\sigma_0)=-4\kappa_{ls}\langle i|\sum_\alpha\{\boldsymbol{l}\cdot\boldsymbol{s}-l_0(\alpha)s_0(\alpha)\}|i\rangle, \tag{30}$$

$$S^{(1)}(\boldsymbol{\sigma})=-4\kappa_{ls}\langle i|\sum_\alpha \boldsymbol{l}(\alpha)\boldsymbol{s}(\alpha)|i\rangle. \tag{31}$$

The NEWSRs have properties of

$$S^{(0)}(\sigma_{-1})-S^{(0)}(\sigma_{+1})=4\langle i|S_0|i\rangle, \tag{32}$$

TABLE I. α and β coefficients.

Mode	α	β
Total	$\langle i\|\boldsymbol{S}^2\|i\rangle/\langle i\|(\boldsymbol{S}^n-\boldsymbol{S}^p)^2\|i\rangle$	$\langle i\|\boldsymbol{S}^2\|i\rangle/\langle i\|(\boldsymbol{S}^n-\boldsymbol{S}^p)^2\|i\rangle$
$\nu=0$	$\langle i\|S_0^2\|i\rangle/\langle i\|(S_0^n-S_0^p)^2\|i\rangle$	$\langle i\|\boldsymbol{S}^2-S_0^2\|i\rangle/2\langle i\|(S_0^n-S_0^p)^2\|i\rangle$
$\|\nu\|=1$	$\langle i\|\boldsymbol{S}^2-S_0^2\|i\rangle/\langle i\|(\boldsymbol{S}^n-\boldsymbol{S}^p)^2-(S_0^n-S_0^p)^2\|i\rangle$	$\langle i\|\boldsymbol{S}^2+S_0^2\|i\rangle/2\langle i\|(\boldsymbol{S}^n-\boldsymbol{S}^p)^2-(S_0^n-S_0^p)^2\|i\rangle$

$$S^{(0)}(\sigma_{-1})+S^{(0)}(\sigma_{+1})=4\langle i|\boldsymbol{S}^2-S_0^2|i\rangle, \tag{33}$$

$$S^{(0)}(\boldsymbol{\sigma})=4\langle i|\boldsymbol{S}^2|i\rangle, \tag{34}$$

$$S^{(0)}(\sigma_0)=4\langle i|S_0^2|i\rangle. \tag{35}$$

Unlike the case of the GT transition, the NEWSRs reflect only the spin structures of the initial states. Using Eqs. (32) and (33), it can be shown that

$$S^{(0)}(\sigma_{+1})=2\langle i|\boldsymbol{S}^2-\left(S_0^2+S_0\right)|i\rangle. \tag{36}$$

Next we consider a special case where the transitions in the $\Delta K=+1$ direction are completely blocked due to the excess of the spin-up liquid with respect to the spin-down liquid, as discussed in Ref. [23]. Using Eq. (36) gives

$$\langle i|\boldsymbol{S}^2|i\rangle=\langle i|\left(S_0^2+S_0\right)|i\rangle. \tag{37}$$

This means that the total spin is aligned with the z direction. Additionally,

$$\langle i|\boldsymbol{S}^2-S_0^2|i\rangle=\langle i|\left(\boldsymbol{S}^n-\boldsymbol{S}^p\right)^2-\left(S_0^n-S_0^p\right)^2|i\rangle \tag{38}$$

because

$$\begin{aligned}\langle i|\boldsymbol{S}^n\cdot\boldsymbol{S}^p-S_0^nS_0^p|i\rangle&=\langle i|S_{+1}^nS_{-1}^p|i\rangle\\&=0.\end{aligned} \tag{39}$$

C. Relation between the Gamow-Teller and isoscalar $M1$ transitions

Combining the GT and $M1$ EWSs can cancel the effect of the one-body spin-orbit part of the Hamiltonian as

$$S^{(1)}(\sigma_{-1}t_\pm)+S^{(1)}(\sigma_{+1}t_\pm)-\frac{1}{2}\left\{S^{(1)}(\sigma_{-1})+S^{(1)}(\sigma_{+1})\right\}=4\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\langle i|\boldsymbol{S}^2+S_0^2|i\rangle, \tag{40}$$

$$S^{(1)}(\sigma_0t_\pm)-\frac{1}{2}S^{(1)}(\sigma_0)=4\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\langle i|\boldsymbol{S}^2-S_0^2|i\rangle, \tag{41}$$

$$S^{(1)}(\boldsymbol{\sigma}t_\pm)-\frac{1}{2}S^{(1)}(\boldsymbol{\sigma})=8\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\langle i|\boldsymbol{S}^2|i\rangle. \tag{42}$$

The right-hand sides of the above equations can be written as combinations of the NEWSRs of the $M1$ transition. Thus, the $\kappa_{\sigma\tau}-\kappa_\sigma$ values can be derived from the sum-rule values.

III. AVERAGE EXCITATION ENERGIES OF GAMOW-TELLER AND $M1$ TRANSITIONS

We define the average excitation energies for the GT transition using the sum rules as

$$E_{\rm GT}^{\rm total}=S^{(1)}(\boldsymbol{\sigma}t^\pm)/S^{(0)}(\boldsymbol{\sigma}t^\pm), \tag{43}$$

$$E_{\rm GT}^{\nu=0}=S^{(1)}(\sigma_0t^\pm)/S^{(0)}(\sigma_0t^\pm), \tag{44}$$

$$E_{\rm GT}^{|\nu|=1}=\frac{S^{(1)}(\sigma_{-1}t^\pm)+S^{(1)}(\sigma_{+1}t^\pm)}{S^{(0)}(\sigma_{-1}t^\pm)+S^{(0)}(\sigma_{+1}t^\pm)}. \tag{45}$$

Similarly, the average excitation energies for the $M1$ transition, $E_{\rm M1}^{\rm total}$, $E_{\rm M1}^{\nu=0}$, and $E_{\rm M1}^{|\nu|=1}$ are defined.

