Coupled-reaction-channel study of the ^{12}C(α, ^{8}Be) reaction and the ^{8}Be + ^{8}Be optical potential

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Background: Given the established 2α structure of ^{8}Be, a realistic model of four interacting α clusters must be used to obtain a ^{8}Be + ^{8}Be interaction potential. Such a four-body problem poses a challenge for the determination of the ^{8}Be + ^{8}Be optical potential (OP) that is still unknown due to the lack of the elastic ^{8}Be + ^{8}Be scattering data.
Purpose: To probe the complex ^{8}Be + ^{8}Be optical potential in the coupled-reaction-channel (CRC) study of the α transfer ^{12}C(α,^{8}Be) reaction measured at $E_\alpha = 65$ MeV and to obtain the spectroscopic information on the $\alpha + {}^8$Be cluster configuration of ^{12}C.
Method: The three- and four-body continuum-discretized coupled-channel (CDCC) methods are used to calculate the elastic $\alpha + {}^8$Be and ^{8}Be + ^{8}Be scattering at the energy around 16 MeV/nucleon, with the breakup effect taken into account explicitly. Based on the elastic cross section predicted by the CDCC calculation, the local equivalent OP's for these systems are deduced for the CRC study of the ^{12}C(α,^{8}Be) reaction.
Results: Using the CDCC-based OP's and α spectroscopic factors given by the cluster model calculation, a good CRC description of the α transfer data for both the ^{8}Be + ^{8}Be and ^{8}Be + ^{8}Be$^*_{2+}$ exit channels is obtained without any adjustment of the (complex) potential strength.
Conclusion: The $\alpha + {}^8$Be and ^{8}Be + ^{8}Be interaction potential can be described by the three- and four-body CDCC methods, respectively, starting from a realistic $\alpha + \alpha$ interaction. The α transfer ^{12}C(α,^{8}Be) reaction should be further investigated not only to probe of the 4α interaction but also the cluster structure of ^{12}C.

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I. INTRODUCTION

The α-cluster structure established for different excited states in several light nuclei like ^{12}C or ^{16}O has inspired numerous experimental and theoretical studies, especially, the direct nuclear reactions measured with ^{12}C as projectile and/or target [1]. Given the cluster states above the α-decay threshold of ^{12}C, some direct reactions with ^{12}C were shown to produce both the free α particle and unstable ^{8}Be in the exit channel [2–4]. Consequently, the knowledge of the $\alpha + {}^8$Be and ^{8}Be + ^{8}Be interaction potentials should be important for the studies of such reactions within the distorted-wave Born approximation (DWBA) or coupled-reaction-channel (CRC) formalism.

Given a well-established 2α-cluster structure of the unbound ^{8}Be nucleus, the ^{8}Be + ^{8}Be interaction potential poses a four-body problem which is a challenge for the determination of the ^{8}Be + ^{8}Be optical potential (OP) that cannot be deduced from a standard optical model (OM) analysis because of the lack of the elastic ^{8}Be + ^{8}Be scattering data. The knowledge about the $\alpha + {}^8$Be and ^{8}Be-nucleus OP's should be also important for the studies of those direct reaction processes that produce ^{8}Be fragments in the exit channel [5–7]. Although the α-nucleus and nucleus-nucleus OP's are proven to be well described by the double-folding model (DFM) using the accurate ground-state densities of the colliding nuclei and a realistic density-dependent nucleon-nucleon (NN) interaction (see, e.g., Refs. [8–12]), the DFM cannot be used to calculate the $\alpha + {}^8$Be and ^{8}Be + ^{8}Be OP's because of a strongly deformed, extended two-center density distribution of ^{8}Be. In general, one could think of the triple- and quadruple folding models for the $\alpha + {}^8$Be and ^{8}Be + ^{8}Be potentials, respectively, but these will surely be complicated and involve much more tedious calculation in comparison with the standard DFM method. Although some phenomenological OP's are available in the literature for ^{7}Li and ^{9}Be, two nuclei neighboring ^{8}Be, a strongly (two-center) deformation of the unstable ^{8}Be nucleus casts doubt on the extrapolated use of these potentials for the ^{8}Be + ^{8}Be and ^{8}Be-nucleus systems.

Given a very loose (unbound) ^{8}Be nucleus that breaks up promptly into 2α particles, we determine in the present work the ^{8}Be + ^{8}Be OP using the continuum-discretized coupled-channel (CDCC) method which was developed to take into account explicitly the breakup of the projectile and/or target. A textbook example is a direct reaction induced by deuteron, which is loosely bound and can be, therefore, easily broken up into a pair of free proton and neutron. Originally, the deuteron breakup states were included in terms of a discretized continuum by the CDCC method (see, e.g., Refs. [13–15] for reviews). In the recent version of the CDCC theory, the continuum of deuteron is approximated by the square-integrable

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functions corresponding to positive energies. As a result, this approach can be well extended to study elastic scattering of exotic nuclei that have rather low breakup energies (typical examples are ^{6}He and ^{11}Be).

The first developments of the CDCC method were done in the framework of a three-body system where the projectile is seen as a two-body nucleus and target is assumed to be structureless, being in its ground state. More recently, four-body calculations were developed, for either a three-body projectile on a structureless target [16] or a two-body projectile and a two-body target [17]. The latter approach is highly time consuming but was successfully applied to study $^{11}\mathrm{Be}+d$ scattering in terms of $^{11}\mathrm{Be} = {}^{10}\mathrm{Be}+n$ and $d = p+n$. The goal of the present study is to determine the $^{8}\mathrm{Be}+{}^{8}\mathrm{Be}$ OP based on the elastic-scattering matrix predicted by the four-body CDCC calculation of four interacting α clusters. While such a 4α model is not appropriate for the spectroscopy of ^{16}O [17], the derived OP for elastic $^{8}\mathrm{Be}+{}^{8}\mathrm{Be}$ scattering is expected to be reliable. The only input for the present 4α CDCC calculation is a realistic $\alpha+\alpha$ interaction potential.

Although ^{8}Be is particle unstable, its half-life around 10^{-16}s is long enough for the ^{8}Be-nucleus OP to contribute significantly to a direct reaction that produces ^{8}Be in the exit channel, like the α transfer $^{12}\mathrm{C}(\alpha,{}^{8}\mathrm{Be})$ reaction. This particular reaction was shown to be a good tool for the study of the high-lying or resonance states of ^{16}O [18,19] and to determine the α cluster configurations of this nucleus [18,20,21]. Because of the unbound structure of ^{8}Be, the direct reaction reactions $A(\alpha,{}^{8}\mathrm{Be})B$ usually have a very low cross section (of a few tens microbarn), but they are extremely helpful for the study of the α-cluster structure of the target nuclei [22]. In particular, the α spectroscopic factors of different cluster states were deduced from these measurements at the α incident energies of 65 to 72.5 MeV.

