Elastic α-^{12}C scattering at low energies with the sharp resonant 0_3^+ state of ^{16}O

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An inclusion of the sharp resonant 0_3^+ state of ^{16}O and the first excited 2_1^+ state of ^{12}C in a study of s-wave elastic α-^{12}C scattering at low energies is investigated in an effective Lagrangian approach. The elastic scattering amplitude is separated into two parts: one is for the sharp resonant 0_3^+ state of ^{16}O parametrized by the Breit-Wigner formula, and the other is for the nonresonant part of the amplitude parametrized by effective range expansion. In the nonresonant part of the amplitude, a contribution from the 2_1^+ state of ^{12}C is included. I discuss a large correlation between a coupling for the 2_1^+ state of ^{12}C and an effective range parameter Q_0 as well as the necessity of including a vertex correction in the initial and final α-^{12}C state interactions. After fixing parameters appearing in the amplitudes by using experimental data, I calculate asymptotic normalization coefficients for the ground 0_1^+ state and the first excited 0_2^+ state of ^{16}O and compare them to previous results found in the literature.

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I. INTRODUCTION

Radiative α capture on ^{12}C, ^{12}C$(\alpha,\gamma)^{16}$O, is a key reaction to determine the ratio of ^{12}C/^{16}O production in stars [1]. Due to subthreshold $l^{\pi}_{i-th} = 1^-_1$ and 2^+_1 states of ^{16}O just below α-^{12}C breakup threshold, the radiative α capture reaction will be dominated by $E1$ and $E2$ transitions while a small contribution comes out of so-called cascade transitions where α and ^{12}C first form an excited bound state of ^{16}O and it subsequently decays down to the ground 0_1^+ state of ^{16}O. Asymptotic normalization coefficients (ANCs) of bound states of ^{16}O play an important role to estimate the radiative α capture rates, equivalent to the astrophysical S factor at Gamow-peak energy, $E_G = 0.3$ MeV, for R-matrix analysis [2]. Values of the ANCs for the subthreshold 1^-_1 and 2^+_1 states of ^{16}O converge in both theory and experiment while scattered values for 3^-_1, 0^+_2, and 0^+_1 states of ^{16}O are found in the literature. During the last half century, numerous experimental and theoretical studies related to the radiative α capture reaction have been carried out. For review, see, e.g., Refs. [3–6] and references therein.

In my recent works, I constructed an effective field theory (EFT) for the ^{12}C$(\alpha,\gamma)^{16}$O reaction at E_G [7]; parameters appearing in an effective Lagrangian were fitted to experimental data for elastic α-^{12}C scattering [8–10] and for the S factor of the ^{12}C$(\alpha,\gamma)^{16}$O reaction through the $E1$ transition [11], and a value of the S_{E1} factor at E_G was estimated in the theory for the first time [11].[1] An EFT may provide us a model-independent method for theoretical calculation at low energies, in which one needs to introduce a separation (momentum) scale between relevant physical degrees of freedom at low energy and irrelevant degrees of freedom at high energy. An effective Lagrangian is constructed using the relevant low-energy degrees of freedom and expanded in terms of the number of derivatives order by order. The irrelevant degrees of freedom are integrated out of the effective Lagrangian, and the effects from those at high energy are presumed to be embedded in the coefficients of the terms appearing in the effective Lagrangian. Those coefficients are possibly determined from its mother theory while they are practically fixed by using experimental data. The derivative expansion scheme provides us a perturbative expansion, which is useful to estimate a theoretical error for a reaction in question. For review for EFTs, see, e.g., Refs. [13–17]. In my subsequent works, I could incorporate the broad resonant 1^-_2 and 3^-_2 states of ^{16}O in the reaction amplitudes using effective range expansion but could not include the sharp resonant 0^+_3 and 2^+_2 states of ^{16}O [10,11]. In this work, I study an inclusion of the sharp resonant 0^+_3 state of ^{16}O, along with the first excited 2^+_1 state of ^{12}C, in the elastic α-^{12}C scattering for the $l = 0$ channel.

Counting rules for a resonant state, when one includes the Coulomb interaction between two charged particles, for a halo-like system are discussed by Higa, Hammer, and van Kolck [18] and for a resonance state that is parametrized as a Breit-Wigner form by Gelman [19]. I follow a prescription suggested by Higa, Hammer, and van Kolck to rewrite a scattering amplitude presented in terms of effective range parameters to an amplitude presented by using the Breit-Wigner formula. I also follow another prescription by Gelman to separate a scattering amplitude into two parts: one is an amplitude for a sharp resonant state, and the other is that for the rest of the nonresonant part of the amplitude. As I discussed in my previous work [8], because of a modification of the counting rules for the elastic α-^{12}C scattering at low energies, I include the terms up to p^6 order in the effective range expansion, where p is the magnitude of relative momentum between α and ^{12}C. In this work, parameters appearing in the amplitudes

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[1]Recently, the EFT was applied to a study of β-delayed α emission from ^{16}N [12].

are fixed by using the following experimental data: binding energies for bound states, a resonant energy and a width for a resonant state of ^{16}O, and phase shift data for elastic α-^{12}C scattering. Because the energy range for the experimental phase shift data reported by Tischhauser *et al.* [20] is $2.6 \leqslant E_\alpha \leqslant 6.62$ MeV, where E_α is the α energy in laboratory frame,[2] I include the first excited 2^+_1 state of ^{12}C whose excited energy is $E(2^+_1) = 4.44$ MeV in the present study. I also include a vertex correction for the initial and final state interactions between α and ^{12}C in a phenomenological way.

