Neutron spin dynamics in polarized targets

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We present the neutron elastic scattering amplitude for an arbitrarily polarized target in an irreducible spherical-tensor representation. The general approach for the description of neutron spin dynamics for propagation through the medium with an arbitrary polarization is discussed in relation to the search for a time-reversal invariance violation in neutron scattering.

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I. INTRODUCTION

With the opportunity to measure time-reversal invariance-violating (TRIV) effects in nuclear reactions by the transmission of polarized neutrons through a polarized target [1–5], it is important to have a complete description of the propagation of neutron spin through an arbitrarily polarized target. These TRIV effects are proportional to the vector polarization of a nuclear target. However, in a general case, to describe the polarization of a target with spin I requires $2I$ tensor momenta [6–8]. Therefore, only for $I = 1/2$ it is sufficient to consider vector polarization (the first-rank tensor) for a complete description of the target polarization. Despite the fact that the recent proposals for searches of TRIV effects in neutron-nucleus scattering (see, for example, Ref. [5] and references therein) demonstrated the existence of a class of experiments that are free from false asymmetries, to design the experiment and to control the possible systematic effects one has to have a detailed description of neutron spin dynamics in targets with arbitrary polarization. The propagation of polarized neutrons through a polarized target in relation to TRIV experiments has been studied in many papers (see, for example, Refs. [5,9–14] and references therein); however, these studies have been done with the focus on the case of a vector-polarized target. However, even for a 100% vector-polarized target with spin $I > 1/2$, the higher-rank tensor polarizations may coexist and be rather large. Therefore, even if these higher-order polarizations cannot mimic TRIV effects, they can change the neutron spin dynamics, which can lead to a suppression of TRIV observables. Thus, the open and important question for the design of new experiments and for future data analysis is how these high-rank tensor polarizations affect neutron spin dynamics inside the polarized target.

In this article we give a systematic approach for the description of low-energy neutron spin propagation in the target with arbitrary polarization using irreducible spherical-tensor representation for target polarizations. Also, as examples, we present the complete expressions for neutron scattering amplitude in irreducible spherical-tensor representation for the case of a nuclear target with spin $I = 7/2$ and the detailed analysis of neutron scattering on ^{139}La. Finally, we apply the developed approach for a general analysis of neutron spin rotation in a target with arbitrary polarization.

II. SCATTERING AMPLITUDE IN IRREDUCIBLE SPHERICAL TENSOR REPRESENTATION

A general expression for the forward elastic scattering amplitude with μ and M, projections of neutron ($s = 1/2$) and target spins on a quantization axis z, can be written as

$$f_{MM'} = \frac{i\pi}{2k} \sum_{JII'Sm_sS'm'_s} Y_{Lm_L}(\theta,\phi)\langle s\mu' IM'|S'm'_s\rangle\langle Sm_s|s\mu IM\rangle \times \langle S'l'\alpha'|R^J|Sl\alpha\rangle(-1)^{J+S'+l'+l}(2J+1) \times \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2S+1)}} \times \langle l0l'0|L0\rangle\langle Lm_LS'm'_s|Sm_s\rangle \begin{Bmatrix} l' & l & L \\ S & S' & J \end{Bmatrix}, \quad (1)$$

where primed parameters correspond to the outgoing channel, and the angles θ and ϕ describe a direction of the neutron momentum $\vec{k}$. The matrix $\hat{R}$ is related to the scattering matrix $\hat{\mathbb{S}}$ as $\hat{R} = \hat{1} - \hat{\mathbb{S}}$, which in the integral of motion representation [15] is

$$\langle S'l'\alpha'|\mathbb{S}^J|Sl\alpha\rangle\delta_{JJ'}\delta_{MM'}\delta(E'-E), \quad (2)$$

where J and M are the total spin and its projection, S is the channel spin, l is the orbital momentum, and α represents the other internal quantum numbers.

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Then, for arbitrary target polarization described by the polarization density matrix $\rho_{MM'}$, the scattering amplitude can be calculated as

$$f = \mathrm{Tr}(f_{MM'}\rho_{MM'}). \tag{3}$$

To describe polarization of the tensor-polarized target, it is convenient to use the expansion [7] of the density polarization matrix in terms of the statistical tensors $t_{q\kappa}$:

$$\rho_{MM'} = \sum_{q\kappa}\sqrt{\frac{2q+1}{2I+1}}\langle IMq\kappa|IM'\rangle t_{q\kappa}. \tag{4}$$

In this expression each tensor $t_{q\kappa}$ corresponds to tensor polarization of the target of rank q, and thus for the case of the unpolarized target all statistical tensors vanish except t_{00}.

For the choice of the spin direction of the target along the quantization axis z, the elastic scattering amplitude (5) can be presented as an expansion in terms of spherical-tensor polarizations t_{q0} with a corresponding weight of w_q:

$$\begin{aligned} f = \frac{i\pi}{2k}\sum_{q=0}^{2I} w_q t_{q0}\sqrt{\frac{2q+1}{2I+1}}\Bigg[\sum_{Jll'Sm_sS'm_s'} &Y_{Lm_L}(\theta,\phi)\langle s\mu' IM|S'm_s'\rangle\,\langle Sm_s|s\mu IM\rangle \\ &\times \langle IMq0|IM\rangle\langle S'l'\alpha'|R^J|Sl\alpha\rangle(-1)^{J+S'+l'+l}(2J+1)\sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2S+1)}} \\ &\times \langle l0l'0|L0\rangle\langle Lm_LS'm_s'|Sm_s\rangle\begin{Bmatrix} l' & l & L \\ S & S' & J\end{Bmatrix}\Bigg]. \end{aligned} \tag{5}$$

For a description of neutron polarization it is convenient to introduce the following function:

$$\begin{aligned} N(\tilde{x},\tilde{y},S,S',M,M') = \frac{1}{|\tilde{x}|^2+|\tilde{y}|^2}\Bigg(&|\tilde{x}|^2\left\langle\frac{1}{2}\frac{1}{2}IM\middle|SM+\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\frac{1}{2}IM'\middle|S'M+\frac{1}{2}\right\rangle\left\langle L0S'M'+\frac{1}{2}\middle|SM+\frac{1}{2}\right\rangle\delta_{m_L,0} \\ &+\tilde{x}\tilde{y}^*\left\langle\frac{1}{2}-\frac{1}{2}IM\middle|SM-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\frac{1}{2}IM'\middle|S'M'+\frac{1}{2}\right\rangle\left\langle L1S'M'-\frac{1}{2}\middle|SM+\frac{1}{2}\right\rangle\delta_{m_L,1} \\ &+\tilde{x}^*\tilde{y}\left\langle\frac{1}{2}-\frac{1}{2}IM\middle|SM-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\frac{1}{2}IM'\middle|S'M+\frac{1}{2}\right\rangle\left\langle L-1S'M'+\frac{1}{2}\middle|SM-\frac{1}{2}\right\rangle\delta_{m_L,-1} \\ &+|\tilde{y}|^2\left\langle\frac{1}{2}-\frac{1}{2}IM\middle|SM-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}-\frac{1}{2}IM'\middle|S'M-\frac{1}{2}\right\rangle\left\langle L0S'M'-\frac{1}{2}\middle|SM-\frac{1}{2}\right\rangle\delta_{m_L,0}\Bigg), \end{aligned} \tag{6}$$

where $\tilde{x}$ and $\tilde{y}$ are the components of neutron spinor $(\begin{smallmatrix}\tilde{x}\\ \tilde{y}\end{smallmatrix})$. To describe an arbitrary orientation of neutron spin in spherical coordinates, one can choose

$$\tilde{x} = \cos(\beta/2)e^{-i\alpha/2}, \quad \tilde{y} = \sin(\beta/2)e^{i\alpha/2}, \tag{7}$$

where β and α are polar and azimuth angles for spin direction relative to the quantization axis z. Then, the elastic scattering amplitude for polarized neutrons and for the spherical-tensor polarization P_q^I [as defined in Eq. (A6)] of the target is

$$\begin{aligned} f(\tilde{x},\tilde{y}) = \frac{i\pi}{2k}\sum_{q=0}^{2I}\frac{P_q^I}{c_q^I}\tilde{\tau}_{q0}\sqrt{2q+1}\Bigg[\sum_{JMll'SS'} &Y_{Lm_L}(\theta,\phi)N(\tilde{x},\tilde{y},S,S',M,M) \\ &\times\langle IMq0|IM\rangle\langle S'l'\alpha'|R^J|Sl\alpha\rangle(-1)^{J+S'+l'+l}(2J+1)\sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2S+1)}} \\ &\times\langle l0l'0|L0\rangle\begin{Bmatrix} l' & l & L \\ S & S' & J\end{Bmatrix}\Bigg] = \sum_{q=0}^{2I}P_q^I f_q, \end{aligned} \tag{8}$$

with c_q^I defined by Eq. (A11).

It should be noted that, usually in nuclear physics, spherical tensors (which we define as $\widetilde{\tau}_{\kappa q}$ in Appendix A) have different normalization [6,8,16] compare to t_{qk} in Eq. (4), defined in Ref. [7]. Therefore for the sake of convenience we present further expressions for amplitudes and angular coefficients in terms of expansions in $\widetilde{\tau}_{q0}$.

