

**Strong decays of strange quarkonia in a corrected  $^3P_0$  model**J. N. de Quadros,<sup>1</sup> D. T. da Silva,<sup>2</sup> M. L. L. da Silva,<sup>2</sup> and D. Hadjimichef<sup>1,\*</sup><sup>1</sup>*Instituto de Física, Universidade Federal do Rio Grande do Sul, Porto Alegre, Rio Grande do Sul, CEP 91501-970, Brazil*<sup>2</sup>*Instituto de Física e Matemática, Universidade Federal de Pelotas, Pelotas, Rio Grande do Sul, CEP 96010-900, Brazil*

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Extensively applied to both light and heavy meson decay and standing as one of the most successful strong decay models is the  $^3P_0$  model, in which  $q\bar{q}$  pair production is the dominant mechanism. In this paper we evaluate strong decay amplitudes and partial widths of strange  $S$  and  $D$  state mesons, namely  $\phi(1020)$ ,  $\phi(1680)$ ,  $\phi(2050)$ ,  $\phi_1(1850)$ ,  $\phi_2(1850)$ , and  $\phi_3(1850)$ , in the bound-state corrected  $^3P_0$  decay model (C  $^3P_0$ ). The C  $^3P_0$  model is obtained in the context of the Fock-Tani formalism, which is a mapping technique.

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The study of strangeonia should enter a new era with the advent of the new Hall D photoproduction facility GlueX at Jefferson Lab [1,2]. The main goal of the GlueX experiment is to search for and study hybrid and exotic mesons which will provide the ideal laboratory for testing QCD in the confinement regime. Another top goal of GlueX is the exploration of the light meson spectrum, where interactions of hadrons with a photon beam can be regarded as a superposition of vector mesons with an important  $s\bar{s}$  component. In this sense, studies of strange final states at GlueX should lead to considerable improvement in our knowledge of the  $s\bar{s}$  spectrum.

Strange quarkonia are light ( $u$ ,  $d$ ,  $s$ ) mesons with at least one strange quark or antiquark in their dominant  $q\bar{q}$  valence component. These are known as kaonia if the dominant valence basis state is  $n\bar{s}$  (where  $n$  should be understood as  $u$ ,  $d$ ), antikaonia if  $s\bar{n}$ , and strangeonia if  $s\bar{s}$ . A principal goal of light meson spectroscopy is the identification of exotica, which are resonances that are not dominantly  $q\bar{q}$  states. These include glueballs, hybrids, and multi-quark systems.

In this sense, a great variety of quark-based models are known that describe with reasonable success single-hadron properties. A natural question that arises is to what extent a model which gives a good description of hadron properties is, at the same time, able to describe the complex hadron-hadron interaction or by the same principles hadron decay. In the direction of clarifying these questions is the successful decay model, the  $^3P_0$  model, which considers only Okubo-Zweig-Iizuka-allowed strong-interaction decays. This model was introduced over thirty years ago by Micu [3] and applied to meson decays in the 1970 by LeYaouanc *et al.* [4]. This description is a natural consequence of the constituent quark model scenario of hadronic states. Since the  $^3P_0$  model precedes QCD and has no clear relation to it, one might

expect that a description of decays in terms of allowed QCD processes such as OGE might be more realistic. There is strong experimental evidence that the  $q\bar{q}$  pair created during the decay does have spin one as is assumed in the  $^3P_0$  decay model.

Barnes *et al.* [5–8] made an extensive survey of meson states in the light of the  $^3P_0$  model. Two basic parameters of their formulation are  $\gamma$  (the interaction strength) and  $\beta$  (the wave function's extension parameter). Although they found the optimum values near  $\gamma = 0.5$  and  $\beta = 0.4$  GeV, for light  $1S$  and  $1P$  decays, these values lead to overestimates of the widths of higher- $L$  states. In this perspective a modified  $q\bar{q}$  pair-creation interaction with  $\gamma = 0.4$  was preferred. The spectrum of meson resonances up to 2 GeV is only moderately well determined. For strangeonia they calculated a set of strong decays of a total of 43 resonances into 525 two-body modes, with 891 numerically evaluated amplitudes for all energetically allowed open-flavor two-body decay modes of all  $n\bar{s}$  and  $s\bar{s}$  strange mesons in the  $1S$ ,  $2S$ ,  $3S$ ,  $1P$ ,  $2P$ ,  $1D$ , and  $1F$  multiplets [7].

In the present work, we concentrate on the  $\phi$  mesons, which are the strange  $S$  and  $D$  states, predicted in the quark model, probable  $s\bar{s}$  resonances expected up to 2.2 GeV. We employ a mapping technique to obtain an effective interaction for meson decay. A particular mapping technique long used in atomic physics [9], the Fock-Tani formalism (FTf), has been adapted, in previous publications [10–15], to describe hadron-hadron scattering interactions with constituent interchange. Now this technique has been extended in order to include meson decay [16,17]. Starting with a microscopic  $q\bar{q}$  pair-creation interaction, in lower order, the  $^3P_0$  results are reproduced. An additional and interesting feature appears in higher orders of the formalism: corrections due to the bound-state nature of the mesons and a natural modification in the  $q\bar{q}$  interaction strength.

In Sec. II we review the basic aspects of the formalism. Section III is dedicated to obtain an effective decay Hamiltonian for a  $\phi$  meson, where in Sec. III A the general amplitudes

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and decay widths are obtained with numerical analysis in Sec. III B. In Sec. V are the conclusions.

## II. MESON MAPPING AND THE C $^3P_0$ MODEL

### A. Review of Fock-Tani formalism

This section reviews the formal aspects of the mapping procedure and how it is implemented to quark-antiquark meson states [10]. In the Fock-Tani formalism one starts with the Fock representation of the system using field operators of elementary constituents which satisfy canonical (anti) commutation relations. Composite-particle field operators are linear combinations of the elementary-particle operators and do not generally satisfy canonical (anti) commutation relations. “Ideal” field operators acting on an enlarged Fock space are then introduced in close correspondence with the composite ones. Next, a given unitary transformation, which transforms the single composite states into single ideal states, is introduced. Application of the unitary operator on the microscopic Hamiltonian, or on other Hermitian operators expressed in terms of the elementary constituent field operators, gives equivalent operators which contain the ideal field operators. The effective Hamiltonian in the new representation has a clear physical interpretation in terms of the processes it describes. Since all field operators in the new representation satisfy canonical (anti)commutation relations, the standard methods of quantum field theory can then be readily applied.

The starting point is the definition of single composite bound states. We write a single-meson state in terms of a meson creation operator  $M_\alpha^\dagger$  as

$$|\alpha\rangle = M_\alpha^\dagger |0\rangle, \quad (1)$$

where  $|0\rangle$  is the vacuum state. The meson creation operator  $M_\alpha^\dagger$  is written in terms of constituent quark and antiquark creation operators  $q^\dagger$  and  $\bar{q}^\dagger$ ,

$$M_\alpha^\dagger = \Phi_\alpha^{\mu\nu} q_\mu^\dagger \bar{q}_\nu^\dagger, \quad (2)$$

where  $\Phi_\alpha^{\mu\nu}$  is the meson wave function and  $q_\mu |0\rangle = \bar{q}_\nu |0\rangle = 0$ . The index  $\alpha$  identifies the meson quantum numbers of space, spin, and isospin. The indices  $\mu$  and  $\nu$  denote the spatial, spin, flavor, and color quantum numbers of the constituent quarks. A sum over repeated indices is implied. It is convenient to work with orthonormalized amplitudes,

$$\langle\alpha|\beta\rangle = \Phi_\alpha^{*\mu\nu} \Phi_\beta^{\mu\nu} = \delta_{\alpha\beta}. \quad (3)$$

The quark and antiquark operators satisfy canonical anticommutation relations,

$$\begin{aligned} \{q_\mu, q_\nu^\dagger\} &= \{\bar{q}_\mu, \bar{q}_\nu^\dagger\} = \delta_{\mu\nu}, \\ \{q_\mu, q_\nu\} &= \{\bar{q}_\mu, \bar{q}_\nu\} = \{q_\mu, \bar{q}_\nu\} = \{q_\mu, \bar{q}_\nu^\dagger\} = 0. \end{aligned} \quad (4)$$

Using these quark anticommutation relations, and the normalization condition of Eq. (3), it is easily shown that the meson operators satisfy the following noncanonical commutation relations:

$$[M_\alpha, M_\beta^\dagger] = \delta_{\alpha\beta} - M_{\alpha\beta}, \quad [M_\alpha, M_\beta] = 0, \quad (5)$$

where

$$M_{\alpha\beta} = \Phi_\alpha^{*\mu\nu} \Phi_\beta^{\mu\sigma} \bar{q}_\sigma^\dagger \bar{q}_\nu + \Phi_\alpha^{*\mu\nu} \Phi_\beta^{\rho\nu} q_\rho^\dagger q_\mu. \quad (6)$$

A transformation is defined such that a single-meson state  $|\alpha\rangle$  is redescribed by an (“ideal”) elementary-meson state by

$$|\alpha\rangle \longrightarrow U^{-1}|\alpha\rangle = m_\alpha^\dagger |0\rangle, \quad (7)$$

where  $m_\alpha^\dagger$  an ideal meson creation operator. The ideal meson operators  $m_\alpha^\dagger$  and  $m_\alpha$  satisfy, by definition, canonical commutation relations

$$[m_\alpha, m_\beta^\dagger] = \delta_{\alpha\beta}, \quad [m_\alpha, m_\beta] = 0. \quad (8)$$

The state  $|0\rangle$  is the vacuum of both  $q$  and  $m$  degrees of freedom in the new representation. In addition, in the new representation the quark and antiquark operators  $q^\dagger, q, \bar{q}^\dagger$ , and  $\bar{q}$  are kinematically independent of the  $m_\alpha^\dagger$  and  $m_\alpha$ ,

$$[q_\mu, m_\alpha] = [q_\mu, m_\alpha^\dagger] = [\bar{q}_\mu, m_\alpha] = [\bar{q}_\mu, m_\alpha^\dagger] = 0. \quad (9)$$

The unitary operator  $U$  of the transformation is

$$U(t) = \exp[tF], \quad (10)$$

where  $F$  is the generator of the transformation and  $t$  a parameter which is set to  $-\pi/2$  to implement the mapping. The generator  $F$  of the transformation is

$$F = m_\alpha^\dagger \tilde{M}_\alpha - \tilde{M}_\alpha^\dagger m_\alpha, \quad (11)$$

where

$$\tilde{M}_\alpha = \sum_{i=0}^n \tilde{M}_\alpha^{(i)}, \quad (12)$$

with

$$\begin{aligned} [\tilde{M}_\alpha, \tilde{M}_\beta^\dagger] &= \delta_{\alpha\beta} + \mathcal{O}(\Phi^{n+1}), \\ [\tilde{M}_\alpha, \tilde{M}_\beta] &= [\tilde{M}_\alpha^\dagger, \tilde{M}_\beta^\dagger] = 0. \end{aligned} \quad (13)$$

It is easy to see from (11) that  $F^\dagger = -F$ , which ensures that  $U$  is unitary. The index  $i$  in Eq. (12) represents the order of the expansion in powers of the wave function  $\Phi$ . The  $\tilde{M}_\alpha$  operator is determined up to a specific order  $n$  consistent with (13).