In the present work, the $\alpha+{}^{8}\mathrm{Be}$ and $^{8}\mathrm{Be}+{}^{8}\mathrm{Be}$ optical potentials deduced from the scattering wave functions given by the 3- and 4α CDCC calculations are used as the core-core and the exit OP, respectively, in the CRC study of $^{12}\mathrm{C}(\alpha,{}^{8}\mathrm{Be})$ reaction measured at 65 MeV [22]. The OP of the entrance channel is calculated in the DFM using the density-dependent CDM3Y6 interaction that was well tested in the mean-field studies of nuclear matter as well as in the OM studies of the elastic α-nucleus scattering [9,11], and it accounts well for the elastic $\alpha+{}^{12}\mathrm{C}$ scattering data measured at 65 MeV [23,24]. The α spectroscopic factors of the $\alpha+{}^{8}\mathrm{Be}$ cluster configurations of ^{12}C are taken from the results of the complex scaling method (CSM) by Kurokawa and Kato [25].

II. THREE- AND FOUR-BODY CDCC METHODS

We discuss here the CDCC method used to determine the $\alpha+{}^{8}\mathrm{Be}$ and $^{8}\mathrm{Be}+{}^{8}\mathrm{Be}$ optical potentials, where the α particles are treated as structureless and interacting with each other through a (real) potential $v_{\alpha\alpha}(r)$. The Hamiltonian of the $\alpha+\alpha$ system is given by

$$H_{\alpha\alpha}(\boldsymbol{r}) = T_r + v_{\alpha\alpha}(r), \tag{1}$$

where T_r is the relative kinetic energy. There are two versions of the $\alpha+\alpha$ potential [26,27] parametrized in terms of Gaussians amenable for the present CDCC calculation. In the present work, we have chosen the $\alpha+\alpha$ potential suggested by Ali and Bodmer [26] (referred to hereafter as AB potential). The AB potential simulates the Pauli blocking effect by a repulsive core that makes this potential much shallower than the deep $\alpha+\alpha$ potential suggested by Buck *et al.* [27]. Both potentials reproduce equally well the $\alpha+\alpha$ phase shifts, and they were shown by Baye [28] to be linked by a supersymmetric transformation. The Buck potential, however, contains some deeply bound states (two states with $\ell=0$ and one with $\ell=2$), which do not have physical meaning but simulate the so-called Pauli forbidden states [29] in the microscopic $\alpha+\alpha$ model. As long as the two-body $\alpha+\alpha$ system is considered, the choice of either AB or Buck potential is not crucial. However, when dealing with more than two α clusters, the forbidden states have to be removed as they produce spurious states in a multicluster system like $\alpha+{}^{8}\mathrm{Be}$ or $^{8}\mathrm{Be}+{}^{8}\mathrm{Be}$. There are two methods to remove the forbidden states in the multicluster systems: either to apply the pseudostate method [30] or to use the supersymmetric transformation [28]. These two techniques, however, give rise to a strong angular-momentum dependence of the $\alpha+\alpha$ potential that cannot be used in most of the multicluster models. Among the 3α models, only the hyperspherical method and Faddeev method are able to use the deep $\alpha+\alpha$ potential with an exact removal of the $\alpha+\alpha$ forbidden states. This is why other studies of the 3α and 4α systems [17,31–33] have used only the ℓ-independent AB potential. In the present work, we perform the CDCC calculation of the $\alpha+{}^{8}\mathrm{Be}$ and $^{8}\mathrm{Be}+{}^{8}\mathrm{Be}$ optical potentials using the AB potential of the $\alpha+\alpha$ interaction, so that the spurious effects arising from the Pauli forbidden states can be avoided.

The present CDCC method is based on the eigenstates $\Phi_\lambda^{\ell m}(\boldsymbol{r})$ of the Hamiltonian (1) which can be written as

$$\Phi_\lambda^{\ell m}(\boldsymbol{r}) = \frac{1}{r}u_\lambda^\ell(r)Y_{\ell m}(\hat{\boldsymbol{r}}),$$

where ℓ is the relative orbital momentum of the $\alpha+\alpha$ system. The radial wave functions $u_\lambda^\ell(r)$ of the two-α state λ are expanded over a basis of N orthonormal functions $\varphi_i(r)$,

$$u_\lambda^\ell(r) = \sum_{i=1}^{N} f_{\lambda,i}^\ell \varphi_i(r), \tag{2}$$

where $f_{\lambda,i}^\ell$ are determined by diagonalizing the eigenvalue problem,

$$\sum_j f_{\lambda,j}^\ell \big(\langle\varphi_i|H_{\alpha\alpha}|\varphi_j\rangle - E_\lambda^\ell\delta_{ij}\big) = 0. \tag{3}$$

The eigenvalues with $E_\lambda^\ell < 0$ correspond to the physical bound states, while those with $E_\lambda^\ell > 0$ are referred to as the pseudostates, which are used in the present CDCC method to simulate the breakup of ^{8}Be. Note that there is only a small number of physical states in a CDCC calculation (often one for exotic nuclei). Although ^{8}Be is unbound, its energy is very close to the $\alpha+\alpha$ threshold, and the lifetime is long enough to use a quasibound approximation for the ground state (g.s.). Equations (2) and (3) are general for any choice of the basis functions $\varphi_i(r)$. We use here a Lagrange-mesh basis [34]

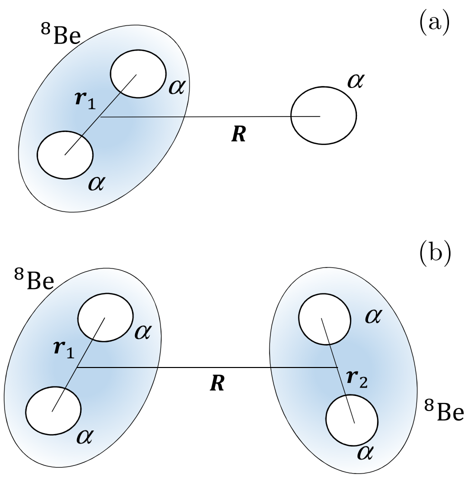

FIG. 1. The α configurations and coordinates used for $\alpha + {}^8$Be (a) and for ^{8}Be + ^{8}Be (b).

derived from the Legendre polynomials, and the calculation of matrix elements in Eq. (3) is fast and accurate. We refer the reader to Ref. [34] for more details and the application of the Lagrange-mesh basis.

