Neutron-proton pairing correction in the extended isovector and isoscalar pairing model

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An extended $\mathrm{O}_{\mathcal{S}T}(8)$ model with multi-j orbits is constructed based on the angular momentum decomposition with "pseudo"-spin $\mathcal{S}$ for valence nucleons in a j orbit. It is shown that the isovector $\mathcal{S}=0$ and $T=1$ pairs are exactly the $J=0$ and $T=1$ pairs in a given j orbit, while the isoscalar $\mathcal{S}=1$ pairs are linear combinations of $J=$ odd pairs, with which the pairing Hamiltonian can be used to estimate isovector and isoscalar pairing interactions. As an example of the model application, some low-lying $J=0^+$ level energies of even-even and odd-odd $A=18$–28 nuclei up to the half-filling in the ds shell above the ^{16}O core are fit by the model and compared with the fitting results of the same Hamiltonian in the $\mathrm{O}_{LST}(8)$ form. It has been verified from the fitting of both the models that the isoscalar pairing interaction can be neglected in the lower-energy part of the spectra of these ds-shell nuclei as far as binding energies and a few $J=0^+$ excited levels of these nuclei are concerned. With the mean-field plus isovector pairing interaction only, neutron-neutron, proton-proton, and neutron-proton pairing contributions at the ground or the lowest $J=0^+$ state of these nuclei are estimated. It is shown that the isovector np pairing contribution to the binding in the odd-odd $N=Z$ nuclei is systematically larger than that in the even-even nuclei. Furthermore, the isoscalar np pair content at the lowest $J=0^+$ state of these nuclei is also estimated. In both the $\mathrm{O}_{\mathcal{S}T}(8)$ and $\mathrm{O}_{LST}(8)$ models, it is clearly shown that the isoscalar pair content in the lowest $J=0^+$ state of the $N=Z$ and $N=Z\pm 2$ nuclei increases with increasing of the valence nucleons, especially in those even-even nuclei, which indicates the isoscalar pairing correlation to be of importance at low-lying states of $N=Z$ and $N=Z\pm 2$ nuclei, especially in those even-even nuclei with more valence nucleons up to the half-filling, even though the isoscalar pairing interaction is negligible.

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I. INTRODUCTION

It is shown from both theoretical and experimental studies that, besides the isovector pairing, isoscalar pairing may also be of importance in $N \approx Z$ nuclei [1–7]. Besides studies in the framework of Hartree-Fock-Bogoliubov theory [1,3], shell-model calculations with effective interactions focusing on the isovector and isoscalar pairing mainly for $N \approx Z$ fp-shell nuclei were carried out extensively [4,7–13]. The α-like quartet structure of the isovector plus isoscalar pairing ground state has also been studied [14–17]. The O(8) algebraic construction of the isovector plus isoscalar pairing realized in the LST-coupling scheme was proposed in Ref. [18], of which the matrix representation was derived explicitly in Refs. [19,20]. Further analysis and applications of the $\mathrm{O}_{LST}(8)$ model were then made in Refs. [21–23]. Exact solution of the charge-independent mean-field with l-orbit-dependent single-particle energies plus isovector and isoscalar pairing was presented [24,25], in which the isovector and isoscalar pairing strengths were assumed to be the same. However, the single-particle energy term in the $\mathrm{O}_{LST}(8)$ model as used in Refs. [19,21–24] cannot properly reproduce the mean-field part of the ground-state energy mainly due to the fact that the model space is restricted within the $L=0$ configuration only, of which the basis vectors are incomplete. An extension of the original $\mathrm{O}_{LST}(8)$ model including $\mathrm{L} \neq 0$ states is necessary, of which, however, the model calculation becomes tedious with other state vectors outside of the $\mathrm{O}_{LST}(8)$ prescription.

In this work, similar to the original $\mathrm{O}_{LST}(8)$ model in the LST-coupling scheme, we consider a different angular momentum decomposition for a valence nucleon in a single-j orbit, which enables us to realize an extended $\mathrm{O}_{\mathcal{S}T}(8)$ model with multi-j orbits. The paper is organized as follows. The construction of the extended $\mathrm{O}_{\mathcal{S}T}(8)$ model is presented in Sec. II. In Sec. III, based on the results shown in Sec. II, the model Hamiltonian in the ds shell is diagonalized in the tensor product subspace of $\mathrm{O}_{\mathcal{S}T}(8)$ in the $\mathrm{O}_{\mathcal{N}}(2)\otimes \mathrm{O}_{\mathcal{S}T}(6)$ basis. Application of the model to even-even and odd-odd ds-shell nuclei in analyzing the isovector and isoscalar pairing correlation is made in Sec. IV. A brief summary is presented in Sec. V.

II. $\mathrm{O}_{\mathcal{S}T}(8)$ IN THE $\mathrm{U}_{\mathcal{S}T}(4) \supset \mathrm{O}_{\mathcal{N}}(2)\otimes \mathrm{SU}_{\mathcal{S}}(2)\otimes \mathrm{SU}_{\mathrm{T}}(2)$ BASIS

Let $\{a^{\dagger}_{jmtm_t}, a_{jmtm_t}\}$ be a set of creation and annihilation operators for a valence nucleon with isospin $t=1/2$ in a j orbit. To realize a similar pair structure of the $\mathrm{O}_{LST}(8)$ model for a j orbit, the angular momentum j is decomposed as $j=2\ell+s$, where $s=1/2$ is the "pseudo"-spin of a valence nucleon in the j orbit, while $\ell=0, 1/2, 1, 3/2, 2, \ldots$ serves as a parameter of the decomposition. Thus, for a given

$j=2\ell+s$, we set

$$\begin{aligned} a^{\dagger}_{j\,j\,tm_t} &= a^{\dagger}_{\ell\ell;s\frac{1}{2};tm_t},\\ a^{\dagger}_{j\,j-1\,tm_t} &= a^{\dagger}_{\ell\ell;s-\frac{1}{2};tm_t},\ldots,a^{\dagger}_{j\,-j+1\,tm_t}=a^{\dagger}_{\ell-\ell;s\frac{1}{2};tm_t},\\ a^{\dagger}_{j\,-j\,tm_t} &= a^{\dagger}_{\ell-\ell;s-\frac{1}{2};tm_t}. \end{aligned} \tag{1}$$

It should be noted that m_ℓ in $a^{\dagger}_{\ell m_\ell;sm_s;tm_t}$ can be taken as one of the $2\ell+1$ values $-\ell,-\ell+1,\ldots,\ell$. Thus, Eq (1) provides the one-to-one correspondence between $\{a^{\dagger}_{\ell m_\ell;sm_s;tm_t}\}$ and $\{a^{\dagger}_{jm\,tm_t}\}$ with $4j+2=8\ell+4$ of them in each set. Furthermore, it should be stated that the decomposition with the "pseudo"-spin and the quasi angular momentum with its quantum number $\ell=(j-1/2)/2$ are not the same as the pseudo-spin and the pseudo-orbital angular momentum decomposition proposed previously [26].

Within the present decomposition scheme, the $\mathcal{S}=0$, $T=1$ and $\mathcal{S}=1$, $T=0$ pair creation operators can be written as

$$P^{\dagger}_{\mu}=\sqrt{\frac{1}{2}}(a^{\dagger}_{\ell st}\,a^{\dagger}_{\ell st})^{01}_{0\mu},\;\; D^{\dagger}_{\mu}=\sqrt{\frac{1}{2}}(a^{\dagger}_{\ell st}\,a^{\dagger}_{\ell st})^{10}_{\mu 0}, \tag{2}$$

respectively, where

$$(a^{\dagger}_{\ell st}\,a^{\dagger}_{\ell st})^{\mathcal{S}T}_{M_{\mathcal{S}}M_T}=\sum_{m_\ell=-\ell}^{\ell}\sum_{m_sm_s'm_tm_t'}\langle sm_s,sm_s'|\mathcal{S}M_S\rangle \times\langle tm_t,tm_t'|TM_T\rangle a^{\dagger}_{\ell m_\ell;sm_s;tm_t}a^{\dagger}_{\ell\,-m_\ell;sm_s';tm_t'}, \tag{3}$$

in which the related Clebsch-Gordan coefficients are involved. Similarly, the number-conserving generators of $U_{\mathcal{S}T}(4)$ can be expressed as

$$\begin{aligned} \hat{n}&=2(a^{\dagger}_{\ell st}\,\tilde{a}_{\ell st})^{00}_{00}, \quad T_\mu=(a^{\dagger}_{\ell st}\,\tilde{a}_{\ell st})^{01}_{0\mu},\\ \mathcal{S}_\mu&=(a^{\dagger}_{\ell st}\,\tilde{a}_{\ell st})^{10}_{\mu 0}, \quad W_{\mu,\mu'}=(a^{\dagger}_{\ell st}\,\tilde{a}_{\ell st})^{11}_{\mu\mu'}, \end{aligned} \tag{4}$$

where the ℓ-part coupling is the same as that given in Eq. (3), and

$$\tilde{a}_{\ell m_\ell;sm_s;tm_t}=(-1)^{1+m_s+m_t}a_{\ell\,-m_\ell;s\,-m_s;t\,-m_t}. \tag{5}$$

Using the correspondence shown in Eq. (1), one can check that $\mathcal{S}=0$, $T=1$ pair operators $P^{\dagger}_{\mu}$ ($\mu=-1,0,1$) are exactly the $J=0$, $T=1$ pair operators in the j orbit with

$$P^{\dagger}_{\mu}=\sqrt{\frac{1}{2}}(a^{\dagger}_{\ell st}\,a^{\dagger}_{\ell st})^{01}_{0\mu}=\frac{\sqrt{2j+1}}{2}(a^{\dagger}_{j,t}\,a^{\dagger}_{j,t})^{01}_{0\mu}. \tag{6}$$

However, due to the special decomposition (1), $\mathcal{S}=1$, $T=0$ pair operators $D^{\dagger}_{\mu}$ are a linear combination of $J=$ odd, $T=0$ pair operators with $J=1,3,\ldots,2j$. For example, when $j=3/2$,

$$\begin{aligned} (a^{\dagger}_{\ell st}\,a^{\dagger}_{\ell st})^{10}_{00}&=\sqrt{\frac{2}{5}}(a^{\dagger}_{j,t}\,a^{\dagger}_{j,t})^{10}_{00}+\sqrt{\frac{8}{5}}(a^{\dagger}_{j,t}\,a^{\dagger}_{j,t})^{30}_{00},\\ (a^{\dagger}_{\ell st}\,a^{\dagger}_{\ell st})^{10}_{\pm 10}&=\sqrt{\frac{6}{5}}(a^{\dagger}_{j,t}\,a^{\dagger}_{j,t})^{10}_{\pm 10}+\sqrt{\frac{4}{5}}(a^{\dagger}_{j,t}\,a^{\dagger}_{j,t})^{30}_{\pm 10}. \end{aligned} \tag{7}$$

Therefore, if $\mathcal{S}=1$, $T=0$ D pair operators are involved in a Hamiltonian, the total angular momentum J and spin S are not good quantum numbers when $j\geqslant 3/2$. The only exception is the $j=1/2$ case, in which $D^{\dagger}_{\mu}=\frac{\sqrt{2j+1}}{2}(a^{\dagger}_{j,t}\,a^{\dagger}_{j,t})^{10}_{\mu 0}$ are just the $J=1$ and $T=0$ pairing operators. Anyway, similar to the $O_{LST}(8)$ case, the pairing operators P_μ, $P^{\dagger}_{\mu}$, D_μ, and $D^{\dagger}_{\mu}$ shown in Eq. (2) and the number-conserving generators of U(4) defined in Eq. (4) obey the same commutation relations as those of generators of O(8), which is called $O_{\mathcal{S}T}(8)$ in the following.

