

Bridging the quartet and pair pictures of isovector proton-neutron pairingV. V. Baran^{1,2,*}, D. R. Nichita^{1,2}, D. Negrea^{1,2}, D. S. Delion^{1,2}, N. Sandulescu² and P. Schuck³¹*Faculty of Physics, University of Bucharest, 405 Atomistilor, POB MG-11, Bucharest-Măgurele RO-077125, Romania*²*“Horia Hulubei” National Institute of Physics and Nuclear Engineering, 30 Reactorului, RO-077125 Bucharest-Măgurele, Romania*³*Université Paris-Saclay, CNRS, IJCLab, IN2P3-CNRS, 91405 Orsay, France
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The formal implications of a quartet coherent-state ansatz for proton-neutron pairing are analyzed. Its nonlinear annihilation operators, which generalize the BCS linear quasiparticle operators, are computed in the quartetting case. Their structure is found to generate nontrivial relationships between the many-body correlation functions. The intrinsic structure of the quartet coherent state is detailed as it hints to the precise correspondence between the quartetting picture and the symmetry restored pair condensate picture for the proton-neutron pairing correlations.

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Introduction. After more than 60 years since pairing effects were first considered in nuclear physics [1], the microscopic pairing models are still facing the challenge of consistently describing the subtle interplay between the isovector ($T = 1$) and the isoscalar ($T = 0$) proton-neutron pairing in nuclear systems [2]. One of the first studies of the isovector pairing Hamiltonian was performed by Beliaev *et al.* in the framework of the generalized BCS approach in which the protons and the neutrons are mixed through the Bogoliubov transformation [3]. Since then, the BCS approach was employed in the majority of studies, and it was further extended to include also the isoscalar proton-neutron pairing interaction [4]. However, as has been noted already by Beliaev *et al.*, the BCS treatment is not complete because “one must take into consideration the quadruple correlations of α -particle-like nucleons in addition to pair correlations.” The first investigation of these correlations has been performed by Soloviev [5], who related them to a four-body interaction term. Later on, the four-body “quartet” correlations have been discussed in relation to the standard two-body isovector pairing interaction by Brémond and Valatin [6] and by Flowers and Vujicic [7]. They proposed a BCS-like function in which the pairs are replaced by quartets, but the calculations with this trial state turned out to be too complicated, and it was never applied to realistic cases. The first proof that the quartets are essential degrees of freedom for the isovector pairing Hamiltonian was given by Dobes and Pittel [8] for the particular case of degenerate shells. They have shown that in this case the exact solution of the isovector pairing Hamiltonian for even-even $N = Z$ systems can be expressed as a quartet condensate with the quartet defined as two isovector pairs coupled to total isospin $T = 0$. Later on quartet condensation models (QCMs) have been proposed for nondegenerate levels and applied for realistic

isovector pairing Hamiltonians [9,10]. Recently it was shown that the exact solution for the nondegenerate levels can be also expressed in terms of quartets [11] and that this solution turns to a quartet condensate in the strong-coupling regime [12]. All these studies have demonstrated that the α -like quartets are indispensable for a proper description of isovector pairing.

At this point it is worth stressing that the quartet condensation in the pairing context mentioned above should not be confused with the other “quartet condensate” concept, based on a similar wave function as the QCM one but dealing with *in medium bound states* of four fermions as, e.g., α particles, and their Bose-Einstein condensation [13,14] in finite nuclei and infinite nuclear matter. Although no α -particle condensate survives at saturation density, one may develop a theory for quartet condensation which in many aspects is similar to the BCS approach for the condensation of pairs [15].

Very recently we introduced such a BCS-like approach involving a quartet coherent state in the context of proton-neutron pairing [16]. The aim of this Rapid Communication is to further explore the implications of a quartet coherent-state ansatz for the proton-neutron pairing problem. This leads us to establish the general relation between the quartet models and the BCS-based models, which was previously investigated only for particular cases [8–10,17,18].

