

Unitary ambiguity of NN contact interactions and the $3N$ force

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We identify a redundancy between two- and three-nucleon contact interactions at the fourth and fifth orders of the chiral expansion, respectively. In particular, we show that tensor-type and spin-orbit three-nucleon contact interactions effectively account for that part of the two-nucleon interaction which depends on the total center-of-mass momentum and is unconstrained by relativity. This might give the chiral effective field theory enough flexibility to successfully address $A = 3$ scattering observables already at N3LO.

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I. INTRODUCTION

The modern understanding of nuclear interactions is based on the chiral effective field theory (ChEFT) framework [1–5]. Compared to more phenomenological approaches, a low-momenta power counting allows us in principle to improve systematically the accuracy of the theoretical description, pursuing the perturbative expansion to higher orders, and at the same time to assess the theoretical uncertainty introduced by the truncation of the series [6,7]. This is made possible by the approximate chiral symmetry of the underlying quantum chromodynamics (QCD), whose dynamical breakdown is responsible for the emergence of pseudo-Goldstone bosons, the pions, that interact weakly at low energy and are much lighter than all other hadrons. Pion exchanges among nucleons determine the longest range component of the nuclear interaction, while the dynamics at shorter distances, unresolved by the effective theory, is described in terms of multinucleon contact interactions. The associated low-energy constants (LECs), being unconstrained by chiral symmetry, need to be determined from experimental data. Their number increases as the perturbative series is pushed to higher orders, but their impact should decrease, provided the expansion is well behaved. Obviously, in order to fully exploit the predictive power of the theory and to put it to a more stringent test, it is important to identify a minimal set of such LECs.

In this paper, we concentrate on a redundancy between two-nucleon (NN) contact couplings which arise at the fourth order of the low-energy expansion (N3LO) and the subleading three-nucleon ($3N$) contact interactions, which were classified in Ref. [8] as consisting of 13 independent operators. The latter arise at the fifth order (N4LO) in the ChEFT [9–17], and as such they should be considered in conjunction with recent accurate versions of the NN interaction developed at N4LO and beyond [7,18–21]. The relevance of these operators has been repeatedly highlighted, in particular for solving long-standing discrepancies in low-energy $N - d$ elastic scattering, like the well-known A_y puzzle [22–24]. As we are

going to show, five of these operators are equivalent to a suitable redefinition of the short-range NN potential, realized by specific unitary transformations of the nuclear Hamiltonian. Such unitary ambiguities are a common feature in all reductions from a quantum-field theoretical Lagrangian to quantum mechanics and affect also pion-mediated interactions [25–27]. They are systematically exploited in the unitary transformation approach to nuclear forces and electroweak currents [28,29] to enforce the renormalizability of nuclear potentials and transition operators [30,31]. We single out in particular two additional transformations that can be used to drop total momentum (\mathbf{P})-dependent NN interactions. Such interactions, which vanish in the center of mass frame, cannot be determined from NN scattering data alone. However, dropping these terms induces $3N$ contact interactions mostly of tensor and spin-orbit type.

The paper is organized as follows. In Sec. II, we classify the most general NN contact unitary transformation at the $O(p^2)$ level. In Sec. III, we relate two of these transformations to the \mathbf{P} -dependent component of the NN interaction and show that the latter depends on two extra LECs, unconstrained by relativity. In Sec. IV, we study the impact of this transformation at the $3N$ level and obtain a reduced form of the subleading $3N$ contact interaction. Finally, the consequences of the above findings on the structure and convergence of the chiral expansion for $3N$ observables are discussed in Sec. V.

II. NN CONTACT UNITARY TRANSFORMATIONS AT $O(p^2)$

Following Ref. [19], we write the most general unitary transformation as $U = \exp(\sum_n \alpha_n T_n)$ with α_n being real parameters and T_n being a complete set of anti-Hermitian operators respecting all underlying symmetries. Since we are interested in purely nucleonic interactions, the generators T_n will only involve local products of nucleon fields, ordered in the low-energy expansion, according to the number of gradients.

The first nontrivial case will consist of two-nucleon operators. Rotational, isospin, parity, and time-reversal symmetry require the presence of at least two gradients. At this level, a complete set consists of the following operators:

$$T_1 = \int d^3\mathbf{x} N^\dagger \overleftrightarrow{\nabla}^i N \nabla^i (N^\dagger N) \sim \mathbf{k} \cdot \mathbf{Q}, \quad (1)$$

$$T_2 = \int d^3\mathbf{x} N^\dagger \overleftrightarrow{\nabla}^i \sigma^j N \nabla^i (N^\dagger \sigma^j N) \sim \mathbf{k} \cdot \mathbf{Q} \sigma_1 \cdot \sigma_2, \quad (2)$$

$$T_3 = \int d^3\mathbf{x} [N^\dagger \overleftrightarrow{\nabla}^i \sigma^i N \nabla^j (N^\dagger \sigma^j N) + N^\dagger \overleftrightarrow{\nabla}^i \sigma^j N \nabla^j (N^\dagger \sigma^i N)] \sim \mathbf{k} \cdot \sigma_1 \mathbf{Q} \cdot \sigma_2 + \mathbf{k} \cdot \sigma_2 \mathbf{Q} \cdot \sigma_1, \quad (3)$$