The $O_{\mathcal{S}T}(8)$ irrep is denoted as $(\Omega-\frac{v}{2},p_1,p_2,p_3)$, where, instead of $\Omega_l=2l+1$ in the original $O_{LST}(8)$ model, $\Omega=j+1/2$ for a given j orbit, v is the $O_{\mathcal{S}T}(8)$ seniority number indicating that there are v nucleons free of the pairs defined in Eq. (3), (p_1,p_2,p_3) is an intrinsic $O_{\mathcal{S}T}(6)$ irrep, which can also be expressed as the corresponding $U_{\mathcal{S}T}(4)$ irrep $[\omega_1,\omega_2,\omega_3,\omega_4]$ satisfying $\sum_{i=1}^{4}\omega_i=v$ and $\omega_1\geqslant\omega_2\geqslant\omega_3\geqslant\omega_4\geqslant 0$ with

$$\begin{aligned} p_1&=\tfrac{1}{2}(\omega_1+\omega_2-\omega_3-\omega_4),\\ p_2&=\tfrac{1}{2}(\omega_1-\omega_2+\omega_3-\omega_4),\\ p_3&=\tfrac{1}{2}(\omega_1-\omega_2-\omega_3+\omega_4). \end{aligned} \tag{8}$$

For the $O_{\mathcal{S}T}(8)$ seniority-zero and -one cases, due to the local isomorphism of $U_{\mathcal{S}T}(4)$ with $O_{\mathcal{N}}(2)\otimes O_{\mathcal{S}T}(6)$, where $\mathcal{N}$ is related to the total number of valence nucleons with $\mathcal{N}=\Omega-\hat{n}/2$ for a given j orbit, the related branching rules of $O_{\mathcal{S}T}(8)\supset U_{\mathcal{S}T}(4)\supset O_{\mathcal{N}}(2)\otimes SU_{\mathcal{S}}(2)\otimes SU_T(2)$ can be expressed as those of $O_{\mathcal{S}T}(8)\supset O_{\mathcal{N}}(2)\otimes O_{\mathcal{S}T}(6)\supset O_{\mathcal{N}}(2)\otimes SU_{\mathcal{S}}(2)\otimes SU_T(2)$:

$$\begin{aligned} &O_{\mathcal{S}T}(8)\quad\downarrow\quad O_{\mathcal{N}}(2)\otimes O_{\mathcal{S}T}(6)\\ &(\Omega,0)=\oplus_{n=\text{even}}^{4\Omega}\oplus_{\sigma_0=0}^{[(\Omega-|\Omega-n/2|)/2]}(\mathcal{N}=\Omega-n/2)\otimes(\Omega-|\Omega-n/2|-2\sigma_0,0),\\ &\left(\Omega-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)=\oplus_{\sigma=0}^{\Omega-1}\oplus_{i=0}^{\Omega-1-\sigma}[\mathcal{N}=\Omega-(2\sigma+4i+1)/2]\otimes\left(\sigma+\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)\\ &\qquad\oplus_{\sigma=0}^{\Omega-1}\oplus_{i=0}^{\Omega-1-\sigma}[\mathcal{N}=\Omega-(2\sigma+4i+3)/2]\otimes\left(\sigma+\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} &O_{\mathcal{S}T}(6)\quad\downarrow\quad SU_{\mathcal{S}}(2)\otimes SU_T(2)\\ &(\sigma,0)=\oplus_{i=0}^{\sigma}\oplus_{q=0}^{[i/2]}(\mathcal{S}=\sigma-i)\otimes(T=i-2q),\\ &\left(\sigma-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)\sim\left(\sigma-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right)=\oplus_{i=1}^{[(2\sigma+1)/2]}\oplus_{q=0}^{[(2i-1)/2]}\left(\mathcal{S}=\sigma-i+\tfrac{1}{2}\right)\otimes\left(T=i-q-\tfrac{1}{2}\right), \end{aligned} \tag{10}$$

where $[y]$ denotes the integer part of y. The closed expressions of the branching rules shown in Eqs. (9) and (10) are consistent with the results presented in Refs. [19,27,28] and can be checked by using the dimension formulas of O(8) and O(6):

$$\dim[(\Omega,0),\mathrm{O}(8)] = \frac{(2\Omega+6)(\Omega+5)!}{6!\Omega!},\quad \dim\left[\left(\Omega-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\mathrm{O}(8)\right] = \frac{(\Omega+5)!}{90(\Omega-1)!},$$
$$\dim[(\sigma,0,0),\mathrm{O}(6)] = \frac{(\sigma+2)(\sigma+3)!}{12\sigma!},\quad \dim\left[\left(\sigma-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\mathrm{O}(6)\right] = \dim\left[\left(\sigma-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right),\mathrm{O}(6)\right] = \frac{(\sigma+3)!}{3!(\sigma-1)!}. \quad (11)$$

III. DIAGONALIZAING THE $\mathrm{O}_{\mathcal{S}T}(8)$ MODEL HAMILTONIAN

The extended charge-independent $\mathrm{O}_{\mathcal{S}T}(8)$ model Hamiltonian is given by

$$\hat{H}_0 = \sum_{i=1}^{p} \epsilon_{j_i}\hat{n}_{j_i} - G_1\sum_{\rho} P_\rho^\dagger P_\rho - G_0\sum_{\rho'} D_{\rho'}^\dagger D_{\rho'}, \qquad (12)$$

where p is the number of j orbits considered, $\hat{n}_{j_i} = \sum_{mm_t} a^\dagger_{j_im;tm_t} a_{j_im;tm_t}$ is the valence-nucleon number operator in the j_i orbit, ϵ_{j_i} is the valence-nucleon single-particle energy of the j_i orbit, $P_\rho^\dagger = \sum_{i=1}^p P_\rho^\dagger(i)$, $D_\rho^\dagger = \sum_{i=1}^p D_\rho^\dagger(i)$, and $P_\rho = \sum_{i=1}^p P_\rho(i)$, $D_\rho = \sum_{i=1}^p D_\rho(i)$ are collective ($\mathcal{S}=0$, $T=1$) and ($\mathcal{S}=1$, $T=0$) pairing operators, $G_1>0$ and $G_0>0$ are isovector and isoscalar pairing interaction strength, respectively. When $G_0=0$, the Hamiltonian (12) is exactly the mean-field plus the $J=0$, $T=1$ pairing one [29–33]. The related $\mathrm{O}_{\mathcal{S}T}(8)$ irrep for a given j orbit in this case can also be decomposed according to the $\mathrm{O}_{\mathcal{S}T}(8)\supset \mathrm{O}_T(5)\otimes\mathrm{O}_{\mathcal{S}}(3)$ branching [28]. In this case, though the Hamiltonian of the $\mathrm{O}_{\mathcal{S}T}(8)$ model is the same as that of the O(5) isovector pairing model, the configuration subspaces of the two models are different. For example, if the Hamitonian is diagonalized in the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero tensor product subspace, which includes both $J=0$, $T=1$ and $J=$ odd, $T=0$ pair states, the eigenstates should be different from those of the same Hamiltonian diagonalized in the seniority-zero O(5) subspace with $J=0$, $T=1$ pair states only. Though the total angular momentum quantum number J turns to be not a good quantum number, the Hamiltonian (12) can be used to estimate $J=0$, $T=1$ and $J=$ odd, $T=0$ pairing strengths, especially at the ground state in even-even and odd-odd nuclei described by the model with any number of j orbits considered, of which the situation is quite similar to the deformed mean-field plus $K=0$ pairing model [34], where K is the quantum number of the angular momentum projection in the intrinsic frame.

The Hamiltonian (12) is diagonalized in the subspace of the tensor product $\otimes_{i=1}^p \mathrm{O}_{\mathcal{S}T}^{(i)}(8)$ basis when p j-orbits are included, in which each copy of the $\mathrm{O}_{\mathcal{S}T}(8)$ irrep is adapted to the $\mathrm{O}_{\mathcal{S}T}(8)\supset \mathrm{U}_{\mathcal{S}T}(4)\supset \mathrm{O}_{\mathcal{N}}(2)\otimes\mathrm{SU}_{\mathcal{S}}(2)\otimes\mathrm{SU}_T(2)$ chain. Though the procedure for the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority nonzero cases is the same, in this work only the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero configuration is considered. Eigenstates of Eq. (12) within the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero subspace are denoted as

$$|\zeta;n,\mathcal{S}M_{\mathcal{S}}\,TM_T\rangle = \sum_{n_i\sigma_i\mathcal{S}_iT_i\xi} C^{\zeta;n_1\sigma_1,\ldots,n_p\sigma_p}_{\mathcal{S}_1,T_1,\ldots,\mathcal{S}_p,T_p,\xi}\left|\begin{matrix}(\Omega_1,0);\ \ldots;\ (\Omega_p,0)\\ n_1\sigma_1;\ \ \ldots;\ n_p\sigma_p;\quad \xi\,\mathcal{S}M_{\mathcal{S}}\,TM_T\\ \mathcal{S}_1,T_1;\ \ldots;\ \mathcal{S}_p,T_p\end{matrix}\right\rangle, \qquad (13)$$

where the eigenstate $|\zeta;n,\mathcal{S}M_{\mathcal{S}}\,TM_T\rangle$ with the total number of valence nucleons $n=\sum_{i=1}^p n_i$, "pseudo"-spin $\mathcal{S}$ and isospin T is expended in terms of the tensor product basis of the p copies of $\mathrm{O}_{\mathcal{S}T}(8)$ irreps $\otimes_{i=1}^p(\Omega_i,0)$ in the $\mathrm{O}_{\mathcal{S}T}(8)\supset \mathrm{U}_{\mathcal{S}T}(4)\supset \mathrm{O}_{\mathcal{N}}(2)\otimes\mathrm{SU}_{\mathcal{S}}(2)\otimes\mathrm{SU}_T(2)$ labeling scheme, ξ is a set of the $\mathcal{S}$ and T multiplicity labels needed in the coupling, $C^{\zeta;n_1\sigma_1,\ldots,n_p\sigma_p}_{\mathcal{S}_1,T_1,\ldots,\mathcal{S}_p,T_p,\xi}$ is the corresponding expansion coefficient, and ζ labels the ζth eigenstate with the same n, $\mathcal{S}$, and T. Matrix elements of each term involved in Eq. (12) under the $\mathrm{O}_{\mathcal{S}T}(8)$ tensor product basis of $\otimes_{i=1}^p(\Omega_i,0)$ in the $\mathrm{O}_{\mathcal{S}T}(8)\supset \mathrm{U}_{\mathcal{S}T}(4)\supset O_{\mathcal{N}}(2)\otimes\mathrm{SU}_{\mathcal{S}}(2)\otimes\mathrm{SU}_T(2)$ labeling scheme can be evaluated by using the results shown in Refs. [19,22,35], of which the explicit expressions are also provided in the Appendix 1.

Using the analytical expressions of the reduced matrix elements $A^{(i)\dagger}$ in the $\mathrm{U}(4)\supset O_{\mathcal{N}}(2)\otimes\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ labeling scheme shown in the Appendix, Sec. 1, where $A^{(i)\dagger}=P^{(i)\dagger}$ or $A^{(i)\dagger}=D^{(i)\dagger}$, one can verify that eigenvalues of the isovector pairing Hamiltonian $\hat{H}_P=\sum_\rho P_\rho^\dagger P_\rho$, those of the isoscalar pairing one $\hat{H}_D=\sum_\rho D_\rho^\dagger D_\rho$, and those of the $\mathrm{U}_{\mathcal{S}T}(4)$-limit one $\hat{H}_{\mathrm{U}_{\mathcal{S}T}(4)}=\sum_\rho(P_\rho^\dagger P_\rho + D_\rho^\dagger D_\rho)$ in the $\mathrm{O}_{\mathcal{S}T}(8)$ tensor product basis adapted to the $\mathrm{U}_{\mathcal{S}T}(4)\supset O_{\mathcal{N}}(2)\otimes\mathrm{SU}_{\mathcal{S}}(2)\otimes\mathrm{SU}_T(2)$ chain in the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero subspace are always integers or 0 in both the original $\mathrm{O}_{\mathrm{LST}}(8)$ and the extended $\mathrm{O}_{\mathcal{S}T}(8)$ models, of which $\mathcal{S}=S=0$ cases of $n\leqslant 6$ particles over three j-orbits with $j_1=1/2$, $j_2=3/2$, and $j_3=5/2$, together with the corresponding ones in the $\mathrm{O}_{LST}(8)$ model for $n\leqslant 6$ particles over two l-orbits with $l_1=0$ and $l_2=5$, are shown in Table I as examples. This feature is quite similar to that in the O(5) isovector pairing model and can be used to check the validity of the computation code. In addition, the dimensions of the model subspaces are greatly reduced in the original $\mathrm{O}_{LST}(8)$ model because the $\mathrm{O}_{LST}(8)$ configuration is restricted within the $L=0$ subspace only. It can be observed from the eigenvalues of $\hat{H}_P$, $\hat{H}_D$, and $\hat{H}_P+\hat{H}_D$ for given n, T, and $\mathcal{S}=S=0$ that the subset of