To describe the amplitude for arbitrary target's and neutron's spin orientations, we use the convention that the direction of the target spin is always parallel to the z axis, and the neutron momentum belongs to the y-z plane, which implies that the angle $\phi = \pi/2$ and the neutron momentum direction is described by the angle θ. Then, assuming that $\vec{\sigma}$, $\vec{I}$, and $\hat{k} = \vec{k}/|\vec{k}|$ are unit vectors in the corresponding directions, we can write the following relations:

$$
\begin{aligned}
&(\hat{k}\cdot\vec{I}) = \cos\theta, \quad (\vec{\sigma}\cdot\vec{I}) = \cos\beta, \quad (\vec{\sigma}\cdot\hat{k}) = \cos\theta\cos\beta + \sin\theta\sin\beta\sin\alpha,\\
&[\hat{k}\times\vec{I}] = \sin\theta, \quad (\vec{\sigma}\cdot[\hat{k}\times\vec{I}]) = \cos\alpha\sin\theta\sin\beta. \qquad (9)
\end{aligned}
$$

Using these relations one can expand the scattering amplitude (8) in terms of irreducible tensors constructed from the products of the neutron spin $\vec{\sigma}$, the target spin $\vec{I}$, and the neutron momentum $\vec{k}$. In general, only the first order of neutron spin and the powers of the target spin up to the value of $(2I)$ can contribute in this expansion; the power of the neutron momentum is not bounded. However, because we consider scattering of low-energy neutrons with only contributions from s and p partial waves (resonances), we can restrict ourselves to terms that have at most two powers of $\vec{k}$. Then, the amplitude (8) can be written as

$$
\begin{aligned}
f &= A' + B'(\vec{\sigma}\cdot\vec{I}) + C'(\vec{\sigma}\cdot\hat{k}) + D'(\vec{\sigma}\cdot[\hat{k}\times\vec{I}]) + H'(\hat{k}\cdot\vec{I}) + K'(\vec{\sigma}\cdot\hat{k})(\hat{k}\cdot\vec{I})\\
&\quad + E'\big[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \tfrac{1}{3}(\hat{k}\cdot\hat{k})(\vec{I}\cdot\vec{I})\big] + F'\big[(\vec{\sigma}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \tfrac{1}{3}(\vec{\sigma}\cdot\hat{k})(\vec{I}\cdot\vec{I})\big]\\
&\quad + G'(\vec{\sigma}\cdot[\hat{k}\times\vec{I}])(\hat{k}\cdot\vec{I}) + B_3'(\vec{\sigma}\cdot\vec{I})\big[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \tfrac{1}{3}(\hat{k}\cdot\hat{k})(\vec{I}\cdot\vec{I})\big] + \cdots, \qquad (10)
\end{aligned}
$$

where the first line contains target spin-independent terms (A', C') and terms proportional to the vector polarization of the target (B', D', H', K'), while the second and the third lines contain E', F', and G' terms, which are proportional to the tensor polarization of the second rank, and the term B_3', which is proportional to the third rank of the target polarization. The terms C', D', H', and F' represent P-odd of the amplitude, and D' and G' terms violate the time-reversal invariance. (The expressions for these coefficients for specific values of the target spins I and their projections M on the axis z are given in Appendix B.)

For parity-conserving parts of the amplitude (8) and, as a consequence, of the amplitude (10), the matrix elements for slow neutrons can be written in the Breit-Wigner resonance approximation as [3]

$$
\begin{aligned}
F^J_{S'SI} &\equiv \langle S'lK|R^J|SlK\rangle\\
&= \sum_K i\frac{\sqrt{\Gamma^n_{l_K}(S'_K)}\sqrt{\Gamma^n_{l_K}(S_K)}}{E - E_K + i\Gamma_K/2} e^{i[\delta_{l_K}(S'_K)+\delta_{l_K}(S_K)]}\\
&\quad - 2ie^{i\delta_{l_K}(S_KS'_K)}\sin\delta_{l_K}(S_KS'_K), \qquad (11)
\end{aligned}
$$

where E_K, Γ_K, and $\Gamma^n_{l_K}$ are the energy, the total width, and the partial neutron width of the Kth nuclear compound resonance; E is the neutron energy; and δ_{l_K} is the potential scattering phase shift. For p-wave resonances we keep only the resonance term, because for low-energy neutrons $\delta_l \sim (kR_0)^{2l+1}$ (where R_0 is nucleus radius), and, as a consequence, the contribution from p-wave potential scattering is negligible.

The matrix elements for parity violating (PV) and TRIV interactions for slow neutrons can be written in the Breit-Wigner resonance approximation with one s-wave resonance and one p-wave resonance as [3,17]

$$
\begin{aligned}
\big(F^J_{S'l'Sl}\big) &\equiv \langle S'l'|R^J|Sl\rangle\\
&= \frac{\sqrt{\Gamma^n_{l'}(S')}(-iv + w)\sqrt{\Gamma^n_l(S)}}{(E - E_l + i\Gamma_l/2)(E - E_{l'} + i\Gamma_{l'}/2)}\\
&\quad \times e^{i[\delta_{l'}(S')+\delta_l(S)]}, \qquad (12)
\end{aligned}
$$

where $l \neq l'$, and v and w are real and imaginary parts of the matrix elements for PV and TRIV mixing between s- and p-wave compound resonances,

$$
v + iw = -\langle\phi_s|V_{\not{P}} + V_{\not{P}\not{T}}|\phi_p\rangle, \qquad (13)
$$

due to $V_{\not{P}}$ (PV) and $V_{\not{P}\not{T}}$ (TRIV) interactions.

In general, the matrix element Eq. (12) has a sum over a number of close resonances, similar to the sum in Eq. (11). However, we are usually interested in a description of symmetry-violating effects in the vicinity of p-wave resonances. In that case, only a contribution from that particular p-wave resonance is important; therefore we can use the two-resonance approximation (12), which resulted from a mixture of the nearest s- and p-wave resonances. It should be noted that, in general, a p-wave resonance can be mixed with two or more s-wave resonances. In that case, Eq. (12) should be modified to the sum of amplitudes over all mixing s-wave resonances. Fortunately, the two-resonance approximation has been proven to be good enough to describe practically all observed PV effects in neutron scattering (see Refs. [18] and references therein).

Because we are interested in applications of our results for the analysis of TRIV effects that are proportional to vector polarization of the target($\vec{\sigma}\cdot[\vec{k}\times\vec{I}]$) correlation), we choose the initial geometry where vector polarization of the target has the simplest form: the neutron spin $\vec{\sigma}$ is parallel to the x axis, the target spin $\vec{I}$ is parallel to the z axis (the quantization axis), and the neutron momentum $\vec{k}$ is going in the direction of the

y axis. Therefore, for our choice of the coordinate system, the angle's values in Eq. (8) are $\theta = \pi/2$ and $\phi = \pi/2$.

III. ANALYSIS OF THE SCATTERING WITH THE TARGET SPIN $I = 7/2$

To understand the structure of the amplitude (10), let us consider an example for a scattering on the target with spin $I = 7/2$. In general, to completely describe polarization of the nucleus with a spin I, we need a set of spherical tensors up to the rank of $q = 2I$, which results in $q = 7$ for $I = 7/2$. However, for low-energy neutron scattering, the tensor structure of the amplitude is much simpler. This is because only s- and p-wave resonances are important, and as a consequence, one cannot have tensor terms in the amplitude built from a momentum vector with a rank higher then 2 (for discussion of the possible contributions from d-wave resonances, see Appendix C). This results in a constraint that the rank of the target spin tensor has a maximum value of $q = 3$.

Evaluation of Eq. (8) for $q = 0$, $q = 1$, $q = 2$, and $q = 3$ with target spin 7/2 (see Appendix D) leads to results that can be summarized, in terms of a linear combination of the tensors already listed in Eq. (10), as

$$\begin{aligned} f_{7/2} = {} & P_0[A' + C'(\vec{\sigma}\cdot\hat{k})] + P_1\{B'(\vec{\sigma}\cdot\vec{I}) + D'(\vec{\sigma}\cdot[\hat{k}\times\vec{I}]) + H'(\hat{k}\cdot\vec{I}) + K'(\vec{\sigma}\cdot\hat{k})(\hat{k}\cdot\vec{I})\} \\ & + P_2\{E'[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \tfrac{1}{3}(\hat{k}\cdot\hat{k})(\vec{I}\cdot\vec{I})] + F'[(\vec{\sigma}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \tfrac{1}{3}(\vec{\sigma}\cdot\hat{k})(\vec{I}\cdot\vec{I})] \\ & + G'(\vec{\sigma}\cdot[\hat{k}\times\vec{I}])(\hat{k}\cdot\vec{I})\} \\ & + P_3\{B_3'((\vec{\sigma}\cdot\vec{I})[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \tfrac{1}{3}(\hat{k}\cdot\hat{k})(\vec{I}\cdot\vec{I})] + \tfrac{2}{5}(\hat{k}\cdot\vec{I})[(\vec{\sigma}\cdot\vec{I})(\hat{k}\cdot\vec{I})\tfrac{1}{3} + (\vec{\sigma}\cdot\hat{k})(\vec{I}\cdot\vec{I})] \\ & - \tfrac{4}{5}(\hat{k}\cdot\vec{I})[(\vec{\sigma}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \tfrac{1}{3}(\vec{\sigma}\cdot\hat{k})(\vec{I}\cdot\vec{I})])\}, \end{aligned} \tag{14}$$

where the primed coefficients are defined by the following expressions:

$$A' = \frac{i}{32k}(7\langle 3,0|R^3|3,0\rangle + 9\langle 4,0|R^4|4,0\rangle + 7\langle 3,1|R^3|3,1\rangle + 9\langle 3,1|R^4|3,1\rangle + 7\langle 4,1|R^3|4,1\rangle + 9\langle 4,1|R^4|4,1\rangle), \tag{15}$$

$$\begin{aligned} C' = {} & \frac{i}{64k}(7(\langle 3,0|R^3|3,1\rangle + \langle 3,1|R^3|3,0\rangle) - 7\sqrt{3}(\langle 3,0|R^3|4,1\rangle + \langle 4,1|R^3|3,0\rangle) \\ & + 3\sqrt{21}(\langle 4,0|R^4|3,1\rangle + \langle 3,1|R^4|4,0\rangle) - 3\sqrt{15}(\langle 4,0|R^4|4,1\rangle + \langle 4,1|R^4|4,0\rangle)), \end{aligned} \tag{16}$$

$$\begin{aligned} B' = {} & -\frac{i}{32k}\bigg(7\langle 3,0|R^3|3,0\rangle - \langle 4,0|R^4|4,0\rangle + \frac{21}{4}\langle 3,1|R^3|3,1\rangle + \frac{21}{4\sqrt{3}}(\langle 3,1|R^3|4,1\rangle + \langle 4,1|R^3|3,1\rangle) \\ & - \frac{9}{20}\sqrt{35}(\langle 3,1|R^4|4,1\rangle + \langle 4,1|R^4|3,1\rangle) - \frac{91}{12}\langle 4,1|R^3|4,1\rangle + \frac{39}{4}\langle 3,1|R^4|3,1\rangle - \frac{63}{20}\langle 4,1|R^4|4,1\rangle\bigg), \end{aligned} \tag{17}$$

$$D' = \frac{1}{16k}\bigg(\frac{7}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle - \langle 4,1|R^3|3,0\rangle) + \sqrt{21}(\langle 4,0|R^4|3,1\rangle - \langle 3,1|R^4|4,0\rangle)\bigg), \tag{18}$$