The above-defined GT and $M1$ average energies are related as

$$E_{\rm GT}^{\rm mode}-\alpha E_{\rm M1}^{\rm mode}=4\frac{\kappa_{\sigma\tau}-\kappa_\sigma}{A}\beta, \tag{46}$$

where the coefficients α and β depend on the mode represented by the superscript "mode" (i.e., total, $\nu=0$, or $|\nu|=1$) and are summarized in Table I.

In the special case where the σ_{+1} transitions are fully blocked, for the $|\nu|=1$ channel,

$$\alpha=1, \tag{47}$$

$$\beta=\frac{\langle i|2S_0^2+S_0|i\rangle}{\langle i|2S_0|i\rangle}, \tag{48}$$

using Eqs. (37) and (38). If S_0 is a good quantum number of the Hamiltonian and large,

$$\beta\sim\langle i|S_0|i\rangle. \tag{49}$$

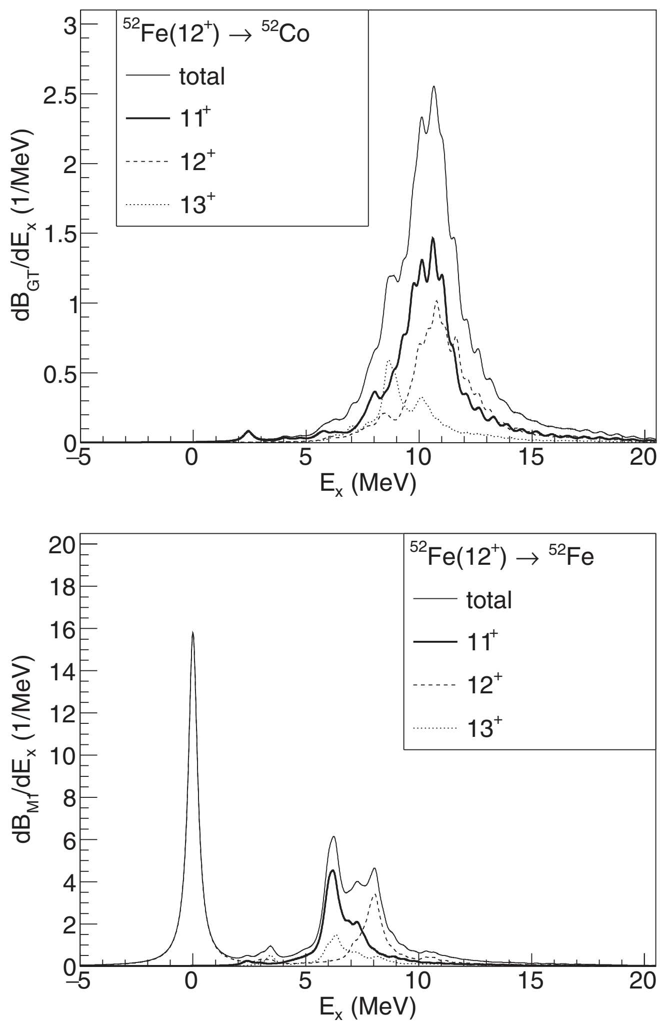

FIG. 1. (top) Gamow-Teller transition strength distributions from the ^{52}Fe(12^+) initial state calculated by the shell model in the full pf-shell-model space with the GXPF1J [32] effective interaction. Transition strengths are averaged by a Lorentzian weighting function with a width of 0.5 MeV. The excitation energy is defined from the initial state. The quenching factor is not adapted. (bottom) Same as the top but for the isoscalar $M1$ transitions from the ^{52}Fe(12^+) state.

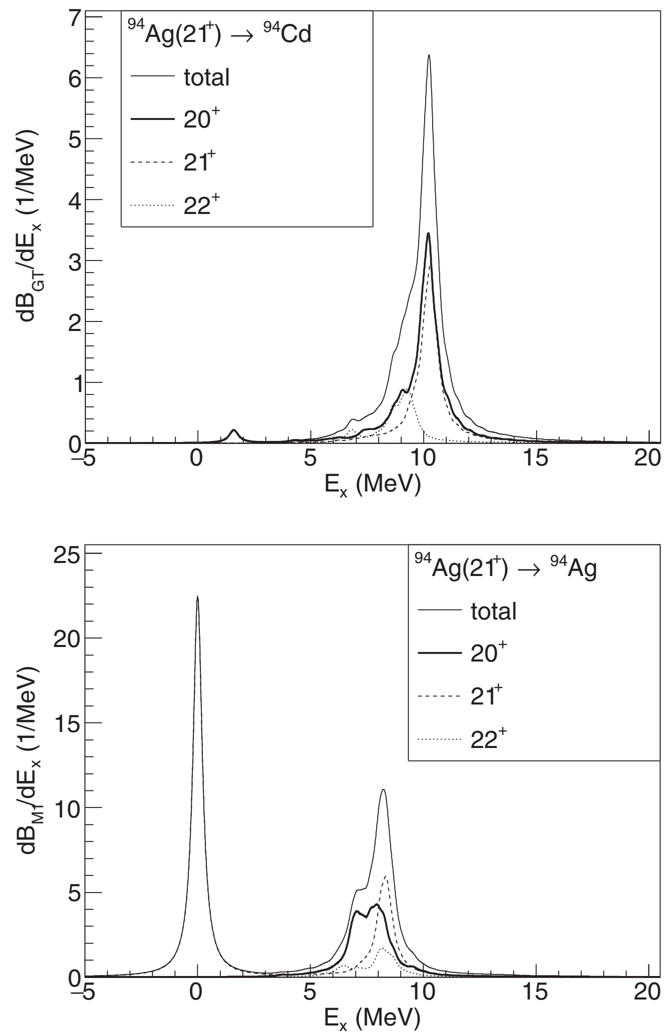

FIG. 2. Same as Fig. 1 but for the transitions from the ^{94}Ag(21^+) initial state. In the shell-model calculation, the $(2p_{1/2}, 1g_{9/2}, 1g_{7/2}, 2d_{5/2})$ shell-model space and a modified PIGD5G3 [33] interaction are employed.

Then the average energy relation is rewritten as

$$E_{\mathrm{GT}}^{\nu=-1} - E_{\mathrm{M1}}^{\nu=-1} = 4\frac{\kappa_{\sigma\tau} - \kappa_{\sigma}}{A}\langle i|S_0|i\rangle. \tag{50}$$

This relation is analogous to the average energy relation derived in Ref. [17] for $N > Z$ 0^+ nuclei.