The $\alpha + {}^8$Be and ^{8}Be + ^{8}Be systems are described by the 3α and 4α Hamiltonians, respectively, as

$$H_3 = H_{\alpha\alpha}(r_1) + T_R + \sum_{i=1}^{2} v_{\alpha\alpha}(S_{1i})$$

$$H_4 = H_{\alpha\alpha}(r_1) + H_{\alpha\alpha}(r_2) + T_R + \sum_{i,j=1}^{2} v_{\alpha\alpha}(|S_{1i} - S_{2j}|), \quad (4)$$

where $\boldsymbol{R}$ is the projectile-target coordinate and $(\boldsymbol{r}_1, \boldsymbol{r}_2)$ are the internal coordinates of the ^{8}Be nuclei, as illustrated in Fig. 1. Coordinates S_{1i} and S_{2j} are expressed as a function of (R, r_1) and of (R, r_2), respectively.

In the CDCC approximation, the $\alpha + {}^8$Be wave function for a total angular moment J and parity π is written as

$$\Psi^{JM\pi}(\boldsymbol{R}, \boldsymbol{r}_1) = \sum_{\lambda\ell L} g^{J\pi}_{\lambda\ell L}(R)\big[\Phi^{\ell}_{\lambda}(\boldsymbol{r}_1) \otimes Y_L(\hat{\boldsymbol{R}})\big]_{JM}, \quad (5)$$

where the summation over the pseudostates λ is truncated at a given energy $E_{\max}$. The ^{8}Be + ^{8}Be wave functions involve four α clusters and can be expressed as

$$\Psi^{JM\pi}(\boldsymbol{R}, \boldsymbol{r}_1, \boldsymbol{r}_2) = \sum_{\lambda_1\ell_1}\sum_{\lambda_2\ell_2}\sum_{IL} g^{J\pi}_{\lambda_1\ell_1\lambda_2\ell_2 IL}(R) \times \big[\big[\Phi^{\ell_1}_{\lambda_1}(\boldsymbol{r}_1) \otimes \Phi^{\ell_2}_{\lambda_2}(\boldsymbol{r}_2)\big]_I \otimes Y_L(\hat{\boldsymbol{R}})\big]_{JM}, \quad (6)$$

where the parity conservation imposes $(-1)^L = (-1)^I$. There are two parameters defining the CDCC basis: the maximum ^{8}Be angular momentum $\ell_{\max}$ and maximum pseudostate energy $E_{\max}$. The physical quantities obtained from the CDCC calculation (scattering matrix, elastic cross section, and local equivalent OP) must be converged with respect to these parameters. In practice, the 4α calculations involve many channels and are, therefore, highly time consuming [17].

The relative radial wave functions $\chi^{J\pi}_c(R)$, with the indices $c = (\lambda\ell L)$ for $\alpha + {}^8$Be and $c = (\lambda_1\ell_1\lambda_2\ell_2 IL)$ for ^{8}Be + ^{8}Be, are obtained from the solutions of the following coupled-channel equations:

$$\left\{-\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dR^2} - \frac{L(L+1)}{R^2}\right] + E_{c_1} + E_{c_2} - E_{\text{c.m.}}\right\}\chi^{J\pi}_c(R) + \sum_{c'} V^{J\pi}_{cc'}(R)\chi^{J\pi}_{c'}(R) = 0, \quad (7)$$

where μ is the reduced mass, $E_{\text{c.m.}}$ is the center-of-mass energy, and E_{c_1} and E_{c_2} are the excitation energies of the two interacting nuclei, separated by the distance R as shown in Fig. 1. The coupling potentials $V^{J\pi}_{cc'}(R)$ are determined by the method explained in Refs. [17,35]. The system of the coupled-channel equations (7) is solved using the R-matrix method, which provides explicitly the scattering matrix and the associated wave functions [36,37]. Although the AB potential [26] is real, it consistently reproduces the experimental $\alpha + \alpha$ phase shifts up to about 20 MeV. In the present CDCC approach, the loss of flux from the elastic-scattering channel is due entirely to the breakup channels, and the local equivalent $\alpha + {}^8$Be and ^{8}Be + ^{8}Be optical potentials are therefore complex. Due to the strong 2α structure of ^{8}Be it is likely that these breakup channels represent the main source of the absorption.

III. RESULTS AND DISCUSSION

A. Local equivalent OP for the $\alpha + {}^8$Be and ^{8}Be + ^{8}Be systems

The main goal of our study is to determine the local (J-independent) equivalent optical potential U for the $\alpha + {}^8$Be and ^{8}Be + ^{8}Be systems at the considered energies, based on the scattering wave functions given by the solutions of the CDCC equations (7). The main requirement for this procedure is that the solutions $\chi^{J\pi}$ of the one-channel OM equation with the optical potential U

$$[E_{\text{c.m.}} - T_R - U(R)]\chi^{J\pi}(R) = 0 \quad (8)$$

give the cross section of the elastic $\alpha + {}^8$Be or ^{8}Be + ^{8}Be scattering close to that given by the 3α or 4α CDCC calculation (7), especially, the cross section at forward angles which is sensitive to the surface part of the 3α or 4α interaction potential. We briefly discuss the two approaches used in the present work for this purpose.

(i) The quantum-mechanically consistent method using the matrix inversion was suggested in Refs. [17,38] to derive a local equivalent potential (LEP) that exactly reproduces the elastic cross section given by the CDCC calculation (7). However, this LEP has two major drawbacks that prevent its further use in the direct nuclear reaction calculation. Namely, the derived LEP strongly depends on the total angular momentum J, and its radial dependence has the singularities caused by the nodes of the scattering wave functions. These problems can be handled by the method proposed in Ref. [38] which averages the obtained LEP over the angular momenta to obtain a smooth J-independent OP without discontinuity that approximately reproduces the CDCC elastic-scattering cross section. The recent four-body CDCC calculation [17] has shown that such an averaging method to a fairly good

TABLE I. WS parameters (9) of $U_{\rm WSD}$ given by the method (ii) based on the OM fit to the elastic ^{8}Be + ^{8}Be and α + ^{8}Be cross sections at $E_{\rm c.m.}$ = 41.3 and 43.3 MeV, predicted by the 4α and 3α CDCC calculations (7), respectively. J_V and J_W are the volume integrals per interacting nucleon pair of Re $U_{\rm WSD}$ and Im $U_{\rm WSD}$, respectively. $J_{V_{\rm LEP}}$ and $J_{W_{\rm LEP}}$ are those of $U_{\rm LEP}$ given by the method (i).