The next step is to obtain the transformed operators in the new representation. The basic operators of the model are expressed in terms of the quark operators. Therefore, in order to obtain the operators in the new representation, one writes

$$q(t) = U^{-1}qU, \quad \bar{q}(t) = U^{-1}\bar{q}U. \quad (14)$$

Once a microscopic interaction Hamiltonian  $H_I$  is defined, at the quark level, a new transformed Hamiltonian can be obtained. This effective interaction, the *Fock-Tani Hamiltonian* ( $\mathcal{H}_{\text{FT}}$ ), is obtained by the application of the unitary operator  $U$  on the microscopic Hamiltonian  $H_I$ , i.e.,  $\mathcal{H}_{\text{FT}} = U^{-1}H_IU$ . The transformed Hamiltonian describes all possible processes involving mesons and quarks. The general structure of  $\mathcal{H}_{\text{FT}}$  is of the following form:

$$\mathcal{H}_{\text{FT}} = \mathcal{H}_q + \mathcal{H}_m + \mathcal{H}_{\text{mq}}, \quad (15)$$

where the first term involves only quark operators, the second one involves only ideal meson operators, and  $\mathcal{H}_{\text{mq}}$  involves quark and meson operators. In  $\mathcal{H}_{\text{FT}}$  there are higher-order terms that provide bound-state corrections to the lower-order

ones. The basic quantity for these corrections is the *bound-state kernel*  $\Delta$  defined as

$$\Delta(\rho\tau; \lambda\nu) = \Phi_{\xi}^{\rho\tau} \Phi_{\xi}^{*\lambda\nu}. \quad (16)$$

The physical meaning of the  $\Delta$  kernel becomes evident, in the sense that it modifies the quark-antiquark interaction strength [10,16]. The following two examples can clarify the physical interpretation.

### 1. First example

Consider that the starting point is a two-body microscopic quark-antiquark Hamiltonian of the form

$$H_{2q} = T(\mu)q_{\mu}^{\dagger}q_{\mu} + T(\nu)\bar{q}_{\nu}^{\dagger}\bar{q}_{\nu} + V_{q\bar{q}}(\mu\nu; \sigma\rho)q_{\mu}^{\dagger}\bar{q}_{\nu}^{\dagger}\bar{q}_{\rho}q_{\sigma} \\ + \frac{1}{2}V_{qq}(\mu\nu; \sigma\rho)q_{\mu}^{\dagger}q_{\nu}^{\dagger}q_{\rho}q_{\sigma} + \frac{1}{2}V_{\bar{q}\bar{q}}(\mu\nu; \sigma\rho)\bar{q}_{\mu}^{\dagger}\bar{q}_{\nu}^{\dagger}\bar{q}_{\rho}\bar{q}_{\sigma}. \quad (17)$$

The transformation  $\mathcal{H}_{\text{FT}} = U^{-1}H_{2q}U$  is implemented again by transforming each quark and antiquark operator in Eq. (17), where a similar structure to Eq. (15) is obtained. In free space, the wave function  $\Phi$  of Eq. (2) satisfies the following equation:

$$H(\mu\nu; \sigma\rho)\Phi_{[\alpha]}^{\sigma\rho} = \epsilon_{[\alpha]}\Phi_{[\alpha]}^{\mu\nu}, \quad (18)$$

where  $H(\mu\nu; \sigma\rho)$  is the Hamiltonian matrix

$$H(\mu\nu; \sigma\rho) = \delta_{\mu[\sigma]\delta_{\nu[\rho]}[T([\sigma]) + T([\rho])] + V_{q\bar{q}}(\mu\nu; \sigma\rho), \quad (19)$$

where  $\epsilon_{[\alpha]}$  is the total energy of the meson. There is no sum over repeated indices inside square brackets.

The effective quark Hamiltonian  $\mathcal{H}_{2q}$  has an identical structure to the microscopic quark Hamiltonian, Eq. (17), except for inclusion of a term corresponding to a modification in the quark-antiquark interaction as follows:

$$\mathcal{H}_{2q} = H_{2q} + \bar{H}_{q\bar{q}}, \quad (20)$$

with

$$\bar{H}_{q\bar{q}} = (-H\Delta - \Delta H + \Delta H\Delta)q_{\mu}^{\dagger}\bar{q}_{\nu}^{\dagger}\bar{q}_{\rho}q_{\sigma}, \quad (21)$$

where the contractions are  $H\Delta \equiv H(\mu\nu; \tau\xi)\Delta(\tau\xi; \sigma\rho)$  and  $\Delta H\Delta \equiv \Delta(\mu\nu; \zeta\eta)H(\zeta\eta; \tau\xi)\Delta(\tau\xi; \sigma\rho)$ . An important property of the bound-state kernel is

$$\Delta(\mu\nu; \sigma\rho)\Phi_{\alpha}^{\sigma\rho} = \Phi_{\alpha}^{\mu\nu}, \quad (22)$$

which follows from the wave function's orthonormalization, Eq. (3). In the case that  $\Phi$  is a solution of Eq. (18), Eq. (20) reduces to

$$\mathcal{H}_{2q} = H_{2q} - \epsilon_{\beta}N_{\beta}, \quad (23)$$

where  $N_{\beta} = M_{\beta}^{\dagger}M_{\beta}$  is the number operator and the following property holds:  $N_{\beta}|\alpha\rangle = |\beta\rangle$ .

The spectrum of the modified quark Hamiltonian  $\mathcal{H}_{2q}$  is positive semidefinite and hence has no bound states [9]. To show this, consider an arbitrary state  $|\alpha\rangle$  formed from a pair quark and antiquark:

$$|\alpha\rangle = \Psi_{\alpha}^{\mu\nu}q_{\mu}^{\dagger}\bar{q}_{\nu}^{\dagger}|0\rangle. \quad (24)$$

The action of the Hamiltonian (23) on this state results in

$$\mathcal{H}_{2q}|\alpha\rangle = (H_{2q} - \epsilon_{\beta}N_{\beta})|\alpha\rangle \\ = H_{2q}|\alpha\rangle - \epsilon_{\beta}\Phi_{\beta}^{*\mu\nu}\Psi_{\alpha}^{\mu\nu}|\beta\rangle. \quad (25)$$

If  $|\alpha\rangle$  is one of the bound eigenstates of the microscopic Hamiltonian then

$$\mathcal{H}_{2q}|\alpha\rangle = (\epsilon_{[\alpha]} - \epsilon_{[\alpha]})|[\alpha]\rangle = 0. \quad (26)$$

On the other hand, if  $\Psi_{\alpha}^{\mu\nu}$  is orthogonal to all bound states  $\Phi_{\alpha}^{\mu\nu}$  then (25) reduces to

$$\mathcal{H}_{2q}|\alpha\rangle = H_{2q}|\alpha\rangle. \quad (27)$$

Let  $\psi_{i\alpha}$  be the continuum (unbound, positive energy) eigenstates of  $H_{2q}$ , with energies  $\epsilon_{i\alpha} \geq 0$ . One can expand any  $\Psi_{\alpha}^{\mu\nu}$  in the form

$$\Psi_{\alpha}^{\mu\nu} = \sum_{\kappa} c_{\kappa}\Phi_{\kappa}^{\mu\nu} + \sum_i c_i\psi_{i\alpha}^{\mu\nu}, \quad (28)$$

where  $(\Phi_{\kappa}, \psi_i) = 0$ . Then by (26) and (27)

$$\mathcal{H}_{2q}|\alpha\rangle = \sum_i \epsilon_{i\alpha}c_i\psi_{i\alpha}^{\mu\nu}|\mu\nu\rangle. \quad (29)$$

Therefore,

$$\langle\alpha|\mathcal{H}_{2q}|\alpha\rangle = \sum_i \epsilon_{i\alpha}|c_i|^2, \quad (30)$$

where it is evident that  $\mathcal{H}_{2q}$  is semidefinite positive and therefore does not have quark-antiquark bound states.

### 2. Second example

Consider in the ideal meson sector  $\mathcal{H}_{\text{m}}$  of equation (15), many approaches similar to the Fock-Tani formalism [10] have obtained, for example, the meson-meson scattering interaction in the Born approximation: resonating group method (RGM) [18], quark Born diagram formalism (QBDF) [19],

$$H_{mm} = T_{mm} + V_{mm}, \quad (31)$$

where  $T_{mm}$  is the kinetic term and  $V_{mm}$  is the meson-meson interaction potential with constituent interchange. This potential is given by

$$V_{mm} = V_{mm}^{\text{dir}} + V_{mm}^{\text{exc}} + V_{mm}^{\text{int}}, \quad (32)$$

where  $V_{mm}^{\text{dir}}$  is the direct potential (no quark interchange),  $V_{mm}^{\text{exc}}$  is the quark exchange term, and  $V_{mm}^{\text{int}}$  is the intra-exchange term. As shown in Refs. [10,20], if one extends the Fock-Tani calculation to higher orders, a new meson-meson Hamiltonian is obtained:

$$\bar{H}_{mm} = H_{mm} + \delta H_{mm}, \quad (33)$$

where  $\delta H_{mm}$  is the bound-state correction Hamiltonian,

$$\delta H_{mm} = \frac{1}{2}\Phi_{\alpha}^{*\mu\nu}\Phi_{\beta}^{*\rho\sigma}H(\mu\nu; \lambda\tau)\Delta(\lambda\tau; \mu'\sigma')\Phi_{\gamma}^{\mu'\sigma'}\Phi_{\gamma}^{\rho\sigma'} \\ + \frac{1}{2}\Phi_{\alpha}^{*\rho\sigma}\Phi_{\beta}^{*\mu\nu}H(\mu\nu; \lambda\tau)\Delta(\lambda\tau; \mu'\sigma')\Phi_{\delta}^{\rho\sigma'}\Phi_{\gamma}^{\mu'\sigma'} \\ + \frac{1}{2}\Phi_{\alpha}^{*\mu\sigma}\Phi_{\beta}^{*\rho\nu}\Delta(\mu\nu; \lambda\tau)H(\lambda\tau; \mu'v')\Phi_{\delta}^{\mu'v'}\Phi_{\gamma}^{\rho\sigma} \\ + \frac{1}{2}\Phi_{\alpha}^{*\rho\nu}\Phi_{\beta}^{*\mu\sigma}\Delta(\mu\nu; \lambda\tau)H(\lambda\tau\mu'v')\Phi_{\delta}^{\rho\sigma}\Phi_{\gamma}^{\mu'v'}. \quad (34)$$

If the wave function  $\Phi$  is chosen to be an eigenstate of the microscopic quark Hamiltonian, then the intra-exchange term  $V_{mm}^{\text{int}}$  is canceled exactly:

$$V_{mm}^{\text{int}} + \delta H_{mm} = 0. \quad (35)$$

In summary, these examples reveal an important and common feature of these corrections to leading order: They modify the microscopic potential in the presence of bound states.