$$\begin{aligned} H' = {} & -\frac{i}{64k}\bigg(21(\langle 3,0|R^3|3,1\rangle + \langle 3,1|R^3|3,0\rangle) - \frac{7}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle + \langle 4,1|R^3|3,0\rangle) \\ & + \sqrt{21}(\langle 4,0|R^4|3,1\rangle + \langle 3,1|R^4|4,0\rangle) + 7\sqrt{15}(\langle 4,0|R^4|4,1\rangle + \langle 4,1|R^4|4,0\rangle)\bigg), \end{aligned} \tag{19}$$

$$\begin{aligned} K' = {} & -\frac{3i}{128k}\bigg(7\langle 3,1|R^3|3,1\rangle - 7\sqrt{3}(\langle 3,1|R^3|4,1\rangle + \langle 4,1|R^3|3,1\rangle) \\ & + \frac{77}{9\sqrt{3}}\langle 4,1|R^3|4,1\rangle - 3\langle 3,1|R^4|3,1\rangle + \frac{9}{5}\sqrt{35}(\langle 3,1|R^4|4,1\rangle + \langle 4,1|R^4|3,1\rangle) - \frac{77}{5}\langle 4,1|R^4|4,1\rangle\bigg), \end{aligned} \tag{20}$$

$$\begin{aligned} E' = {} & \frac{i9}{256k}\bigg(\frac{35}{3}\langle 3,1|R^3|3,1\rangle - \frac{7}{\sqrt{3}}(\langle 3,1|R^3|4,1\rangle + \langle 4,1|R^3|3,1\rangle) - \frac{77}{9}\langle 4,1|R^3|4,1\rangle + \frac{3}{5}\sqrt{35}(\langle 3,1|R^4|4,1\rangle \\ & + \langle 4,1|R^4|3,1\rangle) - 5\langle 3,1|R^4|3,1\rangle + \frac{77}{5}\langle 4,1|R^4|4,1\rangle\bigg), \end{aligned} \tag{21}$$

$$\begin{aligned} F' = {} & \frac{3i}{320k}\bigg(35(\langle 3,0|R^3|3,1\rangle + \langle 3,1|R^3|3,0\rangle) + \frac{35}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle + \langle 4,1|R^3|3,0\rangle) \\ & - 5\sqrt{21}(\langle 4,0|R^4|3,1\rangle + \langle 3,1|R^4|4,0\rangle) - 7\sqrt{15}(\langle 4,0|R^4|4,1\rangle + \langle 4,1|R^4|4,0\rangle)\bigg), \end{aligned} \tag{22}$$

$$G' = -\frac{3}{32\sqrt{5}k}\left(7\sqrt{\frac{5}{3}}(\langle 3,1|R^3|4,1\rangle - \langle 4,1|R^3|3,1\rangle) - 3\sqrt{7}(\langle 3,1|R^4|4,1\rangle - \langle 4,1|R^4|3,1\rangle)\right), \tag{23}$$

$$B_3' = \frac{9i}{256k}\left(7\langle 3,1|R^3|3,1\rangle + (\langle 3,1|R^3|4,1\rangle + \frac{7}{\sqrt{3}}\langle 4,1|R^3|3,1\rangle) + \frac{7}{3}\langle 4,1|R^3|4,1\rangle - 3\sqrt{\frac{7}{5}}(\langle 3,1|R^4|4,1\rangle + \langle 4,1|R^4|3,1\rangle) - 3\langle 3,1|R^4|3,1\rangle - \frac{21}{5}\langle 4,1|R^4|4,1\rangle\right). \tag{24}$$

Because we use a rather particular spin value $I = 7/2$, the above expressions present a pattern for a general amplitude structure.

IV. SCATTERING OF POLARIZED NEUTRONS ON POLARIZED ^{139}La TARGET

Let us apply the obtained results for the case of ^{139}La, which has a spin $I = 7/2$, and consider scattering of polarized neutrons on the polarized ^{139}La target in the vicinity of the p-wave resonance $E_p = 0.734$ eV. The resonance structure [19] of ^{139}La shows that there are two nearest s-wave resonances with $E_{s0} = -48.63$ eV and $E_{s1} = 72.3$ eV, and with spins $J = 4$ and $J = 3$, correspondingly. Because the p-wave resonance has spin $J = 4$, it can be mixed only with the negative-energy $s0$ resonance. Therefore, we can neglect symmetry-violating amplitudes with a total spin $J = 3$. Thus, there are only four parity-violating amplitudes with a total spin $J = 4$:

$$\begin{aligned}
\langle 40|R^4|31\rangle &= \frac{x_S\sqrt{\Gamma^n_{s0}}(-iv + w)\sqrt{\Gamma^n_p}}{(E - E_p + i\Gamma_p/2)(E - E_{s0} + i\Gamma_{s0}/2)}e^{i\delta_{s0}},\\
\langle 40|R^4|41\rangle &= \frac{y_S\sqrt{\Gamma^n_{s0}}(-iv + w)\sqrt{\Gamma^n_p}}{(E - E_p + i\Gamma_p/2)(E - E_{s0} + i\Gamma_{s0}/2)}e^{i\delta_{s0}},\\
\langle 31|R^4|40\rangle &= \frac{x_S\sqrt{\Gamma^n_{s0}}(-iv - w)\sqrt{\Gamma^n_p}}{(E - E_p + i\Gamma_p/2)(E - E_{s0} + i\Gamma_{s0}/2)}e^{-i\delta_{s0}},\\
\langle 41|R^4|40\rangle &= \frac{y_S\sqrt{\Gamma^n_{s0}}(-iv - w)\sqrt{\Gamma^n_p}}{(E - E_p + i\Gamma_p/2)(E - E_{s0} + i\Gamma_{s0}/2)}e^{-i\delta_{s0}}.
\end{aligned} \tag{25}$$

There are six parity-conserving amplitudes,

$$\begin{aligned}
\langle 40|R^4|40\rangle &= i\frac{\Gamma^n_{s0}}{E - E_{s0} + i\Gamma_{s0}/2}e^{2i\delta_{s0}} - 2ie^{i\delta_{s0}}\sin\delta_{s0},\\
\langle 30|R^3|30\rangle &= i\frac{\Gamma^n_{s1}}{E - E_{s1} + i\Gamma_{s1}/2}e^{2i\delta_{s1}} - 2ie^{i\delta_{s1}}\sin\delta_{s1},\\
\langle 31|R^4|31\rangle &= i\frac{x_S^2\Gamma^n_p}{E - E_p + i\Gamma_p/2},\\
\langle 41|R^4|41\rangle &= i\frac{y_S^2\Gamma^n_p}{E - E_p + i\Gamma_p/2},\\
\langle 31|R^4|41\rangle &= i\frac{x_S y_S\Gamma^n_p}{E - E_p + i\Gamma_p/2},\\
\langle 41|R^4|31\rangle &= i\frac{x_S y_S\Gamma^n_p}{E - E_p + i\Gamma_p/2},
\end{aligned} \tag{26}$$

that make main contributions in the vicinity of the p-wave resonance. Also, for slow neutrons we can neglect exponentials with phases in the above expressions.

For the case of phenomenological TRIV and parity-conserving interactions (TVPC), corresponding to the term G', there are four possible amplitudes for the ^{139}La target that contribute to Eq. (23) by the two matrix element differences $(\langle 3,2|R^4_T|4,0\rangle - \langle 4,0|R^4_T|3,2\rangle)$ and $(\langle 3,1|R^4_T|4,1\rangle - \langle 4,1|R^4_T|3,1\rangle)$. Assuming only contributions from a compound resonance mixing (see detailed discussions in Refs. [20–22]), one can write these differences as follows:

$$\begin{aligned}
&(\langle 3,1|R^4_T|4,1\rangle - \langle 4,1|R^4_T|3,1\rangle)\\
&\quad= \frac{iv^{pp}_T\{[\Gamma(3)^n_{p1}\Gamma(4)^n_{p2}]^{1/2} - [\Gamma(4)^n_{p1}\Gamma(3)^n_{p2}]^{1/2}\}}{(E - E_{p1} + i\Gamma_{p1}/2)(E - E_{p2} + i\Gamma_{p2}/2)},\\
&(\langle 3,2|R^4_T|4,0\rangle - \langle 4,0|R^4_T|3,2\rangle)\\
&\quad= \frac{iv^{sd}_T\{[\Gamma(3)^n_d\Gamma(4)^n_s]^{1/2} - [\Gamma(4)^n_d\Gamma(3)^n_s]^{1/2}\}}{(E - E_d + i\Gamma_d/2)(E - E_s + i\Gamma_s/2)}e^{i\delta_s},
\end{aligned} \tag{27}$$

where v^{pp}_T and v^{sd}_T are phenomenological TVPC matrix elements [21], and $\Gamma(S)^n_k$ are partial neutron decay widths for k's resonance (where k is s, $p1$, $p2$, or d) corresponding to the spin channel S. We can see from these expressions that, for the existence of an already very small coefficient G' (due to a naturally small value of phenomenological TVPC interactions [23]), one has to have either an additional p-wave resonance or an additional d-wave resonance. Because we are interested in the region in the vicinity of the p-wave resonance, the possible contributions from the s-d mixture can be neglected; therefore, only the first difference of matrix elements in the above expression can be taken into account. For the completeness of the description of the neutron propagation through the polarized target we provide the expression for the G' coefficient; however, we neglect TVPC correlations in the further analysis. (For experimental constraint of this term from neutron scattering on aligned holmium, see Refs. [24,25].)