Two parameters appearing in the scattering amplitude for the sharp resonance state are fixed by experimental values for the resonant energy and the width for the 0^+_3 state of ^{16}O while two effective parameters appearing in the nonresonant part of the amplitude are fixed by using the binding energies of the ground 0^+_1 state and the first excited 0^+_2 state of ^{16}O. The other four parameters—P_0 and Q_0 for the effective range parameters, a coupling constant $\tilde{g}_0$ for a contribution from the 2^+_1 state of ^{12}C, and a constant R_2 for a vertex correction for the initial and final states of α and ^{12}C—are fitted to experimental phase shift data δ_0 for the elastic α-^{12}C scattering for the $l = 0$ channel. I find that the experimental data, including a sharp peak of the resonant 0^+_3 state of ^{16}O in the energy range $2.6 \leqslant E_\alpha \leqslant 6.62$ MeV, are well reproduced, while a coupling $\tilde{g}_0$ cannot be fixed by using the phase shift data because of a large correlation with Q_0. I then calculate asymptotic normalization coefficients (ANCs) for the 0^+_1 and 0^+_2 states of ^{16}O and compare the results with those found in the previous studies.

The present article is organized as follows. In Sec. II, an effective Lagrangian for the elastic α-^{12}C scattering for the $l = 0$ channel, including the sharp resonant 0^+_3 state of ^{16}O and the first excited 2^+_1 state of ^{12}C, is discussed. In Sec. III, the scattering amplitude for the part of the sharp resonance 0^+_3 state of ^{16}O and for the rest of the nonresonant part of the amplitude are derived. In Sec. IV, four parameters are fixed by using the two binding energies, resonant energy, and width of the 0^+_1, 0^+_2, and 0^+_3 states of ^{16}O, and remaining parameters are fitted to the experimental phase shift data for the elastic α-^{12}C scattering for the $l = 0$ channel. I then calculate the ANCs for the 0^+_1 and 0^+_2 states of ^{16}O and compare the results of the ANCs to those found in the previous works. In Sec. V, results and discussion of this work are presented.

II. EFFECTIVE LAGRANGIAN

An effective Lagrangian to derive a scattering amplitude for s-wave elastic α-^{12}C scattering at low energies, including the sharp resonant 0^+_3 state of ^{16}C and the first excited 2^+_1 state of ^{12}C, may be written as [7–9,21]

$$
\begin{aligned}
\mathcal{L} = {} & \phi_\alpha^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_\alpha} + \cdots \right) \phi_\alpha + \phi_C^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_C} + \cdots \right) \phi_C \\
& + \phi_{C,ij}^{(l=2)\dagger} \left(iD_0 + \frac{1}{2m_C}\vec{D}^2 - \Delta_{(2)} + \cdots \right) \phi_{C,ij}^{(l=2)} \\
& + \sum_{n=0}^{3} C_n^{(rs)} d_{(rs)}^\dagger \left[iD_0 + \frac{\vec{D}^2}{2(m_\alpha + m_C)} \right]^n d_{(rs)} - y_{(rs)} [d_{(rs)}^\dagger (\phi_\alpha \phi_C) + (\phi_\alpha \phi_C)^\dagger d_{(rs)}] \\
& + \sum_{n=0}^{3} C_n^{(nr)} d_{(nr)}^\dagger \left[iD_0 + \frac{\vec{D}^2}{2(m_\alpha + m_C)} \right]^n d_{(nr)} - y_{(nr)} [d_{(nr)}^\dagger (\phi_\alpha \phi_C) + (\phi_\alpha \phi_C)^\dagger d_{(nr)}] \\
& - y'_{(rs)} [d_{(rs)}^\dagger (\phi_\alpha O'_0 \phi_C) + (\phi_\alpha O'_0 \phi_C)^\dagger d_{(rs)}] - y'_{(nr)} [d_{(nr)}^\dagger (\phi_\alpha O'_0 \phi_C) + (\phi_\alpha O'_0 \phi_C)^\dagger d_{(nr)}] \\
& - g_0 \left[d_{(nr)}^\dagger \left(\phi_\alpha O_{2,ij} \phi_{C,ij}^{(l=2)} \right) + \left(\phi_\alpha O_{2,ij} \phi_{C,ij}^{(l=2)} \right)^\dagger d_{(nr)} \right] + \cdots ,
\end{aligned}
\tag{1}
$$