We consider the general isovector pairing Hamiltonian,

$$H = \sum_{i=1}^{N_{\text{lev}}} \epsilon_i N_{i,0} + \sum_{\tau=0,\pm 1} \sum_{i,j=1}^{N_{\text{lev}}} V_{ij} P_{i,\tau}^\dagger P_{j,\tau}, \quad (1)$$

where i, j denote the single-particle fourfold degenerate states and ϵ_i refers to the single-particle energies; a time-conjugated state will be denoted by \bar{i} . The first part is the standard single-particle term whereas the second part is the isovector pairing interaction expressed by the neutron-neutron ($\tau = 1$), proton-proton ($\tau = -1$), and proton-neutron ($\tau = 0$) pairs operators defined by $P_{i,\tau}^\dagger = [c_i^\dagger c_{\bar{i}}^\dagger]_\tau^{T=1}$. In the discussion

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below, we will frequently refer to the set of collective $\pi\pi$, $\nu\nu$, and $\pi\nu$ Cooper pairs $\Gamma_\tau^\dagger(x) \equiv \sum_{i=1}^{N_{\text{lev}}} x_i P_{i,\tau}^\dagger$, which depend on a set of mixing amplitudes x_i , $i = 1, 2, \dots, N_{\text{lev}}$. We denote by $q_i^\dagger = v_{i,\uparrow}^\dagger v_{i,\downarrow}^\dagger \pi_{i,\uparrow}^\dagger \pi_{i,\downarrow}^\dagger$ the isoscalar quartet operator that fills completely the level i .

The BCS-like quartet coherent-state ansatz introduced in Ref. [16] is written in terms of the QCM collective quartet operator $Q^\dagger(x) \equiv \sqrt{3}[\Gamma^\dagger \Gamma^\dagger]^{T=0} \equiv 2\Gamma_1^\dagger(x)\Gamma_{-1}^\dagger(x) - [\Gamma_0^\dagger(x)]^2$ as

$$|\text{QBCS}\rangle = \exp[Q^\dagger]|0\rangle = \sum_n \frac{1}{n!} [Q^\dagger]^n |0\rangle. \quad (2)$$

Below, we will explore in more detail the particular consequences of its coherent-state character.

The Rapid Communication concentrates on formal aspects of pairing and quartetting. We will first show that QBCS of Eq. (2) can be annihilated by a nonlinear transformation of fermion operators (mixing singles and triples and/or doubles with quadruples). This is in analogy to the well-known simpler case where a quasiparticle operator annihilates the BCS state. We will discuss how the former can open very interesting possibilities of calculus with the quartet coherent states. An interesting aspect will be that QBCS can be written as a Hubbard-Stratonovich transformation of a single-particle field. This will help to show that the number projected QBCS is analytically equivalent to the number projected BCS for $T = 0$ states. For $T > 0$ states the equivalence will be shown only numerically, getting very close to 100%.

QBCS annihilation operators. One of the major advantages of the BCS approach is the possibility of describing the paired system in a picture of weakly interacting “quasi-particles”, whose associated operators obey an annihilation condition with respect to the correlated BCS vacuum. Despite its nonlinear character, the above quartet-BCS state still admits a generalized class of annihilation operators due to its coherent-state nature [15]. However, at variance with the linear quasiparticles of the BCS case, the annihilation operators in the quartetting case do not obey simple linear equations of motion. For a specific particle operator c and for a specific pair operator $P = cc$, the general annihilation operators may be computed as

$$\begin{aligned} \alpha &= c + [Q^\dagger, c], \\ \beta &= P + [Q^\dagger, P]_{(2)} + \frac{1}{2}[Q^\dagger, P]_{(3,1)}, \end{aligned} \quad (3)$$

where we used the decomposition $[Q^\dagger, P] \equiv [Q^\dagger, P]_{(2)} + [Q^\dagger, P]_{(3,1)}$ which is of the form $c^\dagger c^\dagger + c^\dagger c^\dagger c^\dagger c$. Explicitly, a protonlike annihilation operator of the QBCS state has the form

$$\alpha_{i,\uparrow} = \pi_{i,\uparrow} - 2x_i \pi_{i,\downarrow}^\dagger \Gamma_1^\dagger(x) + \sqrt{2} x_i v_{i,\downarrow}^\dagger \Gamma_0^\dagger(x), \quad (4)$$

involving the annihilation of a particle and the creation of a particle dressed by a collective pair. Analogous relations hold for the other spin-isospin combinations.

