Nucleon pair shell model in *M* scheme

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The nucleon pair shell model (NPSM) is cast into the so-called *M* scheme for the cases with isospin symmetry and without isospin symmetry. The odd system and even system are treated on the same footing. The uncoupled commutators for nucleon pairs, which are suitable for the *M* scheme, are given. Explicit formula of matrix elements in *M* scheme for overlap, one-body operators, and two-body operators are obtained. It is found that the CPU time used in calculating the matrix elements in *M* scheme is much shorter than that in the *J* scheme of NPSM.

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I. INTRODUCTION

Collective motions in nuclei, such as collective vibration, collective rotation, the backbending phenomenon, giant resonances, etc., of medium and heavy nuclei, are extremely important. How to describe the collective motions of nucleus is a fundamental problem in nuclear structure theory. Since the nuclear shell model [1,2] includes all the degrees of freedom, it can be used to describe the collective phenomena technically [3]. With the development of the computer, the shell model Hamiltonian can be diagonalized in the model space up to about 10^{10} [4]. But even for medium and heavy nuclei, the shell model space is about 10^{14}–10^{18} [5], and the modern computer fails for all of these cases. Therefore, to apply the shell model theory to medium and heavy nuclei, an efficient truncation scheme is necessary. The interacting boson model (IBM) [6] has had great success in nuclear structure theory [7–12]; in the model the valence nucleons pairs are treated as s bosons (with angular momentum $J = 0$) and d bosons (with angular momentum $J = 2$). The vibrational spectrum, rotational spectrum, and γ-unstable spectrum correspond to the U(5), SU(3), and SO(6) limits in the IBM.

In 1993, a new technique, the generalized wick theory, was proposed to calculate the commutators for coupled operators and fermion clusters by Chen *et al.* [13,14]. Based on this new technique, a nucleon pair shell model (NPSM) was proposed [15], in which the building blocks of the configuration space are constructed by nucleon pairs instead of the single valence nucleons. Because of the success of the IBM, the shell model space was truncated to the *SD*-pair subspace, giving the so-called *SD*-pair shell model (SDPSM). Previous works show that the collectivity of the low-lying states can be described very well in the SDPSM [16–18]. The quantum phase transition and the properties of the critical point symmetry can also be reproduced very well in the SDPSM [19]. In 2000, a new version of the NPSM was given by Zhao *et al.* [20], in which the odd and even systems can be treated on the same footing. The NPSM was extended to include isospin symmetry in Ref. [21,22]. The formalism in the NPSM with particle-hole coupling was also developed [23]. However, due to the CPU time in calculating the matrix elements increasing drastically with the number of nucleon pairs, the maximum number of nucleon pairs that the NPSM can handle is five for an identical nucleon system [24]. Therefore, an efficient method to calculate the matrix element is necessary in the NPSM.

In general, the shell model bases are constructed in the so-called *J* scheme [25] or the *M* scheme [26]. Most of the large-scale shell model basis are constructed in the *M* scheme, since it does not need to calculate the $9j$ symbols and coefficients of fractional parentage [3]. The old versions of the NPSM is constructed in the *J* scheme, and one has to recouple and sum over all of the intermediate quantum numbers in calculating

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the matrix elements. This procedure is too time consuming. Because of the advantage of the M scheme, it is interesting to cast the NPSM in the M scheme, and this is the aim of this paper.

The paper is organized as follows. In Sec. III, the NPSM in M scheme for the case without isospin symmetry is given; the NPSM in M scheme for the case with isospin symmetry are presented in Sec. IV, and a brief summary and discussion is given in Sec. VI.

II. THE HAMILTONIAN, *E*2 TRANSITION OPERATOR, AND *M*1 TRANSITION OPERATOR

As in the J scheme, we still use a Hamiltonian consisting of the single-particle energy term H_0 and a residual interaction containing the multipole pairing between like nucleons and the multipole-multipole interaction between all nucleons:

$$\begin{aligned} H &= \sum_{\sigma=\pi,\nu}(H_0(\sigma)+V(\sigma))+\sum_t \kappa_t Q^t_\pi\cdot Q^t_\nu,\\ H_0(\sigma) &= \sum_a \epsilon_a\hat{n}_a,\\ V(\sigma) &= \sum_s G_{s\sigma}A^{s\dagger}\cdot A^s+\sum_t k_{t\sigma}Q^t\cdot Q^t,\\ Q^t &= \sum_{i=1}^n (r_i)^t Y_t(\theta_i\phi_i), \end{aligned} \tag{1}$$

where ϵ_a and $\hat{n}_a$ are the single-particle energy and the number operator respectively, and the pair creation operator is

$$A^{s\dagger}_\nu=\sum_{cd}y_0(cds)(a^\dagger_c\times a^\dagger_d)^s_\nu. \tag{2}$$

Notice that the structure coefficients $y_0(cds)$ depend on the Hamiltonian to be used and are in general different from those in the building blocks, $y(cds)$, in Eq. (8).

The second quantized form of Q^t is given by Eq. (13) with the coefficients $q(cdt)$ equal to

$$\begin{aligned} q(cdt) &= (-)^{c-\frac{1}{2}}\frac{\widehat{\hat{c}\hat{d}}}{\sqrt{20\pi}}C^{t\,0}_{c\frac{1}{2},d-\frac{1}{2}}\Delta_{cdt}\langle Nl_c|r^t|Nl_d\rangle,\\ \Delta_{cdt} &= \frac{1}{2}[1+(-)^{l_c+l_d+t}], \end{aligned} \tag{3}$$

where N is the principal quantum number of the harmonic oscillator wave function, such that the energy is $(N+3/2)\hbar\omega_0$ and l_c and l_d are the orbital angular momentum of the single-particle (s.p.) levels c and d, respectively.

The general form of the two-body realistic interaction in the shell model as shown in the following can also use in this algorithm:

$$\begin{aligned} V = \sum_{JT}\sum_{j_1\leqslant j_2,\,j_3\leqslant j_4}&\frac{V_{JT}(j_1j_2j_3j_4)}{\sqrt{1+\delta_{j_1j_2}}\sqrt{1+\delta_{j_3j_4}}}\\ &\times(A^{JT\dagger}(j_1j_2)\times A^{JT}(j_3j_4))^0. \end{aligned} \tag{4}$$

The $E2$ and $M1$ transition operators are

$$\begin{aligned} E2 &= e_\pi Q^2_\pi+e_\nu Q^2_\nu,\\ T(M1) &= T(M1)_\pi+T(M1)_\nu,\\ T(M1) &= \sqrt{\frac{3}{4\pi}}(g_l\mathbf{L}+g_s\mathbf{S}), \end{aligned} \tag{5}$$

where e_π and e_ν are effective charges of the protons and neutrons, while g_l and g_s are the orbital and spin effective gyromagnetic ratios. The total orbital angular momentum operator $\mathbf{L}$ and total spin $\mathbf{S}$ can be identified with collective dipole operators,

$$\begin{aligned} L_\sigma &\equiv Q^1_\sigma=\sum_{cd}q(cd1)P^1_\sigma(cd),\\ S_\sigma &\equiv Q'^1_\sigma=\sum_{cd}q'(cd1)P^1_\sigma(cd) \end{aligned} \tag{6}$$

with

$$\begin{aligned} q(cd1) &= (-1)^{l+1/2+d}\sqrt{\tfrac{l(l+1)}{3}}\hat{c}\hat{d}\hat{l}\begin{Bmatrix} c & d & 1\\ l & l & \frac{1}{2}\end{Bmatrix},\\ q'(cd1) &= (-1)^{l+1/2+c}\tfrac{1}{\sqrt{2}}\hat{c}\hat{d}\begin{Bmatrix} c & d & 1\\ \frac{1}{2} & \frac{1}{2} & l\end{Bmatrix}. \end{aligned} \tag{7}$$

III. NPSM IN *M* SCHEME WITHOUT ISOSPIN SYMMETRY

In this section, the uncoupled commutators and matrix elements for the one-body operator and two body operator for the case without isospin are given in the M scheme. The odd system and even system are treated on the same footing.

A. Uncoupled commutators for nucleon pairs

As in the old version of the NPSM, the collective nucleon pair with angular momentum r and projection m, designated as $A^{r\dagger}_\nu$, is built from many noncollective pairs $A^r_\nu(cd)^\dagger$ in the single-particle orbits a and b in one major shell,

$$A^{r\dagger}_m=\sum_{ab}y(abr)A^{r\dagger}_m(ab), \tag{8}$$

where $y(abr)$ are structure coefficients satisfying the symmetry

$$y(abr)=-(-)^{a+b+r}y(bar). \tag{9}$$

Noncollective pair $A^{r\dagger}_m(ab)$ is

$$A^{r\dagger}_m(ab)=(a^{a\dagger}\times a^{b\dagger})^r_m=\sum_{m_a,m_b}C^{rm}_{a,m_a,bm_b}a^{a\dagger}_{m_a}a^{b\dagger}_{m_b}, \tag{10}$$

where $a^{j\dagger}_m$ is a single-particle creation operator, which creates a nucleon in the j orbit with projection m, and C^{rm}_{a,m_a,bm_b} is a Clebsch-Gordan coefficient. The time-reversed form of the single annihilaton operator a^j_m is

$$\tilde{a}^j_m=(-)^{j-m}a^j_{-m}. \tag{11}$$

The time-reversed form of a collective pair is

$$\tilde{A}^r_m=\sum_{ab}y(abr)\tilde{A}(ab)=-\sum_{ab}y(abr)(\tilde{a}^a\times\tilde{a}^b)^r_m. \tag{12}$$

A multipole operator or one-body operator Q^t_σ is denoted by

$$Q^t_\sigma = \sum_{cd} q(cdt)P^t_\sigma(cd),$$
$$P^t_\sigma(cd) = (a^{c\dagger} \times \tilde{a}^d)^t_\sigma,$$
$$q(cdt) = -(-)^{c+d} \times q(dct). \quad (13)$$