	$V_v(W_v)$ (MeV)	$R_{v(w)}$ (fm)	$a_{v(w)}$ (fm)	$V_d(W_d)$ (MeV)	$R_{d(s)}$ (fm)	$a_{d(s)}$ (fm)	$-J_V(J_W)$ (MeV fm^3)	$-J_{V_{\rm LEP}}(J_{W_{\rm LEP}})$ (MeV fm^3)
				^{8}Be + ^{8}Be ($E_{\rm c.m.}$ = 41.3 MeV)				
Real	10.42	3.652	0.190	5.785	4.811	0.562	95.88	100.0
Imag.	1.486	7.225	0.183	0.567	2.665	1.564	46.87	29.71
				^{4}He + ^{8}Be ($E_{\rm c.m.}$ = 43.3 MeV)				
Real	1.535	5.182	0.123	5.397	2.584	0.997	111.0	97.62
Imag.	10.09	1.771	0.125	8.201	3.299	0.120	24.56	33.95

approximation determines the local J-independent OP. The complex OP derived using this approach is denoted hereafter as $U_{\rm LEP}$, with its imaginary part $W_{\rm LEP}$ originating from the breakup channels included in the CDCC calculation (7). We have first performed the CDCC calculation (7) for the elastic α + ^{8}Be and ^{8}Be + ^{8}Be scattering at $E_{\rm c.m.}$ = 43.3 and 41.3 MeV, respectively, using the AB potential [26] for the $\alpha + \alpha$ interaction. The maximum angular momentum of the $\alpha + \alpha$ system is $\ell_{\rm max} = 2$, and the maximum pseudostate energy is $E_{\rm max}$ = 10 MeV. These cutoff values were well tested to ensure the convergence of both the elastic cross section and $U_{\rm LEP}$. The complex $U_{\rm LEP}$ for the α + ^{8}Be and ^{8}Be + ^{8}Be systems were obtained first in a Lagrange mesh [38] and then interpolated into the smooth shapes for use as the external input of the complex OP in the CRC calculation.

(ii) The standard OM method can also be used to determine a phenomenological J-independent OP in the conventional Woods-Saxon (WS) form, with its parameters adjusted to obtain a good OM fit to the elastic cross section given by the 3α or 4α CDCC calculation. We have assumed in the present work the following (volume + surface) WS form for the complex OP of the α + ^{8}Be and ^{8}Be + ^{8}Be systems at the energies under study, denoted hereafter as $U_{\rm WSD}$,

$$-U_{\rm WSD}(R) = V_v f_v(R) - 4V_d a_d \frac{df_d(R)}{dR} + i\left[W_v f_w(R) - 4W_d a_s \frac{df_s(R)}{dR}\right],$$

$$\text{where } f_x(R) = \frac{1}{1+\exp[(R-R_x)/a_x]},\ x = v, d, w, s. \quad (9)$$

The elastic ^{8}Be + ^{8}Be and α + ^{8}Be cross sections at $E_{\rm c.m.}$ = 41.3 and 43.3 MeV, predicted by the 4α and 3α CDCC calculations (7), respectively, have been used in the method (ii) as the "experimental data" with the uniform 10% uncertainties for the OM analysis to determine the WS parameters of $U_{\rm WSD}$. We found quite a shallow WS potential that gives a good agreement of the OM result with the CDCC elastic cross section (see the OP parameters in Table I).

The radial shapes of both $U_{\rm LEP}$ and $U_{\rm WSD}$ potentials for the ^{8}Be + ^{8}Be system at $E_{\rm c.m.}$ = 41.3 MeV are shown in Fig. 2, the use of the shallow AB potential of the $\alpha + \alpha$ interaction is shown to result on quite a shallow potential $U_{\rm LEP}$. A moderate oscillation of $U_{\rm LEP}$ is seen at small radii that might originate from the J dependence of the exact LEP discussed above. The best-fit WS complex OP determined by the method (ii) has the strength of Re $U_{\rm WSD}$ enhanced slightly at the surface ($R \approx 5$ fm) and a weak and smooth Im $U_{\rm WSD}$. One can see in Table I that the volume integrals of Re $U_{\rm WSD}$ and Im $U_{\rm WSD}$ are close to those of Re $U_{\rm LEP}$ and Im $U_{\rm LEP}$, which indicates that the OP's given by both methods belong to about the same potential family. The results of the CDCC calculation (7) for the elastic ^{8}Be + ^{8}Be and α + ^{8}Be scattering at $E_{\rm c.m.}$ = 41.3 and 43.3 MeV, respectively, are compared in Fig. 3 with the results of the OM calculation (8) using $U_{\rm LEP}$ and $U_{\rm WSD}$. The OM results given by both OP's agree fairly good with the CDCC prediction at forward angles, while at medium and large angles the phenomenological $U_{\rm WSD}$ determined by the method (ii) better reproduces the CDCC cross sections. The agreement with the CDCC results becomes

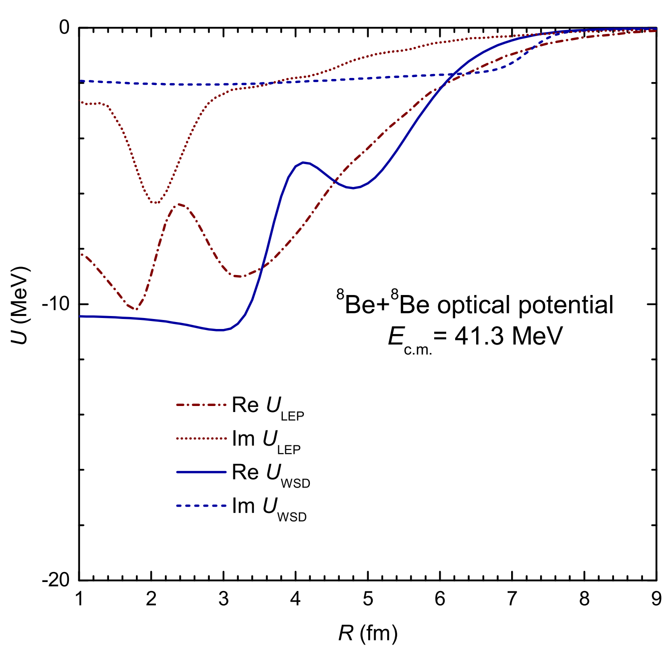

FIG. 2. Complex OP for the elastic ^{8}Be + ^{8}Be scattering at $E_{\rm c.m.}$ = 41.3 MeV given by the method (i) and that in the WS form (9) given by the method (ii).