The specific form of these nonlinear annihilation operators has interesting consequences; one, in particular, is the existence of a nontrivial connection between the two-body normal densities and the four-body anomalous densities. To see this, evaluate the average $\langle \text{QBCS} | \pi_{i,\uparrow}^\dagger \alpha_{i,\uparrow} +$

$\pi_{i,\downarrow}^\dagger \alpha_{i,\downarrow} | \text{QBCS} \rangle = 0$ by using Eq. (4). This leads to the occupations expressed in terms of the quartetting tensor as

$$n_i = \langle \pi_{i,\uparrow}^\dagger \pi_{i,\uparrow} + \pi_{i,\downarrow}^\dagger \pi_{i,\downarrow} \rangle = 2x_i \sum_j x_j \langle [P_i^\dagger P_j^\dagger]^{T=0} \rangle, \quad (5)$$

where the averaging is performed on the QBCS state. It follows that the total number of quartets may be expressed as the average of the collective quartet operator as

$$n_q = \langle \text{QBCS} | Q^\dagger(x) | \text{QBCS} \rangle. \quad (6)$$

This is a generalization of the simple BCS case with the ground-state $|\text{BCS}\rangle = \exp[\Gamma^\dagger(x)]|0\rangle$ and the annihilation operators $\alpha_{i,\uparrow} = c_{i,\uparrow} - x_i c_{i,\uparrow}^\dagger$. Here the occupations may be computed from $0 = \langle c_{i,\uparrow}^\dagger \alpha_{i,\uparrow} \rangle = n_i/2 - x_i \langle c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger \rangle$. It follows that the number of pairs is given by the average of the collective pair operator,

$$n_p = \sum_i n_i/2 = \langle \text{BCS} | \Gamma^\dagger(x) | \text{BCS} \rangle. \quad (7)$$

For the BCS case, we may also introduce the occupation and unoccupation amplitudes and recover the familiar form $n_p = \sum_i x_i \langle c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger \rangle = \sum_i (v_i/u_i) u_i v_i = \sum_i v_i^2$.

Returning to the second class of pairlike QBCS annihilation operators, the expressions resulting from Eq. (3) for each isovector pair $P_{k,\tau}$ are

$$\begin{aligned} \beta_{k,\pm 1} &= P_{k,\pm 1} - x_k^2 P_{k,\mp 1}^\dagger - 2x_k \Gamma_{\mp 1}^\dagger \\ &\quad + x_k \Gamma_{\mp 1}^\dagger N_{k,\pm 1} - x_k \Gamma_0^\dagger T_{k,\mp 1}, \\ \beta_{k,0} &= P_{k,0} + x_k^2 P_{k,0}^\dagger + 2x_k \Gamma_0^\dagger \\ &\quad + x_k \Gamma_1^\dagger T_{k,-1} - x_k \Gamma_{-1}^\dagger T_{k,1} - \frac{1}{2} x_k \Gamma_0^\dagger N_{k,0}, \end{aligned} \quad (8)$$

involving pair creation and annihilation terms, together with a nonlinear pair dressed by the particle number and isospin operators $T_{i,1} = -(\pi_{i,\uparrow}^\dagger v_{i,\uparrow} + \pi_{i,\downarrow}^\dagger v_{i,\downarrow})/\sqrt{2}$ and $T_{i,-1} = -T_{i,1}^\dagger$. Remarkably, there is another nonlinear combination that commutes exactly with the quartet operator. Explicitly, with

$$\eta_k \equiv \Gamma_1^\dagger T_{k,-1} - \Gamma_{-1}^\dagger T_{k,1} + \frac{1}{2} \Gamma_0^\dagger N_{k,0} - \frac{1}{2} x_k P_{k,0}^\dagger \sum_j N_{j,0}, \quad (9)$$

we have $[Q^\dagger, \eta] = 0$ and, thus, $\eta_k |\text{QBCS}\rangle = 0$. Because the isospin operators $T_{\pm 1} = \sum_k T_{k,\pm 1}$, obey $[Q^\dagger, T_{\pm 1}] = 0$, they also annihilate the isospin conserving QBCS state $T_{\pm 1} |\text{QBCS}\rangle = 0$.