TABLE I. Eigenvalues of $\hat{H}_P$, $\hat{H}_D$, and $\hat{H}_P + \hat{H}_D$ in the $O_{\mathcal{ST}}(8)$ model for $n \leqslant 6$ particles over three j-orbits, with $j_1 = 1/2$, $j_2 = 3/2$, and $j_3 = 5/2$, and those of the $O_{LST}(8)$ model over $l_1 = 0$ and $l_2 = 5$ orbits within the seniority-zero subspace of each model with $\mathcal{S} = S = 0$ and isospin T, where the dim[$O_{\mathcal{ST}}(8)$] and the dim[$O_{LST}(8)$] columns provide the corresponding dimensions of the subspaces in diagonalizing the corresponding pairing Hamiltonian, the superscript r of the eigenvalues indicates that the corresponding eigenvalue occurs r times if $r \geqslant 2$, and "—" denotes that the corresponding state does not exist in the $O_{LST}(8)$ model.

n, T	dim[$O_{\mathcal{ST}}(8)$]	E_P	E_D	E_{P+D}	dim[$O_{LST}(8)$]	E_P	E_D	E_{P+D}
2, 1	3	$6, 0^2$	0^3	$6, 0^2$	2	$6, 0$	0^2	$6, 0$
4, 0	11	$13, 7^2\ 0^8$	$13, 7^2, 0^8$	$16, 10^3, 4^2, 0^5$	5	$13, 7, 0^3$	$13, 7, 0^3$	$16, 10^2, 4, 0$
4, 1	3	$6^2, 0$	0^3	$6^2, 0$	1	6	0	6
4, 2	5	$10, 4^2, 0^2$	0^5	$10, 4^2, 0^2$	2	$10, 4$	0^2	$10, 4$
6, 0	2	$6, 0$	$6, 0$	$6, 6$	–			
6, 1	27	$17, 11^4, 7^2$ $5, 4^8, 0^{11}$	$11^3, 6^5, 0^{19}$	$20, 14^4, 12,$ $10^2, 9^2, 8, 6^2$ $5^2, 4^3, 3, 2^2, 0^6$	7	$17, 11^2, 4^3, 0$	$11^2, 6, 0^4$	$20, 14^2, 12, 9, 6, 4$
6, 2	6	$9^2, 5^2, 3, 0$	0^6	$9^2, 5^2, 3, 0$	1	9	0	9
6, 3	6	$12, 6^2, 2^2, 0$	0^6	$12, 6^2, 2^2, 0$	2	$12, 6$	0^2	$12, 6$

the eigenvalues of one of the aforementioned Hamiltonians in the $O_{\mathcal{ST}}(8)$ model overlaps with the whole set of eigenvalues of the corresponding one of the $O_{LST}(8)$ model, especially the highest eigenvalue of one of the Hamiltonians of both the models, which is most important for the ground-state energy of the system, is exactly the same. The fact that there are more eigenvalues of any one of the Hamiltonians in the $O_{\mathcal{ST}}(8)$ model is because the dimension of the $O_{\mathcal{ST}}(8)$ tensor product subspace is always greater than that of the corresponding $O_{LST}(8)$ one. Therefore, the $O_{\mathcal{ST}}(8)$ pairing Hamiltonian in the $\mathcal{S} = 0$ subspace is indeed quite the same as that of the $O_{LST}(8)$ model in the $S = 0$ subspace. Anyway, the $O_{\mathcal{ST}}(8)$ model should provide results similar to those of the $O_{LST}(8)$ model, especially those within the seniority-zero subspace.

IV. MODEL APPLICATIONS TO EVEN-EVEN AND ODD-ODD *ds*-SHELL NUCLEI

In the $O_{\mathcal{ST}}(8)$ seniority-zero subspace as considered, the $O_{\mathcal{ST}}(8)$ model with $G_0 = 0$ is equivalent to the mean-field plus isovector pairing Hamiltonian diagonalized within the O(5) seniority-zero and -nonzero configurations including isoscalar np pairs, while the $O_{LST}(8)$ model is equivalent to the aforementioned calculation restricted within the $L = 0$ subspace. When $G_0 \neq 0$, the total angular momentum J is not a good quantum number of the model. The standard angular momentum projection [36] is required, with which one can calculate the mean value of excitation energies of the model for a given J. Because the eigenenergy of the model increases with $\mathcal{S}$, it can be expected that the lowest mean value of energy with $J = 0$ is mainly contributed from the lowest eigenenergy with $\mathcal{S} = 0$ in the $O_{\mathcal{ST}}(8)$ model, especially when $G_0 > 0$ is small.

As an example of the $O_{\mathcal{ST}}(8)$ model application, some low-lying $J = 0^+$ level energies of even-even and odd-odd $A =$ 18–28 nuclei up to the half-filling in the ds shell above the ^{16}O core is fit by the Hamiltonian (12) with $G_1 = G(1 + x)/2$ and $G_0 = G(1 - x)/2$ in the $O_{\mathcal{ST}}(8)$ seniority-zero and $\mathcal{S} = 0$ subspace, where G is the overall pairing strength, x is within the closed interval $x \in [-1, 1]$. Comparison to the fitting results of the same Hamiltonian in the $O_{LST}(8)$ form is also made. To fit binding energies of these nuclei, in addition to the mean-field plus isovector and isoscalar pairing, the Coulomb energy and the symmetry energy with the isospin-dependent part of the Wigner energy contribution to the binding are considered, leading to the expression of the model Hamiltonian similar to that used in the isovector pairing model [37]:

$$\hat{H} = -BE(^{16}\mathrm{O}) + \bar{\epsilon}(\hat{n})\,\hat{n} + \hat{H}_0 + E_c(A, N, Z) - E_c(16, 8, 8) + \alpha_{\mathrm{sym}}(A, N, Z)\,\mathbf{T}\cdot\mathbf{T}, \tag{14}$$

where $\hat{H}_0$ is the mean-field plus the isovector and isoscalar pairing Hamiltonian of either the $O_{\mathcal{ST}}(8)$ form given by Eq. (12) or the $O_{LST}(8)$ form shown in Refs. [19–22], $BE(^{16}\mathrm{O}) = 127.619$ MeV is the binding energy of the ^{16}O core taken as the experimental value, $\bar{\epsilon}(n)$ is the average binding energy per valence nucleon in the ds shell, of which the valence-nucleon number dependent form is determined from a best fit to binding energies of all ds-shell nuclei considered,

$$E_c(A, N, Z) = 0.699\frac{Z(Z-1)}{A^{1/3}}\left(1 - \frac{0.76}{[Z(Z-1)]^{1/3}}\right)(\mathrm{MeV}) \tag{15}$$

is the Coulomb energy [38], and

$$\alpha_{\mathrm{sym}}(A, N, Z) = \frac{1}{A}\left(134.4 - \frac{203.6}{A^{1/3}}\right)(\mathrm{MeV}) + \delta\alpha(A) \tag{16}$$

is the parameter of the symmetry energy and the isospin-dependent part of the Wigner energy contribution, of which the first term is taken to be the empirical global symmetry energy paramter provided in Ref. [38], while $\delta\alpha(A)$ is adjusted according to the experimental binding energy of the nuclei with given mass number A needed to account for local deviation from the first term when Eq. (14) is used. To get a better fitting quality for low-lying $J = 0^+$ level energies, the overall pairing strength is taken as $G = 1$ MeV in both the models for all the nuclei fitted, which is very close to the value used in Ref. [39] with $G = 20/A$ MeV. The experimentally deduced single-particle energies above the ^{16}O core with $\epsilon_1 = \epsilon_{1s_{1/2}} = -3.27$ MeV, $\epsilon_2 = \epsilon_{0p_{3/2}} = 0.94$ MeV,

TABLE II. The values of the parameters (in MeV) in Eq. (18) and $\delta(A)$ (in MeV) of Eq. (16) used in the overall fitting to the binding energies and low-lying $J = 0^+$ level energies of even-even and odd-odd $A = 18$–26 nuclei.

	a	b	c			
$O_{ST}(8)$	−2.3325	−0.2000	−0.0125			
$O_{LST}(8)$	−3.2325	−0.2000	−0.0125			
A:	18	20	22	24	26	28
$\delta(A)$	−0.025	−0.750	−0.940	−0.500	1.900	−0.005

and $\epsilon_3 = \epsilon_{0d_{5/2}} = -4.14$ MeV [37] are used for the mean-field in the $O_{ST}(8)$ model. In the $O_{LST}(8)$ model, for a given l orbit in the ds shell, by using the angular momentum coupling and recoupling techniques and Wigner-Racah calculus, reduced matrix elements of the single-particle energy term in the $O_{LST}(8)$ tensor product basis can finally be expressed as

$$\langle \alpha_1' S_1' T_1'; \alpha_2' S_2' T_2', S'T' || \sum_j \epsilon_j \hat{n}_j || \alpha_1 S_1 T_1; \alpha_2 S_2 T_2, ST\rangle$$
$$= \prod_{q=1}^{2} \delta_{\alpha_q' \alpha_q} \delta_{S_q' S_q} \delta_{T_q' T_q} \delta_{S'S} \delta_{T'T} \sum_j \epsilon_j \frac{(2j+1)n_l}{2\Omega_l}, \qquad (17)$$

where the sum is over $j = l - 1/2$ and $j = l + 1/2$ if $l \neq 0$, and $j = 1/2$ if $l = 0$, n_l is the total number of particles in the l orbit, for which the $O_{LST}(8)$ single-particle reduced matrix element related to the $O_{LST}(8)$ seniority-one states shown in Ref. [27] and the related $O_{LST}(8) \supset SO_{\mathcal{N}}(2) \otimes O_{ST}(6)$ single-particle isoscalar factors listed in Table 3 of Ref. [27] have been used. Detailed derivation of Eq. (17) is provided in the Appendix, Sec. 2. Equation (17) shows that the results of Refs. [19,22] are consistent with those of the actual ds- and fp-shell model calculations if $\epsilon_l = \sum_j \epsilon_j \frac{2j+1}{2\Omega_l}$ is taken for each l orbit. The best fit of both the models requires a quadratic form of $\overline{\epsilon}(\hat{n})$ with

$$\overline{\epsilon}(\hat{n}) = a + b\hat{n} + c\hat{n}^2. \qquad (18)$$

The parameters in Eq. (18) adopted after the fitting are shown in the first part of Table II. It can be seen that the first constant a in the $O_{ST}(8)$ model is very close to the value of the average binding energy per valence nucleon with $\epsilon_{\rm avg} = -2.301$ MeV used in the O(5) isovector pairing model [37], while a larger value of $|a|$ is needed in the $O_{LST}(8)$ model due to the fact that the model is restricted within the $L = 0$ subspace. The contribution from the second term of Ref. (18) to the binding is related to the two-body interaction, while the third term is related to three-body interaction as further correction. The parameter $\delta(A)$ for both the models used in the fitting is provided in the second part of Table II.

The best fit of both the models requires $x = 1$, which indicates that the isoscalar pairing interaction can be neglected in the lower-energy part of the spectra of these ds-shell nuclei as far as binding energies and a few $J = 0^+$ excited levels of these nuclei are concerned. Though $|x| = 1$ can be taken for the ground state of the $N = Z$ nuclei, $x = +1$ must be taken for excited $J = 0^+$ levels and the adjacent $N \neq Z$ nuclei to keep the fitting quality of both the binding energies and the

TABLE III. Binding energies $BE_{\rm th}$ (in MeV) of 22 even-even and odd-odd nuclei with valence nucleons confined to the ds shell up to the half-filing fitted by the $O_{ST}(8)$ model Hamiltonian (14) and the same Hamiltonian in the $O_{LST}(8)$ form with $x = 1$ and other parameters shown in the text and Table II, where n is the number of valence nucleons in the corresponding nucleus, and the experimental binding energy $BE_{\rm exp}$ (in MeV) of these nuclei is taken from Ref. [41].