### B. Meson decay in the Fock-Tani formalism

In the present calculation, the microscopic interaction Hamiltonian is a pair creation Hamiltonian  $H_{q\bar{q}}$  defined as

$$H_{q\bar{q}} = V_{\mu\nu} q_{\mu}^{\dagger} \bar{q}_{\nu}^{\dagger}, \quad (36)$$

where in Eq. (36) a sum (integration) is again implied over repeated indexes [16]. The pair creation potential  $V_{\mu\nu}$  is given by

$$V_{\mu\nu} \equiv g \delta_{c_{\mu}c_{\nu}} \delta_{f_{\mu}f_{\nu}} \delta(\vec{p}_{\mu} + \vec{p}_{\nu}) \bar{u}_{s_{\mu}}(\vec{p}_{\mu}) v_{s_{\nu}}(\vec{p}_{\nu}), \quad (37)$$

with  $g = 2m_q \gamma$ , where  $\gamma$  is the pair production strength and the indexes  $c_{\mu}$ ,  $f_{\mu}$ ,  $s_{\mu}$  are of color, flavor, and spin. The pair production is obtained from the nonrelativistic limit of  $H_{q\bar{q}}$  involving Dirac quark fields [5]. Applying the Fock-Tani transformation to  $H_{q\bar{q}}$  one obtains the effective Hamiltonian that describes a decay process. In the FTf perspective a new aspect is introduced to meson decay: bound-state corrections. The lowest-order correction is one that involves only one bound-state kernel  $\Delta$ . The bound-state corrected  $C^3P_0$  Hamiltonian is

$$H^{C3P_0} = -\Phi_{\alpha}^{*\rho\xi} \Phi_{\beta}^{*\lambda\tau} \Phi_{\gamma}^{\omega\sigma} V^{C3P_0} m_{\alpha}^{\dagger} m_{\beta}^{\dagger} m_{\gamma}, \quad (38)$$

where  $V^{C3P_0}$  is a condensed notation for

$$V^{C3P_0} = (\bar{\delta} + \bar{\Delta}) V_{\mu\nu}, \quad (39)$$

where

$$\begin{aligned} \bar{\delta} &= \delta_{\mu\lambda} \delta_{\nu\xi} \delta_{\omega\rho} \delta_{\sigma\tau}, \\ \bar{\Delta} &= \frac{1}{4} [\delta_{\sigma\xi} \delta_{\lambda\mu} \Delta(\rho\tau; \omega\nu) + \delta_{\xi\nu} \delta_{\lambda\omega} \Delta(\rho\tau; \mu\sigma) \\ &\quad - 2\delta_{\sigma\xi} \delta_{\lambda\omega} \Delta(\rho\tau; \mu\nu)]. \end{aligned} \quad (40)$$

The first term of (39), involving  $\bar{\delta}$ , is the usual  $^3P_0$  decay potential. The following  $\bar{\Delta}$  term, containing three  $\Delta$ s, is the bound-state correction to the potential. In the ideal meson space the initial and final states involve only ideal meson operators  $|i\rangle = m_{\gamma}^{\dagger}|0\rangle$  and  $|f\rangle = m_{\alpha}^{\dagger} m_{\beta}^{\dagger}|0\rangle$ . The  $C^3P_0$  amplitude is obtained by the following matrix element:

$$\langle f | H^{C3P_0} | i \rangle = \delta(\vec{P}_{\gamma} - \vec{P}_{\alpha} - \vec{P}_{\beta}) h_{fi}^{C3P_0}. \quad (41)$$

The  $h_{fi}^{C3P_0}$  decay amplitude is combined with relativistic phase space, resulting in the differential decay rate

$$\frac{d\Gamma_{\gamma \rightarrow \alpha\beta}}{d\Omega} = 2\pi P \frac{E_{\alpha} E_{\beta}}{M_{\gamma}} w |h_{fi}^{C3P_0}|^2, \quad (42)$$

which, after integration in the solid angle  $\Omega$ , a usual choice for the meson momenta is made:  $\vec{P}_{\gamma} = 0$  ( $P = |\vec{P}_{\alpha}| = |\vec{P}_{\beta}|$ ).

### III. $\phi$ MESON DECAY

The previous section has outlined the essential aspects of the  $C^3P_0$  model and how it is obtained from the Fock-Tani formalism, where the decay Hamiltonian  $H^{C3P_0}$  was deduced. In this section the phenomenological Hamiltonian  $H^{C3P_0}$  will be used in order to evaluate the  $n^3S_1$  decays  $\phi(1020)$ ,  $\phi(1680)$ ,  $\phi(2050)$  with  $n = 1, 2, 3$  and the  $1^3D_J$  decays  $\phi_1(1850)$ ,  $\phi_2(1850)$ , and  $\phi_3(1850)$  mesons.

#### A. Amplitudes and decay widths

In the following decay channels that shall be studied, some have been observed, with no available data, while others are only theoretical [7,21]:

- (a)  $\phi(1020) \rightarrow KK$ ;
- (b)  $\phi(1680) \rightarrow KK, KK^*, \eta\phi$ ;
- (c)  $\ddagger\phi(2050) \rightarrow KK, KK^*, K^*K^*, KK_1(1270), KK_1(1400), KK_0^*(1430), KK_2^*(1430), KK^*(1410), KK(1460), \eta\phi, \eta'\phi, \eta h_1(1380)$ ;
- (d)  $\ddagger\phi_1(1850) \rightarrow K^*K^*$ ;
- (e)  $\ddagger\phi_2(1850) \rightarrow KK, KK^*, K^*K^*, \eta\phi$ ;
- (f)  $\phi_3(1850) \rightarrow KK, KK^*, K^*K^*, KK_1(1270), \eta\phi$ ;

where the  $\ddagger$  symbol indicates an unobserved state,  $\phi_1(1850)$  is sometimes referred only as  $\phi(1850)$ , and  $K^*$  is actually  $K^*(892)$  from the Particle Data Group [21]. In the calculation of the decay amplitudes, the matrix element is given by (41) where the decay Hamiltonian (38) can be split into two parts:  $H^{C3P_0} = H_m + \delta H_m$ . The matrix element of first term, containing  $H_m$ , the term without the bound-state correction, is given by

$$\langle f | H_m | i \rangle = -d_1 - d_2, \quad (43)$$

where

$$\begin{aligned} d_1 &= \Phi_{\alpha}^{*\rho\nu} \Phi_{\beta}^{*\mu\lambda} \Phi_{\gamma}^{\rho\lambda} V_{\mu\nu}, \\ d_2 &= \Phi_{\alpha}^{*\mu\lambda} \Phi_{\beta}^{*\rho\nu} \Phi_{\gamma}^{\rho\lambda} V_{\mu\nu}. \end{aligned} \quad (44)$$

The matrix element of the bound-state correction  $\delta H_m$  is written as

$$\langle f | \delta H_m | i \rangle = - \sum_{k=1}^3 \sum_{j=1}^2 d_j^k, \quad (45)$$

where we introduce the following notation:

$$\begin{aligned} d_1^1 &= \frac{1}{4} \Phi_{\alpha}^{*\rho\sigma} \Phi_{\beta}^{*\mu\tau} \Delta(\rho\tau; \lambda\nu) \Phi_{\gamma}^{\lambda\sigma} V_{\mu\nu}, \\ d_2^1 &= \frac{1}{4} \Phi_{\alpha}^{*\mu\tau} \Phi_{\beta}^{*\rho\sigma} \Delta(\rho\tau; \lambda\nu) \Phi_{\gamma}^{\lambda\sigma} V_{\mu\nu}, \\ d_1^2 &= -\frac{1}{2} \Phi_{\alpha}^{*\rho\sigma} \Phi_{\beta}^{*\lambda\tau} \Delta(\rho\tau; \mu\nu) \Phi_{\gamma}^{\lambda\sigma} V_{\mu\nu}, \\ d_2^2 &= -\frac{1}{2} \Phi_{\alpha}^{*\lambda\tau} \Phi_{\beta}^{*\rho\sigma} \Delta(\rho\tau; \mu\nu) \Phi_{\gamma}^{\lambda\sigma} V_{\mu\nu}, \\ d_1^3 &= \frac{1}{4} \Phi_{\alpha}^{*\rho\nu} \Phi_{\beta}^{*\lambda\tau} \Delta(\rho\tau; \mu\sigma) \Phi_{\gamma}^{\lambda\sigma} V_{\mu\nu}, \\ d_2^3 &= \frac{1}{4} \Phi_{\alpha}^{*\lambda\tau} \Phi_{\beta}^{*\rho\nu} \Delta(\rho\tau; \mu\sigma) \Phi_{\gamma}^{\lambda\sigma} V_{\mu\nu}. \end{aligned} \quad (46)$$

In  $d_1^k$  and  $d_2^k$ , the index  $k = 1, 2, 3$  represents the first, second, and third term of the correction, respectively. As can be seen in equations (43)–(46), the matrix elements depend directly on the wave functions  $\Phi_{\alpha}^{\mu\nu}$  and the potential  $V_{\mu\nu}$ . Considering

as the fundamental degrees of freedom color  $C$ , flavor  $f$ , spin  $\chi$ , and space  $\Phi$ , the meson wave function can be written as product:

$$\Phi_{\alpha}^{\mu\nu} = C^{c_{\mu}c_{\nu}} f_{f_{\alpha}}^{f_{\mu}f_{\nu}} \chi_{S_{\alpha}}^{s_{\mu}s_{\nu}} \Phi_{nl}(\vec{P}_{\alpha} - \vec{p}_{\mu} - \vec{p}_{\nu}), \quad (47)$$

allowing to calculate color, flavor, and spin-space separately. Details of (47) can be found in Appendix B. This factorization of the wave function implies that equations (44) and (46) can also be put in a direct product form of color, flavor, and spin-space:

$$d_1 = d_1^c d_1^f d_1^{s-s}, \quad d_2 = d_2^c d_2^f d_2^{s-s}, \quad (48)$$

and

$$\begin{aligned} d_1^1 &= \frac{1}{4} d_1^{1c} d_1^{1f} d_1^{1s-s}, & d_2^1 &= \frac{1}{4} d_2^{1c} d_2^{1f} d_2^{1s-s}, \\ d_1^2 &= -\frac{1}{2} d_1^{2c} d_1^{2f} d_1^{2s-s}, & d_2^2 &= -\frac{1}{2} d_2^{2c} d_2^{2f} d_2^{2s-s}, \\ d_1^3 &= \frac{1}{4} d_1^{3c} d_1^{3f} d_1^{3s-s}, & d_2^3 &= \frac{1}{4} d_2^{3c} d_2^{3f} d_2^{3s-s}. \end{aligned} \quad (49)$$

It is essential to note that the bound-state kernel definition, Eq. (16), has an implicit contraction in the  $\xi$  index, which physically implies a *sum over all species* condition. In practice, this means that one should sum over intermediate meson bound states. Any of the respective meson multiplet members can be considered in this sum. In our calculations, due to the symmetry of problem, the only possible states will have the following  $n^{2S+1}L_J$  and isospin quantum numbers:  $|1^1S_0\rangle$  and  $I=0$  (type  $\eta, \eta'$ ) or  $|1^3S_1\rangle$  and  $I=0$  (type  $\phi, \omega$ ). The bound-state kernel  $\Delta(\mu\nu; \rho\sigma)$  will then be a sum over  $\eta, \eta', \phi, \omega$  intermediate states, which can be written explicitly as

$$\begin{aligned} \Delta(\mu\nu; \rho\sigma) &= \Delta_{\eta}(\mu\nu; \rho\sigma) + \Delta_{\eta'}(\mu\nu; \rho\sigma) \\ &+ \Delta_{\phi}(\mu\nu; \rho\sigma) + \Delta_{\omega}(\mu\nu; \rho\sigma). \end{aligned} \quad (50)$$

The color amplitude factors of (48) and (49) can be calculated directly with the definition (B11) and the color part of (37), resulting in

$$d_1^c = d_2^c = \frac{1}{\sqrt{3}}. \quad (51)$$

Proceeding similarly, for the bound-state correction one has

$$d_1^{1c} = d_2^{1c} = \frac{d_1^{2c}}{3} = \frac{d_2^{2c}}{3} = d_1^{3c} = d_2^{3c} = \frac{1}{3\sqrt{3}}. \quad (52)$$

This result is independent of which meson is involved, Eqs. (51) and (52) are valid for all decay processes. The flavor factor will be evaluated in the next section for each case. General spin-space amplitude factors can be obtained from the matrix element (48) and (49), apart from a global momentum conservation  $\delta$ . The contribution without the bound-state correction is

$$\begin{aligned} d_1^{s-s} &= -2a_{1ij} \gamma \int d^3K \chi_i^*(\vec{\sigma} \cdot \vec{K}) \chi_j^c \phi^*(2\vec{K} + \vec{P}) \\ &\times \phi^*(2\vec{K} + \vec{P}) \phi(2\vec{K} + 2\vec{P}), \end{aligned} \quad (53)$$

TABLE I. Spectroscopic notation  $n^{2S+1}L_J$ , where  $\ddagger$  is undetected experimentally.

