Rapid Communications

New Rosenbluth formula including the Coulomb correction in the second Born approximation

T. Tamae
Research Center for Electron Photon Science, Tohoku University, Sendai 982-0826, Japan

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The Coulomb correction factor is derived for $M1$ scattering on a pointlike nucleus in the second Born approximation. The Coulomb correction for $M1$ scattering off the proton is larger than that for $C0$ scattering by 2%. The difference affects the determination of the proton radius where an identical Coulomb correction has been applied for both components so far. A new Rosenbluth formula, that includes the Coulomb corrections, is applied to electron scattering on the proton with low-energy electron beams. The Coulomb corrections are essential to discuss the electric and magnetic radii of the proton.

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The proton charge radius r_E has been a topic of great interest since the Lamb shift measurements in muonic hydrogen [1–3] resulted in $r_E = 0.841\,84(67)$ fm, which is smaller by 5% than standard analyses of electron scattering ($r_E = 0.88$ fm) [4–8]. In electron scattering, the information of the proton structure is included in the electric form factor G_E and magnetic form factor G_M.

In the nonrelativistic limit, the electric form factor is obtained by the Fourier transformation of the charge distribution $\rho(r)$ and can be expanded in Taylor series as

$$G_E = 1 - \tfrac{1}{6}\langle r_E^2\rangle Q^2 + \tfrac{1}{120}\langle r_E^4\rangle Q^4 - \cdots . \quad (1)$$

The charge radius r_E is defined by a root-mean-square radius $\langle r_E^2\rangle^{1/2}$. Measurements at very low momentum transfers are necessary to obtain $\langle r_E^2\rangle^{1/2}$ without the influence of higher moments [9]. However, the sensitivity of the form factor to the proton size is very low in the region of low momentum transfer. For instance, it is less than 0.1% at $Q^2 = 0.3$ fm^{-2} for a difference of 0.01 fm of the proton size. The precise separation of Coulomb and magnetic scatterings is necessary to obtain the charge or magnetic radii from the form factors. Typically, they are deduced from measured cross sections using the Rosenbluth formula,

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\left[\frac{G_E^2(Q^2)}{1+\tau} + \left(\frac{\tau}{1+\tau} + 2\tau\tan^2\frac{\theta}{2}\right) \times G_M^2(Q^2)\right], \quad (2)$$

where $(\frac{d\sigma}{d\Omega})_{\text{Mott}}$ is the Mott cross section, G_E and G_M are the electric and magnetic Sachs form factors, respectively, and $\tau = \frac{Q^2}{4M^2}$ (M is the proton mass). This is the formula derived in the plane-wave Born approximation (PWBA). However, higher-order correction is needed to discuss precise values of the form factors.

Furthermore, the Coulomb correction is inevitable to derive precise form factors. The effects of the Coulomb correction have been discussed in different ways [10–13]. So far, the identical Coulomb correction has been applied for both Coulomb and magnetic scatterings. Similarly, the so-called Feshbach correction [14,15] was used in Bernauer *et al.* [13]. This is a correction to the PWBA cross section for the Coulomb scattering off a point charge. The correction factor is obtained by expanding the Mott solution [16] in a power series of $(z\alpha)$ to the next-leading order, where α is the fine-structure constant. However, the assumption that the Coulomb correction is identical for both Coulomb and magnetic scatterings must be reconsidered. This is because the spin states of incoming and outgoing electron waves are the same in Coulomb scattering but different in magnetic scattering [17].