			$BE_{\rm th}$		
Nucleus	n	Isospin	$O_{ST}(8)$	$O_{LST}(8)$	$BE_{\rm exp}$
${}^{18}_{8}O_{10}$	2	$T = 1$	140.000	139.978	139.808
${}^{18}_{9}F_{9}$	2	$T = 1$	137.313	137.292	137.369
${}^{18}_{10}Ne_{8}$	2	$T = 1$	132.035	132.013	132.143
${}^{20}_{8}O_{12}$	4	$T = 2$	151.43	151.550	151.371
${}^{20}_{9}F_{11}$	4	$T = 1$	154.395	154.402	154.403
${}^{20}_{10}Ne_{10}$	4	$T = 0$	160.405	160.323	160.645
${}^{20}_{11}Na_{9}$	4	$T = 1$	146.065	145.465	145.970
${}^{20}_{12}Mg_{8}$	4	$T = 2$	134.139	134.189	134.561
${}^{22}_{8}O_{14}$	6	$T = 3$	161.446	161.799	162.037
${}^{22}_{10}Ne_{12}$	6	$T = 1$	178.228	178.204	177.770
${}^{22}_{11}Na_{11}$	6	$T = 1$	174.144	174.140	174.145
${}^{22}_{12}Mg_{10}$	6	$T = 1$	168.858	168.834	168.581
${}^{22}_{14}Si_{8}$	6	$T = 3$	133.328	133.681	133.276
${}^{24}_{10}Ne_{14}$	8	$T = 2$	191.563	191.871	191.840
${}^{24}_{11}Na_{13}$	8	$T = 1$	192.780	193.519	193.522
${}^{24}_{12}Mg_{12}$	8	$T = 0$	198.845	198.806	198.257
${}^{24}_{13}Al_{11}$	8	$T = 1$	183.583	183.589	183.590
${}^{24}_{14}Si_{10}$	8	$T = 2$	171.484	171.793	172.013
${}^{26}_{12}Mg_{14}$	10	$T = 1$	216.775	217.022	216.681
${}^{26}_{13}Al_{13}$	10	$T = 1$	211.890	212.137	211.894
${}^{26}_{14}Si_{12}$	10	$T = 1$	206.088	206.335	206.042
${}^{28}_{14}Si_{14}$	12	$T = 0$	247.665	248.064	247.737
		$\sigma_{BE} =$	0.32 MeV	0.33 MeV	

low-lying $J = 0^+$ level energies. Because the $O_{ST}(8)$ model Hamiltonian (14) with $x = 1$ is equivalent to the mean-field plus isovector $J = 0$ pairing Hamiltonian diagonalized within the O(5) seniority-zero and -nonzero configurations including isoscalar np pairs, the total angular momentum J is a good quantum number in the eigenstate (13) when $x = 1$. To verify that the eigenstates (13) with $\mathcal{S} = 0$ in this case are those with $J = 0$, numerical diagonalization of the Hamiltonian (14) in the O(5) seniority-zero tensor-product subspace, of which the matrix elements were calculated by using the results shown in Ref. [40], is also performed, which shows that the first few eigenenergies of the Hamiltonian (14) with $\mathcal{S} = 0$ for a given number of particles n and isospin T are exactly the corresponding ones with $J = 0$ in the O(5) seniority-zero subspace. Therefore,

$$|\zeta; n, \mathcal{S} = M_{\mathcal{S}} = 0\, T M_T\rangle \equiv |\zeta; n, J = M_J = 0\, T M_T\rangle \quad (19)$$

when $x = 1$. In fact, there are more eigenstates of Eq. (14) in the O(8) seniority-zero and $\mathcal{S} = 0$ subspace in comparison to

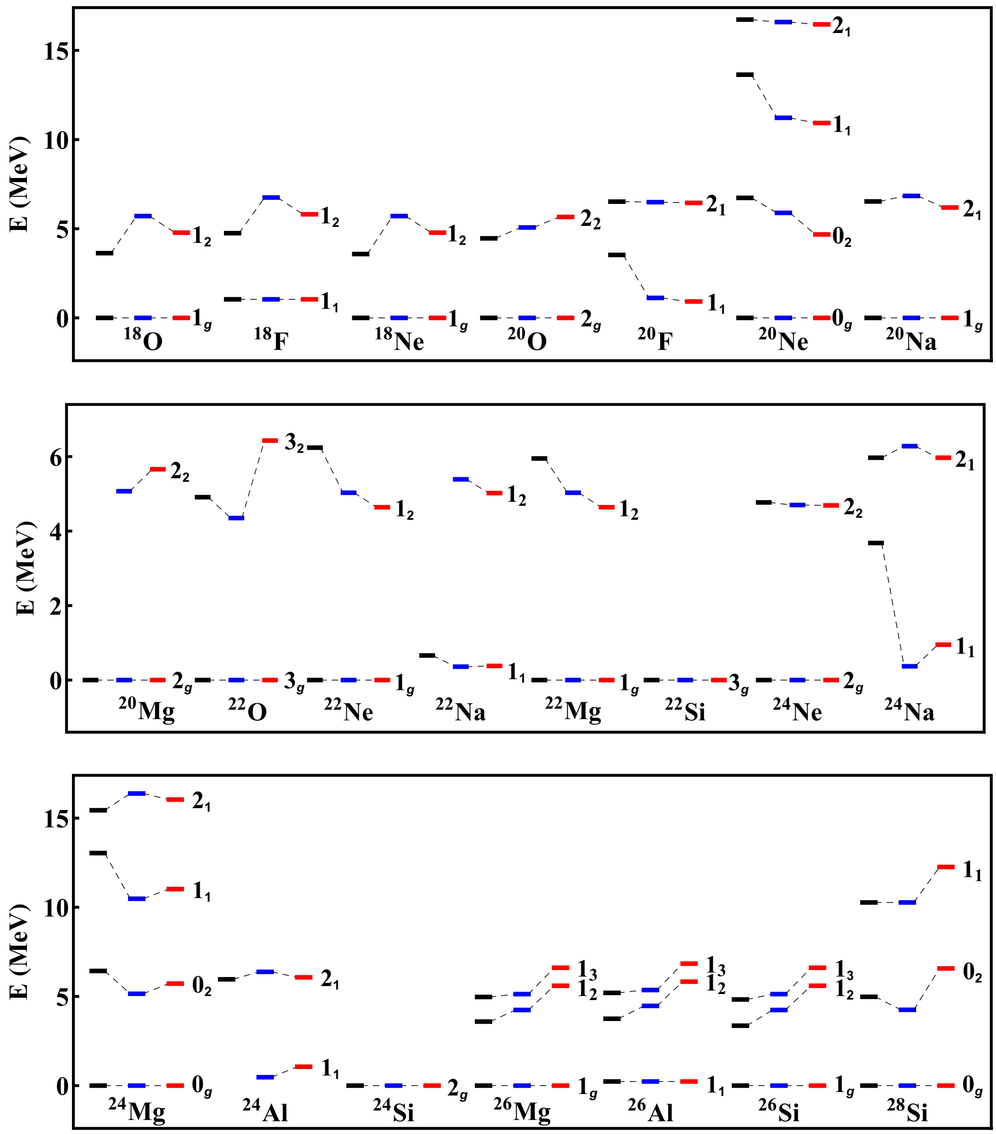

FIG. 1. A few of the lowest $J = 0^+$ level energies of the 22 even-even and odd-odd ds-shell nuclei fitted by both the $O_{ST}(8)$ (middle in blue) and $O_{LST}(8)$ (right in red) models with $x = 1$, where T_ξ on the right of each level denotes the ξth excited level with isospin T, the label g denotes the ground state, and the experimental level energies (left in black) are taken from Ref. [41]. The corresponding numerical data are provided in Table IV.

those in the O(5) seniority-zero subspace. These eigenstates are higher in energy and lie in the O(5) seniority-nonzero subspace, of which the level energies are not considered in comparison to the experimental data here.

The fitting results of the binding energies of even-even and odd-odd $A = 18$–28 nuclei up to the half-filling in the ds shell with the root-mean-square deviation of the $O_{ST}(8)$ model for binding energies $\sigma_{\rm BE} = 0.32$ MeV and that of the $O_{LST}(8)$ model $\sigma_{\rm BE} = 0.33$ MeV are shown in Table III except ^{22}F and ^{22}Al, for which $J = 0^+$ level energies are not available experimentally. Figure 1 shows the lowest experimentally known $J = 0^+$ level energies of these even-even and odd-odd ds-shell nuclei fitted by both the models with the same model parameters as used in fitting the binding energies, of which the corresponding numerical data are provided in Table IV. The root-mean-square deviation of the fitting to these excited $J = 0^+$ level energies is $\sigma_{\rm level} = 1.30$ MeV in the $O_{ST}(8)$ model and $\sigma_{\rm level} = 1.52$ MeV in the $O_{LST}(8)$ model, while the average deviation of the excited level energies $\phi = \sum_i |E^i_{\rm Th} - E^i_{\rm Exp}|/\sum_i E^i_{\rm Exp}$, where the sum runs over all the excited level energies of these nuclei fitted, appears to be $\phi = 16\%$ in the $O_{ST}(8)$ model and $\phi = 21\%$ in the $O_{LST}(8)$ model. Therefore, the overall fitting quality of both the models is quite the same. The fitting results of both the models show that the isovector np pairing interaction, at least, prevails over the isoscalar pairing interaction in the ground state and the low-lying $J = 0^+$ excited states of the ds-shell nuclei. In addition, the fitting quality of the $O_{LST}(8)$ model, which is restricted within the $L = 0$ subspace, is comparable with that of the $O_{ST}(8)$ or the O(5) isovector

TABLE IV. A few of the lowest $J = 0^+$ level energies (in MeV) of the 22 even-even and odd-odd ds-shell nuclei fitted by both the O$_{ST}$(8) and O$_{LST}$(8) models with $x = 1$ as displayed in Fig. 1, where "–" denotes the corresponding level is not observed experimentally, and the model parameters are the same as those used in fitting the binding energies shown in Table III (see text).

^{18}O	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{18}F	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{18}Ne	Exp	O$_{ST}$(8)	O$_{LST}$(8)
0^+ (T_ξ=1_g)	0	0	0	0^+ (T_ξ=1_1)	1.04	1.04	1.04	0^+ (T_ξ=1_g)	0	0	0
0^+ (T_ξ=1_2)	3.63	5.71	4.78	0^+ (T_ξ=1_2)	4.75	6.75	5.81	0^+ (T_ξ=1_2)	3.58	5.71	4.78
^{20}O	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{20}F	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{20}Ne	Exp	O$_{ST}$(8)	O$_{LST}$(8)
0^+ (T_ξ=2_g)	0	0	0	0^+ (T_ξ=1_1)	3.53	1.12	0.92	0^+ (T_ξ=0_g)	0	0	0
0^+ (T_ξ=2_2)	4.46	5.07	5.66	0^+ (T_ξ=2_1)	6.52	6.49	6.45	0^+ (T_ξ=0_2)	6.73	5.89	4.68
								0^+ (T_ξ=1_1)	13.64	11.22	10.93
								0^+ (T_ξ=2_1)	16.73	16.59	16.46
^{20}Na	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{20}Mg	Exp	O$_{ST}$(8)	O$_{LST}$(8)				
0^+ (T_ξ=1_1)	3.09	1.47	0.66	0^+ (T_ξ=2_g)	0	0	0				
0^+ (T_ξ=2_1)	6.53	6.84	6.19	0^+ (T_ξ=2_2)	–	5.07	5.66				
^{22}O	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{22}Ne	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{22}Na	Exp	O$_{ST}$(8)	O$_{LST}$(8)
0^+ (T_ξ=3_g)	0	0	0	0^+ (T_ξ=1_g)	0	0	0	0^+ (T_ξ=1_1)	0.66	0.36	0.38
0^+ (T_ξ=3_2)	4.91	4.35	6.43	0^+ (T_ξ=1_2)	6.24	5.03	4.64	0^+ (T_ξ=1_2)	–	5.39	5.02
^{22}Mg	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{22}Si	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{28}Si	Exp	O$_{ST}$(8)	O$_{LST}$(8)
0^+ (T_ξ=1_g)	0	0	0	0^+ (T_ξ=3_g)	0	0	0	0^+ (T_ξ=0_g)	0	0	0
0^+ (T_ξ=1_2)	5.95	5.03	4.64					0^+ (T_ξ=0_2)	4.98	4.25	6.57
								0^+ (T_ξ=1_1)	10.27	10.27	12.26
^{24}Ne	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{24}Na	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{24}Mg	Exp	O$_{ST}$(8)	O$_{LST}$(8)
0^+ (T_ξ=2_g)	0	0	0	0^+ (T_ξ=1_1)	3.68	0.37	0.95	0^+ (T_ξ=0_g)	0	0	0
0^+ (T_ξ=2_2)	4.77	4.70	4.69	0^+ (T_ξ=2_1)	5.97	6.28	5.97	0^+ (T_ξ=0_2)	6.43	5.15	5.72
								0^+ (T_ξ=1_1)	13.04	10.48	11.02
								0^+ (T_ξ=2_1)	15.44	16.38	16.04
^{24}Al	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{24}Si	Exp.	O$_{ST}$(8)	O$_{LST}$(8)				
0^+ (T_ξ=1_1)	–	0.47	1.06	0^+ (T_ξ=2_g)	0	0	0				
0^+ (T_ξ=2_1)	5.96	6.38	6.07								
^{26}Mg	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{26}Al	Exp	O$_{ST}$(8)	O$_{LST}$(8)	^{26}Si	Exp	O$_{ST}$(8)	O$_{LST}$(8)
0^+ (T_ξ=1_g)	0	0	0	0^+ (T_ξ=1_1)	0.23	0.23	0.23	0^+ (T_ξ=1_g)	0	0	0
0^+ (T_ξ=1_2)	3.59	4.24	5.60	0^+ (T_ξ=1_2)	3.75	4.47	5.83	0^+ (T_ξ=1_2)	3.36	4.24	5.60
0^+ (T_ξ=1_3)	4.97	5.13	6.61	0^+ (T_ξ=1_3)	5.20	5.36	6.84	0^+ (T_ξ=1_3)	4.83	5.13	6.61

pairing model, which indicates the $L = 0$ truncation adopted in the O$_{LST}$(8) model is indeed acceptable for the ds-shell nuclei.