IV. GAMOW-TELLER AND ISOSCALAR $M1$ TRANSITIONS FROM ^{52}Fe AND ^{94}Ag

Figures 1 and 2 show the strength distributions for the GT and $M1$ transitions from the ^{52}Fe(12^+) and ^{94}Ag(21^+) states, respectively. The calculated transition strengths are averaged by a Lorentzian weighting function with a width of 0.5 MeV and the excitation energy is defined from the HSI states.

As discussed in Ref. [23], the GT transition-strength distributions exhibit narrow bumps around 10 MeV. Approximately 70% of the total strengths are concentrated in these bumps, which means the collectivity is strong. The thick solid, dashed, and dotted curves are components corresponding to transitions from the initial state with a total spin of I to the final states with $I' = I - 1, I, I + 1$, respectively.

The $M1$ transition-strength distributions have bumps around 7 MeV. The large peaks at 0 MeV correspond to the elastic channel (i.e., the transitions between the HSI states in the $I' = I$ channel). The energy split between 0 MeV and the bumps around 7 MeV correspond to the spin-orbit splittings. The strengths are fragmented and the collectivity behind the $M1$ transitions is weak.

In the shell-model calculations, normal GT and $M1$ transition operators $\boldsymbol{\sigma} t_\pm$ and $\boldsymbol{\sigma}$ are employed. The transition strengths are given for I', $B_{I'}$. On the other hand, the transition operators used in our sum-rule approach are $\sigma_\nu t_\pm$

TABLE II. Sum-rule values for the GT and $M1$ transitions from the ^{52}Fe(12^+) HSI state.

		Total	$I' = I - 1$	$I' = I$	$I' = I + 1$	$\Delta K = -1$	$\Delta K = 0$	$\Delta K = +1$
GT	S_0	9.024	4.470	3.112	1.442	4.859	2.966	1.199
	S_1 (MeV)	95.740	46.708	34.864	14.169	50.770	33.538	11.433
M1	S_0	30.563	8.539	18.545	3.479	9.282	19.317	1.964
	S_1 (MeV)	136.493	58.718	52.393	25.382	63.824	51.441	21.228

and σ_ν. The transition strengths for each $\Delta K = \nu$, $B_{\Delta K}$, are derived as

$$B_{\Delta K=-1} = \frac{2I+1}{2I-1} B_{I'=I-1}, \tag{51}$$

$$B_{\Delta K=0} = \frac{I+1}{I} B_{I'=I} - \frac{2I+1}{I(2I-1)} B_{I'=I-1}, \tag{52}$$

$$B_{\Delta K=+1} = B_{I'=I+1} - \frac{1}{I} B_{I'=I} + \frac{1}{I(2I-1)} B_{I'=I-1}. \tag{53}$$

Tables II and III summarize the sum-rule values obtained.

Among $\Delta K = -1$, 0, and 1, the $\Delta K = +1$ component has the smallest amount of strength because transitions in the σ_{+1} direction are blocked, as discussed in Ref. [23]. However, the blocking effect is not perfect and $\Delta K = +1$ has a finite amount. This is because the total spin operator is not a good quantum number of the Hamiltonian and the relation $\langle S^2 \rangle = \langle S_0^2 + S_0 \rangle$ cannot be realized in Eq. (36).

Using the properties of NEWSRs [see Eqs. (24)–(27) and (32)–(35)], we derive the expectation values of the spin operators, which are summarized in Table IV. Here, the expectation values for the spin operator S_0 are obtained from the GT and $M1$ transition strengths independently, but have the same value. This means that the projection of the strengths onto ΔK is exact.

Table V summarizes the α and β coefficients with the average excitation energies. Using these values and Eq. (46), several $\kappa_{\sigma\tau} - \kappa_\sigma$ values are deduced and presented in Table V. In principle, all the modes of $\nu = \pm 1$ and 0 should give the same $\kappa_{\sigma\tau} - \kappa_\sigma$ value and, if the nuclear matter property is the same, the value should be similar between the two nuclei. However, the $\nu = 0$ mode gives relatively larger values than the $|\nu| = 1$ mode. Moreover, the results for the $\nu = 0$ mode differ drastically between the ^{52}Fe and ^{94}Ag cases. This is because this mode is insensitive to $\kappa_{\sigma\tau} - \kappa_\sigma$. In Eq. (46), α is large but β is small. Hence, $\kappa_{\sigma\tau} - \kappa_\sigma$ is related to the $M1$ average energy rather than the GT energy, although the $M1$ transition itself does not reflect the strengths of these spin-isospin interactions. Thus, the $\kappa_{\sigma\tau} - \kappa_\sigma$ values given from the $\nu = 0$ mode have large uncertainties. On the other hand, the results obtained from the $|\nu| = 1$ mode agree well between the two nuclei and the obtained values are 20.5 and 20.4 for ^{52}Fe and ^{94}Ag, respectively. The results for the "total" mode are close to those for the $|\nu| = 1$ mode but are less reliable, due to the mixing of the $\nu = 0$ mode. Consequently, our recommended values are the $|\nu| = 1$ results (i.e., $\kappa_{\sigma\tau} - \kappa_\sigma = 20.4 - 20.5$).

V. DISCUSSIONS

Next, we deduce a κ_σ value by using the $\kappa_{\sigma\tau} - \kappa_\sigma$ value of 20.4 – 20.5 with a $\kappa_{\sigma\tau}$ value taken from another work. The $\kappa_{\sigma\tau}$ values have been reported in several works [6,17–21]. All the values are around 20 MeV. In any cases, the present $\kappa_{\sigma\tau} - \kappa_\sigma$ values are slightly smaller than $\kappa_{\sigma\tau}$ by a few MeV. This means that κ_σ is a few 10% smaller than $\kappa_{\sigma\tau}$ and has the same sign or a very small negative value. Taking the $\kappa_{\sigma\tau}$ value of 23 MeV from Ref. [6], we estimate κ_σ as 2.5 MeV.