The coupled commutators between two collective pairs are denoted as

$$[\tilde{A}^r, A^{s\dagger}]^t_\sigma = \sum_{\alpha\beta} C^{t\sigma}_{r\alpha,s\beta}[\tilde{A}^r_\alpha, A^{s\dagger}_\beta]. \quad (14)$$

Some crucial coupled commutators in the NPSM, taken from Ref. [13], are listed in the Appendix. The uncoupled commutator for nucleons pairs, which can be used to constructed the NPSM in M scheme, is obtained from the coupled commutators, and is

$$\left[A^r_\mu, A^{s\dagger}_\nu\right] = (-)^{r-\mu}\sum_{\alpha\beta}\delta_{-\mu\alpha}\delta_{\nu\beta}[\tilde{A}^r_\alpha, A^{s\dagger}_\beta]$$
$$= (-)^{r-\mu}\sum_{t\sigma} C^{t\sigma}_{r-\mu,s\nu}[\tilde{A}^r, A^{s\dagger}]^t_\sigma \quad (15)$$

Based on Eqs. (A1)–(A7), the uncoupled commutators in M scheme can be obtained. The uncoupled commutator between collective pair annihilation operator and collective pair creation operator is given by

$$\left[A^r_\mu, A^{s\dagger}_\nu\right] = (-)^{r-\mu}\sum_{t\sigma} C^{t\sigma}_{r-\mu,s\nu}[\tilde{A}^r, A^{s\dagger}]^t_\sigma$$
$$= 2\delta_{r,s}\delta_{\mu,\nu}\sum_{ab} y(abr)y(abr) - (-)^{r-\mu}\sum_{t\sigma} C^{t\sigma}_{r-\mu,s\nu}P^t_\sigma, \quad (16)$$

where P^t_σ is a new one-body operator, which is given in Eq. (A2). The uncouple commutator for the collective pair and one-body operator would be given by

$$\left[A^r_m, Q^t_\sigma\right] = (-)^{r-m}\sum_{r'm'} C^{r'm'}_{r\,-m,t\sigma}[\tilde{A}^r, Q^t]^{r'}_{m'}$$
$$= \sum_{r'm'} \mathbb{A}^{r'}_{-m'}, \quad (17)$$

where $\mathbb{A}^{r'}_{-m'}$ is a new collective pair, which is

$$\mathbb{A}^{r'}_{-m'} = \sum_{ad} y'(dar')A^{r'}_{-m'}(da),$$
$$y'(dar') = z(dar') - (-)^{a+d+r'}(adr'),$$
$$z(dar') = (-)^{r+r'-m-m'}\hat{r}\hat{t}C^{r'm'}_{r\,-m,t\sigma}$$
$$\times \sum_b y(abr)q(bdt)\begin{Bmatrix} r & t & r' \\ d & a & b\end{Bmatrix}. \quad (18)$$

By using Eqs. (16) and (17) the uncoupled double commutator can be obtained:

$$\left[A^{r_i}_{m_i}, \left[A^{r_k}_{m_k}, A^{s\dagger}_m\right]\right] = (-)^{r_k+r_i-m_k-m_i}\sum_{\substack{t\sigma \\ r'm'}} C^{r'm'}_{r_i-m_i,t\sigma}C^{t\sigma}_{r_k-m_k,sm}[\tilde{A}^{r_i}, [\tilde{A}^{r_k}, A^{s\dagger}]^t]^{r'}_{m'} = \sum_{r'm'}\mathbb{B}^{r'}_{-m'}, \quad (19)$$

where $\mathbb{B}^{r'}_{-m'}$ is a new collective pair, which is

$$\mathbb{B}^{r'}_{-m'} = \sum_{aa'} y'(aa'r'_i)A^{r'}_{-m'}(aa'),\ y'(aa'r'_i) = z(aa'r'_i) - (-)^{a+a'+r'}z(a'ar'_i),$$

$$z(aa'r'_i) = -4\hat{r}_i\hat{r}_k\hat{s}\sum_{t\sigma}\hat{t}(-)^{r_k+r_i+r'-m}C^{r'm'}_{r_i-m_i,t\sigma}C^{t\sigma}_{r_k-m_k,sm}\sum_{bb'} y(a'b'r_i)y(abr_k)y(bb's)\begin{Bmatrix} r_k & s & t \\ a & b' & b\end{Bmatrix}\begin{Bmatrix} r_i & t & r' \\ a & a' & b'\end{Bmatrix}. \quad (20)$$

By using Eq. (17) recursively, the uncoupled double commutator between pair annihilation operator and multipole-multipole interaction operator can be obtained, and is

$$\sum_\sigma (-)^\sigma[[A^r_m, Q^t_\sigma], Q^t_{-\sigma}] = \sum_{r'}(-)^{r-r'}\frac{\hat{r}'}{\hat{r}}\mathbb{A}^r_m, \quad (21)$$

where $\mathbb{A}^r_m$ is a new collective pair, which is given by

$$\mathbb{A}^r_m = \sum_{ab} y'(abr)A^r_m(ab), \quad y'(abr) = [h(abr) - (-)^{a+b+r}h(bar)],$$

$$h(abr) = (-)^{r+r'-m-m'}\hat{r}'\hat{t}C^{r\,-m}_{r'\,\sigma-m,\,t\,-\sigma}\sum_d y''(bdr')q(dat)\begin{Bmatrix} r' & t & r \\ d & b & a\end{Bmatrix}, \quad (22)$$

where the new pair structure $y''(bdr')$ is obtained from Eq. (17).

The uncoupled commutator between single nucleon annihilation operator and one-body operator can be expressed as

$$\left[a^j_{m_0}, Q^t_\sigma\right] = -\sum_{j'm'}(-)^{t-\sigma}C^{j'm'}_{j\,-m_0,t\sigma}\frac{\hat{t}}{\hat{j}'}q(jj't)\,a^{j'}_{-m'}. \quad (23)$$

The uncoupled double commutator for single nucleon can also be obtained, and is

$$\begin{aligned}\left[a^{j}_{m_0},\left[A^{r_k}_{m_k},A^{s\dagger}_{m}\right]\right]&=\sum_{\substack{t\sigma\\ r'm'}}(-)^{r_k+j-m_k+m_0}C^{t\sigma}_{r_k\,-m_k,sm}C^{r'm'}_{j\,-m_0,t\sigma}[a^{j},[A^{r_k},A^{s\dagger}]^{t}]^{r'}_{m'}\\&=4\sum_{\substack{t\sigma\\ r'm'}}(-)^{r_k+t-m}C^{t\sigma}_{r_k\,-m_k,sm}C^{r'm'}_{j\,-m_0,t\sigma}\frac{\hat{r}_k\hat{s}\hat{t}}{\hat{r}'}\sum_{b}y(r'br_k)y(bjs)\begin{Bmatrix}r_k & s & t\\ j & r' & b\end{Bmatrix}a^{r'}_{-m'}.\end{aligned}\tag{24}$$

By recursive applications of Eq. (23), we can obtain the uncoupled double commutator between single nucleon operator and multipole-multipole interaction operator, which is

$$\sum_{\sigma}(-)^{\sigma}\left[\left[a^{j}_{m_0},Q^{t}_{\sigma}\right],Q^{t}_{-\sigma}\right]=\sum_{j'}(-)^{j-j'}\frac{2t+1}{2j+1}q(jj't)q(j'jt)\,a^{j}_{m_0}.\tag{25}$$

B. Commutators in M scheme

The odd system with $2N+1$ nucleons and the even system with $2N$ nucleons are treated on the same footing. The creation operator coupled successively to the total angular momentum projection M is designated by

$$A^{\dagger}(r_0m_0,\dots,r_Nm_N)_M=A^{r_0\dagger}_{m_0}\cdot A^{r_1\dagger}_{m_1}\cdots A^{r_N\dagger}_{m_N},\quad M=\sum_{i=0}^{N}m_i\tag{26}$$

with the convention that

$$A^{r_0\dagger}_{m_0}=\begin{cases}1 & \text{for even system, } m_0\equiv 0,\\ a^{r_0\dagger}_{m_0} & \text{for odd system, } r_0\equiv j.\end{cases}\tag{27}$$

The annihilation operator A_M is defined as

$$A(r_0m_0,\dots,r_Nm_N)_M=A^{r_0}_{m_0}\cdot A^{r_1}_{m_1}\cdots A^{r_N}_{m_N},\quad M=\sum_{i=0}^{N}m_i;\tag{28}$$

the convention of $A^{r_0}_{m_0}$ is similar to that of Eq. (26). Then one can get the commutator between the annihilation operator A_M and the pair creation operator, which is

$$\begin{aligned}[A(r_0m_0,\dots,r_Nm_N)_M,A^{s\dagger}_{m}]=\sum_{k=1}^{N}\Bigg[&\varphi\delta_{r_k,s}\delta_{m_k,m}A(r_0m_0,\dots,r_{k-1}m_{k-1},r_{k+1}m_{k+1},\dots,r_Nm_N)_{M-m}\\&+\sum_{i=k-1}^{0\text{ or }1}\sum_{r_i'm_i'}A(r_0m_0,\dots,r_i'm_i',\dots,r_{k-1}m_{k-1},r_{k+1}m_{k+1},\dots,r_Nm_N)_{M-m}\\&+\sum_{t\sigma}P^{t}_{\sigma}\times A(r_0m_0,\dots,r_{k-1}m_{k-1},r_{k+1}m_{k+1},\dots,r_Nm_N)_{M-m_k}\Bigg],\end{aligned}\tag{29}$$

where $\varphi=2\sum_{ab}y(abr_k)y(abs)$, the summation runs over i from $k-1$, to 0 or 1, corresponding to the odd system or even system respectively, and $A^{r_i'}_{m_i'}$ represents a new collective pair ($i\neq 0$) or a single nucleon ($i=0$),