where ϕ_α (m_α) and ϕ_C (m_C) are scalar fields (masses) of α and ^{12}C, respectively. D^μ is a covariant derivative, $D^\mu = \partial^\mu + i\mathcal{Q}A^\mu$, where $\mathcal{Q}$ is a charge operator and A^μ is the photon field. The dots denote higher-order terms. $\phi_{C,ij}^{(l=2)}$ (Δ_2) is a field (an excited energy) for the first excited 2^+_1 state of ^{12}C. $d_{(rs)}$ and $d_{(nr)}$ are composite fields of ^{16}O consisting of α and ^{12}C for the $l = 0$ channel for the sharp resonant state (rs) and the nonresonant (nr) part, respectively, which are introduced for perturbative expansion around the unitary limit [22–25]. The coupling constants, $C_n^{(rs)}$ and $C_n^{(nr)}$ with $n = 0$, 1, 2, and 3, correspond to effective range parameters of elastic α-^{12}C scattering while the coupling constants $y_{(rs)}$ and $y_{(nr)}$ are redundant,[3] and are conventionally taken as $y_{(rs)} = y_{(nr)} = \sqrt{2\pi/\mu}$. $y'_{(rs)}$ and $y'_{(nr)}$ are higher-order vertex corrections for $d_{(rs,nr)}$-α-^{12}C vertices at next-to-next-to leading order (NNLO). In the following sections, I will not mix these two fields, $d_{(rs)}$ and $d_{(nr)}$, for the sharp resonant part and the nonresonant part of the amplitude through the

[2] One has a relation $E_\alpha = \frac{4}{3}E$, where E is the total kinetic energy of α and ^{12}C in center-of-mass frame.

[3] In the denominator of the elastic scattering amplitudes, the couplings appear in the form, $C_n^{(rs,nr)}/y_{(rs,nr)}^2$ with $n = 0, 1, 2, 3$, and are fitted to, e.g., the effective range parameters, $1/a_0$, r_0, P_0, Q_0, for $l = 0$, respectively. The $y_{(rs,nr)}$ couplings are redundant, and one can arbitrarily fix their values.

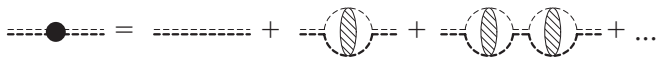

FIG. 1. Diagrams for dressed ^{16}O propagator. A thick (thin) dashed line represents a propagator of ^{12}C (α), and thick and thin double dashed lines with and without a filled circle represent dressed and bare ^{16}O propagators, respectively. A shaded blob represents a set of diagrams consisting of all possible one-potential-photon-exchange diagrams up to infinite order and no potential-photon-exchange one.

$y_{(rs)}$ and $y_{(nr)}$ interactions or the $y'_{(rs)}$ and $y'_{(nr)}$ interactions. In addition, I include the $y'_{(rs)}$ and $y'_{(nr)}$ interactions only in the initial and final state interactions of α and ^{12}C. Those issues will be discussed later. g_0 is a coefficient for the transitions between the s-wave α and first excited 2^+_1 state of ^{12}C and the nonresonant part of the composite ^{16}O field, $d_{(nr)}$. The operators are given as

$$O_{2,ij} = -\frac{\overleftrightarrow{D}_i}{M}\frac{\overleftrightarrow{D}_j}{M} + \frac{1}{3}\delta_{ij}\frac{\overleftrightarrow{D}^2}{M^2},$$
$$O'_0 = -\frac{\overleftrightarrow{D}^2}{M^2}, \quad i\frac{\overleftrightarrow{D}_i}{M} \equiv i\left(\frac{\overrightarrow{D}_C}{m_C} - \frac{\overleftarrow{D}_\alpha}{m_\alpha}\right)_i. \tag{2}$$

III. AMPLITUDES FOR THE ELASTIC SCATTERING

The elastic scattering amplitude A_0 is decomposed into two parts:

$$A_0 = A_0^{(rs)} + A_0^{(nr)}, \tag{3}$$

where $A_0^{(rs)}$ is the amplitude for the sharp resonant 0^+_3 state of ^{16}O parametrized by the Breit-Wigner formula and $A_0^{(nr)}$ is the scattering amplitude for the nonresonant contribution parameterized by the effective range expansion.

Both the scattering amplitudes $A_0^{(rs)}$ and $A_0^{(nr)}$ are calculated from the diagrams depicted in Figs. 1 and 2. For the elastic scattering amplitude for the sharp resonant 0^+_3 state of ^{16}O, I first write it down in terms of the effective range expansion as

$$A_0^{(rs)} = \frac{2\pi}{\mu}\frac{e^{2i\sigma_0}C_\eta^2 F(p)^2}{K_0^r(p) - 2\kappa H(\eta)}, \tag{4}$$

with

$$e^{2i\sigma_0} = \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)},$$
$$C_\eta^2 = \frac{2\pi\eta}{e^{2\pi\eta}-1}, \quad H(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \ln(i\eta), \tag{5}$$

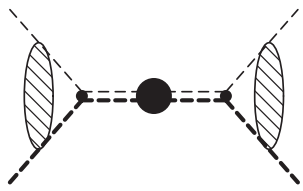

FIG. 2. Diagram of the scattering amplitude. See the caption of Fig. 1 as well.