Additionally, $\kappa_{\sigma\tau} - \kappa_\sigma$ can be derived from the isospin doublets in ^{12}C (i.e., $T = 0$ and $T = 1$ 1^+ states at 12.7 and 15.1 MeV, respectively). Employing the same Hamiltonian and the Tamm-Dancoff approximation, one can show that

$$E(T=1, 1^+) - E(T=0, 1^+) = 2\left(\frac{4}{3}\right)^2 \frac{\kappa_{\sigma\tau} - \kappa_\sigma}{A}. \tag{54}$$

Here, we consider the excitations from the ground state to the two levels occurring within only the $1p$-$1h$ configuration and the p shell. The fraction of 4/3 corresponds to the transition matrix elements. The obtained $\kappa_{\sigma\tau} - \kappa_\sigma$ value is 8 MeV. This value gives a lower limit of $\kappa_{\sigma\tau} - \kappa_\sigma$. In the present approximation, the matrix element is overestimated because the strength fragmentation is not considered. Therefore, the $\kappa_{\sigma\tau} - \kappa_\sigma$ value, 20.4 – 20.5, is consistent with the known energy levels of the ^{12}C $T = 0$ and $T = 1$ 1^+ states.

Microscopically, the spin-isospin interactions represented by κs originate from the short-range parts of the NN interaction in a nuclear medium. In nuclear matter, these short-range parts control the onset of pion condensation [9]. κ_σ can be translated into the so-called Landau-Migdal parameter g_0

TABLE III. Sum-rule values for the GT and $M1$ transitions from the ^{94}Ag(21^+) HSI state.

		Total	$I' = I - 1$	$I' = I$	$I' = I + 1$	$\Delta K = -1$	$\Delta K = 0$	$\Delta K = +1$
GT	S_0	10.924	5.435	3.885	1.604	5.700	3.799	1.425
	S_1 (MeV)	107.444	53.006	39.901	14.543	55.586	39.154	12.704
M1	S_0	38.932	10.303	25.186	3.443	10.806	25.870	2.256
	S_1 (MeV)	171.419	79.504	63.856	28.059	83.382	62.926	25.111

TABLE IV. Expectation values of spin operators for the ^{52}Fe(12$^+$) and ^{94}Ag(21$^+$) HSI state.

Nuclide	$\langle \boldsymbol{S}^2 \rangle$	$\langle S_0^2 \rangle$	$\langle S_0 \rangle_{\mathrm{M1}}$	$\langle (\boldsymbol{S}^n - \boldsymbol{S}^p)^2 \rangle$	$\langle (S_0^n - S_0^p)^2 \rangle$	$\langle S_0 \rangle_{\mathrm{GT}}$
^{52}Fe(12$^+$)	7.641	4.829	1.829	4.512	1.483	1.830
^{94}Ag(21$^+$)	9.733	6.468	2.137	5.462	1.899	2.137

[8,9,26] using the following relation:

$$\kappa_\sigma = g_0 \left(\frac{f_\pi^2}{m_\pi^2} \rho_0 \gamma \right), \qquad (55)$$

where ρ_0 is the nuclear matter density ($\rho_0 = 0.17$ fm^{-3}), $f_\pi^2/m_\pi^2 = 392$ MeV fm^3, and γ is the attenuation factor due to the surface effect. Here the γ value is set to 0.5 [18]. The present value of $\kappa_\sigma = 2.5$ MeV gives $g_0 = 0.075$. We derived this relation according to Ref. [18] [see Eqs. (29), (30), (43), and (44) therein]. We note that the definitions of κs in Ref. [18] differ from ours by a factor of mass number A. The present value is close to zero and much smaller than the prediction based on the framework of the Brueckner theory, which includes three-body forces, ≈0.41 [34]. Here we divided their value of $G_0 \approx 0.82$ by a factor of two according to the difference of the definition of the Landau-Migdal parameters. Our value of g_0 is consistent with that in Ref. [28], $g = 0.05 \pm 0.10$. Here we take into account the difference of the normalization factor C_0 [8]: Our normalization factor is $\frac{f_\pi^2}{m_\pi^2} = 392$ MeV fm^3 as seen in Eq. (55), while their normalization factor is $\frac{\pi^2}{M p_F} = 300$ MeV fm^3. In their definition, our g_0 value corresponds to $g = 0.098$.

Next we considered uncertainties in our results. One is that the Hamiltonian, which was assumed to be the starting point of the sum-rule approach, is very simple. In studies of $N > Z$ 0^+ nuclei, the energy difference between GT and Fermi transitions has large fluctuations. This is because the simple form of the Hamiltonian does not consider the nuclear structures of individual nuclei. Instead, the average of individual nuclei follow the systematics based on the sum-rule approach [21]. Similarly, for HSI states, such fluctuations may be averaged through systematics.

The other uncertainty is the shell-model interactions. These interactions, GXPF1J and PIGD5G3, fit experimental data. In both cases, the data on the $\sigma\sigma$ interaction are scarce, although the energy levels of the high-spin isomers are reproduced [23]. Therefore, it is also possible that the spin residual interactions in the calculations differ from reality. Systematic data on GTR in high-spin isomers are required to clarify these uncertainties.

VI. SUMMARY

We derive a new energy-weighted sum rule for the GT and isoscalar $M1$ transitions from high-spin isomers in $N = Z$ nuclei. The average energy derived from the sum rule is compared with the shell-model calculation performed with modern effective interactions in a large model space. From the comparison, $\kappa_{\sigma\tau} - \kappa_\sigma$ is derived as 20.4–20.5 MeV. Assuming a $\kappa_{\sigma\tau}$ value of 23 MeV, the strength of the spin residual interaction κ_σ is deduced as 2.5 MeV. The present approach can evaluate the short-range part of the spin residual interaction, which is important to describe the onset of the pion condensation in nuclear matter.