Table V shows isovector nn, pp, and np pairing contributions at the ground state or the lowest eigenstate of the O$_{ST}$(8) model with $x = 1$ for these nuclei defined by

$$E_{np}^{(1)} = G\langle\zeta = 1, n, \mathcal{S} = 0\, TM_T|P_0^\dagger P_0|\zeta = 1, n, \mathcal{S} = 0\, TM_T\rangle,$$

$$E_{nn}^{(1)} = G\langle\zeta = 1, n, \mathcal{S} = 0\, TM_T|P_{-1}^\dagger P_{-1}|\zeta = 1, n, \mathcal{S} = 0\, TM_T\rangle,$$

$$E_{pp}^{(1)} = G\langle\zeta = 1, n, , \mathcal{S} = 0\, TM_T|P_1^\dagger P_1|\zeta = 1, n, \mathcal{S} = 0\, TM_T\rangle, \tag{20}$$

where $|\zeta = 1, n, \mathcal{S} = 0\, TM_T\rangle$ is just the lowest $J = 0^+$ state of these nuclei, and the percentage of the isovector np pairing energy contribution to the binding energy with respect to the total isovector pairing energy is obtained by

$$\eta_{np} = E_{np}^{(1)} \big/ \big(E_{np}^{(1)} + E_{nn}^{(1)} + E_{pp}^{(1)}\big). \tag{21}$$

It can be observed that $E_{nn}^{(1)}$ in the $N = Z + 2$ nuclei is the same as $E_{pp}^{(1)}$ in the $Z = N + 2$ mirror nuclei, while $E_{nn}^{(1)} = E_{pp}^{(1)}$ in the $N = Z$ nuclei due to the charge-independent isovector pairing adopted. However, $E_{np}^{(1)} = E_{nn}^{(1)} = E_{pp}^{(1)}$ in even-even $N = Z$ nuclei, while $E_{np}^{(1)} > E_{nn}^{(1)} = E_{pp}^{(1)}$ in odd-odd $N = Z$ nuclei, which leads to the isovector np pairing energy contribution to the binding energy being the largest in odd-odd $N = Z$ nuclei. Besides ^{18}F with $\eta_{np} = 100\%$ because there is no nn or pp pair, η_{np} in the other two odd-odd $N = Z$ nuclei ^{22}Na and ^{26}Al is always greater than 61%, while it is 33.33% in even-even $N = Z$ nuclei, ^{20}Ne, ^{24}Mg, and ^{28}Si.

Moreover, similar to the analysis of the isovector pairing [42], the number of the isoscalar np pairs may be estimated

TABLE V. The isovector np, nn, and pp pairing contributions (in MeV) to the binding energy of the 22 even-even and odd-odd ds-shell nuclei; the percentage of the isovector np pairing energy contribution with respect to the total isovector pairing energy at the ground state or the lowest $J = 0^+$ state defined in Eq. (20); the expectation value of the isoscalar np pairing at the lowest $J = 0^+$ state defined in Eq. (22); and the ratio related to the number of isoscalar np pairs to that of the isovector np pairs in the lowest $J = 0^+$ state defined in Eq. (23).

Nucleus	n	Isospin	$E_{np}^{(1)}$	$E_{nn}^{(1)}$	$E_{pp}^{(1)}$	η_{np}	$\tilde{E}_{np}^{(1)}$ [$\mathrm{O}_{\mathcal{S}T}(8)$]	$\tilde{E}_{np}^{(1)}$ [$\mathrm{O}_{LST}(8)$]	χ_{np}
${}^{18}_{8}\mathrm{O}_{10}$	2	$T=1$	0	5.036	0	0%	0	0	–
${}^{18}_{9}\mathrm{F}_{9}$	2	$T=1$	5.036	0	0	100%	0	0	0
${}^{18}_{10}\mathrm{Ne}_{8}$	2	$T=1$	0	0	5.036	0%	0	0	–
${}^{20}_{8}\mathrm{O}_{12}$	4	$T=2$	0	7.945	0	0%	0	0	–
${}^{20}_{9}\mathrm{F}_{11}$	4	$T=1$	2.568	2.568	0	50%	0	0	0
${}^{20}_{10}\mathrm{Ne}_{10}$	4	$T=0$	3.707	3.707	3.707	33.33%	0.957	0.801	0.51
${}^{20}_{11}\mathrm{Na}_{9}$	4	$T=1$	2.568	0	2.568	50%	0	0	0
${}^{20}_{12}\mathrm{Mg}_{8}$	4	$T=2$	0	0	7.945	0%	0	0	–
${}^{22}_{8}\mathrm{O}_{14}$	6	$T=3$	0	8.666	0	0%	0	0	–
${}^{22}_{10}\mathrm{Ne}_{12}$	6	$T=1$	2.226	7.356	4.444	15.87%	1.582	1.267	0.84
${}^{22}_{11}\mathrm{Na}_{11}$	6	$T=1$	9.573	2.226	2.226	68.25%	1.582	1.267	0.41
${}^{22}_{12}\mathrm{Mg}_{10}$	6	$T=1$	2.226	4.444	7.356	15.87%	1.582	1.267	0.84
${}^{22}_{14}\mathrm{Si}_{8}$	6	$T=3$	0	0	8.666	0%	0	0	–
${}^{24}_{10}\mathrm{Ne}_{14}$	8	$T=2$	1.600	8.393	4.756	10.85%	2.187	1.713	1.17
${}^{24}_{11}\mathrm{Na}_{13}$	8	$T=1$	5.061	4.167	2.681	42.49%	1.978	1.364	0.63
${}^{24}_{12}\mathrm{Mg}_{12}$	8	$T=0$	6.000	6.000	6.000	33.33%	3.148	2.448	0.72
${}^{24}_{13}\mathrm{Al}_{11}$	8	$T=1$	5.061	2.681	4.167	42.49%	1.978	1.364	0.63
${}^{24}_{14}\mathrm{Si}_{10}$	8	$T=2$	1.600	4.756	8.393	10.85%	2.187	1.713	1.17
${}^{26}_{12}\mathrm{Mg}_{14}$	10	$T=1$	3.615	7.917	7.186	19.31%	4.359	3.353	1.10
${}^{26}_{13}\mathrm{Al}_{13}$	10	$T=1$	11.489	3.615	3.615	61.38%	4.359	3.353	0.62
${}^{26}_{14}\mathrm{Si}_{12}$	10	$T=1$	3.615	7.186	7.917	19.31%	4.359	3.353	1.10
${}^{28}_{14}\mathrm{Si}_{14}$	12	$T=0$	6.847	6.847	6.847	33.33%	6.506	4.963	0.97

by the expectation value of $D^\dagger \cdot D$. The expectation values of the isoscalar np pairing at the lowest $J = 0^+$ state defined by

$$\begin{aligned}\tilde{E}_{np}^{(1)}[\mathrm{O}_{\mathcal{S}T}(8)] &= \langle \zeta = 1, n, \mathcal{S} = 0\, T M_T | D^\dagger \\ &\quad \cdot D | \zeta = 1, n, \mathcal{S} = 0\, T M_T \rangle, \\ \tilde{E}_{np}^{(1)}[\mathrm{O}_{LST}(8)] &= \langle \zeta = 1, n, \mathrm{S} = 0\, T M_T | D^\dagger \\ &\quad \cdot D | \zeta = 1, n, \mathrm{S} = 0\, T M_T \rangle, \end{aligned} \tag{22}$$

for the two models, together with the ratio

$$\chi_{np} = \left(G \tilde{E}_{np}^{(1)}[\mathrm{O}_{\mathcal{S}T}(8)] / E_{np}^{(1)} \right)^{1/2}, \tag{23}$$

are also shown in Table V, where χ_{np} roughly estimates the relative ratio of the number of isoscalar np pairs to that of the isovector np pairs in the lowest $J = 0^+$ state. It can be seen that the isoscalar pair content becomes noticeable in the ground state of ^{20}Ne. With further increasing of the number of valence nucleons up to the half-filling, the ratio χ_{np} increases not only in $N = Z$ nuclei but also in adjacent $N = Z \pm 2$ nuclei, especially in those even-even nuclei, while χ_{np} in odd-odd $N = Z$ nuclei is comparatively small. Thus, we conclude that the isoscalar pairing correlation is still important at low-lying states of $N = Z$ and $N = Z \pm 2$ nuclei described by both the O(8) models, especially in those even-even nuclei with more valence nucleons up to the half-filling, even though the isoscalar pairing interaction is negligible.

V. SUMMARY

In this work, similar to the original $\mathrm{O}_{LST}(8)$ model in the LST-coupling scheme, an extended $\mathrm{O}_{\mathcal{S}T}(8)$ model with multi-j orbits is constructed based on the angular momentum decomposition with "pseudo"-spin $\mathcal{S}$ for valence nucleons in a given single-j orbit. It is shown that the $\mathcal{S} = 0$ and $T = 1$ pairs are exactly the $J = 0$ and $T = 1$ pairs in a given j orbit, while isoscalar $\mathcal{S} = 1$ pairs are linear combinations of $J =$ odd pairs, with which the angular momentum of the system is not a conserved quantity. Nevertheless, the new pairing Hamiltonian can be used to estimate $T = 1$ and $T = 0$ pairing interaction contributions, especially at the ground state in even-even and odd-odd nuclei described by the model with any number of j orbits, of which the situation is quite similar to the deformed mean-field plus pairing model [34]. The usefulness of the $\mathrm{O}_{\mathcal{S}T}(8)$ model lies in the fact that the shell-model mean-field plus isovector pairing Hamiltonian diagonalized in the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero subspace is quite similar to the O(5) isovector pairing model diagonalized within the O(5) seniority-zero and -nonzero configurations including isoscalar np pairs, from which the isoscalar pair content of any state in the $\mathrm{O}_{\mathcal{S}T}(8)$ model can easily be estimated. Moreover, the matrix elements of the $\mathrm{O}_{\mathcal{S}T}(8)$ Hamiltonian have closed expressions in the $\mathrm{O}_{\mathcal{S}T}(8) \supset \mathrm{U}_{\mathcal{S}T}(4) \supset O_{\mathcal{N}}(2) \otimes \mathrm{SU}_{\mathcal{S}}(2) \otimes \mathrm{SU}_T(2)$ basis in the O(8) seniority-zero subspace, while more algebraic work is needed for the O(5) isovector pairing model [40], especially when the O(5) senority-nonzero configurations are included.

For multi-j orbits, it is observed that a subset of the eigenvalues of the pure pairing Hamiltonian in any one of the $\mathrm{O}_T(5)$, $\mathrm{U}_{\mathcal{S}T}(4)$, and $\mathrm{O}_{\mathcal{S}}(5)$ limiting cases overlaps with the whole set of eigenvalues of the corresponding $\mathrm{O}_T(5)$, $\mathrm{U}_{ST}(4)$, and $\mathrm{O}_S(5)$ limiting cases [28] of the $\mathrm{O}_{LST}(8)$ model, especially when the lowest eigenvalue of both the models in each limiting case is exactly the same. The fact that there are more eigenvalues of each limiting case in the $\mathrm{O}_{\mathcal{S}T}(8)$ model is because the dimension of the $\mathrm{O}_{\mathcal{S}T}(8)$ tensor product subspace is always greater than that of the $\mathrm{O}_{LST}(8)$ model. Therefore, the $\mathrm{O}_{\mathcal{S}T}(8)$ pairing Hamiltonian in the $\mathcal{S}=0$ subspace is indeed quite the same as that of the $\mathrm{O}_{LST}(8)$ model within the $S=0$ subspace.