where $\psi(z)$ is the digamma function and $\eta = \kappa/p$ where κ is the inverse of the Bohr radius, $\kappa = Z_\alpha Z_C \mu\alpha_E$; Z_α and Z_C are the numbers of protons in α and ^{12}C, μ is the reduced mass of α and ^{12}C, and α_E is the fine-structure constant. One may note that the Coulomb self-energy term, $-2\kappa H(\eta)$, is obtained from a bubble diagram due to propagation of the ground states of α and ^{12}C. Here the self-energy contribution from the 2^+_1 state of ^{12}C is ignored because the amplitude will be rewritten as the Breit-Wigner-like expression below, which has a sharp peak at the resonant energy E_r, and the off-peak energy contribution will be regarded as a higher order correction. The functions $F(p)$ and $K_0^r(p)$ contain dynamics for the elastic scattering through the sharp resonant state. The function $F(p)$ is a vertex correction of the initial and final state interactions between α and ^{12}C while the function $K_0^r(p)$ is a polynomial function expanded around the unitary limit, which is presented in terms of the effective range parameters. Thus, one has

$$F(p) = 1 - \frac{1}{6}R_2 p^2, \tag{6}$$
$$K_0^r(p) = -\frac{1}{a_0^r} + \frac{1}{2}r_0^r p^2 - \frac{1}{4}P_0^r p^4 + Q_0^r p^6, \tag{7}$$

where I have introduced a squared radius-like parameter R_2, $R_2 = -6y'_{(rs)}/(y_{(rs)}\mu^2)$ in Eq. (6), and the coefficients, $C_n^{(rs)}/y_{(rs)}^2$ with $n = 0, 1, 2, 3$, have been replaced by the effective range parameters in Eq. (7).

Following a prescription suggested by Higa, Hammer, and van Kolck [18] to rewrite the amplitude parametrized by the effective range expansion to that by the Breit-Wigner formula, I have

$$A_0^{(rs)} = -\frac{2\pi}{\mu}\frac{e^{2i\sigma_0}}{\sqrt{2\mu E}}\frac{\frac{1}{2}\Gamma(E)F(p)^2}{Z_r D^r(E) + i\frac{1}{2}\Gamma(E)}, \tag{8}$$

with

$$\Gamma(E) = \Gamma_r\frac{e^{2\pi\eta_r}-1}{e^{2\pi\eta}-1}, \quad \eta_r = \frac{\kappa}{\sqrt{2\mu E_r}}, \tag{9}$$
$$Z_r D^r(E) = E - E_r + Z_r\Big\{\mu^2 P_0^r (E-E_r)^2$$
$$-8\mu^3 Q_0^r (E+2E_r)(E-E_r)^2$$
$$+2\kappa\Big[\mathrm{Re}\,H(\eta) - \mathrm{Re}\,H(\eta_r)$$
$$-\frac{\partial}{\partial E}\mathrm{Re}\,H(\eta)\Big|_{E=E_r}(E-E_r)\Big]\Big\}, \tag{10}$$
$$Z_r^{-1} = \frac{\partial}{\partial E}D^r(E)\Big|_{E=E_r}, \quad Z_r = \frac{e^{2\pi\eta_r}-1}{4\pi\kappa}\Gamma_r, \tag{11}$$

where E is the energy of the α-^{12}C system in the center-of-mass frame, $E = p^2/(2\mu)$, and E_r and Γ_r are the energy and the width of the resonant 0^+_3 state of ^{16}O, which are related to two effective range parameters, a_0^r and r_0^r in Eq. (7). P_0^r can be fixed by using a condition that a large contribution from the Coulomb self-energy term at p^4 order is canceled with the P_0^r

term. Thus, I have

$$P_0^r = 24\mu E_r Q_0^r - \frac{\kappa}{\mu^2}\frac{\partial^2}{\partial E^2}\,\mathrm{Re}\,H(\eta)|_{E=E_r}, \tag{12}$$

where Q_0^r can be chosen arbitrarily. Thus, the scattering amplitude $A_0^{(rs)}$ for the sharp resonant state is represented by the three parameters R_2, E_r, and Γ_r.

For the nonresonant part of the amplitude $A_0^{(nr)}$, I have

$$A_0^{(nr)} = \frac{2\pi}{\mu}\frac{e^{2i\sigma_0}C_\eta^2 F(p)^2}{K_0(p) - 2\kappa\left[H(\eta) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta})\right]}, \tag{13}$$

with $\tilde{g_0} = g_0/y_{(nr)}$ and

$$H_2(\tilde{\eta}) = W_2(\tilde{p})H(\tilde{\eta}),$$
$$W_2(\tilde{p}) = \frac{1}{4}(\kappa^2+\tilde{p}^2)(\kappa^2+4\tilde{p}^2), \tag{14}$$
$$\tilde{\eta} = \kappa/\tilde{p}, \quad \tilde{p} = i\sqrt{-2\mu(E-\Delta_2)-i\epsilon}, \tag{15}$$

where Δ_2 is the excitation energy of the 2_1^+ state of ^{12}C, $\Delta_2 = 4.440$ MeV. The second Coulomb self-energy term, $-2\kappa H_2(\tilde{\eta})$, is obtained from a bubble diagram propagating the ground state of α and the excited 2_1^+ state of ^{12}C where those two states are in relative d-wave state and coupled to the s-wave composite ^{16}O field for the nonresonant contribution. Interaction between α and ^{12}C in the function $K_0(p)$ is parametrized by the effective range expansion; one has

$$K_0(p) = -\frac{1}{a_0} + \frac{1}{2}r_0p^2 - \frac{1}{4}P_0p^4 + Q_0p^6. \tag{16}$$