The annihilation of the QBCS state by the operators η_k and $T_{\pm 1}$ leads to the fact that the operators in Eq. (8) are not actually uniquely defined. We could add to any of the β 's an arbitrary combination of η and PT and still obtain a valid pairlike annihilation operator. This freedom could allow for new treatments to be consistently developed for the pairing Hamiltonian, in analogy with Refs. [15,19] as will be explored in future works.

Structure of the QBCS state. Computations with the nonlinear QBCS ansatz are made tractable in Ref. [16] by a linearization procedure for the exponent. The quartet operator is first expressed as the square of a rotated collective pair

γ , $Q = \vec{\gamma}^\dagger \cdot \vec{\gamma}^\dagger$, defined by $\gamma_\tau^\dagger = \sum_{j=1}^{N_{\text{lev}}} x_j p_{j,\tau}^\dagger$, where

$$\begin{aligned} p_{j,1}^\dagger &= i(P_{j,1}^\dagger - P_{j,-1}^\dagger)/\sqrt{2}, & p_{j,2}^\dagger &= (P_{j,1}^\dagger + P_{j,-1}^\dagger)/\sqrt{2}, \\ p_{j,3}^\dagger &= -iP_{j,0}^\dagger. \end{aligned} \quad (10)$$

Note that this choice is not unique. A Hubbard-Stratonovich transformation is then used to represent the quartet coherent state as a combination of general isovector pair BCS states,

$$\begin{aligned} \exp(Q^\dagger) &= \exp(\vec{\gamma}^\dagger \cdot \vec{\gamma}^\dagger) \\ &= \int d^3z \exp(-\vec{z}^2/4 + \vec{z} \cdot \vec{\gamma}^\dagger) \\ &= \int d^3z e^{-\vec{z}^2/4} \prod_{i=1}^{N_{\text{lev}}} (1 + x_i \vec{z} \cdot \vec{p}_i^\dagger + x_i^2 \vec{z}^2 q_i^\dagger/2), \end{aligned} \quad (11)$$

where we omitted the overall normalization factor. In this way, we obtain a superposition of standard BCS states, each factorized as a product over the single-particle levels.

To better understand this specific pattern of partial symmetry breaking, it is instructive to pass to spherical coordinates in Eq. (11) and write the quartet coherent state as

$$\exp(Q^\dagger) = \int_0^\infty dz z^2 e^{-z^2/4} \int_{S^2} d\hat{n} \exp(z\hat{n} \cdot \vec{\gamma}^\dagger). \quad (12)$$

Naturally, the isospin projection is already implemented by the angular integration. To see this, consider the coherent state of the isovector pair $\vec{\gamma}$ integrated over all directions in isospace,

$$\begin{aligned} j_0^\dagger &\equiv \int_{S^2} d\hat{n} \exp(\hat{n} \cdot \vec{\gamma}^\dagger) \\ &= \sum_{k=0}^{\infty} \frac{(\vec{\gamma}^\dagger \cdot \vec{\gamma}^\dagger)^k}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(Q^\dagger)^k}{(2k+1)!} \\ &= j_0(i\sqrt{Q^\dagger}), \end{aligned} \quad (13)$$

which is formally the expansion of a spherical Bessel function of imaginary argument (hence, the name). The basic information about the quartet correlations is, thus, already contained in this simpler ansatz; by projecting onto good particle number, we always recover the QCM state,

$$\mathcal{P}_{n_q} \exp(Q^\dagger)|0\rangle = \mathcal{P}_{n_q} j_0^\dagger|0\rangle = (Q^\dagger)^{n_q}|0\rangle. \quad (14)$$

We interpret now the role of the radial integral in Eq. (12) as just changing the mixing between the components having different particle numbers.

The analytic expressions of the norm function and of the Hamiltonian average on the j_0^\dagger state may be obtained simply by dropping the radial integrals from the QBCS expressions (see Ref. [16], Supplemental Material). Remarkably, identical expressions were reported in Refs. [20,21] in the context of the symmetry restored BCS approach. The definition itself of the j_0^\dagger state hints at a precise relationship with the projected BCS state, which we detail below.