$n^{2S+1}L_J$	Meson
$1^1S_0$	$\eta, \eta', K$
$1^3S_1$	$\phi(1020), K^*$
$1^1P_1$	$h_1(1380)$
$1^3P_0$	$K_0^*(1430)$
$1^3P_2$	$K_2^*(1430)$
$1^3D_1$	$\phi_1(1850)^{\ddagger}$
$1^3D_2$	$\phi_2(1850)^{\ddagger}$
$1^3D_3$	$\phi_3(1850)$
$2^1S_0$	$K(1460)$
$2^3S_1$	$\phi(1680), K^*(1410)$
$3^3S_1$	$\phi(2050)^{\ddagger}$

and the three terms with the bound-state correction are

$$\begin{aligned} d_1^{1s-s} &= -2a_{1ij} \gamma \int d^3K d^3q \chi_i^*(\vec{\sigma} \cdot \vec{K}) \chi_j^c \phi^*(2\vec{q} + \vec{P}) \\ &\times \phi^*(2\vec{K} + \vec{P}) [\phi(\vec{q} + \vec{K} + 2\vec{P}) \phi^*(\vec{q} + \vec{K})]_{\xi} \phi(2\vec{q}), \\ d_1^{2s-s} &= -2a_{2ij} \gamma \int d^3K d^3q \chi_i^*(\vec{\sigma} \cdot \vec{K}) \chi_j^c \phi^*(2\vec{q} + \vec{P}) \\ &\times \phi^*(2\vec{q} + \vec{P}) [\phi(2\vec{q} + 2\vec{P}) \phi^*(2\vec{K})]_{\xi} \phi(2\vec{q}), \\ d_1^{3s-s} &= -2a_{3ij} \gamma \int d^3K d^3q \chi_i^*(\vec{\sigma} \cdot \vec{K}) \chi_j^c \phi^*(2\vec{q} - \vec{P}) \\ &\times \phi^*(2\vec{K} - \vec{P}) [\phi(\vec{q} + \vec{K} - 2\vec{P}) \phi^*(\vec{q} + \vec{K})]_{\xi} \phi(2\vec{q}), \end{aligned} \quad (54)$$

where  $a_{ij} = \chi_{\alpha}^{s_{\rho}s_{\nu}} \chi_{\beta}^{s_{\mu}s_{\lambda}} \chi_{\gamma}^{s_{\rho}s_{\lambda}}$  is a number resulting from the product of the meson's spin wave functions involved in the decay. The coefficients  $a_{1ij}, a_{2ij}$ , and  $a_{3ij}$  are obtained in a similar form and represent the first, second, and third bound-state correction term, respectively. Note that the wave functions in-between brackets in Eq. (54) are related to the bound-state kernel part and therefore it is assumed that there is an implicit sum over species with  $\xi$  assuming the  $\eta, \eta', \phi, \omega$  quantum numbers. The  $d_2^{s-s}$  and  $d_2^{is-s}$  amplitude factors are obtained simply by the changing  $\vec{P} \rightarrow -\vec{P}$  in Eqs. (53) and (54).

## B. Numerical results

### 1. General aspects

In this section, we present the numerical results for the  $\phi_j(M)$  decay widths. The amplitudes can be written in a general form

$$h_{fi}^{C3P0} = \frac{\gamma}{\pi^{1/4}} \mathcal{M}_{fi}, \quad (55)$$

where  $\mathcal{M}_{fi}$  appears in Appendix C. These amplitudes are inserted in Eq. (42) and integrated over the solid angle  $\Omega$ . To calculate  $\mathcal{M}_{fi}$ , the wave function must be determined, knowing the spin and space quantum numbers to be used, which are listed in Table I. The spatial wave functions are considered to be Gaussians characterized by the  $\beta$  parameter, which is the Gaussian's width. Each decay particle has its

TABLE II. Multiplicity factor  $\mathcal{F}$ .

Generic decay	Example	$\mathcal{F}$
$\phi \rightarrow (n\bar{s})(s\bar{n})$	$\phi_3(1850) \rightarrow K^+K^-$	2
$\phi \rightarrow (n\bar{s})(s\bar{n})'$	$\phi(1680) \rightarrow K^+K^{*-}$	4
$\phi \rightarrow (m)_{I=0}(m)_{I=0}^{(1)}$	$\phi(2050) \rightarrow \eta\phi$	1

own  $\beta$ . For example,  $\phi(1020)$  has the width  $\beta_\phi$ ,  $\phi(1680)$  has  $\beta_{\phi_{1680}}$ , and so on. The mesons which are part of the bound-state kernel also have their own widths and are distinguished from others by the notation  $\beta_{\eta_\Delta}$ ,  $\beta_{\eta'_\Delta}$ ,  $\beta_{\phi_\Delta}$ , and  $\beta_{\omega_\Delta}$ . To include all subprocesses in the results, it is necessary to multiply  $\Gamma$  by a multiplicity factor  $\mathcal{F}$ . For example, in  $\phi \rightarrow KK$ , the possible subprocesses are  $\phi \rightarrow K^+K^-$  and  $\phi \rightarrow K^0\bar{K}^0$ . Therefore, the multiplicity factor is  $\mathcal{F} = 2$ . The values of  $\mathcal{F}$  are listed in Table II.

The masses were obtained from Ref. [21], with the exception of  $\phi(2050)$ ,  $\phi_1(1850)$ ,  $\phi_2(1850)$ , which were extracted from Ref. [7]:  $M_{\phi(1020)} = 1.01945$  GeV,  $M_{\phi(1680)} = 1.680$  GeV,  $M_{\phi(2050)} = 2.050$  GeV,  $M_{\phi_1(1850)} = 1.850$  GeV,  $M_{\phi_2(1850)} = 1.850$  GeV,  $M_{\phi_3(1850)} = 1.854$  GeV,  $M_\eta = 0.54785$  GeV,  $M_{\eta'} = 0.95778$  GeV,  $M_K = 0.49367$  GeV,  $M_{K^*} = 0.89166$  GeV,  $M_{K_1(1270)} = 1.272$  GeV,  $M_{K_1(1400)} = 1.403$  GeV,  $M_{K_0^*(1430)} = 1.425$  GeV,  $M_{K_2^*(1430)} = 1.4256$  GeV,  $M_{K^*(1410)} = 1.414$  GeV,  $M_{K(1460)} = 1.460$  GeV,  $M_{h_1(1380)} = 1.386$  GeV.

For the initial- or final-state mesons the Gaussian width parameter is set to its characteristic value used for light mesons; namely,  $\beta_i = 0.4$  GeV [7,16]. The pair production strength parameter  $\gamma$  and the angle  $\theta$  in Eqs. (B9) and (B10) were also set according to Refs. [7,16]  $\gamma = 0.4$  and  $\theta \simeq 35.3^\circ$  ( $\cos \theta = \sqrt{2/3}$ ,  $\sin \theta = \sqrt{1/3}$ ). The  $h_1(1380)$  meson is considered a pure  $n\bar{n}$ ; therefore the coefficients in Eq. (B8) assume the values  $c_1^{h_1} = 1/\sqrt{2}$  and  $c_2^{h_1} = 0$ . The Gaussian widths  $\beta_i$ , remain as free parameters as well as the  $c_1^i$  and  $c_2^i$  coefficients in Eq. (B3) for the bound-state kernel. The parameters  $c_i^{\eta(\eta')}$  and  $c_i^{\eta_\Delta(\eta'_\Delta)}$  shall be a function of the same mixing angle  $\theta_p$ . Similarly with  $c_i^{\phi(\omega)}$  and  $c_i^{\phi_\Delta(\omega_\Delta)}$ , which are defined by an angle  $\theta_v$ . Thus, the free parameters to be adjusted are  $\theta_p, \theta_v, \theta_{v(1680)}, \theta_{v_3}, \beta_{\eta_\Delta}, \beta_{\eta'_\Delta}, \beta_{\phi_\Delta}$ , and  $\beta_{\omega_\Delta}$ . Where  $\theta_v, \theta_{v(1680)}$ , and  $\theta_{v_3}$  are the mixing angles of  $\phi(1020)$ ,  $\phi(1680)$ , and  $\phi_3(1850)$ , respectively.

## 2. S states

$\phi(1020)$  is a natural candidate for the  $s\bar{s}$  state with  $\phi(1680)$  as a radial excitation. One must note that the decay of  $\phi(1680)$  in  $KK$  and  $KK^*$  is sometimes cited as evidence that this state is  $s\bar{s}$ . The free parameters, which shall be varied, will be the wave function width  $\beta$  and the mixing angles. The decay of  $\phi(1020) \rightarrow K^+K^-$  has a partial width of  $2.08 \pm 0.04$  MeV. Following the predicted values for the mixing angles [21], we varied  $\theta_p$  between  $-20^\circ$  and  $-10^\circ$  and  $\theta_v$  between  $26^\circ$  and  $35^\circ$ . The  $\beta_{i_\Delta}$ s were varied from 0.3 to 0.6 GeV.

The two best fits for this channel have values of  $\theta_p = -10^\circ$ ,  $\theta_v = 35^\circ$ ,  $\beta_{\eta_\Delta} = 2\beta_{\eta'_\Delta} = 0.6$  GeV, which we call parametrization (a), resulting in  $\Gamma_{3P_0} = 3.21$  MeV and

TABLE III. Decay width of  $\phi(1680)$  with  $\theta_p = -10^\circ$ ,  $\theta_v = 26^\circ$ ,  $\theta_{v(1680)} = 35^\circ$ , (a)  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 3\beta_{\phi_\Delta} = 2\beta_{\omega_\Delta} = 1.2$  GeV and (b)  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 4\beta_{\phi_\Delta} = 3\beta_{\omega_\Delta} = 1.2$  GeV.