$$T_4 = i\epsilon^{ijk} \int d^3\mathbf{x} N^\dagger \overleftrightarrow{\nabla}^i N N^\dagger \overleftrightarrow{\nabla}^j \sigma^k N \sim i\mathbf{P} \times \mathbf{Q} \cdot (\sigma_1 - \sigma_2), \quad (4)$$

$$T_5 = \int d^3\mathbf{x} [N^\dagger \overleftrightarrow{\nabla}^i \sigma^i N \nabla^j (N^\dagger \sigma^j N) - N^\dagger \overleftrightarrow{\nabla}^i \sigma^j N \nabla^j (N^\dagger \sigma^i N)] \sim (\mathbf{P} \cdot \sigma_1 \mathbf{k} \cdot \sigma_2 - \mathbf{P} \cdot \sigma_2 \mathbf{k} \cdot \sigma_1)/2, \quad (5)$$

where $N^\dagger \overleftrightarrow{\nabla}^i N = N^\dagger (\nabla^i N) - (\nabla^i N^\dagger) N$, and $N(x)$ denotes the nonrelativistic nucleon field operator. We have also introduced the dependence on the initial and final relative momenta \mathbf{p} and \mathbf{p}' , or $\mathbf{k} = \mathbf{p}' - \mathbf{p}$ and $\mathbf{Q} = (\mathbf{p} + \mathbf{p}')/2$, and on the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ of a two-nucleon system. The last two generators, which were not considered in Ref. [19], vanish in the two-nucleon center of mass frame. Their relevance will be clear in the following. In addition, we can also define the corresponding isospin-dependent T_n operators, involving $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$, but using the anticommuting nature of nucleon fields and Fierz reshuffling of spin and isospin indices, one can express them in the chosen basis:

$$T_{1'} = -2T_1 - T_2, \quad (6)$$

$$T_{2'} = -3T_1, \quad (7)$$

$$T_{3'} = -2T_1 + 2T_2 - 3T_3, \quad (8)$$

$$T_{4'} = -2T_5 - T_4, \quad (9)$$

$$T_{5'} = -2T_4 - T_5. \quad (10)$$

When transforming a nuclear Hamiltonian H by the above unitary transformation, one gets additional interactions

$$\begin{aligned} H &\rightarrow U^\dagger H U = H + \sum_n \alpha_n [H, T_n] + \dots \\ &\equiv H + \sum_n \alpha_n \delta_n H + \dots \end{aligned} \quad (11)$$

that amount to a shift of existing LECs, since H already contains all possible interactions allowed by the assumed symmetries. Thus, from the one-body kinetic energy,

$$H_0 = -\frac{1}{2m} \int d^3\mathbf{x} N^\dagger \nabla^2 N, \quad (12)$$

one gets, e.g., using the canonical anticommutation relations¹

$$\begin{aligned} \delta_1 H_0 &= \frac{1}{2m} \int d^3\mathbf{x} [\nabla^i (N^\dagger \overleftrightarrow{\nabla}^i \overleftrightarrow{\nabla}^j N) \nabla^j (N^\dagger N) \\ &\quad - \nabla^i (N^\dagger \overleftrightarrow{\nabla}^i N) \nabla^j (N^\dagger \overleftrightarrow{\nabla}^j N)]. \end{aligned} \quad (13)$$

In the two-nucleon system, the above operator yields an off-shell contribution $\sim (\mathbf{p}^2 - \mathbf{p}'^2)^2$. In Ref. [19], the unitary

transformations corresponding to $n = 1, 2, 3$, were used to absorb three of the $O(p^4)$ NN couplings, reducing their number to 12. As will be shown in Sec. IV, this also implies the appearance of induced subleading $3N$ contact interactions, which can be written as combinations of the 13 operators introduced in Ref. [8]. The remaining transformations, corresponding to $n = 4, 5$, generate total momentum-dependent interactions which vanish in the center of mass frame, e.g., $\delta_4 H_0 \sim i(\mathbf{p}^2 - \mathbf{p}'^2) \mathbf{P} \times (\mathbf{p} + \mathbf{p}') \cdot (\sigma_1 - \sigma_2)$, and will be discussed in the next section.

III. P-DEPENDENT NN CONTACT INTERACTIONS

Total momentum-dependent interactions are strongly constrained by Poincaré symmetry. In the instant form of relativistic dynamics [32], the three-momentum and angular momentum are the same as in the free theory, described by the Dirac Lagrangian starting from the energy-momentum tensor $T^{\mu\nu} = (i/2) \bar{\psi} \gamma^\mu \overleftrightarrow{\partial}^\nu \psi$, while the Hamiltonian H and boost generators \mathbf{K} contain the interactions

$$H = H_0 + H_I, \quad \mathbf{K} = \mathbf{K}_0 + \mathbf{K}_I. \quad (14)$$

Since the free generators already satisfy the Poincaré commutation relations, the interaction terms must satisfy

$$[J^i, K_I^j] = i\epsilon^{ijk} K_I^k, \quad (15)$$

$$[K_I^i, P^j] = i\delta^{ij} H_I, \quad (16)$$

$$[K_I^i, H_0] + [K_0^i, H_I] + [K_I^i, H_I] = 0, \quad (17)$$

$$[K_0^i, K_I^j] + \frac{1}{2} [K_I^i, K_I^j] - i \leftrightarrow j = 0. \quad (18)$$