BCS symmetry restoration for $T = 0$. The generalized BCS equations for isovector pairing in even-even $N = Z$ systems present two degenerate solutions with gap parameters $\Delta_\nu = \Delta_\pi = \Delta$, $\Delta_{\pi\nu} = 0$, and $\Delta_\nu = \Delta_\pi = 0$, $\Delta_{\pi\nu} = \Delta$ (for a proof, see Ref. [17]). The corresponding BCS states are given by

$$\begin{aligned} |\text{BCS}_I\rangle &= \exp[\Gamma_1^\dagger(x)] \exp[\Gamma_{-1}^\dagger(x)]|0\rangle, \\ |\text{BCS}_{II}\rangle &= \exp[\Gamma_0^\dagger(x)]|0\rangle. \end{aligned} \quad (15)$$

Techniques for projecting these solutions onto good particle number and isospin have been developed in Refs. [20–26] with their connection to the quartet models only being mentioned for particular cases in Refs. [8,17,18].

Here, we establish the correspondence in the general case by analytically performing the projection operation on the BCS state, and recovering a version of the j_0^\dagger ansatz of Eq. (13). For simplicity, we consider the axially symmetric state $|\text{BCS}_{II}\rangle$ with $T_z = 0$, and we employ the isospin projection operator [27],

$$\mathcal{P}_{T;T_z=0} = \int_{S^2} d\hat{n} D_{00}^{T*}(\hat{n}) R(\hat{n}) \quad (16)$$

written in terms of a Wigner D matrix and of the rotation operator in isospin space $R(\hat{n})$, which may be factorized as $R(\hat{n}) = \prod_{i=1}^{N_{\text{lev}}} R_i(\hat{n})$. Given the isoscalar character of the fully occupied single-particle level $q_i^\dagger|0\rangle$, the only nontrivial term involves the rotation of the one-pair state. The isospin rotation operator $R_i(\hat{n}) = \exp(-i\varphi\hat{T}_z) \exp(-i\theta\hat{T}_y)$ acting on a $T_z = 0$ pair state is effectively

$$R_i(\hat{n}) P_{i,0}^\dagger R_i(\hat{n})^{-1} = i\hat{n} \cdot \vec{p}_i^\dagger, \quad (17)$$

involving the same rotated pairs \vec{p}_i^\dagger of Eq. (10) used to bring the collective quartet operator to a diagonal form. The isospin rotated BCS state becomes

$$\begin{aligned} R(\hat{n})|\text{BCS}_{II}\rangle &= \prod_{k=1}^{N_{\text{lev}}} (1 + ix_k \hat{n} \cdot \vec{p}_k^\dagger - x_k^2 q_k^\dagger/2)|0\rangle \\ &= \exp(i\hat{n} \cdot \vec{\gamma}^\dagger). \end{aligned} \quad (18)$$

This implies that the isospin projected BCS may be written as

$$\mathcal{P}_{T;T_z=0}|\text{BCS}_{II}\rangle = \int_{S^2} d\hat{n} D_{00}^{T*}(\hat{n}) \exp(i\hat{n} \cdot \vec{\gamma}^\dagger). \quad (19)$$

In particular, the $T = 0$ component is simply

$$\begin{aligned} \mathcal{P}_{T=0}|\text{BCS}_{II}\rangle &= \int_{S^2} d\hat{n} \exp(i\hat{n} \cdot \vec{\gamma}^\dagger) \\ &= \sum_{k=0}^{\infty} \frac{(-\vec{\gamma}^\dagger \cdot \vec{\gamma}^\dagger)^k}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-Q^\dagger)^k}{(2k+1)!} = j_0(\sqrt{Q^\dagger}), \end{aligned} \quad (20)$$

which is nothing else than Eq. (13) evaluated with imaginary mixing amplitudes or, equivalently, originating from the ansatz $\exp(-Q^\dagger)$.