$$A^{r_i'}_{m_i'}=\left[A^{r_i}_{m_i},\left[A^{r_k}_{m_k},A^{s\dagger}_{m}\right]\right];\tag{30}$$

the explicit form of the new pair has already been given in Eqs. (19) and (24), and P^{t}_{σ} has been given in Eq. (16). It is to be noted that the last term in the right-hand side of Eq. (29) is normal ordered and it is unimportant in calculating the matrix elements, since it gives zero when acting to the left on a vacuum state. By using Eq. (29), we have the commutation relation between the pairing interaction and the annihilation operator A_M:

$$\begin{aligned}[A(r_0m_0,\dots,r_Nm_N)_M,A^{s\dagger}\cdot A^{s}]=\sum_{k=1}^{N}\Bigg[&\varphi\delta_{r_k,s}\delta_{m_k,m}A(r_0m_0,\dots,sm_k,\dots,r_Nm_N)_M\\&+\sum_{i=k-1}^{0\text{ or }1}\sum_{r_i'm_i'm}A(r_0m_0,\dots,r_i'm_i',\dots,r_{k-1}m_{k-1},r_{k+1}m_{k+1},\dots,r_Nm_N,sm)_M\\&+\sum_{t\sigma m}P^{t}_{\sigma}\times A(r_0m_0,\dots,r_{k-1}m_{k-1},r_{k+1}m_{k+1},\dots,r_Nm_N,sm)_{M-m_k+m}\Bigg].\end{aligned}\tag{31}$$

It is to be noted the last term in Eq. (31) is normal ordered.

The commutator between one body operator and the annihilation operator A_M is obtained,

$$[A(r_0m_0,\dots,r_Nm_N)_M, Q^t_\sigma] = \sum_{k=N}^{0 \text{ or } 1}\sum_{r'_km'_k} A(r_0m_0, r_1m_1,\dots,r'_km'_k,\dots,r_Nm_N)_{M-\sigma}, \tag{32}$$

where summation runs over k from N to 0 or 1, corresponding to the odd system or even system, respectively, and $A^{r'_k}_{m'_k}$ represents a new collective pair ($k \neq 0$) or a single nucleon ($k = 0$):

$$A^{r'_k}_{m'_k} = \left[A^{r_k}_{m_k}, Q^t_\sigma\right]; \tag{33}$$

the explicit forms of the new pair have already been given in Eqs. (17) and (23).

The commutator between multipole-multipole operator and the annihilation operator A_M is

$$\begin{aligned}\left[A(r_0m_0,\dots,r_Nm_N)_M, \sum_\sigma(-)^\sigma Q^t_\sigma Q^t_{-\sigma}\right] = \sum_{k=N}^{0 \text{ or } 1}\Bigg[&A(r_0m_0, r_1m_1,\dots,(r_km_k)_B,\dots,r_Nm_N)_M\\ &+\sum_{i=k-1}^{0 \text{ or } 1}\sum_{\substack{r'_im'_i\\ r'_km'_k}}\sum_\sigma(-)^\sigma A(r_0m_0, r_1m_1,\dots,r'_im'_i,\dots,r'_km'_k,\dots,r_Nm_N)_M\\ &+\sum_\sigma\sum_{r'_km'_k}(-)^\sigma Q^t_{-\sigma}\times A(r_0m_0, r_1m_1,\dots,r'_km'_k,\dots,r_Nm_N)_{M-\sigma}\Bigg],\end{aligned} \tag{34}$$

where summation range over $k(i)$ from $N(k-1)$ to 0 or 1, corresponding to the odd system or even system, respectively, and $r'_km'_k(r'_im'_i)$ represents a new collective pair ($k \neq 0$) or a single nucleon ($k = 0$):

$$A^{r'_k}_{m'_k} = \left[A^{r_k}_{m_k}, Q^t_\sigma\right], \quad A^{r'_i}_{m'_i} = \left[A^{r_i}_{m_i}, Q^t_{-\sigma}\right]; \tag{35}$$

the explicit forms of the new pair have already given in Eqs. (17) and (23). $(r_km_k)_B$ denotes a new collective pair ($k \neq 0$) or a single nucleon ($k = 0$) obtained by uncoupled double commutators

$$\left(A^{r_k}_{m_k}\right)_B = \sum_\sigma(-)^\sigma\left[\left[A^r_m, Q^t_\sigma\right], Q^t_{-\sigma}\right], \quad \left(A^{r_0}_{m_0}\right)_B = \sum_\sigma(-)^\sigma\left[\left[a^j_{m_0}, Q^t_\sigma\right], Q^t_{-\sigma}\right]; \tag{36}$$

the explicit results have been given in Eqs. (21) and (25). It is to be noted the last term in Eq. (34) is normal ordered.

C. Matrix elements of overlap and interactions

An N pair or N pair plus one single nucleon state in M scheme is designated as

$$|\alpha, M_N> = |r_0m_0, r_1m_1,\dots r_Nm_N; M_N\rangle \equiv A^\dagger(r_0m_0, r_1m_1,\dots r_Nm_N)|0\rangle, \tag{37}$$

where r_0m_0 represents 1 in the even particle number system or a single nucleon the in odd particle number system, and α denotes the additional quantum numbers

$$\alpha = (r_0m_0,\dots,r_N,m_N). \tag{38}$$

It is interesting to note that α is redundant, since it has already been included in the total projection number M.

For proton-neutron coupled system, the basis are constructed by coupling the protons and neutrons to the state with total projection number M,

$$|\alpha, M_nM_p; M\rangle = |\alpha_p, M_p\rangle|\alpha_n, M_n\rangle \tag{39}$$

The overlap between two states is a key quantity, since the matrix elements of one-body and two-body interaction can all be expressed as a summation of the overlaps. From Eq. (29), we can get the overlap between two states,

$$\begin{aligned}\langle 0|A_{M_1}A^\dagger_{M_2}|0\rangle &\equiv \langle r_0\mu_0, r_1\mu_1,\dots r_N\mu_N; M_1|s_0\nu_0, s_1\nu_1,\dots,s_N\nu_N; M_2\rangle\\ &= \sum_{k=1}^{N}\Bigg[2\sum_{ab}y(abr_k)y(abs_N)\delta_{r_k,s_N}\delta_{\mu_k,\nu_N}\langle r_0\mu_0\dots r_{k-1}\mu_{k-1}, r_{k+1}\mu_{k+1}\dots r_N\mu_N; M_1-\nu_N|s_0\nu_0\dots s_{N-1}\nu_{N-1}; M_2-\nu_N\rangle\\ &\quad+\sum_{i=k-1}^{0 \text{ or } 1}\sum_{r'_i\mu'_i}\langle r_0\mu_0\dots r'_i\mu'_i\dots r_{k-1}\mu_{k-1}, r_{k+1}\mu_{k+1}\dots r_N\mu_N; M_1-\nu_N|s_0\nu_0\dots s_{N-1}\nu_{N-1}; M_2-\nu_N\rangle\Bigg].\end{aligned} \tag{40}$$

One can see that although the overlap is still calculated recursively, the most time-consuming factor, the recoupling of the angular momentum, is not needed. The summation over the projection μ_i' of the new pair is redundant, since it is a constant value $\mu_i + \mu_k - \nu_N$. The overlap for one pair state in M scheme is same as that in J scheme, which is

$$\langle r_1\mu_1|s_1\nu_1\rangle = 2\delta_{r_1,s_1}\delta_{\mu_1,\nu_1}\sum_{ab} y(abr_1)y(abs_1). \tag{41}$$

The overlap for one pair plus one single nucleon is given as

$$\begin{aligned}\langle r_0\mu_0, r_1\mu_1|s_0\nu_0, s_1\nu_1\rangle &= 2\delta_{r_1,s_1}\delta_{\mu_1,\nu_1}\delta_{r_0,s_0}\delta_{\mu_0,\nu_0}\sum_{ab} y(abr_1)y(abs_1)\\ &\quad + 4\hat{r}_1\hat{s}_1\sum_{JM} C^{JM}_{r_0\mu_0,r_1\mu_1}C^{JM}_{s_0\nu_0,s_1\nu_1}\sum_a y(as_0r_1)(ar_0s_1)\begin{Bmatrix} s_1 & r_0 & a\\ r_1 & s_0 & J\end{Bmatrix},\end{aligned} \tag{42}$$

where $r_0(s_0)$ denote the single nucleon, and the summation over projection M is redundant, and it should be $\mu_0 + \mu_1$.

The matrix elements of a pair creation operator $A^{s\dagger}_\nu$ between two states differing by one pair is equal to an overlap,

$$\langle r_0\mu_0,\ldots,r_N\mu_N|A^{s\dagger}_\nu|s_0\nu_0,\ldots,s_{N-1}\nu_{N-1}\rangle = \langle r_0\mu_0,\ldots,r_N\mu_N|s_0\nu_0,\ldots,s_{N-1}\nu_{N-1},s\nu\rangle. \tag{43}$$

Using Eq. (31), the matrix elements for the pairing interaction can be written as

$$\begin{aligned}\langle r_0\mu_0, r_1\mu_1,\ldots r_N\mu_N;M|A^{s\dagger}\cdot A^s|s_0\nu_0,s_1\nu_1,\ldots,s_N\nu_N;M\rangle &= \sum_{k=1}^{N}\Bigg[\varphi\delta_{r_k,s}\delta_{\mu_k,\nu}\langle r_0\mu_0\ldots\boldsymbol{s\mu}_k\ldots r_N\mu_N;M|s_0\nu_0\ldots s_N\nu_N;M\rangle\\ &\quad + \sum_{i=k-1}^{0\text{ or }1}\sum_{r_i'\mu_i'm}\langle r_0\mu_0\ldots\boldsymbol{r}_i'\boldsymbol{\mu}_i'\ldots\boldsymbol{sm}\ldots r_N\mu_N;M|s_0\nu_0\ldots s_N\nu_N;M\rangle\Bigg],\end{aligned} \tag{44}$$

where the summation over the projection m in the second term represents the projection of pairing interacting $\sum_m A^{s\dagger}_m A^s_m$, and the summation over the projection μ_i' of the new pair is redundant: it should be a constant value $\mu_i + \mu_k - m$.