The first relation qualifies the interacting boost generator as a vector. The remaining ones are less trivial to satisfy. In the low-energy theory, a nonrelativistic reduction can be used to express these operators in terms of the nonrelativistic nucleon field $N(x)$ as a series containing increasing powers of soft momenta. For example, the free Hamiltonian and boost generators are expanded as

$$\begin{aligned} H_0 &= H_0^{(0)} + H_0^{(2)} + H_0^{(4)} + \dots, \\ \mathbf{K}_0 &= \mathbf{K}_0^{(-1)} + \mathbf{K}_0^{(1)} + \mathbf{K}_0^{(3)} + \dots, \end{aligned} \quad (19)$$

¹It is convenient to use the identity $[AB, C] = A[B, C] - [A, C]B$.

where the superscripts denote the assigned ‘‘soft power.’’ Explicitly,

$$\begin{aligned} H_0^{(0)} &= m \int d^3\mathbf{x} N^\dagger N, & H_0^{(2)} &= -\frac{1}{8m} \int d^3\mathbf{x} N^\dagger \overleftrightarrow{\nabla}^2 N, \dots \\ \mathbf{K}_0^{(-1)} &= m \int d^3\mathbf{x} \mathbf{x} N^\dagger N, \\ \mathbf{K}_0^{(1)} &= -\frac{1}{8m} \int d^3\mathbf{x} \mathbf{x} [N^\dagger \overleftrightarrow{\nabla}^2 N + i\overleftrightarrow{\nabla} \cdot N^\dagger \overleftrightarrow{\sigma} \times \overleftrightarrow{\nabla} N], \dots \end{aligned} \quad (20)$$

Contact interactions in H_I can be classified according to the number of participating nucleons,

$$H_I = H_{NN} + H_{3N} + \dots, \quad (21)$$

and each component can be ordered by the same criterium as

$$H_{NN} = H_{NN}^{(3)} + H_{NN}^{(5)} + H_{NN}^{(7)} + \dots, \quad (22)$$

$$H_{3N} = H_{3N}^{(6)} + H_{3N}^{(8)} + \dots. \quad (23)$$

The first term in H_{NN} contains the two momentum-independent interactions parametrized by the LECs C_S and C_T ,

$$\begin{aligned} H_{NN}^{(3)} &= \frac{1}{2} C_S H_S + \frac{1}{2} C_T H_T \\ &\equiv \int d^3\mathbf{x} \left[\frac{1}{2} C_S N^\dagger N N^\dagger N + \frac{1}{2} C_T N^\dagger \overleftrightarrow{\sigma} N \cdot N^\dagger \overleftrightarrow{\sigma} N \right]. \end{aligned} \quad (24)$$

Starting with the following order, we can have \mathbf{P} -independent or \mathbf{P} -dependent interactions. In $H_{NN}^{(5)}$, the formers are parametrized by the LECs $C_{1,\dots,7}$, while the latter are unambiguously fixed in terms of the leading LECs C_S and C_T as relativistic $1/m$ corrections [33]. At the following order, $H_{NN}^{(7)}$ contains \mathbf{P} -independent interactions depending on the LECs $D_{1,\dots,15}$ and a set of \mathbf{P} -dependent ones which have not yet

been considered in the literature. Most of them take the form of relativistic corrections to lower order interactions, and as such they are fixed unambiguously in terms of the lower order LECs. Instead, we will be concerned with those \mathbf{P} -dependent contributions to $H_{NN}^{(7)}$ which are unconstrained by relativity and depend on extra LECs. As for the $3N$ interactions, their low-energy expansion starts with a momentum-independent term, $H_{3N}^{(6)}$ parametrized by the LEC c_E [15],

$$H_{3N}^{(6)} = \frac{c_E}{2F_\pi^4 \Lambda_\chi} \int d^3\mathbf{x} N^\dagger N N^\dagger \tau^a N N^\dagger \tau^a N, \quad (25)$$

with the pion decay constant F_π and the chiral symmetry breaking scale Λ_χ meant to provide the correct scaling based on naive dimensional analysis [34], and proceeds with the two-derivatives contact interactions parametrized by the LECs $E_{1,\dots,13}$ introduced in Ref. [8],

$$H_{3N}^{(8)} = \int d^3\mathbf{x} \sum_{i=1}^{13} E_i O_i, \quad (26)$$

where the explicit expressions for the operators O_i can be read from Eqs. (14) and (16) of Ref. [8].

In order to satisfy the relation (16), a given interaction in H_I implies a corresponding term in \mathbf{K}_I which we denote as \mathbf{W} , such that

$$H_I = \int d^3\mathbf{x} \mathcal{H}_I(x) \implies \mathbf{W} = \int d^3\mathbf{x} \mathbf{x} \mathcal{H}_I(x). \quad (27)$$