As an example of the $\mathrm{O}_{\mathcal{S}T}(8)$ model application, some low-lying $J=0^+$ level energies of even-even and odd-odd $A=18$–28 nuclei up to the half-filling in the ds shell above the ^{16}O core are fit by the model within the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero and $\mathcal{S}=0$ subspace and compared with the fitting results of the same Hamiltonian in the $\mathrm{O}_{LST}(8)$ form. It has been verified from the fitting of both the models that the isoscalar pairing interaction can be neglected in the ground state and the lower-energy part of the spectra of these ds-shell nuclei as far as binding energies and a few $J=0^+$ excited levels of these nuclei are concerned. Thus, the $\mathrm{O}_{\mathcal{S}T}(8)$ model within the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero subspace in this case is quite the same as the mean-field plus isovector pairing model diagonalized within the O(5) seniority-zero and -nonzero configurations including isoscalar np pairs, while the $\mathrm{O}_{LST}(8)$ model in this case is equivalent to the mean-field plus isovector pairing model within the same configurations restricted to the $L=0$ subspace. With the mean-field plus isovector pairing interaction only, isovector nn, pp, and np pairing contributions at the ground state or the lowest $J=0^+$ state of these nuclei in the $\mathrm{O}_{\mathcal{S}T}(8)$ model are estimated. It is shown that the isovector np pairing contribution to the binding energy in the odd-odd $N=Z$ nuclei is systematically larger than that in the even-even nuclei, which leads to the conclusion that the isovector np pairing is more favored in odd-odd $N=Z$ nuclei. Most importantly, the number of the isoscalar np pairs at the lowest $J=0^+$ state of these nuclei is estimated. It is clearly shown that the isoscalar pair content in the lowest $J=0^+$ state of the $N=Z$ and $N=Z\pm 2$ nuclei increases with increasing of the valence nucleons, especially in those even-even nuclei. It is concluded that the isoscalar pairing correlation is still important at low-lying states of $N=Z$ and $N=Z\pm 2$ nuclei, especially in those even-even nuclei with more valence nucleons up to the half-filling, even though the isoscalar pairing interaction is negligible.

Because the isovector and isoscalar np pair contents are estimated by the expectation values of the corresponding two-body pairing term, the values may be quite different from the actual numbers of isovector and isoscalar np pairs in the ground state of the system. To resolve this issue, one may evaluate these values exactly with the help of the Bargmann variables in representing these pair operators as shown in Ref. [20], for which the analysis and further applications of this model with more j orbits or in other major shells will be a part of our future work.

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APPENDIX: SOME RELEVANT MATRIX ELEMENTS

1. The reduced matrix elements of the pairing operators

Matrix elements of each term involved in the Hamiltonian (12) under the $\mathrm{O}_{\mathcal{S}T}(8)$ tensor product basis of $\otimes_{i=1}^{p}(\Omega_i,0)$ in the $\mathrm{O}_{\mathcal{S}T}(8) \supset \mathrm{U}_{\mathcal{S}T}(4) \supset O_{\mathcal{N}}(2) \otimes \mathrm{SU}_{\mathcal{S}}(2) \otimes \mathrm{SU}_T(2)$ labeling scheme can be evaluated by using the results shown in Refs. [19,22,35]. Specifically, for the $p=3$ case, we have

$$\langle \alpha'_1\mathcal{S}'_1T'_1;\alpha'_2\mathcal{S}'_2T'_2,\ \mathcal{S}'_{12}T'_{12};\alpha'_3\mathcal{S}'_3T'_3;\mathcal{S}'T'||\sum_{i=1}^{p}\epsilon_{j_i}\hat{n}_{j_i}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$=\delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\prod_{q=1}^{3}\delta_{\alpha'_{q'}\alpha_{q'}}\delta_{\mathcal{S}'_q\mathcal{S}_q}\delta_{T'_qT_q}\delta_{\mathcal{S}'_{12}\mathcal{S}_{12}}\delta_{T'_{12}T_{12}}\sum_{i=1}^{p}\epsilon_{j_i}n_i, \tag{A1}$$

where, and in the following, $\alpha_i \equiv \{n_i,\ \sigma_i\}$,

$$\langle \alpha'_1\mathcal{S}'_1T'_1;\alpha'_2\mathcal{S}'_2T'_2,\ \mathcal{S}'_{12}T'_{12};\alpha'_3\mathcal{S}'_3T'_3;\mathcal{S}'T'||P^{(i)\dagger}\cdot P^{(i)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$=\delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\prod_{q'\neq i}^{3}\delta_{\alpha'_{q'}\alpha_{q'}}\prod_{q=1}^{3}\delta_{\mathcal{S}'_q\mathcal{S}_q}\delta_{T'_qT_q}\delta_{\mathcal{S}'_{12}\mathcal{S}_{12}}\delta_{T'_{12}T_{12}}\sum_{\alpha''_iT''_i}\langle\alpha'_i\mathcal{S}_iT_i||P^{(i)\dagger}||\alpha''_i\mathcal{S}_iT''_i\rangle\langle\alpha_1\mathcal{S}_iT_i||P^{(i)\dagger}||\alpha''_i\mathcal{S}_iT''_i\rangle, \tag{A2}$$

for $i = 1,\ 2,\ 3,$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||P^{(1)\dagger}\cdot P^{(2)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_3'\alpha_3}\delta_{T_3'T_3}\prod_{q=1}^{3}\delta_{\mathcal{S}_q'\mathcal{S}_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}\delta_{T_{12}'T_{12}}(-1)^{T_1+T_2+T_{12}}[(2T_1'+1)(2T_2+1)]^{1/2}\begin{Bmatrix}T_1 & T_1' & 1\\ T_2' & T_2 & T_{12}\end{Bmatrix}$$
$$\times\langle\alpha_1'\mathcal{S}_1T_1'||P^{(1)\dagger}||\alpha_1\mathcal{S}_1T_1\rangle\langle\alpha_2\mathcal{S}_2T_2||P^{(2)\dagger}||\alpha_2'\mathcal{S}_2T_2'\rangle, \tag{A3}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||P^{(2)\dagger}\cdot P^{(1)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_3'\alpha_3}\delta_{T_3'T_3}\prod_{q=1}^{3}\delta_{\mathcal{S}_q'\mathcal{S}_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}\delta_{T_{12}'T_{12}}(-1)^{T_1'+T_2'+T_{12}'}[(2T_1+1)(2T_2'+1)]^{1/2}\begin{Bmatrix}T_1' & T_1 & 1\\ T_2 & T_2' & T_{12}'\end{Bmatrix}$$
$$\times\langle\alpha_1\mathcal{S}_1T_1||P^{(1)\dagger}||\alpha_1'\mathcal{S}_1'T_1'\rangle\langle\alpha_2'\mathcal{S}_2T_2'||P^{(2)\dagger}||\alpha_2\mathcal{S}_2T_2\rangle, \tag{A4}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||P^{(1)\dagger}\cdot P^{(3)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_2'\alpha_2}\delta_{T_2'T_2}\prod_{q=1}^{3}\delta_{\mathcal{S}_q'\mathcal{S}_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}(-1)^{T_3+T_{12}+T}[(2T_{12}'+1)(2T_3+1)]^{1/2}\begin{Bmatrix}T_{12} & T_{12}' & 1\\ T_3' & T_3 & T\end{Bmatrix}\langle\alpha_3\mathcal{S}_3T_3||P^{(3)\dagger}||\alpha_3'\mathcal{S}_3T_3'\rangle$$
$$\times(-1)^{T_1'+T_{12}+T_2+1}[(2T_1'+1)(2T_{12}+1)]^{1/2}\begin{Bmatrix}T_{12}' & T_{12} & 1\\ T_1 & T_1' & T_2\end{Bmatrix}\langle\alpha_1'\mathcal{S}_1T_1'||P^{(1)\dagger}||\alpha_1\mathcal{S}_1T_1\rangle, \tag{A5}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||P^{(2)\dagger}\cdot P^{(3)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_1'\alpha_1}\delta_{T_1'T_1}\prod_{q=1}^{3}\delta_{\mathcal{S}_q'\mathcal{S}_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}(-1)^{T_3+T_{12}+T}[(2T_{12}'+1)(2T_3+1)]^{1/2}\begin{Bmatrix}T_{12} & T_{12}' & 1\\ T_3' & T_3 & T\end{Bmatrix}\langle\alpha_3\mathcal{S}_3T_3||P^{(3)\dagger}||\alpha_3'\mathcal{S}_3T_3'\rangle$$
$$\times(-1)^{T_{12}'+T_1+T_2+1}[(2T_2'+1)(2T_{12}+1)]^{1/2}\begin{Bmatrix}T_{12}' & T_{12} & 1\\ T_2 & T_2' & T_1\end{Bmatrix}\langle\alpha_2'\mathcal{S}_2T_2'||P^{(2)\dagger}||\alpha_2\mathcal{S}_2T_2\rangle, \tag{A6}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||P^{(3)\dagger}\cdot P^{(1)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_2'\alpha_2}\delta_{T_2'T_2}\prod_{q=1}^{3}\delta_{\mathcal{S}_q'\mathcal{S}_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}(-1)^{T_3'+T_{12}'+T'}[(2T_{12}+1)(2T_3'+1)]^{1/2}\begin{Bmatrix}T_{12}' & T_{12} & 1\\ T_3 & T_3' & T'\end{Bmatrix}\langle\alpha_3'\mathcal{S}_3T_3'||P^{(3)\dagger}||\alpha_3\mathcal{S}_3T_3\rangle$$
$$\times(-1)^{T_1+T_{12}'+T_2'+1}[(2T_1+1)(2T_{12}'+1)]^{1/2}\begin{Bmatrix}T_{12} & T_{12}' & 1\\ T_1' & T_1 & T_2\end{Bmatrix}\langle\alpha_1\mathcal{S}_1T_1||P^{(1)\dagger}||\alpha_1'\mathcal{S}_1T_1'\rangle, \tag{A7}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||P^{(3)\dagger}\cdot P^{(2)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_1'\alpha_1}\delta_{T_1'T_1}\prod_{q=1}^{3}\delta_{\mathcal{S}_q'\mathcal{S}_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}(-1)^{T_3'+T_{12}'+T'}[(2T_{12}+1)(2T_3'+1)]^{1/2}\begin{Bmatrix}T_{12}' & T_{12} & 1\\ T_3 & T_3' & T'\end{Bmatrix}\langle\alpha_3'\mathcal{S}_3T_3'||P^{(3)\dagger}||\alpha_3\mathcal{S}_3T_3\rangle$$
$$\times(-1)^{T_{12}+T_1'+T_2'+1}[(2T_2+1)(2T_{12}'+1)]^{1/2}\begin{Bmatrix}T_{12} & T_{12}' & 1\\ T_2' & T_2 & T_1\end{Bmatrix}\langle\alpha_2\mathcal{S}_2T_2||P^{(2)\dagger}||\alpha_2'\mathcal{S}_2T_2'\rangle, \tag{A8}$$

and

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||D^{(i)\dagger}\cdot D^{(i)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\prod_{q'\neq i}^{3}\delta_{\alpha_{q'}'\alpha_{q'}}\prod_{q=1}^{3}\delta_{\mathcal{S}_q'\mathcal{S}_q}\delta_{T_q'T_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}\delta_{T_{12}'T_{12}}\sum_{\alpha_i''\mathcal{S}_i''}\langle\alpha_i'\mathcal{S}_iT_i||D^{(i)\dagger}||\alpha_i''\mathcal{S}_i''T_i\rangle\langle\alpha_i\mathcal{S}_iT_i||D^{(i)\dagger}||\alpha_i''\mathcal{S}_i''T_i\rangle \tag{A9}$$