By using Eq. (32), the matrix element of the one-body operator can be written as

$$\langle r_0\mu_0,\ldots,r_N\mu_N;M_1|Q^t_\sigma|s_0\nu_0,\ldots,s_N\nu_N;M_2\rangle = \sum_{k=N}^{0\text{ or }1}\sum_{r_k',\mu_k'}\langle r_0\mu_0,\ldots,\boldsymbol{r}_k'\boldsymbol{\mu}_k',\ldots,r_N\mu_N;M_1-\sigma|s_0\nu_0,\ldots,s_N\nu_N;M_2\rangle, \tag{45}$$

where the summation over the projection μ_k' of new pair is redundant, and it should be a constant value $\mu_k - \sigma$.

The multipole-multipole interaction is given by

$$Q^t\cdot Q^t = \sum_\sigma (-)^\sigma Q^t_\sigma Q^t_{-\sigma}. \tag{46}$$

By using Eq. (34), the matrix elements of the multipole-multipole interaction between like nucleons are

$$\begin{aligned}\langle r_0\mu_0,\ldots,r_N\mu_N;M|Q^t\cdot Q^t|s_0\nu_0,\ldots,s_N\nu_N;M\rangle &= \sum_{k=N}^{0\text{ or }1}\Bigg[\langle r_0m_0,\ldots,(r_km_k)_B,\ldots,r_Nm_N;M|s_0\nu_0,\ldots,s_N\nu_N;M\rangle\\ &\quad + \sum_{i=k-1}^{0\text{ or }1}\sum_{\substack{r_i'm_i'\\ r_k'm_k'}}\sum_\sigma (-)^\sigma\langle r_0m_0,\ldots,r_i'm_i',\ldots,r_k'm_k',\ldots,r_Nm_N;\\ &\quad M|s_0\nu_0,\ldots,s_N\nu_N;M\rangle\Bigg],\end{aligned} \tag{47}$$

where the summation over the projection μ_k' (μ_i') of the new pair is redundant, and it should be a constant value $\mu_k - \sigma$ $(\mu_k + \sigma)$.

The matrix elements of multipole-multipole interaction between proton and neutron can be expressed as the product of matrix elements of the multipole operator for protons and neutrons,

$$\langle M_n, M_p;M|Q^t(\nu)\cdot Q^t(\pi)|M_n', M_p';M\rangle = \sum_\sigma (-)^\sigma\langle\alpha_p, M_p|Q^t_\sigma(\pi)|\alpha_p', M_p'\rangle\langle\alpha_n, M_n|Q^t_{-\sigma}(\nu)|\alpha_n', M_n'\rangle. \tag{48}$$

IV. NPSM IN M SCHEME WITH ISOSPIN SYMMETRY

In this section, the NPSM with isospin symmetry in M scheme is presented. The uncoupled commutators, and matrix elements of one-body operators and two body operators are presented. The odd system and even system are treated on the same footing.

A. Uncoupled commutators for nucleons pairs

We begin by introducing the notation for a system with isospin. The creation operator of a nucleon in a state with total angular momentum j, projection of angular momentum m, isospin t, and projection of isospin τ is designated as $a^{jt\dagger}_{m\tau}$. Now we can build the noncollective pair creation operator,

$$\begin{aligned} A^{JT\dagger}_{m\tau}(ab) &= (a^{a\dagger}\times a^{b\dagger})^{JT}_{m\tau} \\ &= \sum_{\substack{m_a m_b\\ \tau_a\tau_b}} C^{Jm}_{j_a m_a, j_b m_b} C^{T\tau}_{t_a\tau_a, t_b\tau_b} a^{j_a t_a\dagger}_{m_a\tau_a} \times a^{j_b t_b\dagger}_{m_b\tau_b}, \end{aligned} \quad (49)$$

where the superscript a represents j_a and t_a, and $C^{Jm}_{j_a m_a, j_b m_b}$ is the Clebsch-Gordan coefficient. The single annihilation operator can be introduced as $a^{jt}_{m\tau} \equiv (a^{jt\dagger}_{m\tau})^{\dagger}$. The time-reversed form of the single annihilation operator is

$$\tilde{a}^{jt}_{m\tau} = (-)^{j+t-m-\tau} a^{jt}_{-m-\tau}. \quad (50)$$

The commutator between single nucleon operator in coupled form is given by

$$(\tilde{a}^{a},\ a^{b\dagger})^{JT}_{m\tau} = \delta_{J0}\delta_{T0}\delta_{j_a j_b}\delta_{t_a t_b}\sqrt{2}\hat{j}_a. \quad (51)$$

The noncollective pair annihilation operator is given by $A^{JT}_{m\tau}(ab) = (A^{JT\dagger}_{m\tau}(ab))^{\dagger}$, and the corresponding time-reversed form of the pair annihilation operator is designated as

$$\begin{aligned} \tilde{A}^{JT}_{m\tau}(ab) &= (-)^{J+T-m-\tau} A^{JT}_{-m-\tau}(ab) \\ &= -(\tilde{a}^{a}\times\tilde{a}^{b})^{JT}_{m\tau}. \end{aligned} \quad (52)$$

The collective pair creation, annihilation, and time-reserved form operators are

$$\begin{aligned} A^{JT\dagger}_{m\tau} &= \sum_{ab} y(abJT) A^{JT\dagger}_{m\tau}(ab), \\ A^{JT}_{m\tau} &= \sum_{ab} y(abJT) A^{JT}_{m\tau}(ab), \\ \tilde{A}^{JT}_{m\tau} &= \sum_{ab} y(abJT) \tilde{A}^{JT}_{m\tau}(ab), \end{aligned} \quad (53)$$

where $y(abJT)$ are the pair structure coefficients, and have the symmetry

$$y(abJT) = (-)^{J+T-j_a-j_b} y(baJT). \quad (54)$$

The collective multipole operator is defined by

$$\begin{aligned} Q^{JT}_{m\tau} &= \sum_{ab} q(abJT) Q^{JT}_{m\tau}(ab) \\ &= \sum_{ab} q(abJT) P^{JT}_{m\tau}(ab), \\ P^{JT}_{m\tau}(ab) &= (a^{a\dagger}\times\tilde{a}^{b})^{JT}_{m\tau}. \end{aligned} \quad (55)$$

The uncoupled commutators for collective pairs can also be obtainde through coupled commutators. The coupled commutators in the NPSM with isospin symmetry have already been carried out in Ref. [21]. The uncoupled commutator between two collective pairs can be given by

$$\begin{aligned} &\left[A^{rt_1}_{\mu\tau_1}, A^{st_2\dagger}_{\nu\tau_2}\right] \\ &= (-)^{r+t_1+\mu+\tau_1} \sum_{\substack{Jm\\ T\tau}} C^{Jm}_{r-\mu,s\nu} C^{T\tau}_{t_1-\tau_1,t_2\tau_2} [A^{rt_1}, A^{st_2\dagger}]^{JT}_{m\tau} \\ &= 2\delta_{rs}\delta_{t_1t_2}\delta_{\mu\nu}\delta_{\tau_1\tau_2} \sum_{ab} y(abrt_1) y(abst_2) \\ &\quad + 4(-)^{r+t_1+\mu+\tau_1} \sum_{\substack{Jm\\ T\tau}} C^{Jm}_{r-\mu,s\nu} C^{T\tau}_{t_1-\tau_1,t_2\tau_2} P^{JT}_{m\tau}, \end{aligned} \quad (56)$$

where the summation over projections m and τ is redundant, since they are constant values $m = \nu - \mu$ and $\tau = \tau_2 - \tau_1$, and $P^{JT}_{m\tau}$ is a new one-body operator,

$$\begin{aligned} P^{JT}_{m\tau} &= \sum_{da} \hat{r}\hat{s}\hat{t}_1\hat{t}_2 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t_1 \\ t_2 & T & \frac{1}{2} \end{Bmatrix} \sum_{b} y(abrt_1) y(bdst_2) \\ &\quad \times \begin{Bmatrix} j_a & j_b & r_1 \\ s & J & j_d \end{Bmatrix} P^{JT}_{m\tau}(da). \end{aligned} \quad (57)$$

The uncoupled commutator for collective pair and multipole operator is given by

$$\begin{aligned} \left[A^{rt_1}_{\mu\tau_1}, Q^{lt_2}_{\sigma\tau_2}\right] &= -\sum_{\substack{JT\\ m\tau}} (-)^{J+T+r_1+t_1-\sigma-\tau_2} C^{Jm}_{r_1-\mu,l\sigma} C^{T\tau}_{t_1-\tau_1,t_2\tau_2} \\ &\quad \times \mathbb{A}^{JT}_{-m-\tau}, \end{aligned} \quad (58)$$

where $\mathbb{A}^{JT}_{-m-\tau}$ is a new pair, which is

$$\begin{aligned} &\mathbb{A}^{JT}_{-m-\tau} = \sum_{da} y'(daJT) A^{JT}_{-m-\tau}(da), \quad y'(daJT) = z(daJT) + (-)^{J+T-j_a-j_d} z(adJT), \\ &z(daJT) = \hat{r}_1\hat{l}\hat{t}_1\hat{t}_2 \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t_1 \\ t_2 & T & \frac{1}{2} \end{Bmatrix} \sum_{b} y(abr_1t_1) q(bdlt_2) \begin{Bmatrix} j_a & j_b & r_1 \\ j & J & j_d \end{Bmatrix}. \end{aligned} \quad (59)$$