This proves the general equivalence of the projected BCS and QCM approaches for the isovector pairing correlations in the $T = 0$ ground state of $N = Z$ even-even nuclei, i.e.,

$$\mathcal{P}_{T=0}^{N=4n_q} |\text{BCS}\rangle = (Q^\dagger)^{n_q} |0\rangle = |\text{QCM}\rangle. \quad (21)$$

Before detailing with the $N > Z$ case below, we remark the possibility of establishing nontrivial connections between the correlation functions also for the particle number projected QCM state, based on the above annihilation operators. We write Eq. (4) in schematic form $\alpha = c + c^\dagger c^\dagger c^\dagger$ and project the annihilation condition $\alpha \exp[Q^\dagger] |0\rangle = 0$ onto a fixed particle number, which singles out two terms. A proper particlelike annihilation operator for the QCM state may then be expressed in terms of the inverse amplitude coherent quartet, which satisfies $Q(1/x)Q^\dagger(x)|\text{QCM}\rangle = \lambda|\text{QCM}\rangle$ with λ a numerical factor (for details see Appendix A of Ref. [28]). We obtain, e.g., for the protonlike annihilation operator,

$$\left[\pi_{i,\uparrow} + \frac{n_q}{\lambda} [Q^\dagger, \pi_{i,\uparrow}] Q \left(\frac{1}{x} \right) \right] |\text{QCM}\rangle = 0, \quad (22)$$

where the commutator can be read off Eq. (4). In analogy with Eq. (5) for the quartet coherent state, we may obtain a relation between the particle and the quartet densities on the QCM state of the form $\langle \text{QCM} | c^\dagger c^\dagger c^\dagger c^\dagger c | \text{QCM} \rangle = \langle \text{QCM} | c^\dagger c^\dagger c^\dagger c | \text{QCM} \rangle$.

This is perfectly analogous to the simple single-species BCS case where the quasiparticle action on the BCS state $(c_{i,\uparrow} - x_i c_{i,\downarrow}^\dagger) \exp[\Gamma^\dagger(x)] |0\rangle = 0$ may be projected to obtain the nonlinear annihilation relation,

$$\left[c_{i,\uparrow} - \frac{x_i}{N_{\text{lev}} - n + 1} c_{i,\downarrow}^\dagger \Gamma \left(\frac{1}{x} \right) \right] [\Gamma^\dagger(x)]^n |0\rangle = 0. \quad (23)$$

We may then find the connection between the particle and the pair densities on the projected BCS state $|\text{PBCS}\rangle = [\Gamma^\dagger(x)]^n |0\rangle$ as

$$\langle c_{i,\uparrow}^\dagger c_{i,\uparrow} \rangle = \frac{x_i}{N_{\text{lev}} - n + 1} \sum_{j=1}^{N_{\text{lev}}} \frac{1}{x_j} \langle P_i^\dagger P_j \rangle. \quad (24)$$

Similar relationships may be established also for higher-order correlation functions, which could enable new ways of solving the pairing problem, e.g., within the recent many-body bootstrap approach [29,30].

QCM versus projected BCS for $N > Z$.

In the QCM quartetting approach, the states for the $N > Z$ systems are constructed by appending to the $N = Z$ ansatz additional coherent pairs [9]. A state with n_p excess neutron pairs and n_q quartets, having $T = T_z = n_p$ is defined as the particular combination,

$$|\text{QCM}(T = T_z = n_p)\rangle = [\Gamma_1^\dagger(y)]^{n_p} [Q^\dagger(x)]^{n_q} |0\rangle. \quad (25)$$

Here, one allows the extra collective pairs $\Gamma_1^\dagger(y)$ to have a different structure than the pairs $\Gamma^\dagger(x)$ forming the quartets.

The same idea may be applied to the BCS ansatz: Below, we consider the pair condensates of Eq. (16) to have different mixing amplitudes. Note that we also have to append a $\nu\nu$ condensate to the $\pi\nu$ condensate in this $N > Z$ case. In this section, we define $|\text{BCS}_{II}\rangle = \exp[\Gamma_1^\dagger(y)] \exp[\Gamma_0^\dagger(x)] |0\rangle$. We consider as illustrative examples an $N = 4$, $Z = 2$ system and an $N = 6$, $Z = 2$ system. The particle number and isospin projected combinations are