Then, the coefficients in the amplitude (14) for ^{139}La are the following:

$$A'_{\rm La} = \frac{-1}{32k}\left(7\frac{\Gamma^n_{s1}}{E - E_{s1} + i\Gamma_{s1}/2} + 9\frac{\Gamma^n_{s0}}{E - E_{s0} + i\Gamma_{s0}/2} + 9\frac{\Gamma^n_p}{E - E_p + i\Gamma_p/2}\right) + \frac{1}{16}(9a_{s0} + 7a_{s1}), \tag{28}$$

$$B'_{\rm La} = \frac{-1}{32k}\left[7\frac{\Gamma^n_{s0}}{E-E_{s0}+i\Gamma_{s0}/2} - 7\frac{\Gamma^n_{s1}}{E-E_{s1}+i\Gamma_{s1}/2} + \frac{\Gamma^n_p}{E-E_p+i\Gamma_p/2}\left(-\frac{39}{4}x_S^2+\frac{9}{2}\sqrt{\frac{7}{5}}x_Sy_S+\frac{63}{20}y_S^2\right)\right] + \frac{7}{16}(a_{s0}-a_{s1}), \tag{29}$$

$$C'_{\rm La} = \frac{3\sqrt{3}}{32k}\frac{\sqrt{\Gamma^n_{s0}}v\sqrt{\Gamma^n_p}}{(E-E_p+i\Gamma_p/2)(E-E_{s0}+i\Gamma_{s0}/2)} \times (\sqrt{7}x_S-\sqrt{5}y_S), \tag{30}$$

$$D'_{\rm La} = \frac{\sqrt{21}}{8k}\frac{x_S\sqrt{\Gamma^n_{s0}}w\sqrt{\Gamma^n_p}}{(E-E_p+i\Gamma_p/2)(E-E_{s0}+i\Gamma_{s0}/2)}, \tag{31}$$

$$H'_{\rm La} = -\frac{3\sqrt{7}}{32k}\frac{\sqrt{\Gamma^n_{s0}}v\sqrt{\Gamma^n_p}}{(E-E_p+i\Gamma_p/2)(E-E_{s0}+i\Gamma_{s0}/2)} \times (x_S+\sqrt{35}y_S), \tag{32}$$

$$K'_{\rm La} = -\frac{\sqrt{21}}{128k}\frac{\Gamma^n_p}{E-E_p+i\Gamma_p/2} \times \left(3\sqrt{\frac{3}{7}}x_S^2-18\sqrt{\frac{3}{5}}x_Sy_S+\frac{11}{5}\sqrt{21}y_S^2\right), \tag{33}$$

$$E'_{\rm La} = \frac{3\sqrt{21}}{256k}\frac{\Gamma^n_p}{E-E_p+i\Gamma_p/2} \times \left(5\sqrt{\frac{3}{7}}x_S^2-6\sqrt{\frac{3}{5}}x_Sy_S-\frac{11}{5}\sqrt{21}y_S^2\right), \tag{34}$$

$$F'_{\rm La} = -\frac{3\sqrt{21}}{32k}\frac{\sqrt{\Gamma^n_{s0}}v\sqrt{\Gamma^n_p}}{(E-E_p+i\Gamma_p/2)(E-E_{s0}+i\Gamma_{s0}/2)} \times \left(x_S+\sqrt{\frac{7}{5}}y_S\right), \tag{35}$$

$$G'_{\rm La} = \frac{9\sqrt{7}i}{32\sqrt{5}k}\frac{\sqrt{\Gamma^n_{p1}}v_T^{pp}\sqrt{\Gamma^n_p}}{(E-E_p+i\Gamma_p/2)(E-E_{p1}+i\Gamma_{p1}/2)} \times (x_Sy_{S1}-x_{S1}y_S), \tag{36}$$

$$(B'_3)_{\rm La} = \frac{189}{256k}\frac{\Gamma^n_p}{E-E_p+i\Gamma_p/2}\left(\frac{1}{7}x_S^2+\frac{2}{\sqrt{35}}x_Sy_S+\frac{1}{5}y_S^2\right), \tag{37}$$

where a_{s0} and a_{s1} are neutron scattering lengths with the total spins $J = 4$ and $J = 3$, correspondingly.

We can see that not all these correlations are equally important for the analysis of neutron spin propagation through the polarized target for experiments to measure the D' correlation. Correlations A' and B' are related to strong spin-independent and spin-dependent backgrounds, correspondingly, and C' is related to a weak spin-dependent background. However, despite the fact that the values of the H' and K' coefficients are about the same order of magnitude as C', they are proportional to the $(\vec{k}\cdot\vec{I})$ term. Therefore, they can be well controlled by a precise alignment of the target spin to be perpendicular to the direction of the neutron beam. The coefficients E' and F' are also of the order of magnitude of C'; however, they are proportional to the second order of the tensor polarization P_2 and, therefore, can be minimized by creating a pure vector polarization. Term G' is very small as is discussed above, and it is also proportional to the tensor polarization P_2. Moreover, G' is proportional to the $(\vec{k}\cdot\vec{I})$ product, which gives an additional way to suppress it by an alignment of the target spin. Finally, B'_3, being of the same order as E', is proportional to the third order of the tensor polarization P_3, which usually is small.

V. NEUTRON SPIN ROTATION

Using expressions for the scattering amplitude f [see, for example, Eqs. (10), (D2), (D4), (D6), (D8), and (14)], we can describe the transmission of polarized neutrons through polarized medium and in the external magnetic field $\vec{B}$ by Schrödinger's equation (see, for example, Refs. [10,11,26] and references therein) with the following effective Hamiltonian (Fermi potential):

$$H = -\frac{2\pi\hbar^2}{m_n}Nf - \frac{\mu}{2}(\vec{\sigma}\cdot\vec{B}), \tag{38}$$

where m_n is the neutron mass, N is the number of scattering centers per unit volume, and μ is the neutron magnetic moment. Then the evolution operator that relates the initial neutron spinor to the spinor at the distance y is

$$U = e^{-i\frac{Hy}{v\hbar}}, \tag{39}$$

where v is the neutron velocity. Following the approach of Stodolsky [10], by representing the amplitude in the form of a 2×2 matrix in neutron-spin space as $f = a + (\vec{\sigma}\cdot\vec{b})$ and using the formula $\exp(i\vec{\sigma}\cdot\vec{b}) = \cos b + i(\vec{\sigma}\cdot\vec{b})(\sin b)/|b|$, we can calculate coefficients $A(y)$, $B(y)$, etc., which correspond to A', B', etc., in Eqs. (10) and (14) at $y = 0$. For the case of $I \geqslant 3/2$, and in particular for $I = 7/2$, this parametrization leads to

$$a = P_0A' + P_1H'(\hat{k}\cdot\vec{I}) + P_2E'\left[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I}) - \frac{1}{3}\right] - \frac{P_3B'_3}{3},$$

$$\begin{aligned} b_i = {} & P_0C'(\hat{k})_i - \mu_{\rm eff}(\vec{B})_i/2 + P_1\{B'I_i + D'[\hat{k}\times\vec{I}]_i + K'(\hat{k})_i(\hat{k}\cdot\vec{I})\} + P_2\left\{F'\left[I_i(\hat{k}\cdot\vec{I}) - \frac{1}{3}(\hat{k})_i\right]\right. \\ & \left. + G'[\hat{k}\times\vec{I}]_i(\hat{k}\cdot\vec{I})\right\} + P_3B'_3\left\{I_i[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I})] + \frac{2}{5}(\hat{k}\cdot\vec{I})\left[I_i(\hat{k}\cdot\vec{I})\frac{1}{3} + (\hat{k})_i\right]\right. \\ & \left. - \frac{4}{5}(\hat{k}\cdot\vec{I})\left[I_i(\hat{k}\cdot\vec{I}) - \frac{1}{3}(\hat{k})_i\right]\right\}, \end{aligned} \tag{40}$$

where $\mu_{\rm eff} = \mu m_n/(4\pi\hbar^2N)$. (For spin $I = 1$ we should assign $P_3 = 0$ in the above expression, and for the case of $I = 1/2$ we have both $P_2 = 0$ and $P_3 = 0$.)

One can see that, for the perfect alignment of the magnetic field along the z and the neutron momentum along the y axis, Eq. (40) transforms to

$$a = P_0 A' - \frac{P_2 E' + P_3 B'_3}{3}, \quad b_1 = P_1 D' [\hat{k} \times \vec{I}]_1,$$
$$b_2 = P_0 C' - \frac{P_2 F'}{3}, \quad b_3 = -\mu_{\rm eff}(\vec{B})_3/2 + P_1 B'. \quad (41)$$

The parameter a defers from its value in Ref. [10] by the second term, which can be ignored because it is suppressed by a factor $(kR)^2$ and numerically by small values of the tensor polarizations. The parameters b_1 and b_2 are exactly the same as for a "simple" spin amplitude in Ref. [10]. Therefore, the only important correction to the analyses in Refs. [10,26] of spin rotation in the target with $I > 1/2$ is the second term of b_2 (for the target with $I = 1/2$ only vector polarization exists, and $P_2 = P_3 = 0$).

Thus, for the analysis of neutron spin rotation we can use the results of Ref. [10] by changing the parameter b_2 according to Eq. (41) and by changing the coordinate system in Refs. [10,11] as follows: $x \to z$, $y \to x$, and $z \to y$. Alternatively, one can use the explicit expression for the evolution operator (39) as a function of the distance y, which can be written as

$$U(y) = \exp(i\alpha y)\left[\cos(\beta y) + i\frac{(\vec{\sigma}\cdot\vec{b})}{b}\sin(\beta y)\right], \quad (42)$$

where $\alpha = \gamma a$, $\beta_i = \gamma b_i$, $\gamma = 2\pi\hbar N/(m_n v)$, and $\beta = |\vec{\beta}|$.

It should be noted that the parameter a contributes to a general attenuation of the neutron beam, but the value of b is washing out the neutron spin component of the amplitude. Therefore, the smaller value means better sensitivity for spin-related observables (the TRIV effect, in particular). The largest part of b come from the b_3 component, which can be reduced by adjusting the external magnetic field (see Ref. [14] and references therein). The second large part comes from b_2, which also can be reduced by adjusting the value of the second rank of the target polarization P_2.

VI. CONCLUSIONS

We have developed a general systematic approach for the description of low-energy neutron spin propagation in the target with arbitrary polarization using irrecusable spherical-tensor representation for target polarizations. Applying this technique for the case of slow neutrons, when only s- and p-wave resonances are important, we demonstrated that for any value of the target spin only terms up to the third rank of tensor polarization are present in the scattering amplitude, in comparison to the $2I$th rank of tensor polarization in a general case. This is because the low power of momenta (up to a second rank tensor in our case) cannot be coupled to a higher rank of spin tensors. Therefore, even for targets with a large value of spin, the number of irreducible terms in the scattering amplitude is less than or equal to ten. For a target with spin $I = 1/2$ only six irreducible terms exist.

The analysis of neuron spin propagation shows that only seven terms from the ten possible terms are numerically important for the description of the neutron spin propagation, and this number can even be decreased to four for the target with $I = 1/2$.

The obtained results provide the recipe for how to extend the existent "conventional" approach for the description of neutron spin dynamics in a vector-polarized target in the case of an arbitrarily polarized target.

Another important observation is related to the fact that the second-rank tensor polarization of the target can be used for the cancellation of neutron spin rotation due to weak interaction. This, combined with the possible cancellation of the strong spin-spin interaction by an external magnetic field, gives the opportunity for essential increasing of the sensitivity in the search for TRIV.

Finally, using the obtained formalism, we presented the detailed description of neutron spin propagation in arbitrarily polarized ^{139}La, which is one of the candidates for the target for future TRIV experiments.