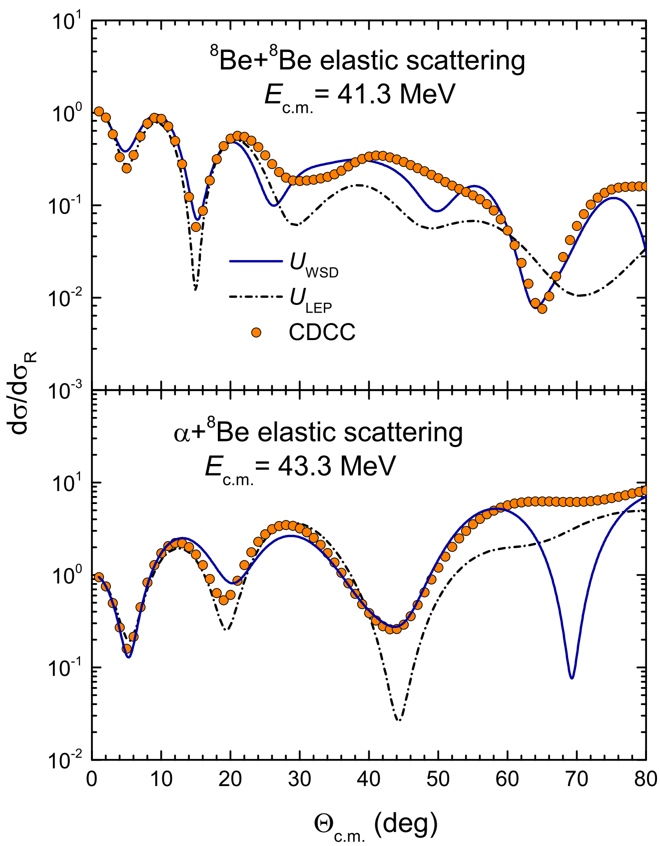

FIG. 3. The CDCC prediction for the elastic $^8\mathrm{Be}+{}^8\mathrm{Be}$ (upper panel) and $\alpha+{}^8\mathrm{Be}$ (lower panel) scattering at $E_{\mathrm{c.m.}}=41.3$ and 43.3 MeV, in comparison with the corresponding OM results given by the optical potentials U_{LEP} and U_{WSD} determined by the methods (i) and (ii), respectively.

worse at large angles, and it might be due to the nonlocality effects.

We note further that the method (i) fails to derive a smooth ℓ-independent U_{LEP} based on the CDCC results obtained with the deep Buck $\alpha+\alpha$ potential. Namely, the obtained ℓ-independent U_{LEP} turns out to be deeper but strongly oscillatory, and it gives the elastic cross section substantially different from that given by the CDCC calculation. Such a failure of the ℓ-independent U_{LEP} based on the Buck potential is presumably caused by the Pauli forbidden states, and this remains an unsolved problem for the present 4α CDCC method. Therefore, we deem hereafter reliable only the CRC results obtained with $\alpha+{}^8\mathrm{Be}$ and $^8\mathrm{Be}+{}^8\mathrm{Be}$ OP's derived based on the CDCC elastic cross section obtained with the AB potential of the $\alpha+\alpha$ interaction.

B. CRC study of the $^{12}\mathrm{C}(\alpha,{}^8\mathrm{Be})$ reaction

The $\alpha+{}^8\mathrm{Be}$ and $^8\mathrm{Be}+{}^8\mathrm{Be}$ optical potentials determined by the methods (i) and (ii) have been further used as the potential inputs for the CRC study of the α transfer $^{12}\mathrm{C}(\alpha,{}^8\mathrm{Be})$ reaction measured at $E_\alpha=65$ MeV [22]. We briefly recall the multichannel CRC formalism, to illustrate how the OP's of the $\alpha+{}^8\mathrm{Be}$ and $^8\mathrm{Be}+{}^8\mathrm{Be}$ systems enter the CRC calculation of

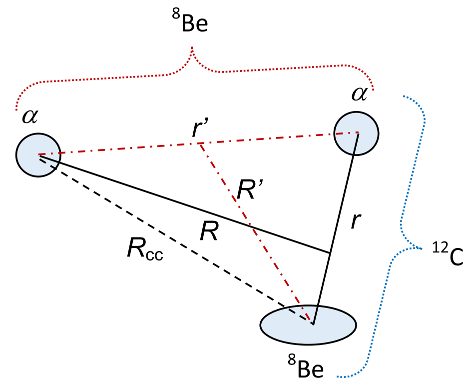

FIG. 4. Cluster configurations in the entrance and exit channels of the $^{12}\mathrm{C}(\alpha,{}^8\mathrm{Be})$ reaction, and the corresponding coordinates used for the inputs of the potentials in the CRC calculation.

the α transfer cross section. In general, the CRC equation for the initial channel β of the transfer reaction can be written as [39,40]

$$[E_\beta-T_\beta-U_\beta(\boldsymbol{R})]\chi_\beta(\boldsymbol{R}) = \sum_{\beta'\neq\beta}\{\langle\beta|W|\beta'\rangle+\langle\beta|\beta'\rangle[T_{\beta'}+U_{\beta'}(\boldsymbol{R}')-E_{\beta'}]\}\chi_{\beta'}(\boldsymbol{R}'). \tag{10}$$