$$\mathcal{P}_{T=T_z=1}^{N=6} |\text{BCS}_I\rangle = (\Gamma_{1,x}^\dagger Q_y^\dagger - 3\Gamma_{1,y}^\dagger [\Gamma_x^\dagger \Gamma_y^\dagger]^{T=0}) |0\rangle, \quad (26a)$$

$$\mathcal{P}_{T=T_z=1}^{N=6} |\text{BCS}_{II}\rangle = (2\Gamma_{1,y}^\dagger Q_x^\dagger - \Gamma_{1,x}^\dagger [\Gamma_x^\dagger \Gamma_y^\dagger]^{T=0}) |0\rangle, \quad (26b)$$

$$\mathcal{P}_{T=T_z=2}^{N=8} |\text{BCS}_I\rangle = (5[\Gamma_{1,y}^\dagger]^2 [\Gamma_y^\dagger \Gamma_x^\dagger]^{T=0} - 2\Gamma_{1,y}^\dagger \Gamma_{1,x}^\dagger Q_y^\dagger) |0\rangle, \quad (26c)$$

$$\mathcal{P}_{T=T_z=2}^{N=8} |\text{BCS}_{II}\rangle = (11[\Gamma_{1,y}^\dagger]^2 Q_x^\dagger + 4[\Gamma_{1,x}^\dagger]^2 Q_y^\dagger - 12\Gamma_{1,x}^\dagger \Gamma_{1,y}^\dagger [\Gamma_x^\dagger \Gamma_y^\dagger]^{T=0}) |0\rangle, \quad (26d)$$

with the notation $\Gamma_x^\dagger = \Gamma^\dagger(x)$, $Q_y^\dagger = Q^\dagger(y)$, etc. Naturally, there are multiple options of coupling various pairs to a given total isospin, and the QCM ansatz of Eq. (25) is just a particular choice. Interestingly, the QCM choice does not appear in all previous expressions.

With the states (26), we performed variation-after-projection calculations for a picket-fence model of eight fourfold degenerate levels of single-particle energies $\epsilon_k = k - 1$ and with a state-independent interaction of strength G . The analytical expressions for the average of the isovector pairing Hamiltonian on the states (26) were derived with the CADABRA2 computer algebra system [31] using the method presented in Refs. [32,33].

In all cases, we obtained a very good agreement between the projected BCS and the QCM results. For the chosen model, the overlaps do not decrease lower than 0.999, and the relative errors in the correlation energies do not exceed 0.5%. We present in Fig. 1 the results for the lightest $N = 4$, $Z = 2$ system; the agreement between projected BCS and QCM improves for heavier systems (we note that the QCM ansatz gives a higher correlation energy in all cases).

Note that even in the case of equal pair and quartet mixing amplitudes ($x = y$) the results are still good: The obtained overlaps with the QCM state (having $x \neq y$) are always greater than 0.98, and the errors in the correlation energies are always smaller than 8%. In this case, all analytical expressions for the projected BCS states reduce to the QCM ansatz of Eq. (25) with $x = y$.

In constructing the QCM ansatz for $N > Z$ systems, Ref. [9] mentions the necessity of a different structure for the excess collective neutron pairs with respect to the collective pairs forming the quartet as to reproduce the Hartree-Fock limit. However, the present results indicate that whereas the $x = y$ choice introduces significant errors, it preserves the correct behavior in the weak pairing regime. Indeed, the Hartree-Fock vacuum may be obtained as a limit of the $x = y$ QCM ansatz by suitably scaling the mixing amplitudes. For the $N = 4$, $Z = 2$ and $N = 6$, $Z = 2$ systems with the scalings $w_\varepsilon = (1/\varepsilon, \varepsilon^2, 0, 0, \dots)$ and $z_\varepsilon = (1/\varepsilon, \varepsilon, \varepsilon, 0, 0, \dots)$,