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APPENDIX A: DOUBLE COMMUTATORS

In this section, the double commutator relations used in the text are derived. For this purpose, instead of the present transition operator of Eq. (1), we introduce operators defined as

$$\hat{O}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \sum_\alpha \boldsymbol{\theta} \cdot \boldsymbol{\sigma}(\alpha) \boldsymbol{\phi} \cdot \boldsymbol{\tau}(\alpha), \qquad (A1)$$

where θ and ϕ are three-dimensional vectors. We define

$$DCT_\xi(\boldsymbol{\theta}', \boldsymbol{\phi}', \boldsymbol{\theta}, \boldsymbol{\phi}) = [\hat{O}^\dagger(\boldsymbol{\theta}', \boldsymbol{\phi}'), [\hat{I}_\xi, \hat{O}(\boldsymbol{\theta}, \boldsymbol{\phi})]]. \qquad (A2)$$

Then

$$DCT[\hat{I}_\xi, \hat{O}_\nu^\pm] = \nabla_{\boldsymbol{\theta}'}^{-\nu} \nabla_{\boldsymbol{\theta}}^{\nu} \nabla_{\boldsymbol{\phi}'}^{\mp} \nabla_{\boldsymbol{\phi}}^{\pm} DCT_\xi(\boldsymbol{\theta}', \boldsymbol{\phi}', \boldsymbol{\theta}, \boldsymbol{\phi}). \qquad (A3)$$

Here $\nabla_{\boldsymbol{\theta}}^\nu$s are the derivative operators with respect to $\boldsymbol{\theta}$ and are defined as

$$\nabla_{\boldsymbol{\theta}}^{\pm 1} = \frac{1}{\sqrt{2}} \left(\frac{d}{d\theta_x} \pm i \frac{d}{d\theta_y} \right), \qquad (A4)$$

TABLE V. Average excitation energies, α and β coefficients, and strengths of spin-isospin interactions.

Nuclide	Mode	$E_{\mathrm{GT}}^{\mathrm{mode}}$ (MeV)	$E_{\mathrm{M1}}^{\mathrm{mode}}$ (MeV)	α	β	$\kappa_{\sigma\tau} - \kappa_\sigma$ (MeV)
^{52}Fe(12$^+$)	Total	10.609	4.466	1.693	1.693	23.4
	$\nu = 0$	11.306	2.663	3.256	0.948	36.1
	$\lvert\nu\rvert = 1$	10.268	7.563	0.928	2.058	20.5
^{94}Ag(21$^+$)	Total	9.836	4.403	1.782	1.782	26.2
	$\nu = 0$	10.306	2.432	3.405	0.860	55.3
	$\lvert\nu\rvert = 1$	9.584	8.306	0.917	2.274	20.4

TABLE VI. Summary of the derivative forms of the S and T functions for the spin part. First column indicates derivative operators $\hat{D}$. Here, the indexes of nucleons, α and α', are omitted. In this table, the first operator has an index of α and the second α'. For the isospin channel, the spin operator σ can be replaced with τ and the variables θ and θ' with ϕ and ϕ'.

$\hat{D}$	$\hat{D}S^{\rm even}_{\sigma\sigma}$	$\hat{D}T^{\rm even}_{\sigma\sigma}$	$\hat{D}S^{\rm odd}_{\sigma\sigma}$
$\nabla^{\pm 1}_{\boldsymbol{\theta}'}\nabla^{\mp 1}_{\boldsymbol{\theta}}$	$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}+\sigma_0\sigma_0}{2}$	$\frac{2-\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}+\sigma_0\cdot\sigma_0}{2}$	$\frac{\sigma_{\pm1}\sigma_{\mp1}-\sigma_{\mp1}\sigma_{\pm1}}{2}$
$\nabla^0_{\boldsymbol{\theta}'}\nabla^0_{\boldsymbol{\theta}}$	$\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}-\sigma_0\sigma_0$	$1-\sigma_0\cdot\sigma_0$	0
$\boldsymbol{\nabla}_{\boldsymbol{\theta}'}\boldsymbol{\nabla}_{\boldsymbol{\theta}}$	$2\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$	$3-\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$	0

and

$$\nabla^0_{\boldsymbol{\theta}} = \frac{d}{d\theta_0}, \tag{A5}$$

while $\nabla^{\pm}_{\boldsymbol{\phi}}$s are derivative operators with respect to $\boldsymbol{\phi}$ and are defined as

$$\nabla^{\pm}_{\boldsymbol{\phi}} = \frac{1}{2}\left(\frac{d}{d\phi_x} \pm i\frac{d}{d\phi_y}\right), \tag{A6}$$

and

$$\nabla^z_{\boldsymbol{\phi}} = \frac{d}{d\phi_z}. \tag{A7}$$

First we derive $DCT_{\sigma\tau}$. We find that

$$\begin{aligned} DCT_{\sigma\tau}(\boldsymbol{\theta}',\boldsymbol{\phi}',\boldsymbol{\theta},\boldsymbol{\phi}) &= 8S^{\rm even}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta})T^{\rm even}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &+ 8S^{\rm even}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi})T^{\rm even}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) \\ &+ 16S^{\rm odd}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta})S^{\rm odd}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &- 8S^{\rm odd}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta})U^{\rm odd}_{\tau}(\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &- 8S^{\rm odd}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi})U^{\rm odd}_{\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) \\ &- 8U^{\rm even}_{\sigma+\tau}(\boldsymbol{\theta}',\boldsymbol{\theta},\boldsymbol{\phi}',\boldsymbol{\phi}). \end{aligned} \tag{A8}$$

Except for the last term, each term is the product of a function of $(\boldsymbol{\theta}',\boldsymbol{\theta})$ and that of $(\boldsymbol{\phi}',\boldsymbol{\phi})$. Each function includes either spin or isospin operators labeled with the subscripts of $\sigma\sigma$ or $\tau\tau$. Due to the separation of the variables and operators, once we have the derivative of each function, $DCT[\hat{I}_{\sigma\sigma},\hat{O}^{\pm}_{\nu}]$ can be easily calculated. The superscripts of the functions indicate whether the functions are even or odd with respect to the exchange of two nucleons (i.e., the exchange between α and α'). Each product is even.