The most general form of the interacting part of the boost generator can be written as $\mathbf{K}_I = \mathbf{W} + \delta\mathbf{W}$, i.e., as the sum of the ‘‘minimal’’ boost \mathbf{W} and of an additional term $\delta\mathbf{W}$ which is translationally invariant ($[P^i, \delta W^j] = 0$) and will be denoted as ‘‘intrinsic,’’ since it is independent of the interacting Hamiltonian. Its low-energy expansion starts with $\delta\mathbf{W}^{(4)}$. At this order, one can list three such operators,

$$\delta W_1^i = \int d^3\mathbf{x} \overleftrightarrow{\nabla} \cdot (N^\dagger \overleftrightarrow{\sigma} N) N^\dagger \sigma^i N, \quad (28)$$

$$\delta W_2^i = i\epsilon^{ijk} \int d^3\mathbf{x} [N^\dagger \overleftrightarrow{\nabla}^j \sigma^k N N^\dagger N + N^\dagger \sigma^j N N^\dagger \overleftrightarrow{\nabla}^k N], \quad (29)$$

$$\delta W_3^i = i\epsilon^{ijk} \int d^3\mathbf{x} [N^\dagger \overleftrightarrow{\nabla}^j \sigma^k N N^\dagger N - N^\dagger \sigma^j N N^\dagger \overleftrightarrow{\nabla}^k N], \quad (30)$$

since the operators involving $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ are Fierz related to the ones above. These intrinsic boosts were ignored in Ref. [33], since they do not play a role at the order considered there, respectively $O(p^4)$ and $O(p^3)$ for the relations (17) and (18),

$$[\mathbf{K}_0^{(-1)}, H_I^{(5)}] + [\mathbf{K}_0^{(1)}, H_I^{(3)}] + [\mathbf{W}^{(4)}, H_0^{(0)}] + [\delta\mathbf{W}^{(4)}, H_0^{(0)}] + [\mathbf{W}^{(2)}, H_0^{(2)}] = 0, \quad (31)$$

$$[\mathbf{K}_0^{(-1)i}, W^{(4)j}] + [\mathbf{K}_0^{(1)j}, W^{(2)i}] + [\mathbf{K}_0^{(-1)j}, \delta W^{(4)i}] - i \leftrightarrow j = 0. \quad (32)$$

Indeed, in the first of the above equations, $\delta\mathbf{W}$ is irrelevant, since it commutes with $H_0^{(0)}$. Moreover, as found in Ref. [33], Eq. (32) without the commutators involving $\delta\mathbf{W}$ is valid as a consequence of Eq. (31). This means that we must have

$$[\mathbf{K}_0^{(-1)i}, \delta W^{(4)j}] - i \leftrightarrow j = 0, \quad (33)$$

which rules out $\delta\mathbf{W}_3$. In other words, only the \mathbf{P} -independent intrinsic boosts $\delta\mathbf{W}_1$ and $\delta\mathbf{W}_2$ are allowed, and we can write the most general intrinsic boost $\delta\mathbf{W}^{(4)}$ in terms of two constants, $\delta\mathbf{W}^{(4)} = \sum_{i=1}^2 \beta_i \delta\mathbf{W}_i$. The two independent intrinsic boost generators are related to the transformations T_4 and T_5 of the previous section by the following relations:

$$[\mathbf{K}_0^{(-1)}, T_5] = -4\delta\mathbf{W}_1, \quad [\mathbf{K}_0^{(-1)}, T_4] = -2\delta\mathbf{W}_2. \quad (34)$$

They start to play a role at the orders $O(p^6)$ and $O(p^5)$ respectively,

$$0 = [\mathbf{W}^{(6)}, H_0^{(0)}] + [\mathbf{W}^{(4)}, H_0^{(2)}] + [\mathbf{K}_0^{(1)}, H_I^{(5)}] + [\mathbf{W}^{(2)}, H_0^{(4)}] + [\mathbf{K}_0^{(3)}, H_I^{(3)}] + [\delta\mathbf{W}^{(4)}, H_0^{(2)}] + [\mathbf{K}_0^{(-1)}, H_I^{(7)}], \quad (35)$$

$$0 = [K_0^{(-1)i}, W^{(6)j}] + [K_0^{(1)i}, W^{(4)j}] + [K_0^{(1)i}, \delta W^{(4)j}] + [K_0^{(3)ij}, W^{(2)j}] - i \leftrightarrow j, \quad (36)$$

which again involve only two-nucleon terms. From these equations the relativistic corrections in $H_I^{(7)}$ can be identified, following the steps of Ref. [33]. Indeed, the interactions in $H_I^{(7)}$, with corresponding minimal boosts $\mathbf{W}^{(6)}$, must satisfy the above constraints. Considering Eq. (35), the first term vanishes, the second and third terms represent $1/m$ corrections to the interactions involving C_i , and the fourth and fifth terms represent $1/m^3$ corrections to the interactions involving C_S and C_T . Ignoring these $1/m$ corrections, we are left either with \mathbf{P} -independent terms in $H_I^{(7)}$, which commute with $\mathbf{K}_0^{(-1)}$, and thus satisfy Eq. (35) with $\delta\mathbf{W}^{(4)} = 0$ (these are the operators multiplied by D_1, \dots, D_{15}) or with \mathbf{P} -dependent terms whose commutator with $\mathbf{K}_0^{(-1)}$ must be compensated by the terms involving $\delta\mathbf{W}^{(4)}$. Thus, there is a one-to-one correspondence between the possible forms of the intrinsic boost $\delta\mathbf{W}^{(4)}$ (which we already classified as $\delta\mathbf{W}_1$ and $\delta\mathbf{W}_2$) and the allowed \mathbf{P} -dependent interactions of $H_I^{(7)}$. It is possible to show that Eq. (36) then follows from Bianchi identities.