By using Eq. (56) and (58) the uncoupled double commutator can be obtained:

$$\left[A^{r_it_i}_{\mu_i\tau_i},\left[A^{r_kt_k}_{\mu_k\tau_k},A^{st\dagger}_{\nu\eta}\right]\right]=-4\sum_{\substack{lm'\\t'\tau'}}\sum_{\substack{Jm\\T\tau}}(-)^{J+T+r_k+r_i+t_k+t_i-\nu-\eta}C^{lm'}_{r_k-\mu_k,s\nu}C^{t'\tau'}_{t_k-\tau_k,t\eta}C^{Jm}_{r_i-\mu_i,lm'}C^{T\tau}_{t_i-\tau_i,t'\tau'}\mathbb{B}^{JT}_{-m-\tau},\tag{60}$$

where the summation over projections m', τ', m, and τ is redundant, and $\mathbb{B}^{JT}_{-m-\tau}$ is a new pair,

$$\mathbb{B}^{JT}_{-m-\tau}=\sum_{aa'}y'(aa'JT)A^{JT}_{-m-\tau}(aa'),$$
$$y'(aa'JT)=z(aa'JT)+(-)^{J+T-j_a-j_{a'}}z(aa'JT),$$
$$z(aa'JT)=\hat{r}_k\hat{r}_i\hat{s}\hat{l}\hat{t}_k\hat{t}_i\hat{t}\hat{t}'\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&t_k\\ t&t'&\frac{1}{2}\end{Bmatrix}\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&t_i\\ t'&T&\frac{1}{2}\end{Bmatrix}\sum_{bb'}y(ab'r_kt_k)y(a'br_it_i)y(b'bst)\begin{Bmatrix}j_a&j_{b'}&r_k\\ s&l&j_b\end{Bmatrix}\begin{Bmatrix}j_{a'}&j_b&r_i\\ l&J&j_a\end{Bmatrix}.\tag{61}$$

By using Eq. (58) recursively, the uncoupled commutator between the collective pair operator and multipole-multipole interaction operator is given as

$$\sum_{\sigma\tau_2}(-)^{\sigma+\tau_2}\left[\left[A^{JT}_{m\tau},Q^{j_2t_2}_{\sigma\tau_2}\right],Q^{j_2t_2}_{-\sigma-\tau_2}\right]=-\sum_{r_1t_1}(-)^{J+T+r_1+t_1}C^{r_1\sigma-m}_{J-m,j_2\sigma}C^{t_1\tau_2-\tau}_{T-\tau,t_2\tau_2}C^{J-m}_{r_1\sigma-m,j_2-\sigma}C^{T-\tau}_{t_1\tau_2-\tau,t_2-\tau_2}\mathbb{A}^{JT}_{m\tau},\tag{62}$$

where $\mathbb{A}^{JT}_{m\tau}$ is a new pair:

$$\mathbb{A}^{JT}_{m\tau}=\sum_{da}y'(daJT)A^{JT}_{m\tau}(da),\quad y'(daJT)=z(daJT)+(-)^{J+T-j_a-j_d}z(adJT),$$
$$z(daJT)=\hat{r}_1\hat{l}\hat{t}_1\hat{t}_2\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&t_1\\ t_2&T&\frac{1}{2}\end{Bmatrix}\sum_b y''(abr_1t_1)q(bdlt_2)\begin{Bmatrix}j_a&j_b&r_1\\ j&J&j_d\end{Bmatrix},\tag{63}$$

where the new pair structure coefficients $y''(abr_1t_1)$ are obtained from Eq. (58).

We can also obtain the uncoupled commutator between the single nucleon operator and multipole operator,

$$\left[a^{j_0t_0}_{m_0\tau_0},Q^{j_2t_2}_{\sigma\tau_2}\right]=(-)^{j_2-\sigma}(-)^{t_2-\tau_2}\sum_{\substack{Jm\\T\tau}}C^{Jm}_{j_0-m_0,j_2\sigma}C^{T\tau}_{t_0-\tau_0,t_2\tau_2}\frac{\hat{j}_2\hat{t}_2}{\hat{J}\hat{T}}q((j_0t_0)(JT)j_2t_2)a^{JT}_{-m-\tau}\tag{64}$$

Base on Eq. (64), the uncoupled commutator between single nucleon operator and multipole-multipole interaction operator is given as,

$$\sum_{\sigma\tau_2}(-)^{\sigma+\tau_2}\left[\left[a^{j_0t_0}_{m_0\tau_0},Q^{j_2t_2}_{\sigma\tau_2}\right],Q^{j_2t_2}_{-\sigma-\tau_2}\right]=\sum_{lt}(-)^{j_0-l}(-)^{t_0-t}\frac{(2j_2+1)(2t_2+1)}{(2j_0+1)(2t_0+1)}q((j_0t_0)(lt)j_2t_2)q((lt)(j_0t_0)j_2t_2)a^{j_0t_0}_{m_0\tau_0}\tag{65}$$

The uncoupled double commutator for single nucleon operator can also be obtained,

$$\left[a^{j_0t_0}_{m_0\tau_0},\left[A^{r_kt_k}_{m_k\tau_k},A^{st\dagger}_{\nu\eta}\right]\right]=4\sum_{\substack{Jm\\T\tau}}\sum_{\substack{lm'\\t'\tau'}}(-)^{l+r_k-\nu}(-)^{t'+t_k-\eta}C^{lm'}_{r_k-m_k,s\nu}C^{t'\tau'}_{t_k-\tau_k,t\eta}C^{Jm}_{j_0-m_0,lm'}C^{T\tau}_{t_0-\tau_0,t'\tau'}\frac{\hat{r}_k\hat{s}\hat{l}}{\hat{J}}\frac{\hat{t}_k\hat{t}\hat{t}'}{\hat{T}}$$
$$\times\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&t_k\\ t&t'&\frac{1}{2}\end{Bmatrix}\sum_b y((JT)br_kt_k)y(b(j_0t_0)st)\begin{Bmatrix}r_k&s&l\\ j_0&J&j_b\end{Bmatrix}a^{JT}_{-m-\tau},\tag{66}$$

where the summation over projections m, m', τ, and τ' is redundant.

B. Commutators in *M* scheme

In this section, the odd system with $2N+1$ nucleons and the even system with $2N$ nucleons are treated on the same footing. The configuration space of the NPSM is constructed by collective pairs. The creation operator with specific total angular momentum projection M and total isospin projection τ are

$$A^\dagger(\mathbb{r}_0,\ldots,\mathbb{r}_N)_{M,\tau}\equiv A^\dagger(r_0m_0t_0\tau_0,\ldots,r_Nm_Nt_0\tau_0)_{M,\tau}=A^{r_0t_0\dagger}_{m_0\tau_0}\cdot A^{r_1t_1\dagger}_{m_1\tau_1}\ldots A^{r_Nt_N\dagger}_{m_N\tau_N},$$
$$M=\sum_{i=0}^N m_i,\quad\tau=\sum_{i=0}^N\tau_i,\tag{67}$$

where the $\mathbb{r}_0$ express all quantum numbers of this pair, and the convention used for the operator $A^{r_0t_0\dagger}_{m_0\tau_0}$ is given as,

$$A^{r_0t_0\dagger}_{m_0\tau_0} = \begin{cases} 1 & \text{for even system, } m_0 \equiv 0,\ \tau_0 \equiv 0, \\ a^{r_0t_0\dagger}_{m_0\tau_0} & \text{for odd system, } r_0 \equiv j. \end{cases} \tag{68}$$

The annihilation operator is

$$A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau} \equiv A(r_0m_0t_0\tau_0, \ldots, r_Nm_Nt_0\tau_0)_{M,\tau} = A^{r_0t_0}_{m_0\tau_0} \cdot A^{r_1t_1}_{m_1\tau_1} \ldots A^{r_Nt_N}_{m_N\tau_N},$$
$$M = \sum_{i=0}^{N} m_i, \quad \tau = \sum_{i=0}^{N} \tau_i, \tag{69}$$

where the convention of $A^{r_0}_{m_0}$ is similar to that of Eq. (67). The time-reversed form of $A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau}$ is redundant in this model.

Then we can obtain the commutator between the annihilation operator$A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau}$

$$\begin{aligned}[A(\mathbb{r}_0, \mathbb{r}_1, \ldots, \mathbb{r}_N)_{M,\tau}\ , A^{st\dagger}_{\nu\eta}] = \sum_{k=1}^{N} \Bigg[& \varphi \delta_{r_k,s}\delta_{m_k,\nu}\delta_{t_k,t}\delta_{t_k,\eta} A(\mathbb{r}_0, \ldots, \mathbb{r}_{k-1}, \mathbb{r}_{k+1}, \ldots, \mathbb{r}_N)_{M-\nu,\tau-\eta} \\ & + \sum_{i=k-1}^{0 \text{ or } 1} \sum_{r_i'm_i'} \sum_{t_i'\tau_i'} A(\mathbb{r}_0, \ldots, \mathbb{r}_i', \ldots, \mathbb{r}_{k-1}, \mathbb{r}_{k+1}, \ldots, \mathbb{r}_N)_{M-\nu,\tau-\eta} \\ & + \sum_{lm'} \sum_{t'\tau'} P^{lt'}_{m'\tau'} \times A(\mathbb{r}_0, \ldots, \mathbb{r}_{k-1}, \mathbb{r}_{k+1}, \ldots, \mathbb{r}_N)_{M-m_k,\tau-\tau_k} \Bigg].\end{aligned} \tag{70}$$

where $\varphi = 2\sum_{ab} y(abr_kt_k)y(abst)$, the summation over i is from $k-1$ to 0 or 1 corresponding to the odd system and even system respectively, and $\mathbb{r}_i'$ is a new collective pair $A^{r_i't_i'}_{m_i'\tau_i'}$ $(i \neq 0)$ or a single nucleon operator $C^{r_0't_0'}_{m_0'\tau_0'}$ $(i = 0)$, and can be obtained from the double commutator by

$$A^{r_i't_i'}_{m_i'\tau_i'} = \big[A^{r_it_i}_{m_i\tau_i}, \big[A^{r_kt_k}_{m_k\tau_k}, A^{st\dagger}_{\nu\eta}\big]\big]; \tag{71}$$

the explicit form of the new pair has been given in Eq. (60) and (66), and $P^{lt'}_{m'\tau'}$ has been given in Eq. (57). It should be noted that the last term in Eq. (70) is normal ordered.