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APPENDIX A: SPIN POLARIZATION

To specify polarization in the spherical representation we use the statistical tensors $\tilde{\tau}_{\kappa q}$, which are defined as the expectation values of irreducible tensor spin operators:

$$\tilde{\tau}_{\kappa q} = \langle \tau_{\kappa q} \rangle. \quad (A1)$$

The spin operator corresponding to the spin j is defined [16] by

$$\left(\tau^j_{kq}\right)_{m'm} = \langle jm'|\tau^j_{kq}|jm\rangle = \sqrt{2k+1}\langle j\, m\, k\, q | j\, m'\rangle. \quad (A2)$$

It should be noted that the tensors $\tilde{\tau}_{\kappa q}$ have different normalization in comparison to the tensors $t_{\kappa q}$ in Eq. (4), which are defined as [7]

$$(t_{kq})_{m'm} = \sqrt{\frac{2k+1}{2j+1}}\langle j\, m\, k\, q | j\, m'\rangle. \quad (A3)$$

We define the population of each magnetic substate m of the spin $\boldsymbol{I}$ as

$$N_m = \langle Im|Im\rangle, \quad \sum_{m=-I}^{I} N_m = 1. \quad (A4)$$

Then we can calculate

$$\overline{\tau}^I_k = \sum_{q=-I}^{I} \langle Iq|\tau^I_{k0}|Iq\rangle, \quad (A5)$$

for k in the range of $1 \leqslant k \leqslant 2I$.

Because the vector polarization P^I_1 is commonly defined to be proportional to $\overline{\tau}^I_1$, we write

$$P^I_1 = c^I_1 \overline{\tau}^I_1, \quad (A6)$$

where the constant c_1^I is obtained from the requirement that $P_1^I = 1$ for $N_I = 1$:

$$c_1^I = \frac{1}{(\tau_{1,0}^I)_{II}}. \tag{A7}$$

Then the vector polarization is uniquely defined as

$$P_1^I = \frac{\overline{\tau}_1^I}{(\tau_{1,0}^I)_{II}}, \tag{A8}$$

and the general tensor polarizations are

$$P_k^I = c_k^I \overline{\tau}_k^I, \tag{A9}$$

with the parameters c_k^I determined from the condition that $P_k^I = 1$ for $N_I = 1$. Thus polarization for arbitrary cases can be uniquely defined as

$$P_k^I = \frac{\overline{\tau}_k^I}{(\tau_{k,0}^I)_{II}}, \tag{A10}$$

with the normalization coefficients

$$\frac{1}{(\tau_{k,0}^I)_{II}} = \frac{1}{\sqrt{(2I+1)(2k+1)}}\sqrt{\frac{(2I-k)!}{(2I)!}}\sqrt{\frac{(2I+k+1)!}{(2I)!}}. \tag{A11}$$

This leads to explicit expressions for P_k^I as follows:

$$P_1^{\frac{1}{2}} = N_{\frac{1}{2}} - N_{-\frac{1}{2}},$$

$$P_1^1 = N_1 - N_{-1},$$

$$P_2^1 = N_1 - 2N_0 + N_{-1},$$

$$P_1^{\frac{3}{2}} = N_{\frac{3}{2}} + \tfrac{1}{3}N_{\frac{1}{2}} - \tfrac{1}{3}N_{-\frac{1}{2}} - N_{-\frac{3}{2}},$$

$$P_2^{\frac{3}{2}} = N_{\frac{3}{2}} - N_{\frac{1}{2}} - N_{-\frac{1}{2}} + N_{-\frac{3}{2}},$$

$$P_3^{\frac{3}{2}} = N_{\frac{3}{2}} - 3N_{\frac{1}{2}} + 3N_{-\frac{1}{2}} - N_{-\frac{3}{2}},$$

$$P_1^2 = N_2 + \tfrac{1}{4}N_1 - \tfrac{1}{4}N_{-1} - N_{-2},$$

$$P_2^2 = N_2 - \tfrac{1}{2}N_1 - N_0 - \tfrac{1}{2}N_{-1} + N_{-1},$$

$$P_3^2 = N_2 - 2N_1 + 2N_{-1} - N_{-1},$$

$$P_4^2 = N_2 - 4N_1 + 6N_0 - 4N_{-1} + N_{-1},$$

$$P_1^{\frac{5}{2}} = N_{\frac{5}{2}} + \tfrac{3}{5}N_{\frac{3}{2}} + \tfrac{1}{5}N_{\frac{1}{2}} - \tfrac{1}{5}N_{-\frac{1}{2}} - \tfrac{3}{5}N_{-\frac{3}{2}} - N_{-\frac{5}{2}},$$

$$P_2^{\frac{5}{2}} = N_{\frac{5}{2}} - \tfrac{1}{5}N_{\frac{3}{2}} - \tfrac{4}{5}N_{\frac{1}{2}} - \tfrac{4}{5}N_{-\frac{1}{2}} - \tfrac{1}{5}N_{-\frac{3}{2}} + N_{-\frac{5}{2}},$$

$$P_3^{\frac{5}{2}} = N_{\frac{5}{2}} - \tfrac{7}{5}N_{\frac{3}{2}} - \tfrac{4}{5}N_{\frac{1}{2}} + \tfrac{4}{5}N_{-\frac{1}{2}} + \tfrac{7}{5}N_{-\frac{3}{2}} - N_{-\frac{5}{2}},$$

$$P_4^{\frac{5}{2}} = N_{\frac{5}{2}} - 3N_{\frac{3}{2}} + 2N_{\frac{1}{2}} + 2N_{-\frac{1}{2}} - 3N_{-\frac{3}{2}} + N_{-\frac{5}{2}},$$

$$P_5^{\frac{5}{2}} = N_{\frac{5}{2}} - 5N_{\frac{3}{2}} + 10N_{\frac{1}{2}} - 10N_{-\frac{1}{2}} + 5N_{-\frac{3}{2}} - N_{-\frac{5}{2}},$$

$$P_1^3 = N_3 + \tfrac{2}{3}N_2 + \tfrac{1}{3}N_1 - \tfrac{1}{3}N_{-1} - \tfrac{2}{3}N_{-2} - N_{-3},$$

$$P_2^3 = N_3 - \tfrac{3}{5}N_1 - \tfrac{4}{5}N_0 - \tfrac{3}{5}N_{-1} + N_{-3},$$

$$P_3^3 = N_3 - N_2 - N_1 + N_{-1} + N_{-2} - N_{-3},$$

$$P_4^3 = N_3 - \tfrac{7}{3}N_2 + \tfrac{1}{3}N_1 + 2N_0 + \tfrac{1}{3}N_{-1} - \tfrac{7}{3}N_{-2} + N_{-3},$$

$$P_5^3 = N_3 - 4N_2 + 5N_1 - 5N_{-1} + 4N_{-2} - N_{-3},$$

$$P_6^3 = N_3 - 6N_2 + 15N_1 - 20N_0 + 15N_{-1} - 6N_{-2} + N_{-3},$$

$$P_1^{\frac{7}{2}} = N_{\frac{7}{2}} + \tfrac{5}{7}N_{\frac{5}{2}} + \tfrac{3}{7}N_{\frac{3}{2}} + \tfrac{1}{7}N_{\frac{1}{2}} - \tfrac{1}{7}N_{-\frac{1}{2}} - \tfrac{3}{7}N_{-\frac{3}{2}} - \tfrac{5}{7}N_{-\frac{5}{2}} - N_{-\frac{7}{2}},$$

$$P_2^{\frac{7}{2}} = N_{\frac{7}{2}} + \tfrac{1}{7}N_{\frac{5}{2}} - \tfrac{3}{7}N_{\frac{3}{2}} - \tfrac{5}{7}N_{\frac{1}{2}} - \tfrac{5}{7}N_{-\frac{1}{2}} - \tfrac{3}{7}N_{-\frac{3}{2}} + \tfrac{1}{7}N_{-\frac{5}{2}} + N_{-\frac{7}{2}},$$

$$P_3^{\frac{7}{2}} = N_{\frac{7}{2}} - \tfrac{5}{7}N_{\frac{5}{2}} - N_{\frac{3}{2}} - \tfrac{3}{7}N_{\frac{1}{2}} + \tfrac{3}{7}N_{-\frac{1}{2}} + N_{-\frac{3}{2}} + \tfrac{5}{7}N_{-\frac{5}{2}} - N_{-\frac{7}{2}},$$

$$P_4^{\frac{7}{2}} = N_{\frac{7}{2}} - \tfrac{13}{7}N_{\frac{5}{2}} - \tfrac{3}{7}N_{\frac{3}{2}} + \tfrac{9}{7}N_{\frac{1}{2}} + \tfrac{9}{7}N_{-\frac{1}{2}} - \tfrac{3}{7}N_{-\frac{3}{2}} - \tfrac{13}{7}N_{-\frac{5}{2}} + N_{-\frac{7}{2}},$$

$$P_5^{\frac{7}{2}} = N_{\frac{7}{2}} - \tfrac{23}{7}N_{\frac{5}{2}} - \tfrac{17}{7}N_{\frac{3}{2}} + \tfrac{15}{7}N_{\frac{1}{2}} - \tfrac{15}{7}N_{-\frac{1}{2}} + \tfrac{17}{7}N_{-\frac{3}{2}} + \tfrac{23}{7}N_{-\frac{5}{2}} - N_{-\frac{7}{2}},$$

$$P_6^{\frac{7}{2}} = N_{\frac{7}{2}} - 5N_{\frac{5}{2}} + 9N_{\frac{3}{2}} - 5N_{\frac{1}{2}} - 5N_{-\frac{1}{2}} + 9N_{-\frac{3}{2}} - 5N_{-\frac{5}{2}} + N_{-\frac{7}{2}},$$

$$P_7^{\frac{7}{2}} = N_{\frac{7}{2}} - 7N_{\frac{5}{2}} + 21N_{\frac{3}{2}} - 35N_{\frac{1}{2}} + 35N_{-\frac{1}{2}} - 21N_{-\frac{3}{2}} + 7N_{-\frac{5}{2}} - N_{-\frac{7}{2}}.$$

APPENDIX B: SOME USEFUL EXPRESSIONS WITH FIXED VALUES OF *M*

For numerical calculations it may be convenient to have expressions for A', B', etc., coefficients for specific values of the projection of the target spins on the z axis. Using Eq. (1), one can get these expressions for a particular value of the target spin I.