Bergstrom [18] has derived a correction factor for $M1$ scattering in the second Born approximation.[1] His result is given in an integral expression, that includes form factors in the integration. If one assumes a point nucleus with charge and magnetism as a target, one can push the calculation forward and obtain the correction factor as a simple form. The magnetic $M1$ cross section in the first Born approximation is given in Bergstrom [18] as

$$\left(\frac{d\sigma}{d\Omega}\right)_{M1}^{\text{first Born}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\left(\frac{Q^2}{2q^2} + \tan^2\frac{\theta}{2}\right)|F_{M1}(q)|^2, \quad (3)$$

[1] Bergstrom treated the Coulomb correction for inelastic magnetic scattering, and his procedure can be used for elastic scattering. Cutler [25] formulated a similar procedure for inelastic Coulomb scattering. It should be noted that the Coulomb correction formula for $C0$ inelastic scattering is different from that for elastic scattering due to the difference of the number of cross terms.

where F_{M1} is the $M1$ form factor and q and Q are three- and four-momentum transfers, respectively. The second Born $M1$ cross section can be expressed using the Coulomb correction factor β_{M1} as

$$\left(\frac{d\sigma}{d\Omega}\right)_{M1}^{\text{second Born}} = (1 + z\alpha\beta_{M1})\left(\frac{d\sigma}{d\Omega}\right)_{M1}^{\text{first Born}}. \tag{4}$$

The correction factor derived by Bergstrom [18] is given as

$$\beta_{M1} = \frac{8\delta^2}{\pi(4+\delta^2)F_{M1}(q)}\int_0^\infty du\, F_{M1}(Eu)F_C(E|\vec{u}-\vec{\delta}|) \times\left[\frac{\delta(8+u^2+\delta^2)}{4u^2(u^2-\delta^2)}\ln\left|\frac{(u+2)}{(u-2)}\right| + \frac{\delta}{4u} - \frac{\delta}{16}\ln\left|\frac{u+2}{u-2}\right| + \frac{1}{u^2}\ln\left|\frac{u+\delta}{u-\delta}\right|\right], \tag{5}$$

where the momentum-transfer vector is rewritten in terms of a unitless vector as

$$\vec{q} = E\vec{\delta}, \quad \delta = 2\sin\frac{\theta}{2}. \tag{6}$$

The momentum transfer $\vec{q}\,'$, which appears in the integration, is also rewritten as

$$\vec{q}\,' = E\vec{u}, \tag{7}$$

and then $F_{M1}(Eu)$ is the $M1$ form factor at a momentum transfer of Eu, and $F_C(E|\vec{u}-\vec{\delta}|)$ is a charge form factor at a momentum transfer of $E|\vec{u}-\vec{\delta}|$. In addition, since the notations of Eqs. (2) and (3) are different, the form factor F_{M1} must be related to the Sachs form factor G_M. Thus, by comparing these equations, taking account of $\frac{Q^2}{q^2} = \frac{1}{1+\tau}$, the $M1$ form factor in Eq. (5) is rewritten as

$$F_{M1}(q) = \frac{1}{\sqrt{2}}\frac{Q}{M}G_M(Q) = \frac{1}{\sqrt{2}}\frac{q}{M}G_M(Q), \tag{8}$$

where the nuclear recoil is ignored, that is, $E_i = E_f = E$, and then $Q = q$, and

$$F_{M1}(Eu) = \frac{1}{\sqrt{2}}\frac{Eu}{M}G_M(Q). \tag{9}$$

As the proton is treated as a point particle with a magnetic moment μ_p, $G_M(Q) = \mu_p$, and $F_C(E|\vec{u}-\vec{\delta}|) = 1$. Although the integration of $\frac{\delta}{4}$ and $\frac{\delta}{16}u\ln|\frac{u+2}{u-2}|$ in Eq. (5) individually diverges, they eventually cancel out each other. The remaining part can be calculated using contour integration. Finally, the correction factor for $M1$ scattering is reduced to a simple form, given as

$$\beta_{M1} = \frac{\pi\sin\frac{\theta}{2}\left(2+\sin\frac{\theta}{2}\right)}{\sin^2\frac{\theta}{2}+1}. \tag{10}$$

The correction factor β_{C0} has been obtained by McKinley, Jr. and Feshbach [14] and Feshbach [15] as

$$\beta_{C0} = \frac{\pi\sin\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2}}. \tag{11}$$

Therefore, using these corrections for Coulomb and magnetic scatterings, a new Rosenbluth formula, which includes the Coulomb correction in the second Born approximation, is given as