Without coupling to the inelatic-scattering channels, the indices β and β' in Eq. (10) stand for the initial $\alpha+{}^{12}\mathrm{C}$ and final $^8\mathrm{Be}+{}^8\mathrm{Be}$ partitions of the α transfer reaction, respectively, as shown in Fig. 4. In the present CRC analysis, the index β' is used to identify both the $^8\mathrm{Be}+{}^8\mathrm{Be}_{\mathrm{g.s.}}$ and $^8\mathrm{Be}+{}^8\mathrm{Be}^*_{2^+}$ exit channels of the final partition. The distorted waves χ_β and $\chi_{\beta'}$ are given by the optical potentials U_β and $U_{\beta'}$ of the $\alpha+{}^{12}\mathrm{C}$ and $^8\mathrm{Be}+{}^8\mathrm{Be}$ systems, respectively. The α transfer proceeds via the transfer interaction W which is determined in the post form [39,40] as

$$W=V_{\alpha-{}^{12}\mathrm{C}}(\boldsymbol{r})+[U_{\alpha+{}^8\mathrm{Be}}(\boldsymbol{R}_{\mathrm{cc}})-U_{{}^8\mathrm{Be}+{}^8\mathrm{Be}}(\boldsymbol{R}')], \tag{11}$$

with the radii of the potentials illustrated in Fig. 4. Here $V_{\alpha-{}^{12}\mathrm{C}}(r)$ is the potential binding the α cluster to the $^8\mathrm{Be}$ core in the ground state of $^{12}\mathrm{C}$. The difference between the core-core OP and that of the final partition, $U_{\alpha+{}^8\mathrm{Be}}(R_{\mathrm{cc}})-U_{{}^8\mathrm{Be}+{}^8\mathrm{Be}}(R')$, is the complex remnant term of W. The CRC equations (10) are solved iteratively using the code FRESCO written by Thompson [41], with the complex (nonlocal) remnant term and boson symmetry of the identical $^8\mathrm{Be}+{}^8\mathrm{Be}$ system properly taken into account. One can see that the $^8\mathrm{Be}+{}^8\mathrm{Be}$ OP enters the CRC calculation of the α transfer $^{12}\mathrm{C}(\alpha,{}^8\mathrm{Be})$ reaction as the input of both the remnant term and the OP of the final partition. Therefore, it can be tested indirectly based on the CRC description of the α transfer data.

The OP of the initial partition $U_{\alpha+{}^{12}\mathrm{C}}$ has its real part given by the double-folding model using the density-dependent CDM3Y6 interaction [11], and imaginary part chosen in the WS shape, with the parameters fine tuned to the best CRC fit of the elastic $\alpha+{}^{12}\mathrm{C}$ scattering data measured at $E_\alpha=65$ MeV [23,24]. A reasonably good CRC description of the elastic $\alpha+{}^{12}\mathrm{C}$ scattering data at $E_\alpha=65$ MeV (see Fig. 5) is

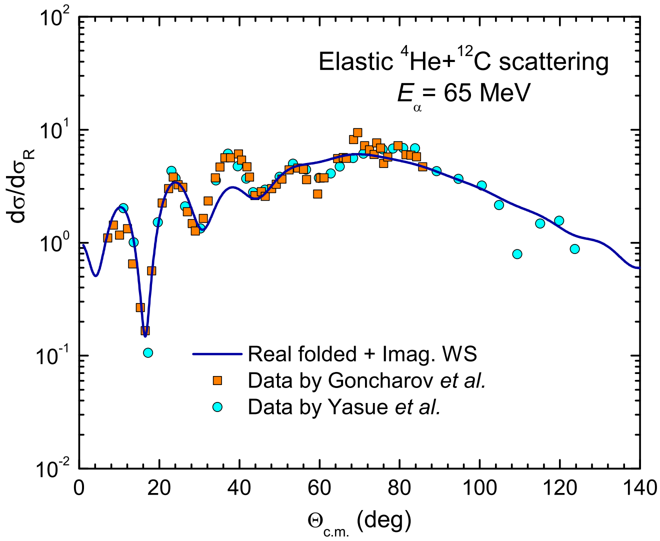

FIG. 5. CRC description of the elastic $\alpha + {}^{12}$C scattering at E_α = 65 MeV given by the (unrenormalized) real folded OP and imaginary OP chosen in the WS form [11], in comparison with the data taken from Refs. [23,24].

achieved without renormalizing the strength of the real OP. In principle, we could also think of using the 4α CDCC method to predict the $\alpha + {}^{12}$C OP at the considered energy. However, Suzuki *et al.* [42] have shown that the use of a *local* $\alpha + \alpha$ potential (that properly reproduces the experimental $\alpha + \alpha$ phase shifts) cannot provide a proper 3α description of both the ground state and 0_2^+ excitation (known as Hoyle state) of ^{12}C. This problem could only be solved by introducing a microscopically founded *nonlocal* $\alpha + \alpha$ force that mimics the interchange of three α clusters in the phase space allowed by the Pauli principle [42]. The use a nonlocal $\alpha + \alpha$ interaction remains beyond the scope of the present four-body CDCC method [17]. On the other hand, the elastic $\alpha + {}^{12}$C scattering at energies above 10 MeV/nucleon is proven to be strongly refractive [9,11,43], with a far-side dominant elastic cross section at large angles typical for the nuclear rainbow, which can be well described by the deep (mean-field type) real OP predicted by the double-folding model [11,43].

For the α transfer reaction, the initial (internal) state of the α cluster bound in ^{12}C is assumed to be 1s state. Then the relative-motion wave function $\Phi_{NL}(\boldsymbol{r})$ of the $\alpha + {}^8$Be configuration in the ^{12}C target (L-wave state) has the number of radial nodes N determined by the Wildermuth condition [39], so that the total number of the oscillator quanta $\mathcal{N}$ is conserved,

$$\mathcal{N} = 2(N-1) + L = \sum_{i=1}^{4} 2(n_i - 1) + l_i, \qquad (12)$$

where n_i and l_i are the principal quantum number and orbital momentum of each constituent nucleon in the α cluster. $\Phi_{NL}(\boldsymbol{r})$ is obtained in the potential model using $V_{\alpha-{}^{12}\mathrm{C}}(r)$ chosen in the WS shape, with its radius and diffuseness fixed as $R = 3.767$ fm and $a = 0.65$ fm, and the WS depth ($V = 51.6$ MeV) adjusted to reproduce the α separation energy of ^{12}C. Because the ground state of ^{8}Be is unbound by 92 keV, we have used in the present CRC calculation a *quasibound* approximation for ^{8}Be similar to that used for the ground state of ^{8}Be in the CDCC calculation as discussed in Sec. II to describe the formation of ^{8}Be on the exit channel of the α transfer reaction. For this purpose, the repulsive core of the AB potential was slightly weakened to give the (1s) state $\Phi_\alpha(\boldsymbol{r}')$ of the α cluster in ^{8}Be a quasibinding energy of 0.01 MeV. The cluster wave functions $\Phi_{NL}(\boldsymbol{r})$ and $\Phi_\alpha(\boldsymbol{r}')$ are used explicitly in the calculation of the complex nonlocal α transfer form factor,

$$\langle \beta'|W|\beta\rangle \sim \langle [\Phi_\alpha(\boldsymbol{r}') \otimes Y_{L_{\beta'}}(\hat{\boldsymbol{R}}')]_{J_{\beta'}}|W|[\Phi_{NL}(\boldsymbol{r}) \otimes Y_{L_\beta}(\hat{\boldsymbol{R}})]_{J_\beta}\rangle, \qquad (13)$$

where L_β and $L_{\beta'}$ are the relative orbital momenta of the initial and final partitions. The wave functions of ^{8}Be core in the initial and ^{4}He core in the final partitions are omitted in (13) because they are spectators and do not contribute to the transfer [40].