For example, for the target spin $I = 1/2$ and the spin projection $M = \pm 1/2$,

$$A'_{1/2\pm1/2} = \frac{i}{8k}(\langle 0,0|R^0|0,0\rangle + 3\langle 1,0|R^1|1,0\rangle + 3\langle 0,1|R^1|0,1\rangle + \langle 1,1|R^0|1,1\rangle + 3\langle 1,1|R^1|1,1\rangle). \tag{B1}$$

For the target spin $I = 1$ and the spin projection $M = \pm 1$,

$$\begin{aligned} A'_{1\pm1} = \frac{i}{120k}\bigg(& 20\left\langle\frac{1}{2},0\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},0\right\rangle + 40\left\langle\frac{3}{2},0\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},0\right\rangle + 20\left\langle\frac{1}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},1\right\rangle \\ &+ 5\sqrt{2}\left\langle\frac{1}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{3}{2},1\right\rangle + 40\left\langle\frac{1}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{1}{2},1\right\rangle - 2\sqrt{5}\left\langle\frac{1}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},1\right\rangle \\ &+ 5\sqrt{2}\left\langle\frac{3}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},1\right\rangle + 25\left\langle\frac{3}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{3}{2},1\right\rangle - 2\sqrt{5}\left\langle\frac{3}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{1}{2},1\right\rangle + 32\left\langle\frac{3}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},1\right\rangle\bigg). \end{aligned} \tag{B2}$$

For the target spin $I = 1$ and the spin projection $M = 0$,

$$\begin{aligned} A'_{10} = \frac{i}{120k}\bigg(& 20\left\langle\frac{1}{2},0\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},0\right\rangle + 40\left\langle\frac{3}{2},0\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},0\right\rangle + 20\left\langle\frac{1}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},1\right\rangle \\ &- 10\sqrt{2}\left\langle\frac{1}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{3}{2},1\right\rangle + 40\left\langle\frac{1}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{1}{2},1\right\rangle + 4\sqrt{5}\left\langle\frac{1}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},1\right\rangle \\ &- 10\sqrt{2}\left\langle\frac{3}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},1\right\rangle + 10\left\langle\frac{3}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{3}{2},1\right\rangle + 4\sqrt{5}\left\langle\frac{3}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{1}{2},1\right\rangle + 56\left\langle\frac{3}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},1\right\rangle\bigg), \end{aligned} \tag{B3}$$

or for the target spin $I > 1$ and the spin projection M,

$$\begin{aligned} A'_{IM} = \frac{i}{4(2I+1)^2k}\bigg\{& 2I(2I+1)\left\langle\left(I-\frac{1}{2}\right),0\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),0\right\rangle + 2(I+1)(2I+1)\left\langle\left(I+\frac{1}{2}\right),0\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),0\right\rangle \\ &+ \frac{6[2I^3+2M^2-I(1+2M^2)]}{(2I-1)}\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle \\ &- \frac{3(I+I^2-3M^2)}{\sqrt{(I+1)(2I-1)}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle + \left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right] \\ &+ \frac{3(I+2I^2+I^3-M^2+IM^2)}{I}\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle \\ &+ \frac{3(I+I^2-3M^2)}{\sqrt{I(2I+3)}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle + \left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right] \\ &+ \frac{3(I^2+I^3+2M^2+IM^2)}{(I+1)}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle \\ &+ \frac{6(1+6I^2+2I^3-4M^2+I(5-2M^2))}{(2I+3)}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle\bigg\}. \end{aligned} \tag{B4}$$

For the target spin $I > 1$ with $M = I$,

$$\begin{aligned} A'_{I} = \frac{i}{4(2I+1)^2k}\bigg\{& 2I(2I+1)\left\langle\left(I-\frac{1}{2}\right),0\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),0\right\rangle \\ &+ 2(I+1)(2I+1)\left\langle\left(I+\frac{1}{2}\right),0\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),0\right\rangle + 6I\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle \\ &+ \frac{3I\sqrt{2I-1}}{\sqrt{I+1}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle + \left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right] \\ &- \frac{3I(2I-1)}{\sqrt{I(2I+3)}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle + \left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right] \end{aligned}$$

$$+\frac{3I^2(3+2I)}{I+1}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle+3(2I^2+I+1)\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle$$
$$+\frac{6(2I^2+5I+1)}{2I+3}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|(I+\frac{1}{2}),1\right\rangle\Bigg\}. \quad \text{(B5)}$$

For the target spin $I=1/2$ and the spin projection M,

$$B'_{1/2M}=-\frac{iM}{4k}(\langle 0,0|R^0|0,0\rangle-\langle 1,0|R^1|1,0\rangle+3\langle 0,1|R^1|0,1\rangle-\langle 1,1|R^0|1,1\rangle). \quad \text{(B6)}$$

For the target spin $I=1$,

$$B'_{1M}=\frac{iM}{120k}\Bigg(-20\left\langle\frac{1}{2},0\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},0\right\rangle+20\left\langle\frac{3}{2},0\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},0\right\rangle-20\left\langle\frac{1}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},1\right\rangle$$
$$-5\sqrt{2}\left\langle\frac{1}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{3}{2},1\right\rangle-40\left\langle\frac{1}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{1}{2},1\right\rangle+2\sqrt{5}\left\langle\frac{1}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},1\right\rangle$$
$$-5\sqrt{2}\left\langle\frac{3}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{1}{2},1\right\rangle+20\left\langle\frac{3}{2},1\middle|R^{\frac{1}{2}}\middle|\frac{3}{2},1\right\rangle+2\sqrt{5}\left\langle\frac{3}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{1}{2},1\right\rangle+4\left\langle\frac{3}{2},1\middle|R^{\frac{3}{2}}\middle|\frac{3}{2},1\right\rangle\Bigg). \quad \text{(B7)}$$

For the target spin $I>1$,

$$B'_{IM}=\frac{iM}{4(2I+1)^2k}\Bigg\{2(2I+1)\left[\left\langle\left(I+\frac{1}{2}\right),0\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),0\right\rangle-\left\langle\left(I-\frac{1}{2}\right),0\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),0\right\rangle\right]$$
$$-\frac{6(2I^2+2I-1-2M^2)}{(2I-1)}\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle$$
$$-\frac{3(2I^2+2I-1-2M^2)}{\sqrt{(I+1)(2I-1)}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle+\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right]$$
$$-\frac{3(1+I+I^2+M^2)}{I}\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle$$
$$+\frac{3(2I^2+2I-1-2M^2)}{\sqrt{I(2I+3)}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle+\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right]$$
$$+\frac{3(1+I+I^2+M^2)}{(I+1)}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle-\frac{6(2I^2+2I-1-2M^2)}{(2I+3)}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle\Bigg\} \quad \text{(B8)}$$

For the target spin $I>1$ and $M=I$,

$$B'_{II}=\frac{i}{4(2I+1)^2k}\Bigg\{2I(2I+1)\left[\left\langle\left(I+\frac{1}{2}\right),0\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),0\right\rangle-\left\langle\left(I-\frac{1}{2}\right),0\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),0\right\rangle\right]$$
$$+3(2I^2+I+1)\left[\frac{I}{I+1}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle-\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right]$$
$$-\frac{3I\sqrt{2I-1}}{\sqrt{I+1}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle+\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right]$$
$$+\frac{3\sqrt{I}(2I-1)}{\sqrt{2I+3}}\left[\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle+\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\right]$$
$$+\frac{6I(2I-1)}{2I+3}\left\langle\left(I+\frac{1}{2}\right),1\middle|R^{I+\frac{1}{2}}\middle|\left(I+\frac{1}{2}\right),1\right\rangle-6I\left\langle\left(I-\frac{1}{2}\right),1\middle|R^{I-\frac{1}{2}}\middle|\left(I-\frac{1}{2}\right),1\right\rangle\Bigg\}. \quad \text{(B9)}$$

For the target spin $I=1/2$ and its projections on the z axis, $M=\pm 1/2$,

$$C'_{1/2}=-\frac{i}{8k}(\langle 0,0|R^0|1,1\rangle-\sqrt{3}\langle 1,0|R^1|0,1\rangle+\sqrt{6}\langle 1,0|R^1|1,1\rangle+\langle 1,1|R^0|0,0\rangle-\sqrt{3}\langle 0,1|R^1|1,0\rangle+\sqrt{6}\langle 1,1|R^1|1,0\rangle). \quad \text{(B10)}$$

For the target spin $I > 1/2$,

$$C'_{IM} = -\frac{i\sqrt{3}}{8(2I+1)^{3/2}k}\Bigg\{-\frac{4(I^2-M^2)}{\sqrt{2I-1}}\Bigg[\Bigg\langle\Big(I-\frac{1}{2}\Big),0\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),1\Bigg\rangle+\Bigg\langle\Big(I-\frac{1}{2}\Big),1\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),0\Bigg\rangle\Bigg]$$
$$+\frac{2(I^2+I+M^2)}{\sqrt{I+1}}\Bigg[\Bigg\langle\Big(I-\frac{1}{2}\Big),0\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),1\Bigg\rangle+\Bigg\langle\Big(I+\frac{1}{2}\Big),1\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),0\Bigg\rangle\Bigg]$$
$$-\frac{2(I^2+I+M^2)}{\sqrt{I}}\Bigg[\Bigg\langle\Big(I+\frac{1}{2}\Big),0\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),1\Bigg\rangle+\Bigg\langle\Big(I-\frac{1}{2}\Big),1\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),0\Bigg\rangle\Bigg]$$
$$+\frac{4[(I+1)^2-M^2]}{\sqrt{2I+3}}\Bigg[\Bigg\langle\Big(I+\frac{1}{2}\Big),0\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),1\Bigg\rangle+\Bigg\langle\Big(I+\frac{1}{2}\Big),1\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),0\Bigg\rangle\Bigg]\Bigg\}. \tag{B11}$$

Then, for the target spin $I > 1/2$ and $M = I$,

$$C'_{II} = \frac{i\sqrt{3}}{4\sqrt{2I+1}k}\Bigg\{\frac{I}{\sqrt{I+1}}\Bigg[\Bigg\langle\Big(I-\frac{1}{2}\Big),0\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),1\Bigg\rangle+\Bigg\langle\Big(I+\frac{1}{2}\Big),1\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),0\Bigg\rangle\Bigg]$$
$$-\sqrt{I}\Bigg[\Bigg\langle\Big(I+\frac{1}{2}\Big),0\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),1\Bigg\rangle+\Bigg\langle\Big(I-\frac{1}{2}\Big),1\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),0\Bigg\rangle\Bigg]$$
$$+\frac{2}{\sqrt{2I+3}}\Bigg[\Bigg\langle\Big(I+\frac{1}{2}\Big),0\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),1\Bigg\rangle+\Bigg\langle\Big(I+\frac{1}{2}\Big),1\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),0\Bigg\rangle\Bigg]\Bigg\} \tag{B12}$$

The general expression for D' is

$$D'_M = \frac{\sqrt{3}M}{4\sqrt{I(I+1)(2I+1)}k}\Bigg\{\sqrt{I+1}\Bigg[\Bigg\langle\Big(I+\frac{1}{2}\Big),0\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),1\Bigg\rangle-\Bigg\langle\Big(I-\frac{1}{2}\Big),1\Bigg|R^{I+\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),0\Bigg\rangle\Bigg]$$
$$+\sqrt{I}\Bigg[\Bigg\langle\Big(I-\frac{1}{2}\Big),0\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I+\frac{1}{2}\Big),1\Bigg\rangle-\Bigg\langle\Big(I+\frac{1}{2}\Big),1\Bigg|R^{I-\frac{1}{2}}\Bigg|\Big(I-\frac{1}{2}\Big),0\Bigg\rangle\Bigg]\Bigg\}, \tag{B13}$$

where M is a projection of spin I on the z axis.