The definitions of the functions are as follows: $S^{\rm even}_{\sigma\sigma}$, $T^{\rm even}_{\sigma\sigma}$, and $S^{\rm odd}_{\sigma\sigma}$ are defined using products of $\boldsymbol{\sigma}(\alpha)$ and $\boldsymbol{\sigma}(\alpha')$ matrices as

$$S^{\rm even}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) = \boldsymbol{\theta}'\cdot\boldsymbol{\theta}\boldsymbol{\sigma}(\alpha)\cdot\boldsymbol{\sigma}(\alpha') - \frac{\boldsymbol{\theta}'\cdot\boldsymbol{\sigma}(\alpha)\boldsymbol{\theta}\cdot\boldsymbol{\sigma}(\alpha')+\boldsymbol{\theta}\cdot\boldsymbol{\sigma}(\alpha)\boldsymbol{\theta}'\cdot\boldsymbol{\sigma}(\alpha')}{2}, \tag{A9}$$

$$T^{\rm even}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) = \boldsymbol{\theta}'\cdot\boldsymbol{\theta} - \frac{\boldsymbol{\theta}'\cdot\boldsymbol{\sigma}(\alpha)\boldsymbol{\theta}\cdot\boldsymbol{\sigma}(\alpha')+\boldsymbol{\theta}\cdot\boldsymbol{\sigma}(\alpha)\boldsymbol{\theta}'\cdot\boldsymbol{\sigma}(\alpha')}{2}, \tag{A10}$$

$$S^{\rm odd}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) = \frac{\boldsymbol{\theta}'\cdot\boldsymbol{\sigma}(\alpha)\boldsymbol{\theta}\cdot\boldsymbol{\sigma}(\alpha')-\boldsymbol{\theta}\cdot\boldsymbol{\sigma}(\alpha)\boldsymbol{\theta}'\cdot\boldsymbol{\sigma}(\alpha')}{2}. \tag{A11}$$

$S^{\rm even}_{\tau\tau}$, $T^{\rm even}_{\tau\tau}$, and $S^{\rm odd}_{\tau\tau}$ have the same forms as $S^{\rm even}_{\sigma\sigma}$, $T^{\rm even}_{\sigma\sigma}$, and $S^{\rm odd}_{\sigma\sigma}$, respectively, but the variables of $(\boldsymbol{\theta}',\boldsymbol{\theta})$ and the operators $(\boldsymbol{\sigma}(\alpha),\ \boldsymbol{\sigma}(\alpha'))$ are replaced with $(\boldsymbol{\phi}',\boldsymbol{\phi})$ and $(\boldsymbol{\tau}(\alpha),\ \boldsymbol{\tau}(\alpha'))$.

$U^{\rm odd}_{\sigma}$, $U^{\rm odd}_{\tau}$, and $U^{\rm even}_{\sigma+\tau}$ are defined as

$$\begin{aligned} U^{\rm odd}_{\sigma} &= i(\boldsymbol{\theta}'\times\boldsymbol{\theta})\cdot(\boldsymbol{\sigma}(\alpha)-\boldsymbol{\sigma}(\alpha')), &\text{(A12)}\\ U^{\rm odd}_{\tau} &= i(\boldsymbol{\phi}'\times\boldsymbol{\phi})\cdot(\boldsymbol{\tau}(\alpha)-\boldsymbol{\tau}(\alpha')), &\text{(A13)}\\ U^{\rm even}_{\sigma+\tau} &= (\boldsymbol{\theta}'\times\boldsymbol{\theta})\cdot\boldsymbol{\sigma}(\alpha)(\boldsymbol{\phi}'\times\boldsymbol{\phi})\cdot\boldsymbol{\tau}(\alpha') \\ &\quad + (\boldsymbol{\phi}'\times\boldsymbol{\phi})\cdot\boldsymbol{\tau}(\alpha)(\boldsymbol{\theta}'\times\boldsymbol{\theta})\cdot\boldsymbol{\sigma}(\alpha'). &\text{(A14)} \end{aligned}$$

These include only first-order terms of spin-isospin matrices.

Using the same functions, DCT_{σ} and DCT_{τ} are derived as

$$\begin{aligned} DCT_{\sigma}(\boldsymbol{\theta}',\boldsymbol{\phi}',\boldsymbol{\theta},\boldsymbol{\phi}) &= -8S^{\rm even}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta})T^{\rm even}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &\quad -8S^{\rm odd}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta})S^{\rm odd}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &\quad +8S^{\rm odd}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta})U^{\rm odd}_{\tau}(\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &\quad +4U^{\rm even}_{\sigma+\tau}(\boldsymbol{\theta}',\boldsymbol{\theta},\boldsymbol{\phi}',\boldsymbol{\phi}), \end{aligned} \tag{A15}$$

$$\begin{aligned} DCT_{\tau}(\boldsymbol{\theta}',\boldsymbol{\phi}',\boldsymbol{\theta},\boldsymbol{\phi}) &= -8S^{\rm even}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi})T^{\rm even}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) \\ &\quad - 8S^{\rm odd}_{\sigma\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta})S^{\rm odd}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &\quad + 8S^{\rm odd}_{\tau\tau}(\boldsymbol{\phi}',\boldsymbol{\phi})U^{\rm odd}_{\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) \\ &\quad + 4U^{\rm even}_{\sigma+\tau}(\boldsymbol{\theta}',\boldsymbol{\theta},\boldsymbol{\phi}',\boldsymbol{\phi}). \end{aligned} \tag{A16}$$

Next we derive $DCT[\hat{I}_i,\hat{O}^{\pm}_{\nu}]$ using Eq. (A3). It is clear that

$$\begin{aligned} \nabla^{-\nu}_{\boldsymbol{\theta}'}\nabla^{\nu}_{\boldsymbol{\theta}}U^{\rm odd}_{\sigma}(\boldsymbol{\theta}',\boldsymbol{\theta}) &= \nabla^{\mp}_{\boldsymbol{\phi}'}\nabla^{\pm}_{\boldsymbol{\phi}}U^{\rm odd}_{\tau}(\boldsymbol{\tau}',\boldsymbol{\tau}) \\ &= \nabla^{-\nu}_{\boldsymbol{\theta}'}\nabla^{\nu}_{\boldsymbol{\theta}}\nabla^{\mp}_{\boldsymbol{\phi}'}\nabla^{\pm}_{\boldsymbol{\phi}}U^{\rm even}_{\sigma+\tau}(\boldsymbol{\theta}',\boldsymbol{\theta},\boldsymbol{\phi}',\boldsymbol{\phi}) \\ &= 0. \end{aligned} \tag{A17}$$

This is because the cross products, $\boldsymbol{\theta}'\times\boldsymbol{\theta}$ and $\boldsymbol{\phi}'\times\boldsymbol{\phi}$ in these functions vanish with the derivatives. As summarized in Tables VI and VII, only the derivatives of $S^{\rm even}_{\sigma\sigma}$, $T^{\rm even}_{\sigma\sigma}$, and $S^{\rm odd}_{\sigma\sigma}$ and its isospin counterparts remain.