$\Gamma$ (MeV)	3P0 (a)	C3P0 (a)	C3P0 (b)	Expt.
$KK$	89.42	87.42	87.42	
$KK^*$	123.28	123.25	122.38	
$\eta\phi$	21.63	26.81	23.49	
$\Gamma^{\text{tot}}$	234.33	237.48	233.29	$150 \pm 50$

$\Gamma_{C3P_0} = 2.81$  MeV. A different parametrization, which we call (b), has  $\theta_p = -10^\circ$ ,  $\theta_v = 26^\circ$ ,  $4\beta_{\eta_\Delta} = 3\beta_{\eta'_\Delta} = 1.2$  GeV, resulting in  $\Gamma_{3P_0} = 2.38$  MeV and  $\Gamma_{C3P_0} = 2.01$  MeV.

The  $\phi(1680)$  meson has a total  $\Gamma_{\text{exp}}^{\text{tot}} = 150 \pm 50$  MeV, the C  $^3P_0$  model's best fit yields a  $\Gamma_{C3P_0}^{\text{tot}} = 233.29$  MeV, which corresponds to the values  $\theta_{v(1680)} = 35^\circ$ ,  $\beta_{\eta_\Delta} = 0.6$  GeV,  $\beta_{\eta'_\Delta} = 0.3$  GeV,  $\beta_{\phi_\Delta} = 0.4$  GeV, and  $\beta_{\omega_\Delta} = 0.6$  GeV. The estimated channels are in Table III.

From these results one can see that, for  $\phi(1020)$ , the decay width is within the experimental range. The same does not happen for  $\phi(1680)$ , where the total decay width is above the experimental value. It should be noted that, in the literature, there are results that indicate higher experimental values:  $211 \pm 14 \pm 19$  MeV in Ref. [22] and  $322 \pm 77 \pm 160$  MeV in Ref. [23]. Another important experimental result are the ratios  $\Gamma_{KK}/\Gamma_{KK^*} = 0.07 \pm 0.01$  and  $\Gamma_{\eta\phi}/\Gamma_{KK^*} \approx 0.37$  [21]. The fit of the C  $^3P_0$  model yields  $\Gamma_{KK}/\Gamma_{KK^*} = 0.71$  and  $\Gamma_{\eta\phi}/\Gamma_{KK^*} = 0.19$ . The  $\theta_{v(1680)}$  angle is a measure of strangeness content of  $\phi(1680)$ . A tentative solution to improve these ratios is to set this angle for values beyond the  $26^\circ$ – $35^\circ$  range. An increase in the angle can lower the total decay width into the range of experimental values ( $150 \pm 50$  GeV) and  $\Gamma_{KK}/\Gamma_{KK^*}$  is also improved. However, the ratio  $\Gamma_{\eta\phi}/\Gamma_{KK^*}$  did not improve. This inconsistency could be an indication that the composition of  $\phi(1680)$  is not well described and it may be a mixture of the states  $1^3D_1$  and  $2^3S_1$ , or it may have hybrid components [21].

$\phi(2050)$  is a  $3^3S_1$   $s\bar{s}$  vector state, to which an estimated mass of 2.05 GeV is assigned, although this state is actually not known at present. It should be important in future spectroscopic studies because with  $1^{--}$  quantum numbers it can be made both in diffractive photoproduction and in  $e^+e^-$  annihilation. As a radial excitation of  $s\bar{s}$  one can use the previous parametrizations. Due to the fact that  $\theta_v$  varies between  $26^\circ$  and  $35^\circ$ , we shall consider the extreme values and consider four sets of parameters. The results of these calculations are presented in Table IV and yields an average total C  $^3P_0$  width of  $\Gamma = 182.35 \pm 3.13$  MeV.

## 3. D states

As is well known,  $\phi_3(1850)$  was first reported in  $K^-p \rightarrow \phi_3\Lambda$  at CERN's bubble-chamber experiment [24]. It was originally reported in  $KK$  and  $KK^*$ , with a total width of  $87_{-23}^{+28}$  MeV and a relative branching fraction of  $B(KK^*/KK) = 0.55_{-0.45}^{+0.85}$  [21]. Following the same strategy

TABLE IV. Decay width of  $\phi(2050)$ , with  $\theta_p = -10^\circ$ ,  $\theta_v = 26^\circ$ , and the parametrizations (a)  $\theta_{v(2050)} = 35^\circ$ ,  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 4\beta_{\phi_\Delta} = 3\beta_{\omega_\Delta} = 1.2$  GeV, (1)  $\theta_{v(2050)} = 26^\circ$ ,  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 3\beta_{\phi_\Delta} = 2\beta_{\omega_\Delta} = 1.2$  GeV, (2)  $\theta_{v(2050)} = 26^\circ$ ,  $2\beta_{\eta_\Delta} = 3\beta_{\eta'_\Delta} = 2\beta_{\phi_\Delta} = 4\beta_{\omega_\Delta} = 1.2$  GeV, (3)  $\theta_{v(2050)} = 35^\circ$ ,  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 3\beta_{\phi_\Delta} = 2\beta_{\omega_\Delta} = 1.2$  GeV, (4)  $\theta_{v(2050)} = 35^\circ$ ,  $2\beta_{\eta_\Delta} = 3\beta_{\eta'_\Delta} = 2\beta_{\phi_\Delta} = 4\beta_{\omega_\Delta} = 1.2$  GeV.

$\Gamma$ (MeV)	3P0 (a)	C3P0 (1)	C3P0 (2)	C3P0 (3)	C3P0 (4)
$KK$	0.06	0.08	0.08	0.06	0.06
$KK^*$	9.88	7.59	7.60	9.89	9.90
$K^*K^*$	58.93	71.36	67.52	59.27	56.08
$KK_1(1270)$	6.55	5.90	6.02	6.37	6.51
$KK_1(1400)$	15.03	16.51	16.82	14.79	15.09
$KK_0^*(1430)$	0.00	0.00	0.00	0.00	0.00
$KK_2^*(1430)$	2.09	2.54	2.47	2.11	2.05
$KK^*(1410)$	46.72	36.33	34.91	47.34	45.50
$KK(1460)$	29.46	32.23	33.91	26.77	28.17
$\eta\phi$	10.44	9.49	9.57	10.11	10.20
$\eta'\phi$	4.61	4.52	4.56	4.48	4.52
$\eta h_1(1380)$	0.00	0.08	0.07	0.00	0.00
$\Gamma^{\text{tot}}$	183.77	186.63	183.53	181.19	178.08

as in the case of the  $S$  states, the  $\phi_3(1850)$  parameters that resulted in the best fit are shown in Table V:

$\theta_p = -10^\circ$ ,  $\theta_v = 26^\circ$ ,  $\theta_{v_3} = 35^\circ$ ,  $\beta_{\eta_\Delta} = 0.6$  GeV,  $\beta_{\eta'_\Delta} = 0.4$  GeV,  $\beta_{\phi_\Delta} = 0.6$  GeV, and  $\beta_{\omega_\Delta} = 0.3$  GeV. The  $C^3P_0$  branching fraction resulted in  $B(KK^*/KK) = 0.13$ , in accordance with the experimental limit.

After fixing these parameters, it was possible to estimate the decay widths for the unobserved mesons  $\phi_1(1850)$ ,  $\phi_2(1850)$ . The estimates are shown in Table VI. Again four sets of parameters were considered in these calculations. The average total  $C^3P_0$  width's estimates are  $\Gamma(\phi_1) = 0.595 \pm 0.058$  MeV and  $\Gamma(\phi_2) = 182.10 \pm 15.87$  MeV.

#### IV. COMPARING $^3P_0$ AND $C^3P_0$ MODELS

In our former study, in Ref. [17], decays in the light  $1S$  and  $1P$  sectors were analyzed and a comparison was made with the usual  $^3P_0$  results. Specifically, the decay processes  $\rho \rightarrow \pi\pi$ ,  $b_1 \rightarrow \omega\pi$ ,  $a_1 \rightarrow \rho\pi$ ,  $a_2 \rightarrow \rho\pi$ ,  $h_1 \rightarrow \rho\pi$ ,  $f_0 \rightarrow$

TABLE V. Decay width of  $\phi_3(1850)$   $\theta_p = -10^\circ$ ,  $\theta_v = 26^\circ$ ,  $\theta_{v_3} = 35^\circ$ , with the parametrizations (a)  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 4\beta_{\phi_\Delta} = 3\beta_{\omega_\Delta} = 1.2$  GeV, (b)  $2\beta_{\eta_\Delta} = 3\beta_{\eta'_\Delta} = 2\beta_{\phi_\Delta} = 4\beta_{\omega_\Delta} = 1.2$  GeV.

$\Gamma$ (MeV)	3P0 (a)	C3P0 (a)	C3P0 (b)	Expt.
$KK$	46.28	46.04	45.76	
$KK^*$	6.02	6.02	6.08	
$K^*K^*$	35.24	35.26	33.58	
$KK_1(1270)$	0.98	0.97	0.92	
$\eta\phi$	0.68	0.84	0.72	
$\Gamma^{\text{tot}}$	89.20	89.03	87.06	$87^{+28}_{-23}$

TABLE VI. Decay widths of  $\phi_1(1850)$  and  $\phi_2(1850)$  with  $\theta_p = -10^\circ$ ,  $\theta_v = 26^\circ$ , and the parametrizations (a)  $\theta_{v_2} = 35^\circ$ ,  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 4\beta_{\phi_\Delta} = 3\beta_{\omega_\Delta} = 1.2$  GeV, (1)  $\theta_{v_2} = 26^\circ$ ,  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 3\beta_{\phi_\Delta} = 2\beta_{\omega_\Delta} = 1.2$  GeV, (2)  $\theta_{v_2} = 26^\circ$ ,  $2\beta_{\eta_\Delta} = 3\beta_{\eta'_\Delta} = 2\beta_{\phi_\Delta} = 4\beta_{\omega_\Delta} = 1.2$  GeV, (3)  $\theta_{v_2} = 35^\circ$ ,  $2\beta_{\eta_\Delta} = 4\beta_{\eta'_\Delta} = 3\beta_{\phi_\Delta} = 2\beta_{\omega_\Delta} = 1.2$  GeV, (4)  $\theta_{v_2} = 35^\circ$ ,  $2\beta_{\eta_\Delta} = 3\beta_{\eta'_\Delta} = 2\beta_{\phi_\Delta} = 4\beta_{\omega_\Delta} = 1.2$  GeV.

	3P0 (a)	C3P0 (1)	C3P0 (2)	C3P0 (3)	C3P0 (4)
$\Gamma(\phi_1)$ (MeV)					
$K^*K^*$	0.54	0.67	0.63	0.56	0.52
$\Gamma(\phi_2)$ (MeV)					
$KK$	0.0	0.00	0.00	0.00	0.00
$KK^*$	135.71	104.31	103.83	135.94	135.31
$K^*K^*$	13.81	16.82	16.32	13.97	13.56
$\eta\phi$	50.74	45.14	46.05	48.12	49.05
$\Gamma^{\text{tot}}(\phi_2)$	200.26	166.27	166.20	198.03	197.92

$\pi\pi$ , and  $f_2 \rightarrow \pi\pi$ . One of the highlights of this study was the fact that all of these channels had experimental data. The model was adjusted in order to minimize  $R$ , defined by

$$R^2 = \sum_{i=1}^7 [a_i(\gamma, \beta) - 1]^2, \quad (56)$$

with  $a_i(\gamma, \beta) = \Gamma_i^{\text{theor}}(\gamma, \beta)/\Gamma_i^{\text{expt}}$ . A comparison of the  $^3P_0$  model with the  $C^3P_0$  was performed by varying the parameters  $\gamma$  and  $\beta$ , then searching for the minimum of (56). It was shown that the inclusion of the correction terms reduced the  $R$  value. A clear demonstration that the bound-state correction globally improves the fit. The average difference between the predictions of  $^3P_0$  and  $C^3P_0$ , in each individual channel, ranged from 1% to 14%. The higher differences were in the channels with lighter mesons in the final state. In the heavier channels, the leading order  $^3P_0$  is dominant and the bound-state corrections represent an actual *next-to-leading-order* correction.