I fix two parameters among the four effective range parameters, a_0, r_0, P_0, and Q_0, by using conditions that the inverse of the scattering amplitude $A_0^{(nr)}$ vanishes at the energies of the ground 0_1^+ state and the first excited 0_2^+ state of ^{16}O. Thus, the denominator of the scattering amplitude vanishes,

$$D_0(p) = K_0(p) - 2\kappa\left[H(\eta) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta})\right] = 0, \tag{17}$$

at $p = i\gamma_0$ and $p = i\gamma_1$, where γ_0 and γ_1 are binding momenta for the 0_1^+ and 0_2^+ states of ^{16}O, respectively; $\gamma_{0,1} = \sqrt{2\mu B_{0,1}}$ where B_0 and B_1 are the binding energies for the 0_1^+ and 0_2^+ states of ^{16}O from the α-^{12}C breakup threshold, respectively. Using the conditions from Eq. (17), I fix two effective range parameters, a_0 and r_0, as

$$\frac{1}{a_0} = \frac{1}{4}\gamma_0^2\gamma_1^2P_0 + \left(\gamma_0^4\gamma_1^2+\gamma_0^2\gamma_1^4\right)Q_0 + \frac{2\kappa}{\gamma_0^2-\gamma_1^2}\left\{\gamma_1^2\left[H(\eta_{b0}) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta}_{b0})\right] - \gamma_0^2\left[H(\eta_{b1}) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta}_{b1})\right]\right\}, \tag{18}$$

$$r_0 = -\frac{1}{2}\left(\gamma_0^2+\gamma_1^2\right)P_0 - 2\left(\gamma_0^4+\gamma_0^2\gamma_1^2+\gamma_1^4\right)Q_0 - \frac{4\kappa}{\gamma_0^2-\gamma_1^2}\left\{\left[H(\eta_{b0}) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta}_{b0})\right] - \left[H(\eta_{b1}) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta}_{b1})\right]\right\}, \tag{19}$$

where $\eta_{b0,b1} = \kappa/(i\gamma_{0,1})$ and $\tilde{\eta}_{b0,b1} = \kappa/(i\sqrt{\gamma_{0,1}^2+2\mu\Delta_2})$. Using the two relations in Eqs. (18) and (19), I rewrite the denominator of the amplitude $D_0(p)$ as

$$D_0(p) = -\frac{1}{4}\left[\gamma_0^2\gamma_1^2 + \left(\gamma_0^2+\gamma_1^2\right)p^2 + p^4\right]P_0 + \left[-\gamma_0^4\gamma_1^2 - \gamma_0^2\gamma_1^4 - \left(\gamma_0^4+\gamma_0^2\gamma_1^2+\gamma_1^4\right)p^2 + p^6\right]Q_0$$
$$-2\kappa\left\{\frac{\gamma_1^2+p^2}{\gamma_0^2-\gamma_1^2}\left[H(\eta_{b0}) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta}_{b0})\right] - \frac{\gamma_0^2+p^2}{\gamma_0^2-\gamma_1^2}\left[H(\eta_{b1}) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta}_{b1})\right] + H(\eta) + \frac{2\tilde{g}_0^2}{3\mu^4}H_2(\tilde{\eta})\right\}, \tag{20}$$

where three constants, P_0, Q_0, $\tilde{g}_0$ in the function $D_0(p)$ and one constant R_2 in the function $F(p)$ for the nonresonant amplitude $A_0^{(nr)}$ will be fixed by using the phase-shift data. One may notice that the same parameter R_2 for both the amplitudes $A_0^{(rs)}$ and $A_0^{(nr)}$ is used because the parameter R_2 commonly appears in the initial and final state interactions between α and ^{12}C. Thus, there are six parameters $\{P_0, Q_0, \tilde{g}_0, R_2, E_r, \Gamma_r\}$ in the scattering amplitude A_0.

The ANCs $|C_b|_0$ and $|C_b|_1$ for the 0_1^+ and 0_2^+ states of ^{16}O are calculated by using the formula

$$|C_b|_n = \Gamma(1+\eta_{bn})F(i\gamma_n)\left[(-1)^n\frac{\partial D_0(p)}{\partial p^2}\bigg|_{p^2=-\gamma_n^2}\right]^{-1/2}, \tag{21}$$

with $n = 0$ or $n = 1$, where I have included the vertex correction $F(p)$ in the expression for the ANCs, which is found in Ref. [26].

IV. NUMERICAL RESULTS

To fix the coefficients appearing in the scattering amplitude A_0 for the $l = 0$ channel, the data for the phase shift δ_0 reported by Tischhauser *et al.* [20] are employed, as mentioned before. The elastic scattering amplitude for $l = 0$ in terms of the phase shift δ_0 is given as

$$A_0 = \frac{2\pi}{\mu}\frac{e^{2i\sigma_0}}{p\cot\delta_0 - ip}.$$

Because the scattering amplitude A_0 is represented by two terms, $A_0^{(rs)}$ and $A_0^{(nr)}$, as given in Eq. (3), the parameters are fitted to the data by using a relation for the squared amplitude as

$$\frac{1}{p^2}\sin^2\delta_0 = \left|\frac{1}{p\cot\delta_0 - ip}\right|^2 = \frac{\mu^2}{4\pi^2}\left|A_0^{(rs)} + A_0^{(nr)}\right|^2.$$

As mentioned above, the six parameters $\{P_0, Q_0, \tilde{g}_0, R_2, E_r, \Gamma_r\}$ remain in A_0 while the resonant

TABLE I. Values and errors of five parameters $\{P_0, Q_0, R_2, E_r, \Gamma_r\}$ fitted to the phase shift δ_0 of elastic α-^{12}C scattering for $l = 0$ using some values of $\tilde{g}_0$: $\tilde{g}_0 = 0, 20, 40, 60$. The ANCs, $|C_b|_0$ and $|C_b|_1$, for the 0_1^+ and 0_2^+ states of ^{16}O are calculated by using each set of the fitted values of the parameters.