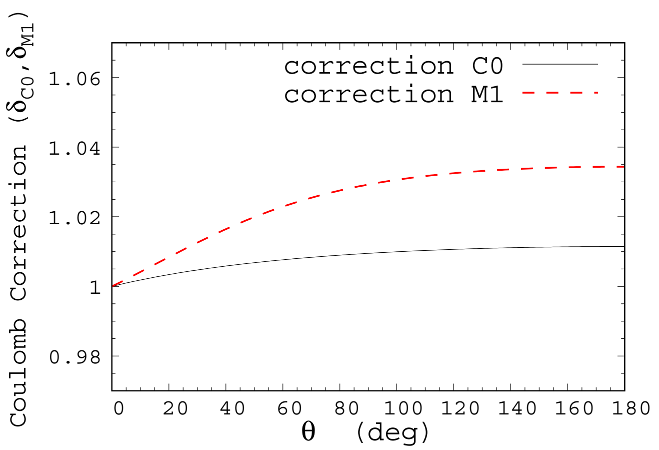

FIG. 1. The second Coulomb corrections of $C0$ and $M1$, $\delta_{C0} = 1 + z\alpha\beta_{C0}$, $\delta_{M1} = 1 + z\alpha\beta_{M1}$ $(z = 1)$.

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\left[(1+z\alpha\beta_{C0})\frac{G_E^2(Q)}{1+\tau} + (1+z\alpha\beta_{M1})\left(\frac{\tau}{1+\tau}+2\tau\tan^2\frac{\theta}{2}\right)G_M^2(Q)\right]. \tag{12}$$

The Coulomb corrections of $C0$ and $M1$, $\delta_{C0} = 1 + z\alpha\beta_{C0}$, $\delta_{M1} = 1 + z\alpha\beta_{M1}$ are shown for $z = 1$ in Fig. 1; the correction for $M1$ is 2% larger than that for $C0$. The influence of the corrections on the problem to determine the proton radius is discussed below.

In the PWBA, the reduced cross section defined as

$$\sigma_{\text{red}} = \varepsilon(1+\tau)\left(\frac{d\sigma}{d\Omega}\right)\Big/\left(\frac{d\sigma}{d\Omega}\right)_M \tag{13}$$

is linear against the polarization,

$$\varepsilon = \left\{1 + 2(1+\tau)\tan^2\frac{\theta}{2}\right\}^{-1}, \tag{14}$$

$$\sigma_{\text{red}} = \varepsilon G_E^2(Q) + \tau G_M^2(Q). \tag{15}$$

However, in Eq. (12), the different Coulomb corrections for the Coulomb and magnetic terms induce the nonlinearity because the correction factors are functions of the scattering angle. The larger correction factor for $M1$ alters the magnetic form factor and affects the charge form factor through the nonlinearity. The nonlinearity is removed by replacement, such as

$$\sigma_{\text{red}} \to \sigma_{\text{red}}^{\text{second}} = \frac{\sigma_{\text{red}}}{(1+z\alpha\beta_{M1})}, \tag{16}$$

and

$$\varepsilon \to \varepsilon_{\text{second}} = \frac{(1+z\alpha\beta_{C0})}{(1+z\alpha\beta_{M1})}\varepsilon. \tag{17}$$

Most recently, Xiong *et al.* [19] obtained a small value of the proton size $r_E = 0.83$ fm from an experiment at very