In the present work, we studied the strange  $S$  and  $D$  states where data from individual channels are, in general, still not known. In the best situation, only the total  $\Gamma$  has an experimental value. Again a comparison was made between the theoretical predictions for  $^3P_0$  and  $C^3P_0$ . For example, in the decay of  $\phi(1020) \rightarrow KK$ , two fits were presented with a difference of 14% between  $^3P_0$  and  $C^3P_0$ . Again, the impact of the correction was larger in a channel with lighter mesons. A qualitative interpretation consists in observing that the tightly bound quark-antiquark pair of lower states are affected to a larger extent by the bound-state correction when compared with their radial excitation. In the heavier channels, as seen in Tables III to VI, this effect is clear and the discrepancy falls again to about 1%. This is similar to the case of the heavier mesons of  $1S$  and  $1P$  sectors studied in Ref. [17].

#### V. CONCLUSIONS

In this paper, we have concentrated on the  $\phi$  mesons, which are the strange  $S$  and  $D$  states, predicted in the quark model, as probable  $s\bar{s}$  resonances expected up to 2.2 GeV.

The central body of this study was to employ the Fock-Tani formalism, a field-theoretic mapping technique in order to obtain an effective interaction for meson decay. We have outlined the essential aspects of the C  $^3P_0$  model, the bound-state corrected  $^3P_0$  model, and how it is obtained from the Fock-Tani formalism, where the decay Hamiltonian  $H^{C^3P_0}$  was deducted.

This work is intended as a modest guide for future experimental studies of meson spectroscopy that may become possible with the advent of the new Hall D photoproduction facility GlueX at Jefferson Lab. The main goal of the GlueX experiment is to search for and study hybrid and exotic mesons. In this sense we have studied  $6s\bar{s}$  states, in which three were unobserved, presenting interesting issues for future experimental studies involving the conventional quark model states. The predicted total widths for these new, rather narrow states, that have not been identified, are  $\Gamma(\phi(2050)) = 182.35 \pm 3.13$  MeV,  $\Gamma(\phi_1) = 0.595 \pm 0.058$  MeV, and  $\Gamma(\phi_2) = 182.10 \pm 15.87$  MeV.

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### APPENDIX A: PHYSICAL NATURE OF BOUND-STATE CORRECTIONS

The bound-state corrections are an essential aspect of the Fock-Tani formalism, sometimes called *orthogonality corrections* because they are equivalent to the description of the continuum states by orthogonalized plane waves with the projection to bound-states subtracted [9,25,26]. Consider, for example, an ideal two-meson state (which shall be represented by a round ket)

$$|\alpha\beta\rangle = m_\alpha^\dagger m_\beta^\dagger |0\rangle. \quad (\text{A1})$$

The norm of (A1) can be calculated by using (8):

$$\langle\rho\sigma|\alpha\beta\rangle = \delta_{\alpha\rho}\delta_{\beta\sigma} + \delta_{\alpha\sigma}\delta_{\beta\rho}. \quad (\text{A2})$$

The same calculation can be done for the physical two-meson state

$$|\alpha\beta\rangle = M_\alpha^\dagger M_\beta^\dagger |0\rangle. \quad (\text{A3})$$

Defining the norm function as  $N(\rho\sigma;\alpha\beta) \equiv \langle\rho\sigma|\alpha\beta\rangle$  and using (5), one obtains

$$N(\rho\sigma;\alpha\beta) = \delta_{\alpha\rho}\delta_{\beta\sigma} - N_E(\rho\sigma;\alpha\beta) + (\rho \leftrightarrow \sigma), \quad (\text{A4})$$

where

$$N_E(\rho\sigma;\alpha\beta) = \Phi_\rho^{*\xi\tau} \Phi_\sigma^{*\omega\lambda} \Phi_\alpha^{\xi\lambda} \Phi_\beta^{\omega\tau}. \quad (\text{A5})$$

The presence of  $N_E$  in Eq. (A4) has its origin in the composite nature of the meson operator  $M_\alpha$  and implies that the two-meson state is not normalized as in Eq. (A2). A correctly normalized state would be written as

$$\overline{|\alpha\beta\rangle} = N^{-\frac{1}{2}}(\alpha\beta;\alpha'\beta')|\alpha'\beta'\rangle. \quad (\text{A6})$$

Now, consider the following state:

$$|\mu\nu\alpha\rangle = q_\mu^\dagger \bar{q}_\nu^\dagger M_\alpha^\dagger |0\rangle, \quad (\text{A7})$$

which by a similar procedure can be normalized, resulting in

$$\overline{|\mu\nu\alpha\rangle} = N_q^{-\frac{1}{2}}(\mu\nu\alpha;\mu'\nu'\alpha')|\mu'\nu'\alpha'\rangle, \quad (\text{A8})$$

where

$$N_q(\mu\nu\alpha;\mu'\nu'\alpha') = \delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{\alpha\alpha'} - N_E^q(\mu\nu\alpha;\mu'\nu'\alpha'), \quad (\text{A9})$$

with

$$N_E^q(\mu\nu\alpha;\mu'\nu'\alpha') = \delta_{\mu\mu'}\Phi_{\alpha'}^{*\xi\nu}\Phi_{\alpha'}^{\xi\nu'} + \delta_{\nu\nu'}\Phi_{\alpha'}^{*\mu\tau}\Phi_{\alpha'}^{\mu'\tau} - \Phi_{\alpha'}^{*\mu\nu}\Phi_{\alpha'}^{\mu'\nu'}. \quad (\text{A10})$$

A decay in which  $A \rightarrow BC$  is described by an amplitude obtained evaluating the following matrix element  $\langle BC|V|A\rangle$ . In second quantization this is written as

$$\begin{aligned} \langle BC|V|A\rangle &= \langle 0|M_\alpha M_\beta V_{\mu\nu} q_\mu^\dagger \bar{q}_\nu^\dagger M_\gamma^\dagger |0\rangle \\ &= V_{\mu\nu} \langle\alpha\beta|\mu\nu\gamma\rangle. \end{aligned} \quad (\text{A11})$$

According to what was shown, the state vectors in the decay amplitude (A11) should be replaced by a normalized version

$$\begin{aligned} \overline{\langle BC|V|A\rangle} &= V_{\mu\nu} \overline{\langle\alpha\beta|\mu\nu\gamma\rangle} \\ &= V_{\mu\nu} N^{-\frac{1}{2}}(\alpha\beta;\beta'\alpha') \langle\alpha'\beta'|\mu'\nu'\gamma'\rangle \\ &\quad \times N_q^{-\frac{1}{2}}(\mu'\nu'\gamma';\mu\nu\gamma). \end{aligned} \quad (\text{A12})$$

Defining a potential norm function as

$$\begin{aligned} N_V(\mu\nu\alpha'\beta';\mu'\nu'\gamma') &\equiv V_{\mu\nu} \langle\alpha'\beta'|\mu'\nu'\gamma'\rangle \\ &= V_1 - V_3, \end{aligned} \quad (\text{A13})$$

where  $V_i$ , with one or three wave functions is given by

$$\begin{aligned} V_1 &= \Phi_{\alpha'}^{*\mu'\nu'} \delta_{\beta'\gamma'} V_{\mu\nu} + (\alpha' \leftrightarrow \beta'), \\ V_3 &= \Phi_{\alpha'}^{*\mu'\tau} \Phi_{\beta'}^{*\omega\nu'} \Phi_{\gamma'}^{\omega\tau} V_{\mu\nu} + (\alpha' \leftrightarrow \beta'). \end{aligned} \quad (\text{A14})$$

The complete evaluation of the normalized decay amplitude is reduced to the expansion of (A12):

$$\begin{aligned} \overline{\langle BC|V|A\rangle} &= N^{-\frac{1}{2}}(\alpha\beta;\beta'\alpha') N_V(\mu\nu\alpha'\beta';\mu'\nu'\gamma') \\ &\quad \times N_q^{-\frac{1}{2}}(\mu'\nu'\gamma';\mu\nu\gamma). \end{aligned} \quad (\text{A15})$$

For example, in the lowest order of the expansion,

$$N^{-\frac{1}{2}} \approx 1 + \frac{1}{2}N_E, \quad N_q^{-\frac{1}{2}} \approx 1 + \frac{1}{2}N_E^q, \quad (\text{A16})$$

one obtains

$$\overline{\langle BC|V|A\rangle} \approx -\Phi_\alpha^{*\rho\xi} \Phi_\beta^{*\lambda\tau} \Phi_\gamma^{\omega\sigma} V(\rho\xi\lambda\tau;\omega\sigma) + (\alpha \leftrightarrow \beta), \quad (\text{A17})$$

where

$$V(\rho\xi\lambda\tau;\omega\sigma) = (\bar{\delta} + \bar{\Delta}_f) V_{\mu\nu}, \quad (\text{A18})$$



with

$$\begin{aligned}\bar{\delta} &= \delta_{\mu\lambda}\delta_{\nu\xi}\delta_{\omega\rho}\delta_{\sigma\tau}, \\ \bar{\Delta}_f &= f[\delta_{\sigma\xi}\delta_{\lambda\mu}\Delta(\rho\tau; \omega\nu) + \delta_{\xi\nu}\delta_{\lambda\omega}\Delta(\rho\tau; \mu\sigma) \\ &\quad - 2\delta_{\sigma\xi}\delta_{\lambda\omega}\Delta(\rho\tau; \mu\nu)].\end{aligned}\quad (\text{A19})$$

In Eq. (A19),  $f$  is combinatorial factor with the value  $f = 1$  related to the truncation of (A16) in the lowest order. When higher orders are considered, new contributions change this factor to the Fock-Tani value  $f = 1/4$  of (40).

In summary, the essence of the Fock-Tani formalism is to move the bound-state information from the state vector (1), written in second quantization, into the interaction (38). As shown in the *First* and *Second examples*, this action has different impacts in the physical system. The new state vector is now the ideal state vector (7) which satisfies canonical commutation relations (8).

If one chooses not to use the Fock-Tani formalism the decay amplitude can be evaluated directly by calculating the  $\langle BC|V|A\rangle$  matrix element. As a first approximation, neglecting the meson's composite structure,  ${}^3P_0$  gives a correct leading-order contribution to  $\langle BC|V|A\rangle$ . To go beyond this lowest order, one needs to calculate  $\langle BC|V|A\rangle$  and expand the kernels  $N$ ,  $N_q$ , and  $N_V$  in Eq. (A15), introducing the overlap effects due to the extended nature of the meson, which constitute the bound-state corrections.