It should be noted that for the fixed value M some of the above coefficients contain a mixture from different types of the target polarizations. It can be clearly seen from the case of $M = I$ that the 100% population of this level contributes to all possible tensor polarizations P_q up to the rank $q = 2I$ (see Appendix A).

APPENDIX C: *d*-WAVE CONTRIBUTIONS

To estimate contributions from d waves, let us consider that E' and F' coefficients obtained the scattering amplitude related to the tensor polarization t_{20} with d waves:

$$f_{20}(\theta,\phi,x,y) = \frac{i\pi}{2k}\sqrt{\frac{5}{2I+1}}t_{20}\sum_{JMll'SS'm}Y_{Lm}(\theta,\phi)N(x,y)\langle IM20|IM\rangle$$
$$\times\langle S'l'\alpha'|R^J|Sl\alpha\rangle(-1)^{J+S'+l'+l}(2J+1)\sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2S+1)}}\langle l0l'0|L0\rangle\begin{Bmatrix} l' & l & L\\ S & S' & J\end{Bmatrix}. \tag{C1}$$

The expression for E' for the tensor-polarized $\widetilde{\tau}_{20}$ target with $I = 7/2$ is

$$E'_{7/2} = -\frac{i}{512\sqrt{2}k}\bigg(5\sqrt{\frac{21}{2}}\langle 3, 1|R^3|3, 1\rangle - 15\sqrt{\frac{3}{14}}\langle 3, 1|R^4|3, 1\rangle - 11\sqrt{\frac{7}{6}}\langle 4, 1|R^3|4, 1\rangle - 3\sqrt{\frac{7}{2}}(\langle 3, 1|R^3|4, 1\rangle + \langle 4, 1|R^3|3, 1\rangle) + 9\sqrt{\frac{3}{10}}(\langle 3, 1|R^4|4, 1\rangle + \langle 4, 1|R^4|3, 1\rangle) + \frac{33}{5}\sqrt{\frac{21}{2}}\langle 4, 1|R^4|4, 1\rangle + \frac{55}{\sqrt{42}}\langle 3, 2|R^3|3, 2\rangle + \frac{225}{7}\sqrt{\frac{3}{14}}\langle 3, 2|R^4|3, 2\rangle + 5\sqrt{14}(\langle 3, 0|R^3|3, 2\rangle + \langle 3, 2|R^3|3, 0\rangle) + 3\sqrt{6}(\langle 4, 0|R^4|3, 2\rangle + \langle 3, 2|R^4|4, 0\rangle) - \sqrt{42}(\langle 3, 0|R^3|4, 2\rangle + \langle 4, 2|R^3|3, 0\rangle) - \frac{15}{\sqrt{14}}(\langle 3, 2|R^3|4, 2\rangle + \langle 4, 2|R^3|3, 2\rangle) - \frac{3}{7}\sqrt{\frac{33}{14}}(\langle 3, 2|R^4|4, 2\rangle + \langle 4, 2|R^4|3, 2\rangle) + 3\sqrt{66}(\langle 4, 0|R^4|4, 2\rangle + \langle 4, 2|R^4|4, 0\rangle) - \frac{11}{\sqrt{42}}\langle 4, 2|R^3|4, 2\rangle + \frac{195}{7}\sqrt{\frac{3}{14}}\langle 4, 2|R^4|4, 2\rangle\bigg). \tag{C2}$$

One can see that the E' coefficient is equal to zero at s-wave resonances and depends on p-wave and d-wave resonances. However, contributions from d-wave resonances are suppressed in the low-energy region by a factor (kR) in comparison to p-wave resonances (where R is a nuclear radius). Moreover, they behave in the vicinity of p-wave resonances as a flat (energy-independent) background. Therefore d-wave contributions are negligible in the vicinity of low-energy p-wave resonances.

The expression for F' for the tensor-polarized target with $I = 7/2$ is

$$F'_{7/2} = -\frac{i}{128k}\bigg(\sqrt{\frac{21}{2}}(\langle 3, 0|R^3|3, 1\rangle + \langle 3, 1|R^3|3, 0\rangle) - \frac{3}{\sqrt{2}}(\langle 4, 0|R^4|3, 1\rangle + \langle 3, 1|R^4|4, 0\rangle) + \sqrt{\frac{7}{2}}(\langle 3, 0|R^3|4, 1\rangle + \langle 4, 1|R^3|3, 0\rangle) - 3\sqrt{\frac{7}{10}}(\langle 4, 0|R^4|4, 1\rangle + \langle 4, 1|R^4|4, 0\rangle) + \frac{15}{\sqrt{14}}(\langle 3, 1|R^3|3, 2\rangle + \langle 3, 2|R^3|3, 1\rangle) - 10\sqrt{\frac{2}{21}}(\langle 4, 1|R^3|3, 2\rangle + \langle 3, 2|R^3|4, 1\rangle) + \frac{81}{7\sqrt{14}}(\langle 3, 1|R^4|3, 2\rangle + \langle 3, 2|R^4|3, 1\rangle) + \frac{9}{7}\sqrt{\frac{2}{5}}(\langle 4, 1|R^4|3, 2\rangle + \langle 3, 2|R^4|4, 1\rangle) - \sqrt{\frac{6}{7}}(\langle 3, 1|R^3|4, 2\rangle + \langle 4, 2|R^3|3, 1\rangle) + \frac{5}{\sqrt{14}}(\langle 4, 1|R^3|4, 2\rangle + \langle 4, 2|R^3|4, 1\rangle) + \frac{12}{7}\sqrt{\frac{22}{7}}(\langle 3, 1|R^4|4, 2\rangle + \langle 4, 2|R^4|3, 1\rangle) - \frac{39}{7}\sqrt{\frac{11}{10}}(\langle 4, 1|R^4|4, 2\rangle + \langle 4, 2|R^4|4, 1\rangle)\bigg). \tag{C3}$$

We can see that the F' coefficient is defined by P-odd mixtures of s-wave and p-wave resonances and of p-wave and d-wave resonances. Therefore, contributions from d-wave resonances are suppressed in low-energy regions by a factor $(kR)^2$ and can be neglected.

APPENDIX D: NONZERO CONTRIBUTIONS IN A SPHERICAL TENSOR EXPANSION OF THE AMPLITUDE IN EQ. (8)

The f_0 for $q = 0$ reads as follows:

$$f_0 = \frac{i}{64k}\{2(7\langle 3, 0|R^3|3, 0\rangle + 9\langle 4, 0|R^4|4, 0\rangle + 7\langle 3, 1|R^3|3, 1\rangle + 9\langle 3, 1|R^4|3, 1\rangle + 7\langle 4, 1|R^3|4, 1\rangle + 9\langle 4, 1|R^4|4, 1\rangle) + [7(\langle 3, 0|R^3|3, 1\rangle + \langle 3, 1|R^3|3, 0\rangle) - 7\sqrt{3}(\langle 3, 0|R^3|4, 1\rangle + \langle 4, 1|R^3|3, 0\rangle) + 3\sqrt{21}(\langle 4, 0|R^4|3, 1\rangle + \langle 3, 1|R^4|4, 0\rangle) - 3\sqrt{15}(\langle 4, 0|R^4|4, 1\rangle + \langle 4, 1|R^4|4, 0\rangle)] \times [\cos(\beta)\cos(\theta) + \sin(\alpha)\sin(\beta)\sin(\theta)]\} \tag{D1}$$

or

$$f_0 = \frac{i}{64k}\{2(7\langle 3,0|R^3|3,0\rangle + 9\langle 4,0|R^4|4,0\rangle + 7\langle 3,1|R^3|3,1\rangle + 9\langle 3,1|R^4|3,1\rangle + 7\langle 4,1|R^3|4,1\rangle + 9\langle 4,1|R^4|4,1\rangle) + [7(\langle 3,0|R^3|3,1\rangle + \langle 3,1|R^3|3,0\rangle) - 7\sqrt{3}(\langle 3,0|R^3|4,1\rangle + \langle 4,1|R^3|3,0\rangle) + 3\sqrt{21}(\langle 4,0|R^4|3,1\rangle + \langle 3,1|R^4|4,0\rangle) - 3\sqrt{15}(\langle 4,0|R^4|4,1\rangle + \langle 4,1|R^4|4,0\rangle)](\vec{\sigma}\cdot\hat{k})\}. \tag{D2}$$