$\tilde{g}_0$	0	20	40	60
P_0 (fm^3)	−0.03573(3)	−0.03575(3)	−0.03578(3)	−0.03584(3)
Q_0 (fm^5)	0.002055(9)	0.002894(9)	0.005408(9)	009598(9)
R_2 (fm^2)	0.976(14)	0.954(15)	0.869(15)	0.731(16)
E_r (MeV)	4.88763(6)	4.88763(6)	4.88760(6)	4.88758(6)
Γ_r (MeV)	0.00168(4)	0.00167(4)	0.00164(4)	0.00157(4)
$\|C_b\|_0$ (fm$^{-1/2}$)	41.0(1)	30.9(0)	20.2(0)	14.2(0)
$\|C_b\|_1$ (fm$^{-1/2}$)	443(3)	278(1)	166(0)	115(0)

energy E_r and its width Γ_r for the sharp resonant 0_3^+ state of ^{16}O are experimentally known as $E_r = 4.887(2)$ MeV and $\Gamma_r = 1.5(5)$ keV [27]. [The corresponding laboratory energy of E_r is $E_{\alpha,r} = \frac{4}{3}E_r = 6.516(3)$ MeV.] I include them in the fitting because more precise adjustment is necessary to reproduce the sharp peak appearing in the data. R_2 basically accounts for the slowly varying shape of the phase shift in the high-energy region, $5.5 \leqslant E_\alpha \leqslant 6.62$ MeV. $\tilde{g}_0$ is a dimensionless parameter and represents a contribution from the first excited 2_1^+ state of ^{12}C. As will be discussed below, I find that the parameter $\tilde{g}_0$ is strongly correlated to Q_0 and cannot be determined from the phase shift data. Because there is no restriction for $\tilde{g}_0$, I give some values for $\tilde{g}_0$ (here values of $\tilde{g}_0$ are arbitrarily chosen as $\tilde{g}_0 = 0, 20, 40, 60$) and fit the remaining five parameters $\{P_0, Q_0, R_2, E_r, \Gamma_r\}$ to the phase shift data employing a standard χ^2 fit[4] where the number of the data is $N = 351$ [20].

In Table I, values and errors of the five parameters $\{P_0, Q_0, R_2, E_r, \Gamma_r\}$ using the four values of $\tilde{g}_0$, $\tilde{g}_0 = 0, 20, 40, 60$, fitted to the phase shift data δ_0 are displayed. Values and errors of the ANCs, $|C_b|_0$ and $|C_b|_1$, for the ground 0_1^+ state and the first excited 0_2^+ state of ^{16}O calculated by using each set of the fitted parameters are also displayed in the table. I obtain almost the same χ^2 values for the four fittings, $\chi^2/N = 1.60$, and one can see that the errors of those fitted parameters almost do not change for the four cases. One can see that the fitted values of E_r and Γ_r almost do not change either for all the $\tilde{g}_0$ values and agree well with the experimental data within the error bars. A similar tendency can be seen for the fitted values of P_0 as well. On the other hand, one can notice that remarkable $\tilde{g}_0$ dependence for the values of Q_0; $\tilde{g}_0$ and Q_0 are strongly correlated with each other and cannot simultaneously be fitted by using the phase shift data. Minor but significant $\tilde{g}_0$ dependence can be seen for the values of R_2. Because the self-energy contribution, $-2\kappa H_2(\tilde{\eta})$, from the first excited 2_1^+ state of ^{12}C appears out of the d-wave coupling while R_2 accounts for the nonresonant shape of the phase shift data at high energies, $5.5 < E_\alpha < 6.62$ MeV, those contributions may become competitive at high energies. In Fig. 3, a curve for the squared amplitude, $\sin^2\delta_0/p^2$, is plotted as a function of E_α by using the fitted values of the parameters; those four sets of the fitted parameters displayed in the table give almost the same curve plotted in the figure. The experimental data are included in the figure as well. One can see the calculated curve well reproduces the experimental data.