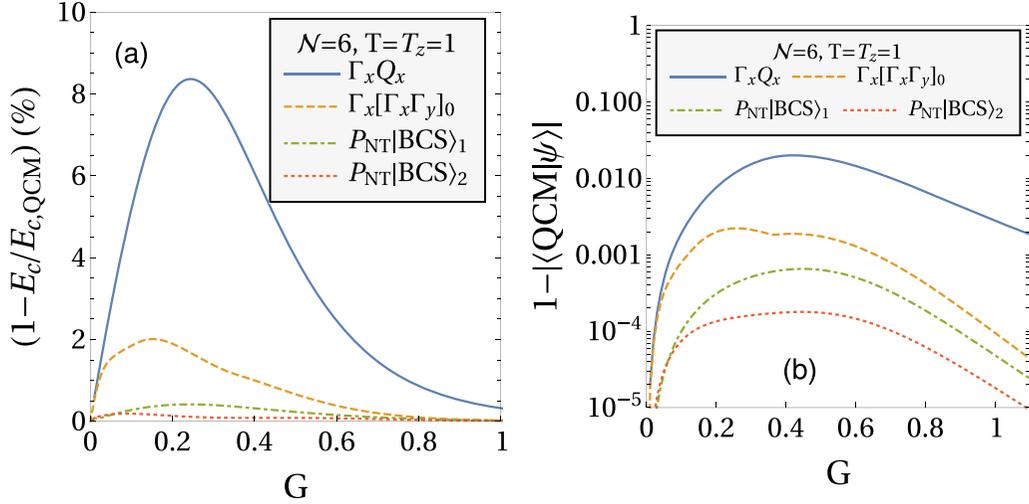


FIG. 1. Error in the correlation energy (a) $E_c = E(G) - E(G=0)$ and overlaps (b) for the states (26) relative to the QCM state (25), versus the interaction strength G (in units of the level spacing). The results are for two protons and four neutrons on eight equidistant levels. To indicate the results we use the notations: “ $\Gamma_x Q_x$ ” for the state $\Gamma_1^\dagger(x)Q^\dagger(x)|0\rangle$ of Eq. (25) with equal pair and quartet amplitudes $x = y$; “ $\Gamma_x[\Gamma_x \Gamma_y]_0$ ” for state $\Gamma_1^\dagger(x)[\Gamma^\dagger(x)\Gamma^\dagger(y)]^{T=0}|0\rangle$; “ $P_{NT}|BCS\rangle_{1,2}$ ” for the projected BCS states of Eqs. (26a) and (26b).

we obtain

$$\begin{aligned} \Gamma_1^\dagger(w_\varepsilon)Q^\dagger(w_\varepsilon) &\propto q_1^\dagger P_{2,1}^\dagger + O(\varepsilon), \\ [\Gamma_1^\dagger(z_\varepsilon)]^2 Q^\dagger(z_\varepsilon) &\propto q_1^\dagger P_{2,1}^\dagger P_{3,1}^\dagger + O(\varepsilon), \end{aligned} \quad (27)$$

which reduce to the exact Hartree-Fock state in the $\varepsilon \rightarrow 0$ limit.

Summary and conclusions. We presented an attempt at bridging the descriptions of the proton-neutron isovector pairing correlations in the symmetry preserving quartet picture and in the mean-field pair-condensate picture.

For both the coherent and the projected states, the non-linear annihilation operators are shown to generate nontrivial connections between the many-body correlation functions. A possible application of these relations would be to consider the novel quantum many-body bootstrap approach [29,30] and to implement the condensate property of the ansatz in terms of these constraints for the correlation functions. This would enable a numerically unified description, based on a quartet coherent state of both nuclear matter and finite nuclei. The same framework could be generalized to quartetting

in condensed-matter systems, e.g., to the study of biexciton condensation in semiconductors or trapped fermionic atoms in optical lattices.

Then, inspired by the structure of the quartet coherent state, we have shown that the QCM ansatz for the ground state of even-even $N = Z$ systems can be obtained by projecting out the particle number and the isospin from a proton-neutron BCS state. For the $N > Z$ systems the P_{NT} BCS and QCM states are not analytically equivalent. However, their overlaps are very close to one. The numerical P_{NT} BCS calculations indicate that the particular way of coupling various pairs to the total isospin of the $N > Z$ system does not influence much the final results as long as the trial states obeys the correct symmetry constraints. An interesting question is whether these facts hold in the case of an isovector-isoscalar pairing Hamiltonian. This issue we intend to address in a future study.

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