## APPENDIX B: WAVE FUNCTIONS

The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad (\text{B1})$$

and the corresponding spinors are

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_1^c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_2^c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.\quad (\text{B2})$$

The meson flavor components listed in the decay channels (a)–(f) depend on the isospin  $I$  and strangeness  $s$ :

(1)  $I = 0$

$$\phi, \eta, \eta', h_1 \rightarrow \frac{c_1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) + c_2|s\bar{s}\rangle.\quad (\text{B3})$$

(2)  $I = 1/2, s = +1$

$$\begin{aligned}K^+ &\rightarrow -|u\bar{s}\rangle, & I_z &= +1/2, \\ K^0 &\rightarrow -|d\bar{s}\rangle, & I_z &= -1/2.\end{aligned}\quad (\text{B4})$$

(3)  $I = 1/2, s = -1$

$$\begin{aligned}\bar{K}^0 &\rightarrow -|s\bar{d}\rangle, & I_z &= +1/2, \\ K^- &\rightarrow |s\bar{u}\rangle, & I_z &= -1/2.\end{aligned}\quad (\text{B5})$$

The SU(3) mixing coefficients  $c_1$  and  $c_2$ , in Eq. (B3), for the pseudoscalar mesons  $\eta$  and  $\eta'$  are related through the

angle  $\theta_p$

$$\begin{aligned}|\eta\rangle &= \frac{c_1(\theta_p)}{\sqrt{2}}[|u\bar{u}\rangle + |d\bar{d}\rangle] - c_2(\theta_p)|s\bar{s}\rangle, \\ |\eta'\rangle &= \frac{c_2(\theta_p)}{\sqrt{2}}[|u\bar{u}\rangle + |d\bar{d}\rangle] + c_1(\theta_p)|s\bar{s}\rangle.\end{aligned}\quad (\text{B6})$$

Similarly, the coefficients of the vector mesons  $\phi(1020)$  and  $\omega(782)$  are related through angle  $\theta_v$ :

$$\begin{aligned}|\phi\rangle &= \frac{c_1(\theta_v)}{\sqrt{2}}[|u\bar{u}\rangle + |d\bar{d}\rangle] - c_2(\theta_v)|s\bar{s}\rangle, \\ |\omega\rangle &= \frac{c_2(\theta_v)}{\sqrt{2}}[|u\bar{u}\rangle + |d\bar{d}\rangle] + c_1(\theta_v)|s\bar{s}\rangle,\end{aligned}\quad (\text{B7})$$

where

$$\begin{aligned}c_1(\theta_i) &= \frac{\cos\theta_i}{\sqrt{3}} - \sqrt{\frac{2}{3}}\sin\theta_i, \\ c_2(\theta_i) &= \sqrt{\frac{2}{3}}\cos\theta_i + \frac{\sin\theta_i}{\sqrt{3}}.\end{aligned}\quad (\text{B8})$$

Equations (B4) and (B5) are also valid mesons to  $K_1, K^*, K_0^*$ , and  $K_2^*$ . In particular,  $K_1(1270)$  and  $K_1(1400)$  are related by mixing angle  $\theta$ :

$$\begin{aligned}|K_1(1270)\rangle &= +\cos\theta|1^1P_1\rangle + \sin\theta|1^3P_1\rangle, \\ |K_1(1400)\rangle &= -\sin\theta|1^1P_1\rangle + \cos\theta|1^3P_1\rangle.\end{aligned}\quad (\text{B9})$$

For antikaons there is a change in the sign:

$$\begin{aligned}|\bar{K}_1(1270)\rangle &= -\cos\theta|1^1P_1\rangle + \sin\theta|1^3P_1\rangle, \\ |\bar{K}_1(1400)\rangle &= +\sin\theta|1^1P_1\rangle + \cos\theta|1^3P_1\rangle.\end{aligned}\quad (\text{B10})$$

The color wave function is the same for all mesons and is given by

$$C^{c_\mu c_\nu} = \frac{1}{\sqrt{3}}\delta^{c_\mu c_\nu}, \quad c_k = 1, 2, 3.\quad (\text{B11})$$

The spin wave functions can be singlet or triplet:

Singlet ( $S = 0$ )

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad S_z = 0.\quad (\text{B12})$$

Triplet ( $S = 1$ )

$$\begin{aligned}|\uparrow\uparrow\rangle, & & S_z &= +1, \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), & & S_z &= 0, \\ |\downarrow\downarrow\rangle, & & S_z &= -1.\end{aligned}\quad (\text{B13})$$

The spatial wave functions are harmonic-oscillator functions, as they describe color confinement and provide analytical amplitudes

$$\Phi_{nl}(\vec{P}_\alpha - \vec{p}_\mu - \vec{p}_\nu) = \delta(\vec{P}_\alpha - \vec{p}_\mu - \vec{p}_\nu)\phi_{nl}(\vec{p}_\mu - \vec{p}_\nu),\quad (\text{B14})$$

where  $\phi_{nl}(\vec{p}_\mu - \vec{p}_\nu)$  is given by

$$\phi_{nl}(\vec{p}_\mu - \vec{p}_\nu) = \left(\frac{1}{2\beta}\right)^l N_{nl} |\vec{p}_\mu - \vec{p}_\nu|^l \phi(\vec{p}_\mu - \vec{p}_\nu) \mathcal{L}_{n-1}^{l+\frac{1}{2}} \left[ \frac{(\vec{p}_\mu - \vec{p}_\nu)^2}{4\beta^2} \right] Y_{lm}(\Omega_{\vec{p}_\mu - \vec{p}_\nu}), \quad (\text{B15})$$

where  $p_{\mu(\nu)}$  is the internal momentum,  $Y_{lm}$  spherical harmonic, and  $\beta$  the Gaussian width parameter. The momentum wave function  $\phi(\vec{p}_\mu - \vec{p}_\nu)$ , the normalization constant  $N_{nl}$ , and the Laguerre polynomials  $\mathcal{L}_{n-1}^{l+\frac{1}{2}}(p)$ , which depend on the radial  $n$  and orbital  $l$  quantum numbers, are all defined as

$$\begin{aligned} N_{nl} &= \left[ \frac{2(n-1)!}{\beta^3 \Gamma(n+l+1/2)} \right]^{\frac{1}{2}}, \\ \mathcal{L}_{n-1}^{l+\frac{1}{2}}(p) &= \sum_{k=0}^n \frac{(-)^k \Gamma(n+l+1/2)}{k!(n-k-1)! \Gamma(k+l+3/2)} p^k, \\ \phi(\vec{p}_\mu - \vec{p}_\nu) &= \exp \left[ -\frac{(\vec{p}_\mu - \vec{p}_\nu)^2}{8\beta^2} \right], \end{aligned} \quad (\text{B16})$$

where  $n = 1, 2, \dots$  and  $l = 0, 1, \dots$

### APPENDIX C: AMPLITUDES

In this Appendix, we present the results of the algebraic decay amplitudes of  $h_{fi}$ , for subprocesses of (a)–(f), obtained with the  $C^3P_0$  model. Defining

$$\begin{aligned} e_1(p, \beta_A, \beta_B, \beta_C) &= e^{-\frac{(\beta_B^2 + \beta_C^2)p^2}{8[\beta_C^2 \beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]}}, \\ e_2(p, \beta_A, \beta_B, \beta_C, \beta) &= e^{-\frac{[(2\beta_A^2 + \beta^2)\beta_B^2 + 2\beta^2 + \beta_C^2(2(\beta_A^2 + \beta_B^2) + 5\beta^2)]p^2}{8\beta^2[\beta_C^2(\beta_B^2 + 2\beta^2) + \beta_A^2(\beta_B^2 + \beta_C^2 + 2\beta^2)]}}, \end{aligned} \quad (\text{C1})$$

where  $\beta_A$  is the width of the Gaussian initial state of the meson,  $\beta_B$  and  $\beta_C$  of the final state mesons and  $\beta$  for the bound-state correction mesons, one has

$$\mathcal{M}_{fi}^{\phi(1020) \rightarrow KK} = C_{10}^{\phi(1020)} Y_{11}(\Omega_p), \quad (\text{C2})$$

$$\mathcal{M}_{fi}^{\phi(1680) \rightarrow KK} = C_{10}^{\phi(1680)KK} Y_{11}(\Omega_p), \quad (\text{C3})$$

$$\mathcal{M}_{fi}^{\phi(1680) \rightarrow KK^*} = C_{11}^{\phi(1680)KK^*} Y_{10}(\Omega_p), \quad (\text{C4})$$

$$\mathcal{M}_{fi}^{\phi(1680) \rightarrow \eta\phi} = C_{11}^{\phi(1680)\eta\phi} Y_{10}(\Omega_p), \quad (\text{C5})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK} = C_{10}^{\phi(2050)KK} Y_{11}(\Omega_p), \quad (\text{C6})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK^*} = C_{11}^{\phi(2050)KK^*} Y_{10}(\Omega_p), \quad (\text{C7})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow K^*K^*} = C_{12}^{\phi(2050)K^*K^*} Y_{1-1}(\Omega_p), \quad (\text{C8})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK_1(1270)} = C_{01}^{\phi(2050)KK_1(1270)} Y_{00}(\Omega_p) + C_{21}^{\phi(2050)KK_1(1270)} Y_{20}(\Omega_p), \quad (\text{C9})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK_1(1400)} = C_{01}^{\phi(2050)KK_1(1400)} Y_{00}(\Omega_p) + C_{21}^{\phi(2050)KK_1(1400)} Y_{20}(\Omega_p), \quad (\text{C10})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK_0^*(1430)} = C_{20}^{\phi(2050)KK_0^*(1430)} Y_{21}(\Omega_p), \quad (\text{C11})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK_2^*(1430)} = C_{22}^{\phi(2050)KK_0^*(1430)} Y_{2-1}(\Omega_p), \quad (\text{C12})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK^*(1410)} = C_{11}^{\phi(2050)KK^*(1410)} Y_{10}(\Omega_p), \quad (\text{C13})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow KK(1460)} = C_{10}^{\phi(2050)KK(1460)} Y_{11}(\Omega_p), \quad (\text{C14})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow \eta\phi} = C_{11}^{\phi(2050)\eta\phi} Y_{10}(\Omega_p), \quad (\text{C15})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow \eta'\phi} = C_{11}^{\phi(2050)\eta'\phi} Y_{10}(\Omega_p), \quad (\text{C16})$$

$$\mathcal{M}_{fi}^{\phi(2050) \rightarrow \eta h_1(1380)} = \mathcal{C}_{01}^{\phi(2050)\eta h_1} Y_{00}(\Omega_p) + \mathcal{C}_{21}^{\phi(2050)\eta h_1} Y_{20}(\Omega_p), \quad (\text{C17})$$

$$\mathcal{M}_{fi}^{\phi_1 \rightarrow K^* K^*} = \mathcal{C}_{12}^{\phi_1 K^* K^*} Y_{1-1}(\Omega_p) + \mathcal{C}_{32}^{\phi_1 K^* K^*} Y_{3-1}(\Omega_p), \quad (\text{C18})$$

$$\mathcal{M}_{fi}^{\phi_2 \rightarrow KK} = \mathcal{C}_{30}^{\phi_2 KK} Y_{32}(\Omega_p), \quad (\text{C19})$$

$$\mathcal{M}_{fi}^{\phi_2 \rightarrow KK^*} = \mathcal{C}_{11}^{\phi_2 KK^*} Y_{11}(\Omega_p) + \mathcal{C}_{31}^{\phi_2 KK^*} Y_{31}(\Omega_p), \quad (\text{C20})$$