for $i = 1,\ 2,\ 3,$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||D^{(1)\dagger}\cdot D^{(2)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_3'\alpha_3}\delta_{\mathcal{S}_3'\mathcal{S}_3}\prod_{q=1}^{3}\delta_{T_q'T_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}\delta_{T_{12}'T_{12}}(-1)^{\mathcal{S}_2+\mathcal{S}_1+\mathcal{S}_{12}}[(2\mathcal{S}_1'+1)(2\mathcal{S}_2+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_1 & \mathcal{S}_1' & 1\\ \mathcal{S}_2' & \mathcal{S}_2 & \mathcal{S}_{12}\end{Bmatrix}$$
$$\times\langle\alpha_1'\mathcal{S}_1'T_1||D^{(1)\dagger}||\alpha_1\mathcal{S}_1T_1\rangle\langle\alpha_2\mathcal{S}_2T_2||D^{(2)\dagger}||\alpha_2'\mathcal{S}_2'T_2\rangle, \tag{A10}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||D^{(2)\dagger}\cdot D^{(1)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_3'\alpha_3}\delta_{\mathcal{S}_3'\mathcal{S}_3}\prod_{q=1}^{3}\delta_{T_q'T_q}\delta_{\mathcal{S}_{12}'\mathcal{S}_{12}}\delta_{T_{12}'T_{12}}(-1)^{\mathcal{S}_2'+\mathcal{S}_1'+\mathcal{S}_{12}'}[(2\mathcal{S}_1+1)(2\mathcal{S}_2'+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_1' & \mathcal{S}_1 & 1\\ \mathcal{S}_2 & \mathcal{S}_2' & \mathcal{S}_{12}'\end{Bmatrix}$$
$$\times\langle\alpha_1\mathcal{S}_1T_1||D^{(1)\dagger}||\alpha_1'\mathcal{S}_1'T_1\rangle\langle\alpha_2'\mathcal{S}_2'T_2||D^{(2)\dagger}||\alpha_2\mathcal{S}_2T_2\rangle, \tag{A11}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||D^{(1)\dagger}\cdot D^{(3)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_2'\alpha_2}\delta_{\mathcal{S}_2'\mathcal{S}_2}\prod_{q=1}^{3}\delta_{T_q'T_q}\delta_{T_{12}'T_{12}}(-1)^{\mathcal{S}_3+\mathcal{S}_{12}+\mathcal{S}}[(2\mathcal{S}_{12}'+1)(2\mathcal{S}_3+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12} & \mathcal{S}_{12}' & 1\\ \mathcal{S}_3' & \mathcal{S}_3 & \mathcal{S}\end{Bmatrix}\langle\alpha_3\mathcal{S}_3T_3||D^{(3)\dagger}||\alpha_3'\mathcal{S}_3'T_3\rangle$$
$$\times(-1)^{\mathcal{S}_1'+\mathcal{S}_{12}+\mathcal{S}_2+1}[(2\mathcal{S}_1'+1)(2\mathcal{S}_{12}+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12}' & \mathcal{S}_{12} & 1\\ \mathcal{S}_1 & \mathcal{S}_1' & \mathcal{S}_2\end{Bmatrix}\langle\alpha_1'\mathcal{S}_1'T_1||D^{(1)\dagger}||\alpha_1\mathcal{S}_1T_1\rangle, \tag{A12}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||D^{(2)\dagger}\cdot D^{(3)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_1'\alpha_1}\delta_{\mathcal{S}_1'\mathcal{S}_1}\prod_{q=1}^{3}\delta_{T_q'T_q}\delta_{T_{12}'T_{12}}(-1)^{\mathcal{S}_3+\mathcal{S}_{12}+\mathcal{S}}[(2\mathcal{S}_{12}'+1)(2\mathcal{S}_3+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12} & \mathcal{S}_{12}' & 1\\ \mathcal{S}_3' & \mathcal{S}_3 & \mathcal{S}\end{Bmatrix}\langle\alpha_3\mathcal{S}_3T_3||D^{(3)\dagger}||\alpha_3'\mathcal{S}_3'T_3\rangle$$
$$\times(-1)^{\mathcal{S}_{12}'+\mathcal{S}_1+\mathcal{S}_2+1}[(2\mathcal{S}_2'+1)(2\mathcal{S}_{12}+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12}' & \mathcal{S}_{12} & 1\\ \mathcal{S}_2 & \mathcal{S}_2' & \mathcal{S}_1\end{Bmatrix}\langle\alpha_2'\mathcal{S}_2'T_2||D^{(2)\dagger}||\alpha_2\mathcal{S}_2T_2\rangle, \tag{A13}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||D^{(3)\dagger}\cdot D^{(1)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_2'\alpha_2}\delta_{\mathcal{S}_2'\mathcal{S}_2}\prod_{q=1}^{3}\delta_{T_q'T_q}\delta_{T_{12}'T_{12}}(-1)^{\mathcal{S}_3'+\mathcal{S}_{12}'+\mathcal{S}'}[(2\mathcal{S}_{12}+1)(2\mathcal{S}_3'+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12}' & \mathcal{S}_{12} & 1\\ \mathcal{S}_3 & \mathcal{S}_3' & \mathcal{S}'\end{Bmatrix}\langle\alpha_3'\mathcal{S}_3'T_3||D^{(3)\dagger}||\alpha_3\mathcal{S}_3T_3\rangle$$
$$\times(-1)^{\mathcal{S}_1+\mathcal{S}_{12}'+\mathcal{S}_2'+1}[(2\mathcal{S}_1+1)(2\mathcal{S}_{12}'+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12} & \mathcal{S}_{12}' & 1\\ \mathcal{S}_1' & \mathcal{S}_1 & \mathcal{S}_2\end{Bmatrix}\langle\alpha_1\mathcal{S}_1T_1||D^{(1)\dagger}||\alpha_1'\mathcal{S}_1'T_1\rangle, \tag{A14}$$

$$\langle\alpha_1'\mathcal{S}_1'T_1';\alpha_2'\mathcal{S}_2'T_2',\ \mathcal{S}_{12}'T_{12}';\alpha_3'\mathcal{S}_3'T_3';\mathcal{S}'T'||D^{(3)\dagger}\cdot D^{(2)}||\alpha_1\mathcal{S}_1T_1;\alpha_2\mathcal{S}_2T_2,\ \mathcal{S}_{12}T_{12};\alpha_3\mathcal{S}_3T_3;\mathcal{S}T\rangle$$
$$= \delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T}\delta_{\alpha_1'\alpha_1}\delta_{\mathcal{S}_1'\mathcal{S}_1}\prod_{q=1}^{3}\delta_{T_q'T_q}\delta_{T_{12}'T_{12}}(-1)^{\mathcal{S}_3'+\mathcal{S}_{12}'+\mathcal{S}'}[(2\mathcal{S}_{12}+1)(2\mathcal{S}_3'+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12}' & \mathcal{S}_{12} & 1\\ \mathcal{S}_3 & \mathcal{S}_3' & \mathcal{S}'\end{Bmatrix}\langle\alpha_3'\mathcal{S}_3'T_3||D^{(3)\dagger}||\alpha_3\mathcal{S}_3T_3\rangle$$
$$\times(-1)^{\mathcal{S}_{12}+\mathcal{S}_1'+\mathcal{S}_2'+1}[(2\mathcal{S}_2+1)(2\mathcal{S}_{12}'+1)]^{1/2}\begin{Bmatrix}\mathcal{S}_{12} & \mathcal{S}_{12}' & 1\\ \mathcal{S}_2' & \mathcal{S}_2 & \mathcal{S}_1\end{Bmatrix}\langle\alpha_2\mathcal{S}_2T_2||D^{(2)\dagger}||\alpha_2'\mathcal{S}_2'T_2\rangle. \tag{A15}$$

For any j orbit, within the $\mathrm{O}_{\mathcal{S}T}(8)$ seniority-zero subspace, the "pseudo"-spin and isospin reduced matrix elements of $A^{\mathcal{S}_0T_0\dagger}$, where $P^\dagger = A^{01\dagger}$ and $D^\dagger = A^{10\dagger}$, can be expressed as [19,20]

$$\langle(\Omega,0)n\sigma'\mathcal{S}T||A^{\mathcal{S}_0T_0\dagger}||(\Omega,0)n'\sigma\mathcal{S}'T'\rangle$$
$$= \delta_{n'n-2}\frac{1}{2}\left(\delta_{\sigma'\sigma+1}\sqrt{\frac{\sigma'(\Omega-\sigma'+\mathcal{N}+2)(\Omega+\sigma'-\mathcal{N}+4)}{\sigma'+2}}+\delta_{\sigma'\sigma-1}\sqrt{\frac{(\sigma'+4)(\Omega+\sigma'+\mathcal{N}+6)(\Omega-\sigma'-\mathcal{N})}{\sigma'+2}}\right)$$
$$\times\left\langle\begin{matrix}[\sigma\sigma] & [11]\\ \mathcal{S}'T' & \mathcal{S}_0T_0\end{matrix}\middle|\begin{matrix}[\sigma'\sigma']\\ \mathcal{S}T\end{matrix}\right\rangle, \tag{A16}$$

where $\mathcal{N} = \Omega - \frac{n}{2}$, and $\langle\begin{smallmatrix}[\sigma\sigma] & [11]\\ \mathcal{S}'T' & \mathcal{S}_0T_0\end{smallmatrix}|\begin{smallmatrix}[\sigma'\sigma']\\ \mathcal{S}T\end{smallmatrix}\rangle$ is the $\mathrm{SU}(4)\supset\mathrm{SU}_{\mathcal{S}}(2)\otimes\mathrm{SU}_T(2)$ isoscalar factors [35] with

$$\left\langle\begin{matrix}[\sigma\sigma] & [11]\\ \mathcal{S}'T' & \mathcal{S}_0T_0\end{matrix}\middle|\begin{matrix}[\sigma'\sigma']\\ \mathcal{S}T\end{matrix}\right\rangle = -\sqrt{\frac{(\mathcal{S}+1)(\sigma-\mathcal{S}+T+2)(\sigma-\mathcal{S}-T+1)}{2(\sigma+1)(\sigma+2)(2\mathcal{S}+1)}}\delta_{\mathcal{S}'\mathcal{S}+1}\delta_{T'T}\delta_{\mathcal{S}_01}\delta_{T_00}\delta_{\sigma'\sigma+1}$$
$$-\sqrt{\frac{(T+1)(\sigma+\mathcal{S}-T+2)(\sigma-\mathcal{S}-T+1)}{2(\sigma+1)(\sigma+2)(2T+1)}}\delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T+1}\delta_{\mathcal{S}_00}\delta_{T_01}\delta_{\sigma'\sigma+1}$$
$$+\sqrt{\frac{T(\sigma+\mathcal{S}+T+3)(\sigma-\mathcal{S}+T+2)}{2(\sigma+1)(\sigma+2)(2T+1)}}\delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T-1}\delta_{\mathcal{S}_00}\delta_{T_01}\delta_{\sigma'\sigma+1}$$

$$+\sqrt{\frac{\mathcal{S}(\sigma+\mathcal{S}+T+3)(\sigma+\mathcal{S}-T+2)}{2(\sigma+1)(\sigma+2)(2\mathcal{S}+1)}}\delta_{\mathcal{S}'\mathcal{S}-1}\delta_{T'T}\delta_{\mathcal{S}_0 1}\delta_{T_0 0}\delta_{\sigma'\sigma+1}$$
$$+\sqrt{\frac{(\mathcal{S}+1)(\sigma+\mathcal{S}-T+2)(\sigma+\mathcal{S}+T+3)}{2(\sigma+2)(\sigma+3)(2\mathcal{S}+1)}}\delta_{\mathcal{S}'\mathcal{S}+1}\delta_{T'T}\delta_{\mathcal{S}_0 1}\delta_{T_0 0}\delta_{\sigma'\sigma-1}$$
$$-\sqrt{\frac{(T+1)(\sigma-\mathcal{S}+T+2)(\sigma+\mathcal{S}+T+3)}{2(\sigma+2)(\sigma+3)(2T+1)}}\delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T+1}\delta_{\mathcal{S}_0 0}\delta_{T_0 1}\delta_{\sigma'\sigma-1}$$
$$+\sqrt{\frac{T(\sigma-\mathcal{S}-T+1)(\sigma+\mathcal{S}-T+2)}{2(\sigma+2)(\sigma+3)(2T+1)}}\delta_{\mathcal{S}'\mathcal{S}}\delta_{T'T-1}\delta_{\mathcal{S}_0 0}\delta_{T_0 1}\delta_{\sigma'\sigma-1}$$
$$-\sqrt{\frac{\mathcal{S}(\sigma-\mathcal{S}-T+1)(\sigma-\mathcal{S}+T+2)}{2(\sigma+2)(\sigma+3)(2\mathcal{S}+1)}}\delta_{\mathcal{S}'\mathcal{S}-1}\delta_{T'T}\delta_{\mathcal{S}_0 1}\delta_{T_0 0}\delta_{\sigma'\sigma-1}. \tag{A17}$$