In view of Eqs. (34), the above statement can be immediately understood: A unitary transformation involving T_4 and T_5 will generate from H_0 some \mathbf{P} -dependent interaction terms of $H_I^{(7)}$. At the same time, as is clear from Eq. (34), from the free boost generator \mathbf{K}_0 one gets the interacting intrinsic boosts that exactly compensate for these terms, such that the Poincaré commutation relations (35) and (36) remain satisfied, as they should by unitarity. This means that there are two \mathbf{P} -dependent NN contact interactions in $H_I^{(7)}$ which are completely unconstrained, depending on two free LECs. A possible parametrization of the resulting \mathbf{P} -dependent NN potential is in terms of two extra LECs, D_{16} and D_{17} , such that

$$\begin{aligned} V_{NN}(\mathbf{P}) &= iD_{16}\mathbf{k} \cdot \mathbf{Q} \mathbf{Q} \times \mathbf{P} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) + D_{17}\mathbf{k} \cdot \mathbf{Q} (\mathbf{k} \times \mathbf{P}) \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \\ &= i[(D_{16} - D_{17}) + D_{17}\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2]\mathbf{k} \cdot \mathbf{Q} \mathbf{Q} \times \mathbf{P} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2), \end{aligned} \quad (37)$$

where the last equality follows from the Fierz identities. These interactions change the spin of the NN system and they are both linear in \mathbf{P} . They act differently in the two isospin channels.²

The \mathbf{P} -dependent potential (37) bears some resemblance with the Thomas precession term [35–37], which is a relativistic \mathbf{P} -dependent interaction determined by the center of mass potential v as

$$\delta v_{\text{TP}} = \frac{i}{8m^2} [(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \times \mathbf{P} \cdot \mathbf{p}, v] \quad (38)$$

and is part of the terms discussed in the previous paragraph. We emphasize once more that the \mathbf{P} -dependent interaction (37) is of a different nature, and the corresponding LECs are not fixed as $1/m$ corrections to lower order interactions. For example, if we take for v in Eq. (38) the subleading two-nucleon contact potential, defined as customary

$$v = C_1 k^2 + C_2 k^2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + C_3 k^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_4 k^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + C_5 S_{12}(\mathbf{k}) + C_6 S_{12}(\mathbf{k}) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + iC_7 \mathbf{S} \cdot \mathbf{Q} \times \mathbf{k}, \quad (39)$$

with $S_{12}(\mathbf{k}) = 3\boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} - k^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ and $\mathbf{S} = (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)/2$, then the induced Thomas precession potential is

$$\begin{aligned} \delta v_{\text{TP}} &= \frac{i}{8m^2} \mathbf{P} \times \mathbf{k} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \{k^2 [C_1 - C_3 + C_5 + (C_2 - C_4 + C_6) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2] \\ &\quad - 4Q^2 [C_3 + 3C_4 - C_5 - 3C_6 + (C_3 - C_4 - C_5 + C_6) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2]\} + \frac{3i}{4m^2} [\mathbf{P} \cdot \mathbf{Q} \mathbf{k} \times \mathbf{Q} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) - \mathbf{k} \cdot \mathbf{Q} \mathbf{P} \times \mathbf{Q} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)] \\ &\quad \times [C_5 + 3C_6 + (C_5 - C_6) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2] - \frac{iC_7}{16m^2} \mathbf{P} \times \mathbf{Q} \cdot \mathbf{k} \mathbf{Q} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) (1 - \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2). \end{aligned} \quad (40)$$

Thus, we see that part of the interactions in Eq. (37) are indeed also generated by the Thomas precession, with fixed coefficients, while the LECs D_{16} and D_{17} are completely unconstrained. They cannot be determined from NN scattering data, but only in $A > 2$ systems, or as a high-order contribution to the two-nucleon electromagnetic current. Their contribution vanishes on shell and, similarly to the three combinations of the D_i already identified in Ref. [19], they can be absorbed by a unitary transformation. However, in so doing, one obtains at the same time an induced $3N$ interaction, as will be discussed in the next section.

²Notice that, away from the center of mass system, the usual relation $(-1)^{L+S+T} = -1$ between the orbital angular momentum L , the spin S , and the isospin T of the NN pair is not necessarily satisfied, due to the presence of the angular momentum associated to the overall motion.

IV. IMPACT ON THE $3N$ SECTOR

If H in Eq. (11) is a two-nucleon operator, then the unitary transformations defined in Sec. II generate three-nucleon operators. At the leading order, we have the following contributions induced by the transformations of H_S and H_T appearing in Eq. (24):

$$\delta_1 H_S = -4 \int d^3 \mathbf{x} \nabla^i (N^\dagger N) \nabla^i (N^\dagger N) (N^\dagger N), \quad (41)$$

$$\delta_1 H_T = -4 \int d^3 \mathbf{x} \nabla^i (N^\dagger \sigma^j N) \nabla^i (N^\dagger N) (N^\dagger \sigma^j N), \quad (42)$$

$$\delta_2 H_S = \delta_1 H_T, \quad (43)$$

$$\delta_2 H_T = 4 \int d^3 \mathbf{x} [i\epsilon^{ijk} \nabla^l (N^\dagger \sigma^i N) N^\dagger \overleftrightarrow{\nabla}^l \sigma^j N N^\dagger \sigma^k N - \nabla^j (N^\dagger \sigma^i N) \nabla^j (N^\dagger \sigma^i N) N^\dagger N], \quad (44)$$