By using Eq. (70), the commutation relation between general pairing interaction and $A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau}$ is obtained by

$$\begin{aligned}[A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau}\ , A^{st\dagger} \cdot A^{st}] = \sum_{k=1}^{N} \Bigg[& \varphi \delta_{r_k,s}\delta_{m_k,m}\delta_{t_k,t}\delta_{t_k,\eta} A(\mathbb{r}_0, \ldots, \mathbb{s}, \ldots, \mathbb{r}_N)_{M,\tau} \\ & + \sum_{i=k-1}^{0 \text{ or } 1} \sum_{r_i'm_i'm} \sum_{t_i'\tau_i'\eta} A(\mathbb{r}_0, \ldots, \mathbb{r}_i', \ldots, \mathbb{r}_{k-1}, \mathbb{s}, \mathbb{r}_{k+1}, \ldots, \mathbb{r}_N)_{M,\tau} \\ & + \sum_{lm'm} \sum_{t'\tau'\eta} P^{lt'}_{m'\tau'} \times A(\mathbb{r}_0, \ldots, \mathbb{r}_{k-1}, \mathbb{s}, \mathbb{r}_{k+1}, \ldots, \mathbb{r}_N)_{M-m_k+m,\tau-\tau_k+\eta} \Bigg],\end{aligned} \tag{72}$$

where the last term in right-hand side of equation is normal ordered.

The commutator between the one body operator and $A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau}$ is obtained, which is

$$[A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_M, Q^{jt}_{\sigma\eta}] = \sum_{k=N}^{0 \text{ or } 1} \sum_{r_k'm_k'} \sum_{t_k'\tau_k'} A(\mathbb{r}_0, \mathbb{r}_1, \ldots, \mathbb{r}_k', \ldots, \mathbb{r}_N)_{M-\sigma,\tau-\eta}, \tag{73}$$

where summation over k is from N to 0 or 1 corresponding to the odd system or even system respectively. $\mathbb{r}_k'$ represents a new collective pair $A^{r_k't_k'}_{m_k'\tau_k'}$ $(k \neq 0)$ or a single nucleon operator $C^{r_0't_0'}_{m_0'\tau_0'}$ $(k = 0)$,

$$A^{r_k't_k'}_{m_k'\tau_k'} = \big[A^{r_kt_k}_{m_k\tau_k}, Q^{jt}_{\sigma\eta}\big]. \tag{74}$$

The explicit form of the new pair has been given in Eqs. (58) and (64).

The commutator between the general multipole-multipole operator and $A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau}$ is obtained by

$$\left[A(\mathbb{r}_0, \ldots, \mathbb{r}_N)_{M,\tau}, \sum_{\sigma\eta}(-)^{\sigma+\eta}Q^{jt}_{\sigma\eta}Q^{jt}_{-\sigma-\eta}\right] = \sum_{k=N}^{0\text{ or }1}\Bigg[A(\mathbb{r}_0, \mathbb{r}_1, \ldots, (\mathbb{r}_k)_B, \ldots, \mathbb{r}_N)_M$$
$$+ \sum_{i=k-1}^{0\text{ or }1}\sum_{\substack{r'_i m'_i\\ r'_k m'_k}}\sum_{\substack{t'_i\tau'_i\\ t'_k\tau'_k}}\sum_{\sigma\eta}(-)^{\sigma+\eta}A(\mathbb{r}_0, \mathbb{r}_1, \ldots, \mathbb{r}'_i, \ldots, \mathbb{r}'_k, \ldots, \mathbb{r}_N)_{M,\tau}$$
$$+ \sum_{r'_k m'_k}\sum_{t'_k\tau'_k}\sum_{\sigma\tau}(-)^{\sigma+\eta}Q^{jt}_{-\sigma-\tau} \times A(\mathbb{r}_0, \mathbb{r}_1, \ldots, \mathbb{r}'_k, \ldots, \mathbb{r}_N)_{M-\sigma,\tau-\eta}\Bigg]. \quad (75)$$

where summation over $k(i)$ is from $N(k-1)$ to 0 or 1 corresponding to the odd system or even system respectively. $\mathbb{r}'_k(\mathbb{r}'_i)$ represents a new collective pair ($k \neq 0$) or a single nucleon ($k = 0$),

$$A^{r'_k t'_k}_{m'_k\tau'_k} = \left[A^{r_k t_k}_{m_k\tau_k}, Q^{jt}_{\sigma\eta}\right], \quad A^{r'_i t'_i}_{m'_i\tau'_i} = \left[A^{r_i t_i}_{m_i\tau_i}, Q^{jt}_{-\sigma-\eta}\right]. \quad (76)$$

The explicit form of the new pair has been given in Eqs. (58) and (64). $(\mathbb{r}_k)_B$ denotes a new collective pair ($k \neq 0$) or a single nucleon ($k = 0$), and can be obtained by uncoupled double commutators

$$\left(A^{r_k t_k}_{m_k\tau_k}\right)_B = \sum_{\sigma\eta}(-)^{\sigma+\eta}\left[\left[A^{r_k t_k}_{m_k\tau_k}, Q^{jt}_{\sigma\eta}\right], Q^{jt}_{-\sigma-\eta}\right], \quad \left(A^{r_0 t_0}_{m_0\tau_0}\right)_B = \sum_{\sigma\eta}(-)^{\sigma+\eta}\left[\left[a^{j_0 t_0}_{m_0\tau_0}, Q^{jt}_{\sigma\eta}\right], Q^{jt}_{-\sigma-\eta}\right]. \quad (77)$$

The explicit results have been carried out in Eq. (62) and (65). It should be noted that the last term in Eq. (75) is normal ordered.

C. Matrix elements of overlap and interactions

An N-pair or N-pair plus one single nucleon state in M scheme is designated as

$$|\alpha, M\tau\rangle = |\mathbb{r}_0, \mathbb{r}_1, \ldots, \mathbb{r}_N; M\tau\rangle$$
$$\equiv A^\dagger(\mathbb{r}_0, \mathbb{r}_1, \ldots, \mathbb{r}_N)|0\rangle, \quad (78)$$

where $\mathbb{r}_k$ represents a pair with angular momentum r_k, angular momentum projection m_k, isospin t_k, and isospin projection τ_k, $\mathbb{r}_0$ denotes 1 in the even system or a single nucleon in the odd system, and α denotes the additional quantum numbers

$$\alpha = (r_0 m_0 t_0 \tau_0, \ldots, r_N, m_N t_N \tau_N) \quad (79)$$

It is interesting to note that α is redundant: it contains $4N$ quantum numbers and has already been included in the information of total projection M and τ.

The overlap matrix element is a key quantity, since the one- and two-body interaction matrix elements can be expressed as a summation of the overlaps. From Eq. (70), the overlap can be obtained by

$$\langle 0|A_{M_l\tau_l}A^\dagger_{M_r\tau_r}|0\rangle = \langle\mathbb{r}_0, \ldots, \mathbb{r}_N; M_l\tau_l|\mathbb{s}_0, \ldots, \mathbb{s}_N; M_r\tau_r\rangle \equiv \langle r_0\mu_0 t_0\tau_0, \ldots, r_N\mu_N t_N\tau_N; M_l\tau_l|s_0\nu_0 h_0\eta_0, \ldots, s_N\nu_N h_0\eta_0; M_r\tau_r\rangle$$
$$= \sum_{k=1}^{N}\Bigg[2\sum_{ab}y(abr_k t_k)y(abs_N h_N)\delta_{r_k,s_N}\delta_{\mu_k,\nu_N}\delta_{t_k,h_N}\delta_{\tau_k,\eta_N}$$
$$\times \langle\mathbb{r}_0\ldots\mathbb{r}_{k-1}, \mathbb{r}_{k+1}\ldots\mathbb{r}_N; M_l-\nu_N, \tau_l-\eta_N|\mathbb{s}_0\ldots\mathbb{s}_{N-1}; M_r-\nu_N, \tau_r-\eta_N\rangle$$
$$+ \sum_{i=k-1}^{0\text{ or }1}\sum_{r'_i\mu'_i}\sum_{t'_i\tau'_i}\langle\mathbb{r}_0\ldots\mathbb{r}'_i\ldots\mathbb{r}_{k-1}, \mathbb{r}_{k+1}\ldots\mathbb{r}_N; M_1-\nu_N, \tau_l-\eta_N|\mathbb{s}_0\ldots\mathbb{s}_{N-1}; M_r-\nu_N, \tau_r-\eta_N\rangle\Bigg], \quad (80)$$

where r_k and s_k denote the angular moments of the kth pair, and μ_k and ν_k denote its projection of angular moment, t_k and h_k represent the isospin, and τ_k and η_k stand for its projection. Although the overlap matrix element formula in M scheme is still written in a recursive way, it does not need to recouple the angular momentum and isospin. It is easy to show the total angular momentum projection M_l should be equal to M_r, and total projection of isospin τ_l should be equal to τ_r. Notice that the summation over the projections of the new pair, μ'_i and τ'_i, are redundant, and should be constant values $\mu_i + \mu_k - \nu_N$ and $\tau_i + \tau_k - \eta_N$, respectively, since projection is a scalar. The overlap for one pair is given by using Eq. (80), and is

$$\langle\mathbb{r}_1|\mathbb{s}_1\rangle = 2\delta_{r_1,s_1}\delta_{\mu_1,\nu_1}\delta_{t_1,h_1}\delta_{\tau_1,\eta_1}\sum_{ab}y(abr_1 t_1)y(abs_1 h_1). \quad (81)$$