By using Eq. (A16), the quantities appearing in Eqs. (A2) and (A9) can be expressed explicitly as

$$\begin{aligned}
&\sum_{n_i''\sigma_i''T_i''}\langle\Omega_i n_i'\sigma_i'\mathcal{S}_i T_i||P^{(i)\dagger}||\Omega_i n_i''\sigma_i''\mathcal{S}_i T_i''\rangle\langle\Omega_i n_i\sigma_i\mathcal{S}_i T_i||P^{(i)\dagger}||\Omega_i n_i''\sigma_i''\mathcal{S}_i T_i''\rangle\\
&=\delta_{n_i n_i'}\delta_{\sigma_i',\sigma_i-2}\sqrt{\frac{(\sigma_i+\mathcal{S}_i+T_i+2)(\sigma_i+\mathcal{S}_i-T_i+1)(\sigma_i-\mathcal{S}_i+T_i+1)(\sigma_i-\mathcal{S}_i-T_i)}{1024\,(\sigma_i+2)(\sigma_i+1)^2\sigma_i}}\\
&\times\sqrt{(n_i-2\sigma_i+4)(n_i+2\sigma_i+8)(4\Omega_i-n_i+2\sigma_i+8)(4\Omega_i-n_i-2\sigma_i+4)}\\
&+\delta_{n_i n_i'}\delta_{\sigma_i',\sigma_i+2}\sqrt{\frac{(\sigma_i+\mathcal{S}_i+T_i+4)(\sigma_i+\mathcal{S}_i-T_i+3)(\sigma_i-\mathcal{S}_i+T_i+3)(\sigma_i-\mathcal{S}_i-T_i+2)}{1024\,(\sigma_i+2)(\sigma_i+3)^2(\sigma_i+4)}}\\
&\times\sqrt{(n_i-2\sigma_i)(n_i+2\sigma_i+12)(4\Omega_i-n_i+2\sigma_i+12)(4\Omega_i-n_i-2\sigma_i)}+\delta_{n_i n_i'}\delta_{\sigma_i',\sigma_i}\\
&\times\frac{(\sigma_i+3)(2\sigma_i+n_i+8)(4\Omega_i-n_i-2\sigma_i+4)(\sigma_i^2+\sigma_i-\mathcal{S}_i^2-\mathcal{S}_i+T_i^2+T_i)+(\sigma_i+1)(n_i-2\sigma_i)(4\Omega_i+2\sigma_i-n_i+12)\big(\sigma_i^2+7\sigma_i-\mathcal{S}_i^2-\mathcal{S}_i+T_i^2+T_i+12\big)}{32(\sigma_i+1)(\sigma_i+2)(\sigma_i+3)},
\end{aligned}\tag{A18}$$

$$\begin{aligned}
&\sum_{n_i''\sigma_i''\mathcal{S}_i''}\langle\Omega_i n_i'\sigma_i'\mathcal{S}_i T_i||D^{(i)\dagger}||\Omega_i n_i''\sigma_i''\mathcal{S}_i'' T_i\rangle\langle\Omega_i n_i\sigma_i\mathcal{S}_i T_i||D^{(i)\dagger}||\Omega_i n_i''\sigma_i''\mathcal{S}_i'' T_i\rangle\\
&=-\delta_{n_i n_i'}\delta_{\sigma_i',\sigma_i-2}\sqrt{\frac{(\sigma_i+\mathcal{S}_i+T_i+2)(\sigma_i+\mathcal{S}_i-T_i+1)(\sigma_i-\mathcal{S}_i+T_i+1)(\sigma_i-\mathcal{S}_i-T_i)}{1024\,(\sigma_i+2)(\sigma_i+1)^2\sigma_i}}\\
&\times\sqrt{(n_i-2\sigma_i+4)(n_i+2\sigma_i+8)(4\Omega_i-n_i+2\sigma_i+8)(4\Omega_i-n_i-2\sigma_i+4)}\\
&-\delta_{n_i n_i'}\delta_{\sigma_i',\sigma_i+2}\sqrt{\frac{(\sigma_i+\mathcal{S}_i+T_i+4)(\sigma_i+\mathcal{S}_i-T_i+3)(\sigma_i-\mathcal{S}_i+T_i+3)(\sigma_i-\mathcal{S}_i-T_i+2)}{1024\,(\sigma_i+2)(\sigma_i+3)^2(\sigma_i+4)}}\\
&\times\sqrt{(n_i-2\sigma_i)(n_i+2\sigma_i+12)(4\Omega_i-n_i+2\sigma_i+12)(4\Omega_i-n_i-2\sigma_i)}+\delta_{n_i n_i'}\delta_{\sigma_i',\sigma_i}\\
&\times\frac{(\sigma_i+3)(2\sigma_i+n_i+8)(4\Omega_i-n_i-2\sigma_i+4)\big(\sigma_i^2+\sigma_i+\mathcal{S}_i^2+\mathcal{S}_i-T_i^2-T_i\big)+(\sigma_i+1)(n_i-2\sigma_i)(4\Omega_i+2\sigma_i-n_i+12)\big(\sigma_i^2+7\sigma_i+\mathcal{S}_i^2+\mathcal{S}_i-T_i^2-T_i+12\big)}{32(\sigma_i+1)(\sigma_i+2)(\sigma_i+3)}.
\end{aligned}\tag{A19}$$

Though the expressions of the diagonal parts of the reduced matrix elements are quite different, it can be checked that Eqs. (A18) and (A19) are consistent with the results shown in Ref. [22], in which some typos of Ref. [19] were corrected. The above results are also valid in the $O_{LST}(8)$ model with $\Omega = 2l+1$ for a given l orbit.

2. Matrix elements of the single-particle energy term in the $O_{LST}(8)$ model

In the $O_{LST}(8)$ model, the single-particle energy term can be expressed as

$$\sum_i\sum_{j_i m m_t}\epsilon_{j_i}a^\dagger_{j_i m t m_t}a_{j_i m t m_t}=\sum_i\sum_{j_i}\epsilon_{j_i}\sqrt{(2t+1)(2j_i+1)}\big(a^\dagger_{(l_i s)j_i t}\times\tilde{a}_{(l_i s)j_i t}\big)^{00}_{00}=\sum_i U^{00}(i), \tag{A20}$$

where $s = t = 1/2$ are the spin and the isospin of the valence nucleons, respectively, and i runs over all l orbits considered, which obviously is a total angular momentum and isospin scalar in the j-coupling scheme. Because the orbital angular momentum is always zero, the basis vector $|\alpha_1 S_1 T_1; \alpha_2 S_2 T_2; ST\rangle$ in the $O_{LST}(8)$ tensor product subspace is equivalent to the corresponding one in the j-coupling scheme,

$$|\alpha_1 \mathrm{S}_1 T_1; \alpha_2 \mathrm{S}_2 T_2; ST\rangle \equiv |\alpha_1(0\mathrm{S}_1)J_1 = \mathrm{S}_1, T_1; \alpha_2(0\mathrm{S}_2)J_2 = \mathrm{S}_2, T_2; J = \mathrm{S}, T\rangle, \tag{A21}$$

where $\alpha_i \equiv \{n_i, \sigma_i\}$. Therefore, the reduced matrix elements of the angular momentum and the isospin scalar in the ith l orbit, $U^{00}(i)$, can be expressed as

$$\langle \alpha_1' S_1' T_1'; \alpha_2' S_2' T_2'; S'T'||U^{00}(i)||\alpha_1 S_1 T_1; \alpha_2 S_2 T_2; ST\rangle = \delta_{S'S}\delta_{T'T}\prod_{q=1}^{2}\delta_{T_q T_q'}\delta_{S_q S_q'}\prod_{q'\neq i}^{2}\delta_{\alpha_{q'}'\alpha_{q'}}\langle \alpha_i'(0S_i)S_i T_i||U^{00}(i)||\alpha_i(0S_i)S_i T_i\rangle. \tag{A22}$$

The reduced matrix element $\langle \alpha_i'(0S_i)S_iT_i||U^{00}(i)||\alpha_i(0S_i)S_iT_i\rangle$ can further be expressed as

$$\begin{aligned}
&\langle \alpha_i'(0S_i)S_iT_i||U^{00}(i)||\alpha_i(0S_i)S_iT_i\rangle \\
&= \sum_{j_i}\epsilon_{j_i}(2j_i+1)\sum_{\alpha_i''S_i''T_i''J_i''}\frac{2J_i''+1}{2l_i+1}\begin{Bmatrix} j_i & s & l_i \\ S'' & J_i'' & S_i\end{Bmatrix}^2 \langle \alpha_i' 0 S_i T_i||a^\dagger_{l_i st}||\alpha_i'' l_i S_i'' T_i''\rangle\langle \alpha_i 0 S_i T_i||a^\dagger_{l_i st}||\alpha_i'' l_i S_i'' T_i''\rangle \\
&= \sum_{j_i}\epsilon_{j_i}\frac{2j_i+1}{(2l_i+1)(2s+1)}\sum_{\alpha_i''S_i''T_i''}\langle \alpha_i' 0 S_i T_i||a^\dagger_{l_i st}||\alpha_i'' l_i S_i'' T_i''\rangle\langle \alpha_i 0 S_i T_i||a^\dagger_{l_i st}||\alpha_i'' l_i S_i'' T_i''\rangle,
\end{aligned} \tag{A23}$$

where the sum rule of the $6j$-symbol for J_i'' is used. Because $a^\dagger_{l_i st}$ is the O(8) $(1/2) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ tensor operator, therefore α_i'' must belong to the O(8) seniority-one irrep $(\Omega_i - \frac{1}{2}) \equiv (\Omega_i - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Thus, by using the Racah factorization lemma, the reduced matrix element $\langle \alpha_i 0 S_i T_i||a^\dagger_{l_i st}||\alpha_i'' l_i S_i'' T_i''\rangle$ can further be expressed as

$$\langle \alpha_i 0 S_i T_i||a^\dagger_{l_i st}||\alpha_i'' l_i S_i'' T_i''\rangle = \delta_{n_i'' n_i - 1}\left\langle \Omega_i|||a^\dagger_{l_i}|||\left(\Omega_i - \frac{1}{2}\right), l_i\right\rangle\left\langle \begin{matrix}(\Omega_i - 1/2) & (1/2) \\ n_i - 1\,[\sigma_i''] & 1[1]\end{matrix}\middle| \begin{matrix}\Omega_i \\ n_i\,[\sigma_i\sigma_i]\end{matrix}\right\rangle\left\langle \begin{matrix}[\sigma_i''] & [1] \\ S_i''T_i'' & st\end{matrix}\middle|\begin{matrix}[\sigma_i\sigma_i] \\ S_i\, T_i\end{matrix}\right\rangle, \tag{A24}$$

where $[\sigma_i'']$ stands for the possible SU(4) irrep involved, and the O(8) single-particle reduced matrix element is given by [27]

$$\langle \Omega|||a^\dagger_{l_i}|||(\Omega - \tfrac{1}{2}), l_i\rangle = -\sqrt{4(2l_i+1)}. \tag{A25}$$

After substituting Eq. (A24) into Eq. (A23) and summing over n_i'', S_i'', and T_i'', Eq. (A23) can be simplified as

$$\langle \alpha_i'(0S_i)S_iT_i||U^{00}(i)||\alpha_i(0S_i)S_iT_i\rangle = \delta_{\sigma_i'\sigma_i}\delta_{n_i'n_i}\sum_{j_i}\epsilon_{j_i}2(2j_i+1)\sum_{[\sigma_i'']}\left\langle \begin{matrix}(\Omega_i - 1/2) & (1/2) \\ n_i - 1 \quad [\sigma_i''] & 1[1]\end{matrix}\middle|\begin{matrix}\Omega_i \\ n_i \quad [\sigma_i\sigma_i]\end{matrix}\right\rangle^2. \tag{A26}$$

Finally, using the single-particle isoscalar factors of O(8) $\supset$ U(4) shown in Table 3 of Ref. [27], one obtains

$$\langle \alpha_i'(0S_i)S_iT_i||U^{00}(i)||\alpha_i(0S_i)S_iT_i\rangle = \delta_{\sigma_i\sigma_i'}\delta_{n_i'n_i}\sum_{j_i}\epsilon_{j_i}\frac{(2j_i+1)n_i}{2\Omega_i}, \tag{A27}$$

where $n_i \equiv n_{l_i}$ is the number of valence nucleons in the l_i orbit. By substituting Eq. (A27) into Eq. (A22), the final result is used in Sec. IV for the single-particle energy term in the $\mathrm{O}_{LST}(8)$ model.

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