The CRC calculation of the α transfer ^{12}C(α,^{8}Be) reaction requires the input of the spectroscopic factor of the α cluster in ^{8}Be which is naturally assumed to be unity and that of the cluster configuration $\alpha + {}^8$Be in ^{12}C. The latter is determined as $S_\alpha = |A_{NL}|^2$, where the spectroscopic amplitude A_{NL} is given by the dinuclear overlap

$$\langle {}^8\mathrm{Be}\,|\,{}^{12}\mathrm{C}\rangle = A_{NL}\Phi_{NL}(\boldsymbol{r}). \qquad (14)$$

Because two different exit channels of the α transfer ^{12}C(α,^{8}Be) reaction were identified, with the emitting ^{8}Be being in the ground state and excited 2^+ state [22], one needs to evaluate (14) for the two configurations $\alpha + {}^8\mathrm{Be}_{\mathrm{g.s.}}$ and $\alpha + {}^8\mathrm{Be}^*_{2^+}$, which are associated with $\Phi_{N=3,L=0}(\boldsymbol{r})$ (S wave) and $\Phi_{N=2,L=2}(\boldsymbol{r})$ (D wave). In general, one can treat these two S_α values as free parameters to be adjusted by the best DWBA or CRC fit to the α transfer data. Instead of this procedure, we have adopted in the present work the S_α values predicted for these configurations by Kurokawa and Kato using the CSM method [25]. Namely, $S_\alpha(\mathrm{g.s.}) \approx 0.36$ and $S_\alpha(2^+) \approx 0.38$, which are rather close to the spectroscopic factors predicted recently by other cluster models [44,45]. Note that the S_α values used in our CRC calculation are also close to those extracted from the DWBA analysis of the ^{8}Be transfer reaction ^{24}Mg(α, ^{12}C) ^{16}O [46]. The same ^{8}Be + ^{8}Be OP has been used for both $^8\mathrm{Be} + {}^8\mathrm{Be}_{\mathrm{g.s.}}$ and $^8\mathrm{Be} + {}^8\mathrm{Be}^*_{2^+}$ exit channels of the final partition (see more discussion below).

The CRC results for the α transfer ^{12}C(^{4}He, ^{8}Be) reaction at E_α = 65 MeV to the ground state of ^{8}Be are compared with the measured data [22] in Fig. 6. One can see that the CDCC-based optical potentials of the ^{8}Be + ^{8}Be partition [U_{LEP} and U_{WSD} determined by the methods (i) and (ii), respectively] give a good CRC description of the α transfer data without any adjustment of its strength, using $S_\alpha(\mathrm{g.s.}) = |A_{30}|^2 \approx 0.36$ taken from the results of the CSM calculation [25]. With a better OM description of the CDCC elastic cross section given by the U_{WSD} potential (see Fig. 3), the CRC cross section given by U_{WSD} also agrees slightly better the measured α transfer data.

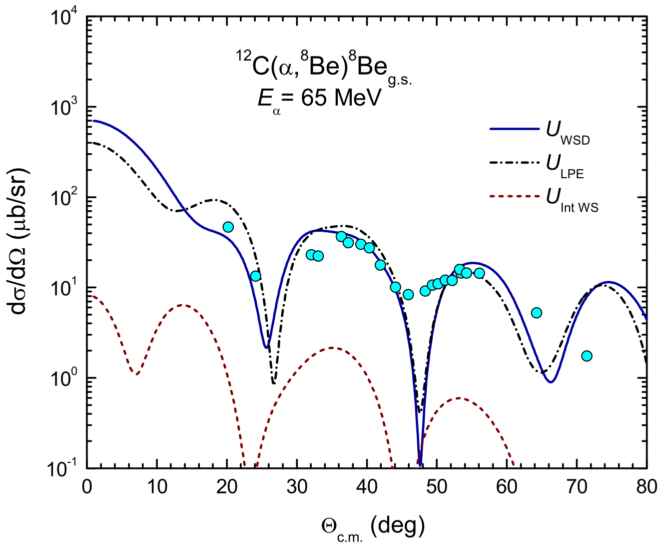

FIG. 6. CRC description of the α transfer reaction ^{12}C$(\alpha,{}^{8}$Be$)\,^{8}$Be$_{\text{g.s.}}$ measured at $E_\alpha = 65$ MeV [22], using the α spectroscopic factor $S_\alpha(\text{g.s.}) \approx 0.36$ taken from the CSM calculation [25]. The CRC results obtained with the CDCC-based optical potentials U_{WSD} and U_{LEP} for the ^{8}Be + ^{8}Be partition are shown as the solid and dash-dotted lines, respectively. The dashed line is the CRC result obtained with the WS potential U_{IntWS}, interpolated from the OP's adopted for the ^{7,8}Be + ^{9}Be systems at the nearby energies [5].

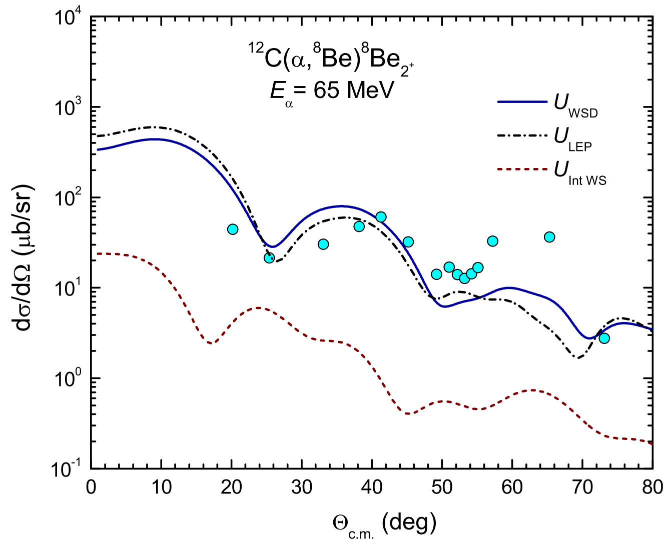

FIG. 7. The same as Fig. 6 but for the α transfer reaction with one emitting ^{8}Be nucleus being in its 2^+ state ($E_x \approx 2.94$ MeV), using the α spectroscopic factor $S_\alpha(2^+) \approx 0.38$ taken from the CSM calculation [25].