TABLE VII. Same as Table VI but for the isospin part.

$\hat{D}$	$\hat{D}S^{\rm even}_{\tau\tau}$	$\hat{D}T^{\rm even}_{\tau\tau}$	$\hat{D}S^{\rm odd}_{\tau\tau}$
$\nabla^{\pm}_{\boldsymbol{\phi}'}\nabla^{\mp}_{\boldsymbol{\phi}}$	$\frac{\boldsymbol{\tau}\cdot\boldsymbol{\tau}+\tau_0\tau_0}{4}$	$\frac{2-\boldsymbol{\tau}\cdot\boldsymbol{\tau}+\tau_0\cdot\tau_0}{4}$	$\frac{\tau_{\pm}\tau_{\mp}-\tau_{\mp}\tau_{\pm}}{8}$
$\nabla^0_{\boldsymbol{\phi}'}\nabla^0_{\boldsymbol{\phi}}$	$\frac{\boldsymbol{\tau}\cdot\boldsymbol{\tau}-\tau_0\tau_0}{4}$	$\frac{1-\tau_0\cdot\tau_0}{4}$	0
$\boldsymbol{\nabla}_{\boldsymbol{\phi}'}\boldsymbol{\nabla}_{\boldsymbol{\phi}}$	$\frac{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}{2}$	$\frac{3-\boldsymbol{\tau}\cdot\boldsymbol{\tau}}{4}$	0

For the operator $\hat{O}_\nu^\pm$, using Eq. (A3) and Tables VI and VII gives

$$\begin{aligned} DCT[I_\sigma(\alpha,\alpha'),\hat{O}_\nu^\pm] &= -8\nabla_{\theta'}^{-\nu}\nabla_{\theta}^{\nu}S_{\sigma\sigma}^{\rm even}\nabla_{\phi'}^{\mp}\nabla_{\phi}^{\pm}T_{\tau\tau}^{\rm even} \\ &\quad -8\nabla_{\theta'}^{-\nu}\nabla_{\theta}^{\nu}S_{\sigma\sigma}^{\rm odd}\nabla_{\phi'}^{\mp}\nabla_{\phi}^{\pm}S_{\tau\tau}^{\rm odd} \\ &= -\{\boldsymbol{\sigma}(\alpha)\cdot\boldsymbol{\sigma}(\alpha')+\sigma_0(\alpha)\sigma_0(\alpha')\} \\ &\quad \times\{2-\boldsymbol{\tau}(\alpha)\cdot\boldsymbol{\tau}(\alpha')+\tau_0(\alpha)\tau_0(\alpha')\} \\ &\quad -\frac{1}{2}\{\sigma_{\pm1}(\alpha)\sigma_{\mp1}(\alpha')-\sigma_{\mp1}(\alpha)\sigma_{\pm1}(\alpha')\} \\ &\quad \times\{\tau_\pm(\alpha)\tau_\mp(\alpha')-\tau_\mp(\alpha)\tau_\pm(\alpha')\}. \end{aligned} \tag{A18}$$

APPENDIX B: PROPERTIES OF DOUBLE COMMUTATORS

In this section, we show three properties of the double commutators used in the text.

First, from Eqs. (A8), (A15), and (A16),

$$DCT_\sigma + DCT_\tau + DCT_{\sigma\tau} = 0. \tag{B1}$$

Hence,

$$\sum_\xi DCT[\hat{I}_\xi,\hat{O}_\nu^\pm] = 0. \tag{B2}$$

Second,

$$\begin{aligned} &-16\nabla_{\theta'}^{-1}\nabla_{\theta}^{+1}\nabla_{\phi'}^{\mp}\nabla_{\phi}^{\pm}S_{\sigma\sigma}^{\rm odd}S_{\tau\tau}^{\rm odd} \\ &\quad = DCT[\hat{I}_\sigma,\hat{O}_{+1}^\pm] - DCT[\hat{I}_\sigma,\hat{O}_{-1}^\pm], \end{aligned} \tag{B3}$$

and

$$\begin{aligned} &-16\nabla_{\theta'}^{-\nu}\nabla_{\theta}^{\nu}\nabla_{\phi'}^{-}\nabla_{\phi}^{+}S_{\sigma\sigma}^{\rm odd}S_{\tau\tau}^{\rm odd} \\ &\quad = DCT[\hat{I}_\tau,\hat{O}_\nu^+] - DCT[\hat{I}_\tau,\hat{O}_\nu^-]. \end{aligned} \tag{B4}$$

Due to spin and isospin symmetries, the expectation values of the right-hand sides of Eqs. (B3) and (B4) should equal zero for states with a total spin of zero and those with a total isospin of zero. Therefore, these odd terms do not contribute to the case where either the total spin or the total isospin of the initial state is zero. In other words, they have to be considered in a system with both isospin and spin excesses such as neutron-rich high-spin isomeric states.

Last, one can show that

$$\langle i|DCT_\sigma|i\rangle = 0 \tag{B5}$$

and

$$\langle i|DCT_\tau|i\rangle = 0 \tag{B6}$$

for an initial state with a total spin of zero and that with a total isospin of zero, respectively.

APPENDIX C: EXPRESSION OF GAMOW-TELLER NON-ENERGY-WEIGHTED SUM RULE WITH TOTAL SPIN OPERATORS

In this section, we derive Eqs. (25)–(27). For simplicity, we show Eq. (27), from which the other two equations can easily be derived by taking into account the contributions from the x and y directions in the same manner as z (i.e., σ_0).