Regarding estimate of the ANCs, $|C_b|_0$ and $C_b|_1$, for the ground 0_1^+ state and the first excited 0_2^+ state of ^{16}O, respectively, I find the results for the ANCs are significantly sensitive to the $\tilde{g}_0$ values; those two ANCs decrease as the $\tilde{g}_0$ value increases. In addition, a value of $|C_b|_1$ is about one order of magnitude larger than that of $|C_b|_0$ for a given value of $\tilde{g}_0$. In Fig. 4, a curve for $D_0(p)/F(p)^2$ is plotted as a function of E_α; two filled (red) squares in the figure denote the binding energies of the two bound states. Those points are fixed in Eq. (17), and the ANCs are calculated from the slope of the curve at those points by using the relation in Eq. (21). Because the function by which the curve is plotted is given as a polynomial function whose coefficients are the effective range parameters, the slope of the curve becomes steep when the magnitude of E_α becomes large. Thus, the slope at the ground 0_1^+ state is steeper than that of the 0_2^+ state, i.e., $|C_b|_0$ is smaller than $|C_b|_1$. In addition, when a value of $\tilde{g}_0$ becomes larger, the contribution from the higher order terms in the polynomial function [in the $-2\kappa H_2(\tilde{\eta})$ function, compared to

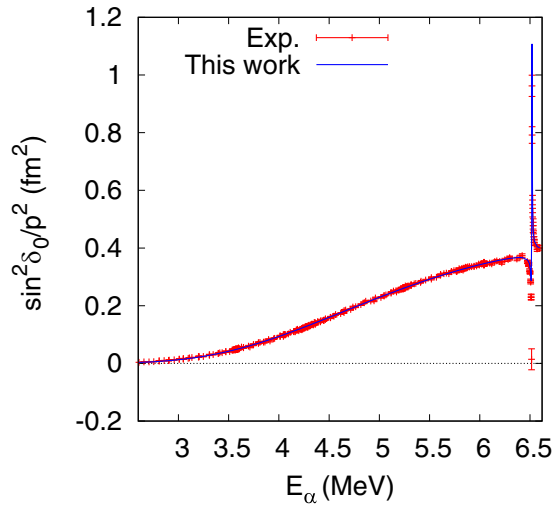

FIG. 3. Squared amplitude, $\sin^2\delta_0/p^2$, as a function of E_α calculated by using the fitted values of the parameters in Table I. Experimental data are included in the figure as well.

[4]I employ a Python package, `emcee` [28], for the fitting.

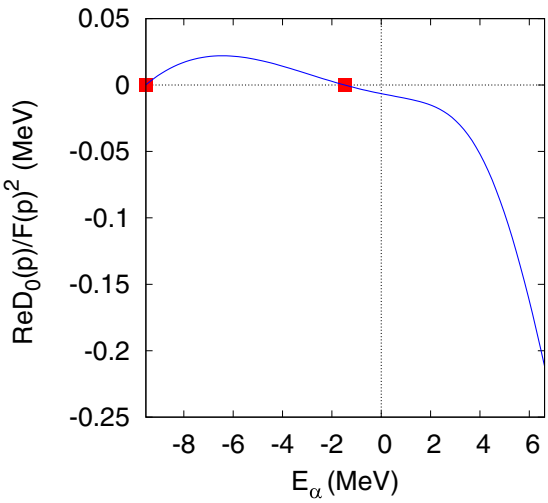

FIG. 4. Real part of the denominator of the dressed ^{16}O propagator for $l = 0$, $D_0(p)$, including vertex form factors $F(p)^2$ is plotted as a function of E_α. Filled (red) squares denote the binding energies for the 0_1^+ and 0_2^+ states of ^{16}O.

those in the $-2\kappa H_0(\eta)$ term] become larger; thus both the ANCs, $|C_b|_0$ and $|C_b|_0$, become smaller. Though there is no clear clue for a value of $\tilde{g}_0$, one can fix it by using a value of one of the two ANCs, and then the other one can be predicted.

I now discuss values of the ANCs found in the previous studies and compare them to the present results. For the ANC, $|C_b|_1$, for the first excited 0_2^+ state of ^{16}O, I have $|C_b|_1 =$ 443–115 fm$^{-1/2}$ for $\tilde{g}_0 = 0$–60. One can find in the literature [6] two groups for values of $|C_b|_1$: a large value group and a small value group. For the large value group, one may find three results, which are about more than 4 times larger than the present result: $|C_b|_1 = (15.6 \pm 1.0) \times 10^2$ fm$^{-1/2}$ obtained from the α transfer reaction ^{6}Li(^{12}C, d) ^{16}O reported by Avila *et al*, [29], 1800 fm$^{-1/2}$ from the R-matrix analysis for broad level structure of ^{16}O by deBoer *et al.* [30], and 1560 fm$^{-1/2}$ from the R-matrix analysis for the ^{12}C(α, γ) ^{16}O reaction by deBoer *et al.* [6]. For the small value group, one finds two results, which agree with the present result: $|C_b|_1 = 44^{+270}_{-44}$ fm$^{-1/2}$ from the study of the 0_2^+ state cascade transition in the ^{12}C(α, γ) ^{16}O reaction by Schürmann *et al.* [31] and 405.7 fm$^{-1/2}$ from the so-called Δ method based on the effective range theory by Orlov, Irgaziev, and Nabi [32].