Since there is only one pair, the overlap formula is equivalent to the formulas for the case of $N = 1$ in J scheme. The overlap for the one pair plus one single nucleon is given as

$$\begin{aligned}\langle \mathbb{r}_0, \mathbb{r}_1; M\tau|\mathbb{s}_0, \mathbb{s}_1; M\tau\rangle = {}& 2\delta_{r_1,s_1}\delta_{\mu_1,\nu_1}\delta_{t_1,h_1}\delta_{\tau_1,\eta_1}\sum_{ab} y(abr_1t_1)y(abs_1h_1)\\ &+4\sum_{\substack{lm\\ t'\tau'}}(-)^{l+r_1-\nu_1}(-)^{t'+t_1-\eta_1}C^{lm}_{r_1-\mu_1,s_1\nu_1}C^{t'\tau'}_{t_1-\tau_1,h_1\eta_1}C^{s_0-\nu_0}_{r_0-\mu_0,lm}C^{h_0-\eta_0}_{t_0-\tau_0,t'\tau'}\frac{\hat{r}_1\hat{s}_1\hat{l}}{\hat{s}_0}\frac{\hat{t}_1\hat{h}_1\hat{t}'}{\hat{h}_0}\\ &\times\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & t_1\\ h_1 & t' & \frac{1}{2}\end{Bmatrix}\sum_b y\Big((s_0h_0)br_1t_1\Big)y\Big(b(r_0t_0)s_1h_1\Big)\begin{Bmatrix} r_1 & s_1 & l\\ r_0 & s_0 & j_b\end{Bmatrix},\end{aligned}\tag{82}$$

where $\mathbb{r}_0(\mathbb{s}_0)$ denotes the single nucleon.

It is easy to show that the matrix elements of a pair creation operator $A^{s\dagger}_{\nu}$ between two states differing by one pair is equal to an overlap,

$$\langle \mathbb{r}_0, \mathbb{r}_1, \dots, \mathbb{r}_N|A^{sh\dagger}_{\nu\eta}|\mathbb{s}_0, \mathbb{s}_1, \dots, \mathbb{s}_{N-1}\rangle = \langle \mathbb{r}_0, \mathbb{r}_1, \dots, r_N\mu_N|\mathbb{s}_0, \mathbb{s}_1, \dots, \mathbb{s}_{N-1}, \mathbb{s}\rangle. \tag{83}$$

Using Eq. (72), we can have the matrix elements of the pairing interaction,

$$\begin{aligned}\langle \mathbb{r}_0, \mathbb{r}_1, \dots \mathbb{r}_N; M\tau|A^{st\dagger}\cdot A^{st}|\mathbb{s}_0, \mathbb{s}_1, \dots, \mathbb{s}_N; M\tau\rangle = \sum_{k=1}^{N}\Bigg[&\varphi\delta_{r_k,s}\delta_{\mu_k,m}\delta_{t_k,t}\delta_{\tau_k,\eta}\langle \mathbb{r}_0\dots\mathbb{s}\dots\mathbb{r}_N; M\tau|\mathbb{s}_0\dots\mathbb{s}_N; M\tau\rangle\\ &+\sum_{i=k-1}^{0\text{ or }1}\sum_{r_i'\mu_i'm}\sum_{t_i'\tau_i'\eta}\langle \mathbb{r}_0\dots\mathbb{r}_i'\dots\mathbb{s}\dots\mathbb{r}_N; M\tau|\mathbb{s}_0\dots\mathbb{s}_N; M\tau\rangle\Bigg],\end{aligned}\tag{84}$$

where the summation over the projections m and η in the second term stand for the projection of pairing interacting $G_{st}\sum_{m\eta}A^{st\dagger}_{m\eta}A^{st}_{m\eta}$, and the summation over the projections μ_i' and τ_i' of the new pair is redundant: they should be constant values $\mu_i + \mu_k - m$ and $\tau_i + \tau_k - \eta$, respectively.

By using Eq. (73), for the one-body operator matrix element we have

$$\langle \mathbb{r}_0\dots\mathbb{r}_N; M_1\tau_1|Q^{jt}_{\sigma\eta}|\mathbb{s}_0\dots\mathbb{s}_N; M_2\tau_2\rangle = \sum_{k=N}^{0\text{ or }1}\sum_{r_k',\mu_k'}\sum_{t_k',\tau_k'}\langle \mathbb{r}_0\dots\mathbb{r}_k'\dots\mathbb{r}_N; M_1-\sigma,\ \tau_1-\eta|\mathbb{s}_0\dots\mathbb{s}_N; M_2\rangle, \tag{85}$$

where the summation over the projections μ_k' and τ_k' of the new pair is redundant: they should be constant values $\mu_k - \sigma$ and $\tau_k - \eta$, respectively.

By using Eq. (75), we can obtain the matrix elements of the multipole-multipole interaction between like nucleons,

$$\begin{aligned}\langle \mathbb{r}_0, \dots, \mathbb{r}_N; M\tau|Q^{jt}\cdot Q^{jt}|\mathbb{s}_0\dots\mathbb{s}_N; M\tau\rangle = \sum_{k=N}^{0\text{ or }1}\Bigg[&\langle \mathbb{r}_0\dots(\mathbb{r}_k)_B\dots\mathbb{r}_N; M\tau|\mathbb{s}_0\dots\mathbb{s}_N; M\tau\rangle\\ &+\sum_{i=k-1}^{0\text{ or }1}\sum_{\substack{r_i'm_i'\\ r_k'm_k'}}\sum_{\substack{t_i'\tau_i'\\ t_k'\tau_k'}}\sum_{\sigma\eta}(-)^{\sigma+\eta}\langle \mathbb{r}_0\dots\mathbb{r}_i'\dots\mathbb{r}_k'\dots\mathbb{r}_N; M\tau|\mathbb{s}_0\dots\mathbb{s}_N; M\tau\rangle\Bigg],\end{aligned}\tag{86}$$

where the summation over the projections μ_k' (μ_i') and τ_k' (τ_i') of the new pair is redundant, since they should be a constant values $\mu_k - \sigma$ ($\mu_k + \sigma$) and $\tau_k - \eta$ ($\tau_k + \eta$), respectively.

V. DISCUSSION

It is known that all the matrix elements in the NPSM in J scheme can be expressed as the summation of overlaps between two N-pair bases. The formulas to calculate the overlaps have to sum over all the intermediate angular momentum quantum numbers J_i and the quantum numbers of the new pairs. Therefore, the CPU time used in calculating the overlaps increases drastically with the number of nucleon pairs.

From our previous discussion, it is known that the formula in calculating matrix elements in M scheme does not contain the summation of the intermediate quantum number; it is interesting to see the validity of the NPSM in M scheme. To this end the average CPU time used in computing the overlap versus number of pairs is presented in Fig. 1, in which the average CPU time of computing 16 overlaps between the basis constructed by identical $D(J = 2)$ collective pairs is presented.

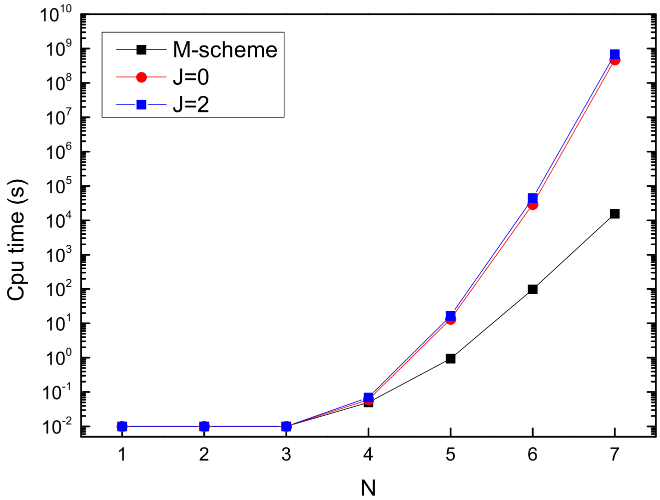

FIG. 1. The average CPU time used in computing one overlap matrix element against the number of pairs is presented. The overlap in J scheme is calculated in the subspace with total angular momenta $J = 0$ and $J = 2$. The overlap in M scheme is for the states with $M = 0$.

As an example, only the results for the systems without isospin symmetry are presented here. Since the CPU time is too small to be obtained for the systems with the number of the collective pairs smaller than 4, we set them all to be equal to 10^{-2} s per matrix element. For the system with $N = 7$, the average CPU time in calculating the overlap in J scheme is too time consuming to be obtained; its CPU time is estimated through the recursive formula of the overlap and the average CPU time of the overlap for the system with $N = 6$. Figure 1 shows that the average CPU time of the overlap matrix element in M scheme is indeed much smaller that that in J scheme. For example, the average CPU time for the system with $N = 6$ is about 10^2 seconds in the M scheme, while it is about 10^5 seconds in the J scheme.

To see the difference of the configuration space between M scheme and J scheme, the numbers of states for the system with $N = 5$ collective $S(J = 0)$ and $D(J = 2)$ pairs are presented in Table I. It can be seen that the numbers of states in J scheme are much smaller than those in the M scheme.