Because the ^{8}Be + ^{8}Be OP is unknown so far, a practical assumption is to estimate it from the phenomenological OP's adopted for the neighboring ^{9,7}Be isotopes. For example, the proton transfer reaction ^{7}Li(^{10}B, ^{9}Be) ^{8}Be was measured by Romanyshyn *et al.* [5], and a deep WS potential was deduced for the real OP of the ^{8}Be + ^{9}Be system at $E_{\text{c.m.}} = 31.7$ MeV from the DWBA analysis of transfer data. Interestingly, these authors also found that the ^{8}Be + ^{9}Be OP is quite close to that adopted earlier for the ^{7}Be + ^{9}Be system (see Fig. 10 of Ref. [5]). Therefore, one might expect the ^{8}Be + ^{8}Be OP to be close to the WS optical potentials adopted for the ^{7,8}Be + ^{9}Be systems. To explore the reliability of this practical approach, we have interpolated the ^{8}Be + ^{8}Be OP from those of the ^{7,8}Be + ^{9}Be systems adopted in Ref. [5] and denoted it as U_{IntWS}, with $V_v = 155.0$ MeV, $R_v = 3.152$ fm, $a_v = 0.768$ fm and $W_v = 13.5$ MeV, $R_w = 5.6$ fm, $a_w = 0.768$ fm. The use of U_{IntWS} in the CRC calculation of the ^{12}C$(\alpha,{}^{8}$Be) reaction completely fails to account for the data (see dashed lines in Fig. 6). In fact, the spectroscopic factor $S_\alpha(\text{g.s.})$ taken from Ref. [25] must be scaled by a factor of 25, so that the CRC cross section obtained with U_{IntWS} can be comparable with the measured α transfer data.

We have also performed the CRC calculation of the ^{12}C$(\alpha,{}^{8}$Be$)\,^{8}$Be* reaction at $E_\alpha = 65$ MeV with one emitting ^{8}Be nucleus being in its 2^+ state ($E_x \approx 2.94$ MeV). Although the 2^+ state of ^{8}Be is a broad resonance, its 2α-cluster structure remains similar to that of the ground state, and the α transfer cross section measured for the 2^+ state is of about the same strength as that measured for the ground state as shown in Figs. 6 and 7. The α spectroscopic factors S_α predicted by the CSM calculation [25] are also close for both the ground and 2^+ states. It is, therefore, reasonable to use the same CDCC-based ^{8}Be + ^{8}Be OP for the partition ^{8}Be + ^{8}Be$^*_{2+}$ in the exit channel. The results of the CRC calculation are compared with the data [22] in Fig. 7, and one can see that the (unrenormalized) CDCC-based ^{8}Be + ^{8}Be OP also delivers a good description of the α transfer data using $S_\alpha(2^+) = |A_{22}|^2 \approx 0.38$ given by the CSM calculation [25]. The use of U_{IntWS} for the ^{8}Be + ^{8}Be OP in the CRC calculation also strongly underestimates the α transfer data (see dashed line in Fig. 7).

In conclusion, a good CRC description of the measured ^{12}C$(\alpha,{}^{8}$Be) data has been obtained with the ^{8}Be + ^{8}Be OP's determined by the methods (i) and (ii) from the elastic ^{8}Be + ^{8}Be cross section given by the 4α CDCC calculation and with the α spectroscopic factors given by the CSM calculation [25]. The fact that no adjustment of the potential strength of U_{LEP} and U_{WSD} was necessary suggests that the 4α CDCC method is a reliable approach to study the ^{8}Be + ^{8}Be system. These results also show that, despite the short lifetime of ^{8}Be, the α transfer ^{12}C(^{4}He, ^{8}Be) reaction is very sensitive to the OP of the ^{8}Be + ^{8}Be partition. The measured α transfer data clearly prefer the shallow OP based on the result of the 4α CDCC calculation using the AB potential of the $\alpha + \alpha$ interaction [26] over the deep WS potential interpolated from those adopted for the ^{7,8}Be + ^{9}Be systems [5].

IV. SUMMARY

The three-body and recently suggested four-body CDCC methods [17] have been used to predict the elastic α + ^{8}Be and ^{8}Be + ^{8}Be scattering at $E_{\text{c.m.}} = 43.3$ and 41.3 MeV, respectively, using the $\alpha + \alpha$ interaction suggested by Ali and Bodmer [26] that well reproduces the experimental $\alpha + \alpha$ phase shifts. The elastic cross sections predicted by the CDCC calculation were used to determine the local OP's of the $\alpha + {}^8$Be and ^{8}Be + ^{8}Be systems.

The CDCC-based $\alpha + {}^8$Be and ^{8}Be + ^{8}Be OP's are further used as the inputs of the core-core OP and that of the final partition, respectively, in the CRC study of the α transfer ^{12}C(α,^{8}Be) reaction measured at $E_\alpha = 65$ MeV [22], with the emitting ^{8}Be being in both the ground state and 2^+ state. These α transfer data are well reproduced by the CRC results obtained with the CDCC-based OP's and α spectroscopic factors of the ^{8}Be + ^{8}Be$_{\rm g.s.}$ and ^{8}Be + ^{8}Be$^*_{2^+}$ configurations in ^{12}C taken from the CSM cluster calculation [25].

As alternative to the shallow (surface-type) ^{8}Be + ^{8}Be OP determined from the elastic ^{8}Be + ^{8}Be cross section predicted by the 4α CDCC calculation, a deep WS potential with parameters interpolated from the OP's adopted for the ^{7,8}Be + ^{9}Be systems at the nearby energies [5] has been used in the CRC calculation, and it grossly underestimates the α transfer data. This might due to the fact that ^{7}Be and ^{9}Be are well-bound nuclei, and the breakup effect is therefore much weaker than that of the unbound ^{8}Be.

We conclude that the $\alpha + {}^8$Be and ^{8}Be + ^{8}Be optical potentials can be determined from the elastic-scattering cross section predicted, respectively, by the 3α and 4α CDCC calculations using the realistic $\alpha + \alpha$ interaction that properly reproduces the experimental $\alpha + \alpha$ phase shifts [26]. The present CRC study should motivate further theoretical and experimental studies of the α transfer ^{12}C(α,^{8}Be) reaction as a probe of the 4α interaction and the α-cluster structure of ^{12}C.

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