The f_1 for $q = 1$ reads as follows:

$$\begin{aligned} f_1 = &-\frac{i}{32k}\bigg(7\langle 3,0|R^3|3,0\rangle - 7\langle 4,0|R^4|4,0\rangle + \frac{21}{4}\langle 3,1|R^3|3,1\rangle + \frac{7}{4}\sqrt{3}(\langle 3,1|R^3|4,1\rangle + \langle 4,1|R^3|3,1\rangle) \\ &- \frac{9}{20}\sqrt{35}(\langle 3,1|R^4|4,1\rangle + \langle 4,1|R^4|3,1\rangle) - \frac{91}{12}\langle 4,1|R^3|4,1\rangle + \frac{39}{4}\langle 3,1|R^4|3,1\rangle - \frac{63}{20}\langle 4,1|R^4|4,1\rangle\bigg)\cos(\beta) \\ &+ \frac{1}{16k}\bigg(\frac{7}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle - \langle 4,1|R^3|3,0\rangle) + \sqrt{21}(\langle 4,0|R^4|3,1\rangle - \langle 3,1|R^4|4,0\rangle)\bigg)\cos(\alpha)\sin(\beta)\sin(\theta) \\ &- \frac{i}{64k}\bigg(21(\langle 3,0|R^3|3,1\rangle + \langle 3,1|R^3|3,0\rangle) - \frac{7}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle + \langle 4,1|R^3|3,0\rangle) \\ &+ \sqrt{21}(\langle 4,0|R^4|3,1\rangle + \langle 3,1|R^4|4,0\rangle) + 7\sqrt{15}(\langle 4,0|R^4|4,1\rangle + \langle 4,1|R^4|4,0\rangle)\bigg)\cos(\theta) \\ &- \frac{3i}{128k}\bigg(7\langle 3,1|R^3|3,1\rangle - 7\sqrt{3}(\langle 3,1|R^3|4,1\rangle + \langle 4,1|R^3|3,1\rangle) \\ &+ \frac{77}{9\sqrt{3}}\langle 4,1|R^3|4,1\rangle - 3\langle 3,1|R^4|3,1\rangle + \frac{9}{5}\sqrt{35}(\langle 3,1|R^4|4,1\rangle + \langle 4,1|R^4|3,1\rangle) \\ &- \frac{77}{5}\langle 4,1|R^4|4,1\rangle\bigg)[\cos(\beta)\cos(\theta) + \sin(\alpha)\sin(\beta)\sin(\theta)]\cos(\theta) \end{aligned} \tag{D3}$$

or

$$\begin{aligned} f_1 = &-\frac{i}{32k}\bigg(7\langle 3,0|R^3|3,0\rangle - 7\langle 4,0|R^4|4,0\rangle + \frac{21}{4}\langle 3,1|R^3|3,1\rangle + \frac{7}{4}\sqrt{3}(\langle 3,1|R^3|4,1\rangle + \langle 4,1|R^3|3,1\rangle) \\ &- \frac{9}{20}\sqrt{35}(\langle 3,1|R^4|4,1\rangle + \langle 4,1|R^4|3,1\rangle) - \frac{91}{12}\langle 4,1|R^3|4,1\rangle + \frac{39}{4}\langle 3,1|R^4|3,1\rangle - \frac{63}{20}\langle 4,1|R^4|4,1\rangle\bigg)(\vec{\sigma}\cdot\vec{I}) \\ &+ \frac{1}{16k}\bigg(\frac{7}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle - \langle 4,1|R^3|3,0\rangle) + \sqrt{21}(\langle 4,0|R^4|3,1\rangle - \langle 3,1|R^4|4,0\rangle)\bigg)(\vec{\sigma}\cdot[\hat{k}\times\vec{I}]) \\ &- \frac{i}{64k}\bigg(21(\langle 3,0|R^3|3,1\rangle + \langle 3,1|R^3|3,0\rangle) - \frac{7}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle + \langle 4,1|R^3|3,0\rangle) \\ &+ \sqrt{21}(\langle 4,0|R^4|3,1\rangle + \langle 3,1|R^4|4,0\rangle) + 7\sqrt{15}(\langle 4,0|R^4|4,1\rangle + \langle 4,1|R^4|4,0\rangle)\bigg)(\hat{k}\cdot\vec{I}) \\ &- \frac{3i}{128k}\bigg(7\langle 3,1|R^3|3,1\rangle - 7\sqrt{3}(\langle 3,1|R^3|4,1\rangle + \langle 4,1|R^3|3,1\rangle) \\ &+ \frac{77}{9\sqrt{3}}\langle 4,1|R^3|4,1\rangle - 3\langle 3,1|R^4|3,1\rangle + \frac{9}{5}\sqrt{35}(\langle 3,1|R^4|4,1\rangle + \langle 4,1|R^4|3,1\rangle) - \frac{77}{5}\langle 4,1|R^4|4,1\rangle\bigg)(\vec{\sigma}\cdot\hat{k})(\hat{k}\cdot\vec{I}). \end{aligned} \tag{D4}$$

The f_2 for $q = 2$ reads as follows:

$$\begin{aligned} f_2 = &\frac{3i}{320k}\bigg(35(\langle 3,0|R^3|3,1\rangle + \langle 3,1|R^3|3,0\rangle) + \frac{35}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle + \langle 4,1|R^3|3,0\rangle) \\ &- 5\sqrt{21}(\langle 4,0|R^4|3,1\rangle + \langle 3,1|R^4|4,0\rangle) - 7\sqrt{15}(\langle 4,0|R^4|4,1\rangle + \langle 4,1|R^4|4,0\rangle)\bigg) \\ &\times\bigg(\cos(\beta)\cos(\theta) - \frac{1}{3}[\cos(\beta)\cos(\theta) + \sin(\alpha)\sin(\beta)\sin(\theta)]\bigg) \end{aligned}$$

$$+\frac{i9}{256k}\left(\frac{35}{3}\langle 3,1|R^3|3,1\rangle-\frac{7}{\sqrt{3}}(\langle 3,1|R^3|4,1\rangle+\langle 4,1|R^3|3,1\rangle)\right.$$
$$-\frac{77}{9}\langle 4,1|R^3|4,1\rangle+\frac{3}{5}\sqrt{35}(\langle 3,1|R^4|4,1\rangle+\langle 4,1|R^4|3,1\rangle)$$
$$\left.-5\langle 3,1|R^4|3,1\rangle+\frac{77}{5}\langle 4,1|R^4|4,1\rangle\right)\left(\cos^2(\theta)-\frac{1}{3}\right)-\frac{3}{32\sqrt{5}k}\left(7\sqrt{\frac{5}{3}}(\langle 3,1|R^3|4,1\rangle-\langle 4,1|R^3|3,1\rangle)\right.$$
$$\left.-3\sqrt{7}(\langle 3,1|R^4|4,1\rangle-\langle 4,1|R^4|3,1\rangle)\right)\sin(\beta)\sin(\theta)\cos(\theta)\cos(\alpha) \tag{D5}$$

or

$$f_2=\frac{3i}{320k}\left(35(\langle 3,0|R^3|3,1\rangle+\langle 3,1|R^3|3,0\rangle)+\frac{35}{\sqrt{3}}(\langle 3,0|R^3|4,1\rangle+\langle 4,1|R^3|3,0\rangle)\right.$$
$$\left.-5\sqrt{21}(\langle 4,0|R^4|3,1\rangle+\langle 3,1|R^4|4,0\rangle)-7\sqrt{15}(\langle 4,0|R^4|4,1\rangle+\langle 4,1|R^4|4,0\rangle)\right)$$
$$\times\left[(\vec{\sigma}\cdot\vec{I})(\hat{k}\cdot\vec{I})-\frac{1}{3}(\vec{\sigma}\cdot\hat{k})(\vec{I}\cdot\vec{I})\right]+\frac{i9}{256k}\left(\frac{35}{3}\langle 3,1|R^3|3,1\rangle-\frac{7}{\sqrt{3}}(\langle 3,1|R^3|4,1\rangle+\langle 4,1|R^3|3,1\rangle)\right.$$
$$-\frac{77}{9}\langle 4,1|R^3|4,1\rangle+\frac{3}{5}\sqrt{35}(\langle 3,1|R^4|4,1\rangle+\langle 4,1|R^4|3,1\rangle)$$
$$\left.-5\langle 3,1|R^4|3,1\rangle+\frac{77}{5}\langle 4,1|R^4|4,1\rangle\right)\left[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I})-\frac{1}{3}(\hat{k}\cdot\hat{k})(\vec{I}\cdot\vec{I})\right]$$
$$-\frac{3}{32\sqrt{5}k}\left(7\sqrt{\frac{5}{3}}(\langle 3,1|R^3|4,1\rangle-\langle 4,1|R^3|3,1\rangle)-3\sqrt{7}(\langle 3,1|R^4|4,1\rangle-\langle 4,1|R^4|3,1\rangle)\right)(\vec{\sigma}\cdot[\hat{k}\times\vec{I}])(\hat{k}\cdot\vec{I}). \tag{D6}$$

The f_3 for $q=3$ reads as follows:

$$f_3=-\frac{3i}{512k}\left(7\langle 3,1|R^3|3,1\rangle+\frac{7}{\sqrt{3}}(\langle 3,1|R^3|4,1\rangle+\langle 4,1|R^3|3,1\rangle)+\frac{7}{3}\langle 4,1|R^3|4,1\rangle-3\sqrt{\frac{7}{5}}(\langle 3,1|R^4|4,1\rangle+\langle 4,1|R^4|3,1\rangle)\right.$$
$$\left.-3\langle 3,1|R^4|3,1\rangle-\frac{21}{5}\langle 4,1|R^4|4,1\rangle\right)[\cos(\beta)+3\cos(\beta)\cos(2\theta)-4\cos(\theta)\sin(\alpha)\sin(\beta)\sin(\theta)] \tag{D7}$$

or

$$f_3=\frac{9i}{256k}\left(7\langle 3,1|R^3|3,1\rangle+\frac{7}{\sqrt{3}}(\langle 3,1|R^3|4,1\rangle+\langle 4,1|R^3|3,1\rangle)+\frac{7}{3}\langle 4,1|R^3|4,1\rangle-3\sqrt{\frac{7}{5}}(\langle 3,1|R^4|4,1\rangle+\langle 4,1|R^4|3,1\rangle)\right.$$
$$\left.-3\langle 3,1|R^4|3,1\rangle-\frac{21}{5}\langle 4,1|R^4|4,1\rangle\right)\left\{(\vec{\sigma}\cdot\vec{I})\left[(\hat{k}\cdot\vec{I})(\hat{k}\cdot\vec{I})-\frac{1}{3}(\hat{k}\cdot\hat{k})(\vec{I}\cdot\vec{I})\right]\right.$$
$$\left.\times+\frac{2}{5}(\hat{k}\cdot\vec{I})\left[(\vec{\sigma}\cdot\vec{I})(\hat{k}\cdot\vec{I})\frac{1}{3}+(\vec{\sigma}\cdot\hat{k})(\vec{I}\cdot\vec{I})\right]-\frac{4}{5}(\hat{k}\cdot\vec{I})\left[(\vec{\sigma}\cdot\vec{I})(\hat{k}\cdot\vec{I})-\frac{1}{3}(\vec{\sigma}\cdot\hat{k})(\vec{I}\cdot\vec{I})\right]\right\}. \tag{D8}$$

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