For the ANC $|C_b|_0$, for the ground 0_1^+ state of ^{16}O, I have $|C_b|_0 = 41.0$–14.2 fm$^{-1/2}$ for $\tilde{g}_0 = 0$–60. One can also find two groups for values of $|C_b|_0$, a large value group and a small value group, in the literature. For the large value group, one may find two results, which are one or two orders of magnitude larger than the present result: 709 fm$^{-1/2}$ from $E2$ interference effects in the ^{12}C(α, γ) ^{16}O reaction by Sayre *et al.* [33] and 4000 fm$^{-1/2}$ (WS1), 1200 fm$^{-1/2}$ (WS2), and 750 fm$^{-1/2}$ (FP) from a study of ^{12}C(^{16}O, ^{12}C) ^{16}O reaction, where the results depend on the use of nuclear potentials: the Wood-Saxon 1 and 2 potentials (WS1, WS2) and the folding potential (FP) by Morais and Lichtenthäler [34]. For the small value result, one may find four results, which agree with the present ones: $|C_b|_0 = 13.9(24)$ fm$^{-1/2}$ from the continuum discretized coupled channels (CDCC) study for a resonant breakup of ^{16}O by Adhikari and Basu [35], 20.33 fm$^{-1/2}$ from the effective range expansion by Orlov, Irgaziev, and Nikitina [36], 21.76 fm$^{-1/2}$ from the Δ method based on the effective range expansion by Orlov, Irgaziev, and Nabi [32], and 58 fm$^{-1/2}$ from the R-matrix analysis for the ^{12}C(α, γ) ^{16}O reaction by deBoer *et al.* [6].

V. RESULTS AND DISCUSSION

In this work, I studied an inclusion of the sharp resonant 0_3^+ state of ^{16}O and the first excited 2_1^+ state of ^{12}C for elastic α-^{12}C scattering for the $l = 0$ channel up to the energy at which the sharp resonant 0_3^+ state of ^{16}O appears. I separate the scattering amplitude into two parts: one is an amplitude for the sharp resonant state, and the other is for the rest of the nonresonant part of the amplitude. The resonant part of the amplitude is presented in the Breit-Wigner-like form while the nonresonant part of the amplitude is parametrized by the effective range expansion. A contribution from a bubble diagram due to the propagation of α and the 2_1^+ state of ^{12}C is included in the nonresonant part of the amplitude. I also include a vertex correction for the initial and final state interactions of α and ^{12}C. Four parameters appearing in the amplitude are fixed by using the binding energies for the 0_1^+ and 0_2^+ states and the resonant energy and the width for the 0_3^+ state of ^{16}O, while the remaining four parameters, P_0, Q_0, $\tilde{g}_0$, and R_2, are fitted to the experimental phase shift data of the elastic α-^{12}C scattering for the $l = 0$ channel. I find a large correlation between Q_0 and $\tilde{g}_0$ and that a value of $\tilde{g}_0$, which represents a contribution from the 2_1^+ state of ^{12}C, is not fixed from the phase shift data, while a vertex correction, R_2, for the initial and final states of α and ^{12}C is found to be crucial to reproduce the phase shift data in the energy range $5.5 \leqslant E_\alpha \leqslant 6.62$ MeV. I then calculate the ANCs for the 0_1^+ and 0_2^+ states of ^{16}O. I find that the numerical results of the ANCs significantly depend on the values of $\tilde{g}_0$ while the values of the ANC for the 0_2^+ are about one order of magnitude larger than those for the 0_1^+ state. The obtained results for the ANCs are compared to those in the literature. Scattered values of the ANCs in the previous results are found, and those can be separated into two groups, a large value group and a small value group, for both the ANCs. The present results reasonably well agree with those of the small value groups for both the ANCs.

As one might have noticed, I did not mix the composite ^{16}O fields, $d_{(rs)}$ and $d_{(nr)}$, for the sharp resonant amplitude and the nonresonant part of the amplitude. Those two fields can be mixed in the amplitudes through the α-^{12}C propagation; in the α-^{12}C bubble diagram, the α-^{12}C state is created through the $y_{(rs)}$ or $y_{(nr)}$ interaction, and, after a propagation of α and ^{12}C, they are destroyed through the $y_{(rs)}$ or $y_{(nr)}$ interaction. Here I have assumed a naive counting rule in which the sharp resonant part of the amplitude becomes a leading order (LO) contribution near the resonant energy while at the off-resonant energy the resonant part of the amplitude is suppressed and the nonresonant part of the amplitude becomes a LO contribution. As a part of the higher order corrections at NNLO, I phenomenologically included it as a vertex correction, the

R_2 term, in the initial and final state interactions between α and ^{12}C. I found that the correction is crucial to reproduce the phase shift data in the energy region $5.5 \leqslant E_\alpha \leqslant 6.62$ MeV, close to the sharp resonant energy $E_\alpha(0_3^+) = 6.52$ MeV. A complete treatment for the terms at NNLO would be interesting for a future work.

I also found a significant $\tilde{g}_0$ dependence in the numerical results for the ANCs for the 0_1^+ and 0_2^+ states of ^{16}O while a value of $\tilde{g}_0$ could not be fixed from the phase shift data of the elastic α-^{12}C scattering. As mentioned above, a value of $\tilde{g}_0$ can be fixed by using an experimental datum of one of the two ANCs, and then one can predict the other one of the two ANCs, though the values of the ANCs in the literature are significantly scattered. Another way to fix $\tilde{g}_0$ is to use experimental data for inelastic α-^{12}C scattering, $\alpha + {}^{12}\mathrm{C}(0_1^+) \rightarrow \alpha + {}^{12}\mathrm{C}^*(2_1^+)$. For a better understanding of the present situation, further studies for the ANCs for the 0_1^+ and 0_2^+ states of ^{16}O, both experimentally and theoretically, would be required

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