$$\delta_3 H_S = -4 \int d^3 \mathbf{x} [\nabla^i (N^\dagger \sigma^j N) \nabla^j (N^\dagger N) N^\dagger \sigma^i N + \nabla^i (N^\dagger \sigma^i N) \nabla^j (N^\dagger N) N^\dagger \sigma^j N], \quad (45)$$

$$\delta_3 H_T = 4 \int d^3 \mathbf{x} [i\epsilon^{ijk} \nabla^l (N^\dagger \sigma^l N) (N^\dagger \overleftrightarrow{\nabla}^l \sigma^j N) N^\dagger \sigma^k N + i\epsilon^{ijk} \nabla^l (N^\dagger \sigma^l N) N^\dagger \overleftrightarrow{\nabla}^i \sigma^j N N^\dagger \sigma^k N - \nabla^i (N^\dagger \sigma^j N) \nabla^j (N^\dagger \sigma^i N) N^\dagger N - \nabla^i (N^\dagger \sigma^i N) \nabla^j (N^\dagger \sigma^j N) N^\dagger N], \quad (46)$$

$$\delta_4 H_S = 4i\epsilon^{ijk} \int d^3 \mathbf{x} [\nabla^i (N^\dagger N) N^\dagger \overleftrightarrow{\nabla}^j N N^\dagger \sigma^k N - \nabla^i (N^\dagger N) N^\dagger \overleftrightarrow{\nabla}^j \sigma^k N N^\dagger N], \quad (47)$$

$$\delta_4 H_T = -4 \int d^3 \mathbf{x} [i\epsilon^{ijk} \nabla^l (N^\dagger \sigma^l N) N^\dagger \overleftrightarrow{\nabla}^j \sigma^k N N^\dagger \sigma^l N + i\epsilon^{ijk} \nabla^l (N^\dagger \sigma^j N) N^\dagger \overleftrightarrow{\nabla}^k N N^\dagger N + N^\dagger \overleftrightarrow{\nabla}^i N N^\dagger \overleftrightarrow{\nabla}^j \sigma^i N N^\dagger \sigma^j N - N^\dagger \overleftrightarrow{\nabla}^i \sigma^i N N^\dagger \overleftrightarrow{\nabla}^j N N^\dagger \sigma^j N], \quad (48)$$

$$\delta_5 H_S = 0, \quad (49)$$

$$\delta_5 H_T = -4 \int d^3 \mathbf{x} [i\epsilon^{ijk} \nabla^l (N^\dagger \sigma^l N) N^\dagger \overleftrightarrow{\nabla}^j \sigma^i N N^\dagger \sigma^k N - i\epsilon^{ijk} \nabla^l (N^\dagger \sigma^l N) N^\dagger \overleftrightarrow{\nabla}^i \sigma^j N \cdot N^\dagger \sigma^k N + \nabla^i (N^\dagger \sigma^i N) \nabla^j (N^\dagger \sigma^j N) N^\dagger N - \nabla^i (N^\dagger \sigma^j N) \nabla^j (N^\dagger \sigma^i N) N^\dagger N]. \quad (50)$$

These operators can be expressed in terms of the basis $\{O_i\}$ defined in Eq. (26), using the identities derived in Ref. [8]. As a result, the general unitary transformation of the two-nucleon Hamiltonian in Eq. (24) produces the following contribution,

$$\left[H_{NN}, \sum_{n=1}^5 \alpha_n T_n \right] = \int d^3 \mathbf{x} \sum_{i=1}^{13} \delta E_i O_i, \quad (51)$$

with

$$\delta E_1 = \alpha_1 (C_S + C_T) + \alpha_2 (C_S - 2C_T), \quad (52)$$

$$\delta E_2 = 3\alpha_2 C_T + 2\alpha_3 C_T - 4\alpha_4 C_T + 2\alpha_5 C_T, \quad (53)$$

$$\delta E_3 = 2\alpha_1 C_T + \alpha_2 (2C_S - C_T) + \frac{2}{3}\alpha_3 (2C_S - C_T) + 4\alpha_4 C_T - 2\alpha_5 C_T, \quad (54)$$

$$\delta E_4 = \frac{2}{3}\alpha_1 C_T + \frac{1}{3}\alpha_2 (2C_S - 7C_T) - \frac{2}{3}\alpha_3 C_T + \frac{4}{3}\alpha_4 C_T - \frac{2}{3}\alpha_5 C_T, \quad (55)$$

$$\delta E_5 = 2\alpha_1 C_T + 2\alpha_2 (C_S - 2C_T) + \frac{2}{3}\alpha_3 (2C_S - C_T) + 4\alpha_4 C_T - 2\alpha_5 C_T, \quad (56)$$

$$\delta E_6 = \frac{2}{3}\alpha_1 C_T + \frac{2}{3}\alpha_2 (C_S - 2C_T) - \frac{2}{3}\alpha_3 C_T + \frac{4}{3}\alpha_4 C_T - \frac{2}{3}\alpha_5 C_T, \quad (57)$$

$$\delta E_7 = 8\alpha_4 C_T, \quad (58)$$

$$\delta E_8 = \frac{1}{3}\delta E_7, \quad (59)$$

$$\delta E_9 = 3\alpha_1 C_T + 3\alpha_2 (C_S - 2C_T) + 2\alpha_3 (C_S - 2C_T) - \alpha_4 (C_S - 5C_T) - 4\alpha_5 C_T, \quad (60)$$