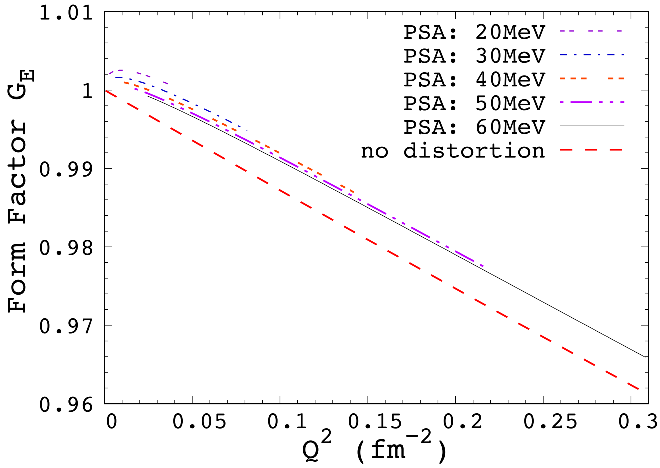

FIG. 2. Form factors calculated by the phase-shift analysis (PSA) at $\theta = 30^\circ$–150° and that obtained by the Fourier transformation of the charge density (no distortion) for $\langle r_E^2\rangle^{1/2} = 0.88$ fm. The form factors (PSA) calculated with different electron energies do not scale and are different from the result with no distortion.

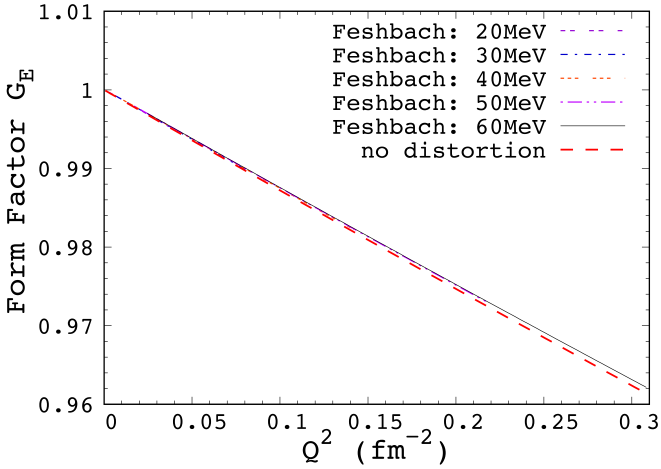

FIG. 3. Form factors corrected using the Feshbach correction. The results for different electron energies align on a line. They are close to the line without the Coulomb distortion.

forward angles (0.7°–7.0°) performed at the Jefferson Laboratory (JLab). In this experimental condition, the contribution of the magnetic scattering can be ignored, and the influence of the Coulomb distortion is negligible. Experiments in different kinematic conditions may be necessary to check the result. However, the effects, which can be ignored in the experiment at JLab, are essential in experiments using lower-energy electron beams.

Bernauer *et al.* [7,13] obtained $r_E = 0.88$ and $r_M = 0.78$ fm from 1400 cross sections measured at Mainz using electron beams of 180–855 MeV. They applied the Feshbach correction for both Coulomb and magnetic parts. In the $G_M/(\mu_p G_{\text{std. dipole}})$, they observed a wiggle of 1% around 0.2 (GeV/c), which gives a small r_M. On the other hand, Arrington [11] pointed out that the two-photon exchange correction will modify G_M and bring r_M larger. The possibility that the wiggle may be attributed to the influence of the Coulomb distortion on magnetic scattering should be considered. Using Eq. (12) gives a smaller magnetic form factor and, then, a larger magnetic radius.

The Tohoku group [20,21] plans a precise measurement in the region of low-momentum transfer using low-energy electron beams of $E_e = 20$–60 MeV. First, the effect of β_{C0} is investigated. The predicted Coulomb distortion on Coulomb scattering is large as shown in Fig. 2. In the figure, the charge form factor, which has been deduced from the cross section calculated using a phase-shift program DREPHA [22], are compared with the result with no distortion. The description of the phase-shift calculation is shown in Yennie *et al.* [23]. These have been calculated using the same charge distribution: dipole form factor of $\langle r_E^2\rangle^{1/2} = 0.88$ fm. A wiggle is found in the phase-shift calculations similar to that observed at Mainz in the magnetic form factor. This shows that the Coulomb distortion is too large to define the proton size using the expression of Eq. (1) because it changes the form factor gradient around $Q = 0$ significantly. This situation is improved by applying the Feshbach correction. The form factors corrected by the Feshbach correction are shown in Fig. 3, using

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{PSA}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} (1 + \alpha\beta_{C0})\frac{G_E^2(Q^2)}{1+\tau}. \tag{18}$$

Results calculated at different electron energies align on a line, and the difference to the PWBA (no distortion) is small. The root-mean-square radius obtained by fitting with a Taylor series is 0.871 fm. It shows the Coulomb distortion is mostly removed by the Feshbach correction, but there remains a small difference. The difference is discussed later. It has been confirmed that the form factors calculated using dipole and Gaussian charge distributions are almost the same at these low-momentum transfers, and the cross section is determined by only $\langle r_E^2\rangle^{1/2}$.