It is known that, to calculate the matrix elements for the case with $N = 5$ in J scheme in the NPSM in the SD-pair subspace, the CPU time is about one month. To see the validity of the NPSM in M scheme, as an example, the property of the low-lying states of ^{150}Nd (five proton valence pairs and four neutron valence pairs) are studied in SD-pair subspace. A Hamiltonian with pairing and quadrupole-quadrupole interactions is adopted:

$$\begin{aligned}
\hat{H} &= \sum_{\sigma=\pi,\nu} \hat{H}_\sigma - \kappa Q_\pi^2 \cdot Q_\nu^2, \\
\hat{H}_\sigma &= H_{0\sigma} - G_{0\sigma} A^{(0)\dagger} A^{(0)} - G_{2\sigma} A^{(2)\dagger} A^{(2)\dagger} - \kappa_\sigma Q^2 Q^2, \\
H_{0\sigma} &= \sum_a \epsilon_a C_a^\dagger C_a, \\
A^{(0)\dagger} &= \sum_a \frac{\hat{a}}{2} (C_a^\dagger \times C_a^\dagger)^0, \\
A^{(2)\dagger} &= \sum_{ab} q(ab2) (C_a^\dagger \times C_b^\dagger)^2,
\end{aligned} \tag{87}$$

where $H_{0\sigma}$ is the single-particle energy term, and G_0, G_2, κ are the monopole pairing, the quadrupole pairing, and the quadrupole-quadrupole interaction strength, respectively. As an approximation, the S-pair structure coefficient is chosen to be $y(aa0) = \hat{j}_a \frac{\nu_a}{\mu_a}$, where ν_a and μ_a are occupied and empty amplitudes obtained by solving the BCS equation, while the D-pair structure coefficients are obtained from the commutator

$$A^{2\dagger} = D^\dagger = \tfrac{1}{2}[Q^2, S^\dagger]. \tag{88}$$

The single-particle energies, adopted as ^{133}Sn and ^{133}Sb experimental data [27–29] for neutron and proton, are listed in Table II. By fitting the experimental $E_{2_1^+}$, $E_{4_1^+}$, and $E_{0_2^+}$ energies, the model parameters are fixed as $G_{0\pi} = 0.14$ MeV, $G_{2\pi} = 0.07$ MeV$/r_0^4$, $G_{0\nu} = 0.12$ MeV, $G_{2\nu} = 0.04$ MeV$/r_0^4$, $\kappa_\pi = \kappa_\nu = 0$, and $\kappa = 0.17$ MeV$/r_0^4$. As shown in the Fig. 2,

TABLE I. Number of states for 5 collective pairs in SD subspace. The left column is for the case in J scheme, while the right one is for the case in M scheme.

J scheme		M scheme			
J	Number of states	M	Number of states	M	Number of states
10	1	10	1	−10	1
8	2	9	1	−9	1
7	1	8	3	−8	3
6	4	7	4	−7	4
5	2	6	8	−6	8
4	6	5	10	−5	10
3	2	4	16	−4	16
2	7	3	18	−3	18
0	5	2	25	−2	25
		1	25	−1	25
		0	30		

TABLE II. Adopted single-particle energies $\epsilon_\sigma(\sigma = \pi$ or $\nu)$ for protons (50–82 shell) and neutrons (82-126 shell) (in MeV).

j_ν	$1f_{7/2}$	$0h_{9/2}$	$1f_{5/2}$	$2p_{3/2}$	$2p_{1/2}$	$0i_{13/2}$
ϵ_ν	0	1.561	2.005	0.854	1.656	1.8
j_π	$0g_{7/2}$	$1d_{5/2}$	$1d_{3/2}$	$2s_{1/2}$	$0h_{11/2}$	
ϵ_π	0	0.963	2.69	2.99	2.76	

the low-lying spectrum of ^{150}Nd can be reproduced very well. By fitting $B(E2; 2_1^+ \to 0_1^+)$ of the experimental value, the effective charge is fixed as $e_\pi = 2.3e$ for protons and $e_\nu = 1.5e$ for neutrons, respectively. Some $B(E2)$ transition values are listed in Table III, from which one can see that our results are close to IBM results, and the experimental data can be produced approximately.

From above analysis, one can see that although the numbers of states in *M* scheme are larger than those in *J* scheme, and the CPU times are much smaller in *M* scheme than those in *J* scheme. One can also see that it is a challenge to use the *M* scheme to study nuclei with more valance nucleons.

VI. SUMMARY

The NPSM is cast in the *M* scheme for both even and odd systems under the assumptions that all nucleon pairs are collective and for given angular momentum *r* there is only one collective pair $A_r^\dagger$. The cases with isospin symmetry and without isospin symmetry are all discussed. If there are more than one type of collective pairs with given angular momentum *s*, we only need to introduce an additional label to distinguish them. The NPSM in *M* scheme can also be easily extended to include the noncollective pairs. The CPU time used in calculating the matrix elements in *M* scheme is much shorter than that used in the *J* scheme of NPSM, which make it possible to study the medium-heavy nuclei in the NPSM.

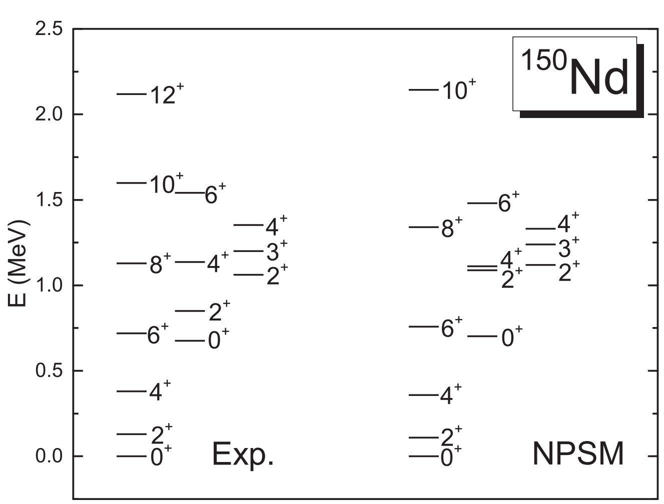

FIG. 2. The low-lying spectrum of ^{150}Nd. The experimental data are obtained from Ref. [27].

TABLE III. $B(E2)$ values for ^{150}Nd.

	$B(E2; J_i \to J_f$ (eb)^2		
$J_i \to J_f$	$B(E2)_{\rm expt}$ [a]	$B(E2)_{NPSM}$ [b]	$B(E2)_{sdg\text{-}IBM}$ [c]
$2_1^+ \to 0_1^+$	0.563 ± 0.002	0.573	0.560
$4_1^+ \to 2_1^+$	0.819 ± 0.038	0.804	0.810
$6_1^+ \to 4_1^+$	0.980 ± 0.09	0.844	0.883
$0_2^+ \to 2_1^+$	0.208 ± 0.009	0.016	0.071
$2_3^+ \to 4_1^+$	0.095 ± 0.028	0.0	0.033
$2_2^+ \to 2_1^+$	0.034 ± 0.007	0.001	0.073
$2_2^+ \to 0_1^+$	0.015 ± 0.0009	0.0	0.012

[a]Reference [27,30].
[b]Present calculation.
[c]Reference [31].

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APPENDIX: SOME CRUCIAL COUPLED COMMUTATORS IN THE NPSM

Some crucial coupled commutators in the NPSM, taken from Ref. [13], are listed in the following. The coupled commutator between collective pair annihilation operator and collective pair creation operator is given by

$$[\tilde{A}^r, A^{s\dagger}]_\sigma^t = 2\hat{r}\delta_{rs}\delta_{t0}\sum_{ab} y(abr)y(abs) - P_\sigma^t, \qquad (A1)$$

where P_σ^t is a new one-body operator,

$$P_\sigma^t = 4\hat{r}\hat{s}\sum_{abd} y(abr)y(bds)\begin{Bmatrix} r & s & t \\ d & a & b\end{Bmatrix}(a^{d\dagger} \times \tilde{a}^a)_\sigma^t. \qquad (A2)$$

The coupled commutator for collective pair and multipole operator would be given by

$$[\tilde{A}^r, Q^t]_{m'}^{r'} = \tilde{A}_{m'}^{r'}, \qquad (A3)$$

where $\tilde{A}_{m'}^{r'}$ is a new collective pair annihilation operator,

$$\tilde{A}_{m'}^{r'} = \sum_{ad} y'(dar')\tilde{A}_{m'}^{r'}(da),$$
$$y'(dar') = z(dar') - (-)^{a+d+r'}z(adr'),$$
$$z(dar') = \hat{r}\hat{t}\sum_b y(abr)q(bdt)\begin{Bmatrix} r & t & r' \\ d & a & b\end{Bmatrix}. \qquad (A4)$$

By using Eqs. (A1) and (A3) the coupled double commutator can be obtained,

$$[A^{r_i}, [A^{r_k}, A^{s\dagger}]^t]^{r_i'} = \mathbb{B}^{r_i'}, \qquad (A5)$$

where $\mathbb{B}^{r'_i}$ is a new collective pair,

$$\mathbb{B}^{r'_i} = \sum_{aa'} y'(aa'r'_i)\tilde{A}^{r'_i}(aa'),$$

$$y'(aa'r'_i) = z(aa'r'_i) - (-)^{a+a'+r'} z(a'ar'_i),$$

$$z(aa'r'_i) = -4\hat{r}_i\hat{r}_k\hat{s}\hat{t}\sum_{bb'} y(a'b'r_i)y(abr_k)y(bb's)\begin{Bmatrix} r_k & s & t \\ a & b' & b \end{Bmatrix}\begin{Bmatrix} r_i & t & r' \\ a & a' & b' \end{Bmatrix}, \tag{A6}$$

where $\hat{t} = \sqrt{2t+1}$. The coupled commutator between single-particle and one-body operators was given by Eq. (2.11a) in Ref. [20], and is

$$[\tilde{a}^j, Q^t]^{j'}_{m'} = (-)^{t-j-j'} q(j, j', t)\frac{\hat{t}}{\hat{j}'}\tilde{a}^{r'}_{m'}. \tag{A7}$$

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