$$\delta E_{10} = \alpha_1 C_T + \alpha_2 (C_S - 2C_T) - \frac{1}{3}\alpha_4 (3C_S - 7C_T), \quad (61)$$

$$\delta E_{11} = 3\alpha_1 C_T + 3\alpha_2 (C_S - 2C_T) + 2\alpha_3 (C_S - 2C_T) + \alpha_4 (C_S - 5C_T) + 4\alpha_5 C_T, \quad (62)$$

$$\delta E_{12} = \alpha_1 C_T + \alpha_2 (C_S - 2C_T) + \frac{1}{3}\alpha_4 (3C_S - 7C_T), \quad (63)$$

$$\delta E_{13} = -8\alpha_4 C_T + 4\alpha_5 C_T, \quad (64)$$

which amounts to a shift of the 13 subleading LECs E_i in Eq. (26),

$$H_{3N}^{(8)} \rightarrow \int d^3\mathbf{x} \sum_{i=1}^{13} (E_i + \delta E_i) O_i \equiv \int d^3\mathbf{x} \sum_{i=1}^{13} E_i^{(\alpha)} O_i. \quad (65)$$

Therefore, for all nonzero values of C_S and C_T , the five parameters α_n defining the unitary transformation can be chosen so that five of the 13 subleading LECs can be eliminated, i.e., $E_i^{(\alpha)} = 0$. The sum in Eq. (26) can then be restricted, e.g., to $i = 2, \dots, 9$. This happens at the price of considering a more elaborate NN interaction, comprising all of the N4LO LECs $D_{1,\dots,15}$ as well as the ones parametrizing the \mathbf{P} -dependent NN interaction.

Another point of view can be adopted. It is generally accepted that the $3N$ interaction is parameter-free at N3LO [16,17], the LECs E_i of Eq. (26) contributing only at N4LO. However, the (N3LO) effect of the \mathbf{P} -dependent or the off-shell component of the NN interaction is equivalent to the above contact $3N$ interaction. Thus, the shifts δE_i in Eq. (65) are to be regarded as an effect at N3LO. In other words, the five LECs parametrizing the N3LO NN off-shell interaction can be fitted to observables of the $3N$ system and interpreted as a $3N$ interaction.

We emphasize in particular the role of the \mathbf{P} -dependent NN interaction, which has never been taken into account,

since it cannot be determined from the NN scattering data. Even if the complete NN interaction (including all of the LECs $D_{1,\dots,15}$) is used in $3N$ calculations, the necessity to discard the \mathbf{P} -dependent terms leads to the appearance of a subleading $3N$ contact interaction already at N3LO. The exact form of this interaction can be read from Eqs. (52)–(64) by inspecting the terms proportional to α_4 and α_5 , which can be regarded as a sort of LECs. In other words, supplementing the NN interaction with its \mathbf{P} -dependent component, Eq. (37), the corresponding interactions can be absorbed by a unitary transformation with parameters

$$\alpha_4 = -\frac{m}{8}D_{16}, \quad \alpha_5 = -\frac{m}{4}D_{17}, \quad (66)$$

i.e., by the transformation of the kinetic energy operator of Eq. (12),

$$[H_0, \alpha_4 T_4 + \alpha_5 T_5] = -V_{NN}(\mathbf{P}), \quad (67)$$

that in turn generates the $3N$ couplings E_i 's according to Eqs. (52)–(64).

Thus, we can say that, contrary to the commonly accepted wisdom, the $3N$ force is not parameter-free at N3LO, but depends on five LECs. Three of them are combinations of the LECs $D_{1,\dots,15}$, if one removes them from the NN potential, as done in Ref. [19]. Two more correspond to the new LECs D_{16} and D_{17} , which can be viewed as contributions to the subleading $3N$ contact potential,

$$V_{3N} = \frac{m}{8}D_{16} \left[C_S(O_9 + O_{10} - O_{11} - O_{12}) - C_T \left(4O_3 + \frac{4}{3}O_4 + 4O_5 + \frac{4}{3}O_6 + 8O_7 + \frac{8}{3}O_8 + 5O_9 - \frac{7}{3}O_{10} - 5O_{11} - \frac{7}{3}O_{12} - 8O_{13} \right) \right] - \frac{m}{2}D_{17}C_T \left(O_2 - O_3 - \frac{1}{3}O_4 - O_5 - \frac{1}{3}O_6 - 2O_9 + 2O_{11} + 2O_{13} \right). \quad (68)$$

Due to the fact that, on phenomenological grounds, $C_S \gg C_T$, the main effect of the new \mathbf{P} -dependent NN interactions amounts in this limit to the first line of the above equation and involves a single LEC mD_{16} . On the other hand, the large numerical coefficients multiplying C_T in δE_7 , i.e., most notably for the spin-orbit operator O_7 , might explain its instrumental role in the resolution of the A_y puzzle of low-energy $N-d$ scattering [22–24,38].