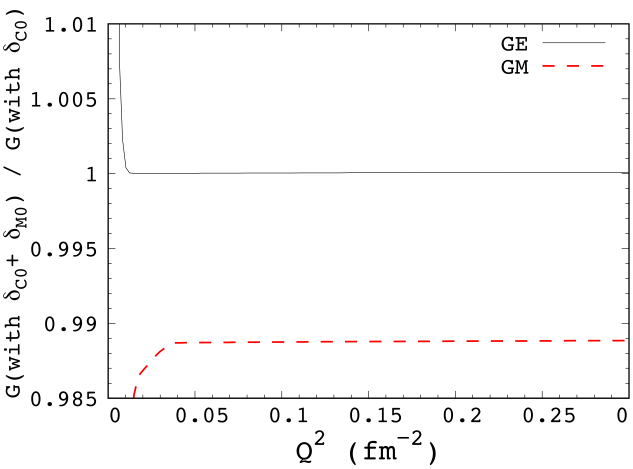

FIG. 4. Ratio of form factors in cases where the $C0$ and $M1$ corrections are applied as in Eq. (12), and only the Feshbach correction is applied to the whole cross section. The kinematic condition is the same as shown in Fig. 2.

Next, the effect of β_{M1} is investigated. Figure 4 demonstrates the ratio of form factors calculated using two different formulas. One is the formula shown in Eq. (12) where individual corrections are applied for $C0$ and $M1$ scatterings. In the other, only the Feshbach correction is applied to the whole cross section as performed by Bernauer *et al.* [13]. In the calculation, pseudodata created using Eq. (12) for $E_e =$ 20–60 MeV and $\theta = 30$–150° are separated into the Coulomb and magnetic parts by assuming that δ_{M1} is equal to δ_{C0}. The use of the wrong Coulomb correction for $M1$ scattering affects G_M by more than 1%. The Coulomb correction for the magnetic part does not affect the charge form factor at $q^2 >$ 0.011 fm^{-2}, namely, the influence is 0.01% or less. However, the cross section calculated using Eq. (12) is slightly nonlinear against the polarization ε. The influence of the nonlinearity appears at low-momentum transfers. Therefore, the use of correct Coulomb corrections is necessary to obtain the charge radius from the gradient of G_E in these low-momentum transfers. Furthermore, the determination of the magnetic radius and the verification of the Coulomb corrections are expected in the experiments using the low-energy electron beams in addition to the cross-check of the proton charge radius.

The approximations of the second-order Coulomb corrections analyzed in the present Rapid Communication are summarized below.

(1) The corrections are in an order of $(z\alpha)$, and they are good for the proton but not sufficient for heavy nuclei.

(2) The Coulomb distortion of electron waves is calculated using the one-photon exchange potential induced by a pointlike nucleus. A numerical calculation is necessary if the extended nucleus is considered.

(3) The recoil of the nucleus and the electron mass are ignored. In the phase-shift calculation, the effects are partly recovered by the calculation in the center-of-mass system [24].

(4) Influence of the Coulomb corrections is 3% at most. It is small, but the nonlinearity induced from the corrections makes large errors when the Rosenbluth fit is made using Eq. (2) in a limited range of polarization ϵ.

The precise determination of the charge and magnetic radii of the proton is a challenging project. It needs high accuracy that has never tried in this field. In the present investigation, it is unclear which of the problems of the phase-shift calculation mentioned above or the accuracy of the Feshbach correction accounts for a small difference shown in Fig. 3. In order to accomplish our goal, everything including formulas used for analysis must be considered if it is equal to the usage.

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