First we evaluate the left-hand side of Eq. (27) as

$$\begin{aligned} S^{(0)}(\sigma_0 t_\mp) &= \langle i|\sum_\alpha t_\pm(\alpha)t_\mp(\alpha)|i\rangle \\ &\quad + \langle i|\sum_{\alpha\neq\beta}\sigma_0(\alpha)t_\pm(\alpha)\sigma_0(\beta)t_\mp(\beta)|i\rangle. \end{aligned} \tag{C1}$$

Here the first term on the left-hand side is the transition amplitude where the transition operator acts on the same nucleon (i.e., diagonal part). In this term, the square of the σ_0 operator is replaced with unity. The second term corresponds to the nondiagonal part where two different nucleons are involved.

Here, as the initial state $|i\rangle$, we consider a Slater determinant

$$|i\rangle = A|abcd\ldots\rangle, \tag{C2}$$

where $|a\rangle$, $|b\rangle$, etc. are single-particle-orbit wave functions and A is the antisymmetrization operator.

The diagonal part is evaluated as

$$\sum_a \langle a|\left(\frac{1}{2}\pm t_z\right)|a\rangle, \tag{C3}$$

where the left-hand side is evaluated as N and Z for the $\sigma_0 t_-$ and $\sigma_0 t_+$ operators, respectively.

The nondiagonal part is evaluated as

$$\begin{aligned} &\sum_{a\neq b}\langle a|\sigma_0 t_\pm|a\rangle\langle b|\sigma_0 t_\mp|b\rangle - \sum_{a\neq b}\langle a|\sigma_0 t_\pm|b\rangle\langle b|\sigma_0 t_\mp|a\rangle \\ &\quad = -\sum_{\substack{a\,\in\,p(n)\\ b\,\in\,n(p)}}\langle a|\sigma_0 t_\pm|b\rangle\langle b|\sigma_0 t_\mp|a\rangle \\ &\quad = -\sum_{j_\nu,j'_\pi}\langle j_\nu|\sigma_0|j'_\pi\rangle\langle j'_\pi|\sigma_0|j_\nu\rangle. \end{aligned} \tag{C4}$$

Here, the first and the other terms in the first line are due to the antisymmetrization of the wave functions, i.e., the direct and exchange terms, respectively. Because of the isospin lower and upper operators, the direct term and the exchange term for nucleon pairs with the same isospin direction (i.e., proton-proton and neutron-neutron pairs) vanish, as shown in the second line, where the summation runs over all the proton (neutron) wave functions for the single-particle orbit a in the Slater determinant and over all the neutron (proton) wave functions for b, in the case of the $\sigma_0 t_-$ ($\sigma_0 t_+$) operator. The last line is same as the second line but is expressed with the single-particle-orbit wave function for neutrons and protons, $|j_\nu\rangle$ and $|j_\pi\rangle$, respectively.

Combining the above results, we find that

$$S^{(0)} = \begin{Bmatrix} N \\ Z \end{Bmatrix} - \sum_{j_\nu, j'_\pi} \langle j_\nu|\sigma_0|j'_\pi\rangle\langle j'_\pi|\sigma_0|j_\nu\rangle, \tag{C5}$$

where the first term is N and Z for the cases of the $\sigma_0 t_-$ and $\sigma_0 t_+$ transition operators, respectively.

Next we evaluate the right-hand side of Eq. (27). The expectation values of the total spin operators are evaluated as

$$\langle i|(S_0^n)^2|i\rangle = \frac{1}{4}N + \frac{1}{4}\sum_{j_\nu, j_{\nu'}} \langle j_\nu|\sigma_0|j_\nu\rangle\langle j_{\nu'}|\sigma_0|j_{\nu'}\rangle - \frac{1}{4}\sum_{j_\nu, j_{\nu'}} \langle j_\nu|\sigma_0|j_{\nu'}\rangle\langle j_{\nu'}|\sigma_0|j_\nu\rangle, \tag{C6}$$

$$\langle i|(S_0^p)^2|i\rangle = \frac{1}{4}Z + \frac{1}{4}\sum_{j_\pi, j_{\pi'}} \langle j_\pi|\sigma_0|j_\pi\rangle\langle j_{\pi'}|\sigma_0|j_{\pi'}\rangle - \frac{1}{4}\sum_{j_\pi, j_{\pi'}} \langle j_\pi|\sigma_0|j_{\pi'}\rangle\langle j_{\pi'}|\sigma_0|j_\pi\rangle, \tag{C7}$$

$$\langle i|S_0^n S_0^p|i\rangle = \frac{1}{4}\sum_{j_\pi, j_\nu} \langle j_\nu|\sigma_0|j_\nu\rangle\langle j_\pi|\sigma_0|j_\pi\rangle. \tag{C8}$$

Using these properties, the right-hand side of Eq. (27) is

$$\begin{aligned} 2\langle i|(S_0^n - S_0^p)^2|i\rangle = {} & \frac{1}{2}(N+Z) \\ & - \frac{1}{2}\sum_{j_\nu, j_{\nu'}} \langle j_\nu|\sigma_0|j_{\nu'}\rangle\langle j_{\nu'}|\sigma_0|j_\nu\rangle \\ & - \frac{1}{2}\sum_{j_\pi, j_{\pi'}} \langle j_\pi|\sigma_0|j_{\pi'}\rangle\langle j_{\pi'}|\sigma_0|j_\pi\rangle \\ & + \frac{1}{2}\left(\sum_{j_\nu} \langle j_\nu|\sigma_0|j_\nu\rangle - \sum_{j_\pi} \langle j_\pi|\sigma_0|j_\pi\rangle\right)^2. \end{aligned} \tag{C9}$$

Comparing Eqs. (C5) and (C9) for $T = 0$ initial states gives

$$S^{(0)} = 2\langle i|(S_0^n - S_0^p)^2|i\rangle, \tag{C10}$$

because the last term of Eq. (C9) vanishes and the other terms become the same as the right-hand side of Eq. (C5). Here we are using the rotational symmetry in the isospin space of $T = 0$ states. We note that the same result can be easily obtained also from the NEWSR for $\boldsymbol{\sigma}\tau_z$ transition operator derived in Ref. [31], after taking into account the isospin-spatial rotational symmetry between the $\tau_{\pm 1}$ and τ_z operators.

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