Notice that the order mismatch between N3LO and N4LO is removed in the Weinberg counting, i.e., $m \sim O(\Lambda_\chi^2/p)$ [19]. On the other hand, if one explicitly includes a factor of $1/m$ in the contact LECs parametrizing the NN potential, in order to give the latter the same scaling as the kinetic energy, then the unitary transformation becomes independent of m , as in Eq. (66), corresponding to the possibility of removing the m dependence from the nonrelativistic theory altogether. The potential in Eq. (68) would also scale as $1/m$ in this case, and no enhancement would be formally obtained as compared to the leading $3N$ contact interaction, provided a factor of $1/m$ is also attached to c_E in Eq. (25).

The same unitary transformation also affects another class of contributions to the $3N$ interaction, with one pion exchange/contact topology, of the same type as the so-called c_D term [15]. Indeed, if H in Eq. (11) is the πN Hamiltonian,

$$H_{\pi N} = \frac{g_A}{2F_\pi} \int d^3\mathbf{x} N^\dagger \nabla \pi^a \cdot \sigma \tau^a N, \quad (69)$$

the induced Hamiltonian is a πNN coupling, which gives rise to a $3N$ potential of the following form:

$$\begin{aligned} V_{3NF} = & -\frac{g_A}{F_\pi} \sum_{i \neq j \neq k} \frac{\mathbf{k}_k \cdot \sigma_k \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k}{k_k^2 + m_\pi^2} \{ \alpha_1 \mathbf{k}_j \cdot \mathbf{k}_k \mathbf{k}_k \cdot \sigma_i + \alpha_2 [\mathbf{k}_j \cdot \mathbf{k}_k \mathbf{k}_k \cdot \sigma_j + 2i \mathbf{k}_j \cdot (\mathbf{Q}_i - \mathbf{Q}_j) \mathbf{k}_k \cdot \sigma_i \times \sigma_j] \\ & + (\alpha_3 + \alpha_5) [k_k^2 \mathbf{k}_j \cdot \sigma_j - 2i \mathbf{k}_j \cdot \sigma_j \mathbf{k}_k \cdot \mathbf{Q}_i \times \sigma_i + 2i \mathbf{Q}_j \cdot \sigma_j \mathbf{k}_k \cdot \mathbf{k}_j \times \sigma_i] \\ & + (\alpha_3 - \alpha_5) [\mathbf{k}_j \cdot \mathbf{k}_k \mathbf{k}_k \cdot \sigma_j + 2i \mathbf{k}_j \cdot \sigma_j \mathbf{k}_k \cdot \mathbf{Q}_j \times \sigma_i - 2i \mathbf{Q}_i \cdot \sigma_j \mathbf{k}_k \cdot \mathbf{k}_j \times \sigma_i] \\ & - 2\alpha_4 [\mathbf{k}_k \cdot \sigma_i \mathbf{k}_k \cdot \mathbf{Q}_j \times \sigma_j - 2i \mathbf{k}_k \cdot \mathbf{Q}_i (\mathbf{k}_k \cdot \mathbf{Q}_i \mathbf{Q}_j \cdot \sigma_i - \mathbf{k}_k \cdot \mathbf{Q}_j \mathbf{Q}_i \cdot \sigma_i)] \}. \end{aligned} \quad (70)$$

Similarly to the purely contact ones, these contributions, which are nominally N4LO, are enhanced by a factor of m , and therefore promoted to N3LO. To the best of our knowledge, they were not considered in the literature before. It will be interesting to study their effects in $3N$ scattering observables.

It is worth mentioning that no genuine three-nucleon unitary transformation can be used to the same purpose. For instance, taking

$$T_{3N} = i \int d^3\mathbf{x} N^\dagger \vec{\sigma} N \times N^\dagger \vec{\sigma} N \cdot N^\dagger \vec{\sigma} N, \quad (71)$$

then $[H_0, T_{3N}] = 0$, due to the antisymmetry with respect to the nucleon labels.

V. CONCLUDING REMARKS

By examining the most general NN contact unitary transformation, we have identified the precise relation between the subleading contact $3N$ interaction and specific off-shell or \mathbf{P} -dependent components of the NN interaction. Provided the full NN interaction up to $O(p^4)$ is determined from experimental data, including the off-shell components encoded in the LECs D_i and the \mathbf{P} -dependent contributions (which might require considering electromagnetic observables depending

on the same LECs), then five of the $3N$ subleading LECs E_i become redundant. Alternatively, one can disregard these contributions in the two-nucleon systems, at the price of a more involved $3N$ contact interaction. We have identified in particular the two unitary transformations that allow us to drop the free LECs parametrizing the \mathbf{P} -dependent component of the NN interaction, Eq. (37), which cannot be determined from a fit to NN scattering data alone. These interactions would contribute in larger systems, like $A = 3$, together with “drift terms” representing relativistic $1/m$ corrections, but they are never considered in actual calculations. Their effect can be traded with two specific combinations of the subleading $3N$ contact operators, which can be read from Eqs. (52)–(64) as the contributions proportional to α_4 and α_5 , the latter given in turn by Eqs. (66). We notice that these unitary transformations reshuffle the individual terms of the low-energy expansion. In particular, by absorbing the N3LO NN contact LECs D_i ’s, their effect is attributed to the N4LO $3N$ contact LECs E_i ’s and give rise additionally to a $3N$ force of one pion exchange/contact type. Therefore, we argue that this procedure could modify the expected convergence pattern of the chiral series and explain the difficulty of the N3LO $3N$ chiral interactions to address the $A = 3$ scattering observables [39], justifying the observed prominent role of the spin-orbit and tensor $3N$ contact interactions [24].

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