$$\mathcal{M}_{fi}^{\phi_2 \rightarrow K^* K^*} = \mathcal{C}_{12}^{\phi_2 K^* K^*} Y_{10}(\Omega_p) + \mathcal{C}_{32}^{\phi_2 K^* K^*} Y_{30}(\Omega_p), \quad (\text{C21})$$

$$\mathcal{M}_{fi}^{\phi_2 \rightarrow \eta \phi} = \mathcal{C}_{11}^{\phi_2 \eta \phi} Y_{11}(\Omega_p) + \mathcal{C}_{31}^{\phi_2 \eta \phi} Y_{31}(\Omega_p), \quad (\text{C22})$$

$$\mathcal{M}_{fi}^{\phi_3 \rightarrow KK} = \mathcal{C}_{30}^{\phi_3 KK} Y_{33}(\Omega_p), \quad (\text{C23})$$

$$\mathcal{M}_{fi}^{\phi_3 \rightarrow KK^*} = \mathcal{C}_{31}^{\phi_3 KK^*} Y_{32}(\Omega_p), \quad (\text{C24})$$

$$\mathcal{M}_{fi}^{\phi_3 \rightarrow K^* K^*} = \mathcal{C}_{12}^{\phi_3 K^* K^*} Y_{11}(\Omega_p) + \mathcal{C}_{32}^{\phi_3 K^* K^*} Y_{31}(\Omega_p), \quad (\text{C25})$$

$$\mathcal{M}_{fi}^{\phi_3 \rightarrow KK_1(1270)} = \mathcal{C}_{21}^{\phi_3 KK_1(1270)} Y_{22}(\Omega_p) + \mathcal{C}_{41}^{\phi_3 KK_1(1270)} Y_{42}(\Omega_p), \quad (\text{C26})$$

$$\mathcal{M}_{fi}^{\phi_3 \rightarrow \eta \phi} = \mathcal{C}_{31}^{\phi_3 \eta \phi} Y_{32}(\Omega_p), \quad (\text{C27})$$

where  $\mathcal{C}_{LS}$  are given by  $\phi(1020) \rightarrow K^+ K^-$ :

$$\begin{aligned} \mathcal{C}_{10}^{\phi(1020)} = & (c_1^\phi - c_2^\phi) \{ f_1(p_{KK}, \beta_\phi, \beta_K) e_1(p_{KK}, \beta_\phi, \beta_K, \beta_K) \\ & - c_1^{\eta_\Delta} c_2^{\eta_\Delta} f_2(p_{KK}, \beta_\phi, \beta_K, \beta_{\eta_\Delta}) e_2(p_{KK}, \beta_\phi, \beta_K, \beta_K, \beta_{\eta_\Delta}) \\ & - c_1^{\eta'_\Delta} c_2^{\eta'_\Delta} f_2(p_{KK}, \beta_\phi, \beta_K, \beta_{\eta'_\Delta}) e_2(p_{KK}, \beta_\phi, \beta_K, \beta_K, \beta_{\eta'_\Delta}) \}, \end{aligned} \quad (\text{C28})$$

$$f_1(p, \beta_A, \beta_B) = \frac{8p\beta_A^{3/2}(\beta_B^2 + \beta_A^2)}{\sqrt{3}(\beta_B^2 + 2\beta_A^2)^{5/2}}, \quad (\text{C29})$$

$$f_2(p, \beta_A, \beta_B, \beta) = \frac{16p\beta_A^{3/2}\beta_B^3(\beta_B^2 + \beta_A^2)(\beta_B^2 + \beta^2)}{3\sqrt{3}[\beta_B^4 + 2\beta^2\beta_A^2 + 2\beta_B^2(\beta^2 + \beta_A^2)]^{5/2}}. \quad (\text{C30})$$

$\phi(1680) \rightarrow K^+ K^-$ :

$$\begin{aligned} \mathcal{C}_{10}^{\phi(1680)KK} = & (c_1^{\phi(1680)} - c_2^{\phi(1680)}) \{ f_3(p_{KK}, \beta_{\phi(1680)}, \beta_K) e_1(p_{KK}, \beta_{\phi(1680)}, \beta_K, \beta_K) \\ & + c_1^{\eta_\Delta} c_2^{\eta_\Delta} f_4(p_{KK}, \beta_{\phi(1680)}, \beta_K, \beta_{\eta_\Delta}) e_2(p_{KK}, \beta_{\phi(1680)}, \beta_K, \beta_K, \beta_{\eta_\Delta}) \\ & + c_1^{\eta'_\Delta} c_2^{\eta'_\Delta} f_4(p_{KK}, \beta_{\phi(1680)}, \beta_K, \beta_{\eta'_\Delta}) e_2(p_{KK}, \beta_{\phi(1680)}, \beta_K, \beta_K, \beta_{\eta'_\Delta}) \}, \end{aligned} \quad (\text{C31})$$

$$f_3(p, \beta_A, \beta_B) = \frac{4\sqrt{2}p\beta_A^{3/2}[-3\beta_B^6 + \beta_B^4\beta_A^2 + 20\beta_B^2\beta_A^4 + 12\beta_A^6 - 2\beta_A^2(\beta_B^2 + \beta_A^2)p^2]}{3(\beta_B^2 + 2\beta_A^2)^{9/2}}, \quad (\text{C32})$$

$$\begin{aligned} f_4(p, \beta_A, \beta_B, \beta) = & \frac{8\sqrt{2}p\beta_B^3\beta_A^{3/2}(\beta_B^2 + \beta^2)}{9[\beta_B^4 + 2\beta^2\beta_A^2 + 2\beta_B^2(\beta^2 + \beta_A^2)]^{9/2}} \{ [3\beta_B^4(\beta_B^2 + 2\beta^2) - 7\beta_B^4\beta_A^2 - 6(\beta_B^2 + \beta^2)\beta_A^4] \\ & [\beta_B^4 + 2\beta^2\beta_A^2 + 2\beta_B^2(\beta^2 + \beta_A^2)] + 2(\beta_B^2 + \beta^2)^2\beta_A^2(\beta_B^2 + \beta_A^2)p^2 \}. \end{aligned} \quad (\text{C33})$$

$\phi(1680) \rightarrow K^+ K^{*-}$ :

$$\begin{aligned} \mathcal{C}_{11}^{\phi(1680)KK^*} = & (c_1^{\phi(1680)} + c_2^{\phi(1680)}) \{ f_5(p_{KK^*}, \beta_{\phi(1680)}, \beta_K, \beta_K^*) e_1(p_{KK^*}, \beta_{\phi(1680)}, \beta_K, \beta_K^*) \\ & + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_6(p_{KK^*}, \beta_{\phi(1680)}, \beta_K, \beta_K^*, \beta_{\phi_\Delta}) e_2(p_{KK^*}, \beta_{\phi(1680)}, \beta_K, \beta_K^*, \beta_{\phi_\Delta}) \\ & + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_6(p_{KK^*}, \beta_{\phi(1680)}, \beta_K, \beta_K^*, \beta_{\omega_\Delta}) e_2(p_{KK^*}, \beta_{\phi(1680)}, \beta_K, \beta_K^*, \beta_{\omega_\Delta}) \}, \end{aligned} \quad (\text{C34})$$

$$\begin{aligned} f_5(p, \beta_A, \beta_B, \beta_C) = & -\frac{\sqrt{2}p\beta_B^{3/2}\beta_C^{3/2}\beta_A^{3/2}}{3[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^{9/2}} \{ 12\beta_B^6\beta_C^6 - 2\beta_B^4\beta_C^4(\beta_B^2 + \beta_C^2)\beta_A^2 - 20\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)^2\beta_A^4 \\ & - 6(\beta_B^2 + \beta_C^2)^3\beta_A^6 + p^2(\beta_B^2 + \beta_C^2)^2\beta_A^2[2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2)\beta_A^2] \}, \end{aligned} \quad (\text{C35})$$

$$\begin{aligned}
f_6(p, \beta_A, \beta_B, \beta_C, \beta) = & -\frac{2\sqrt{2}p\beta_B^{3/2}\beta_C^{3/2}\beta_A^{3/2}}{9[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2) + 2(\beta_C^2 + \beta_A^2)\beta^2]^{9/2}} (2\{\beta_C^2\beta_A^2 + \beta_B^2[\beta_C^2 + \beta_A^2] + 2[\beta_C^2 + \beta_A^2]\beta^2\} \\
& \times \{\beta_B^4[6\beta_C^4 - 7\beta_C^2\beta_A^2 - 3\beta_A^4] - \beta_B^2[7\beta_C^4\beta_A^2 + 6\beta_C^2\beta_A^4 + 2(-9\beta_C^4 + 7\beta_C^2\beta_A^2 + 6\beta_A^4)\beta^2] \\
& - 3[4\beta_C^2\beta_A^4\beta^2 + 4\beta_A^4\beta^4 + \beta_C^4(\beta_A^4 - 4\beta^4)]\} + \beta_A^2\{\beta_B^2 + \beta_C^2 + 2\beta^2\}^2\{\beta_C^2\beta_A^2 + \beta_B^2[2\beta_C^2 + \beta_A^2] \\
& + 2[\beta_C^2 + \beta_A^2]\beta^2\}p^2). \tag{C36}
\end{aligned}$$

$\phi(1680) \rightarrow \eta\phi$  :

$$\begin{aligned}
C_{11}^{\phi(1680)\eta\phi} = & -2\{[2c_1^\eta c_1^{\phi_{1680}} c_1^\phi + c_2^\eta c_2^{\phi_{1680}} c_2^\phi]f_5(p_{\eta\phi}, \beta_{\phi_{1680}}, \beta_\eta, \beta_\phi)e_1(p_{\eta\phi}, \beta_{\phi_{1680}}, \beta_\eta, \beta_\phi) \\
& + [2c_1^\eta c_1^{\phi_{1680}} c_1^\phi (c_1^{\phi_\Delta})^2 + c_2^\eta c_2^{\phi_{1680}} c_2^\phi (c_2^{\phi_\Delta})^2]f_6(p_{\eta\phi}, \beta_{\phi_{1680}}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta})e_2(p_{\eta\phi}, \beta_{\phi_{1680}}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta}) \\
& + [2c_1^\eta c_1^{\phi_{1680}} c_1^\phi (c_1^{\omega_\Delta})^2 + c_2^\eta c_2^{\phi_{1680}} c_2^\phi (c_2^{\omega_\Delta})^2]f_6(p_{\eta\phi}, \beta_{\phi_{1680}}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta})e_2(p_{\eta\phi}, \beta_{\phi_{1680}}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta})\}. \tag{C37}
\end{aligned}$$

$\phi(2050) \rightarrow K^+K^-$  :

$$\begin{aligned}
C_{10}^{\phi(2050)KK} = & (c_1^{\phi_{2050}} - c_2^{\phi_{2050}})\{f_7(p_{KK}, \beta_{\phi_{2050}}, \beta_K)e_1(p_{KK}, \beta_{\phi_{2050}}, \beta_K, \beta_K) \\
& + c_1^{\eta_\Delta} c_2^{\eta_\Delta} f_8(p_{KK}, \beta_{\phi_{2050}}, \beta_K, \beta_{\eta_\Delta})e_2(p_{KK}, \beta_{\phi_{2050}}, \beta_K, \beta_K, \beta_{\eta_\Delta}) \\
& + c_1^{\eta'_\Delta} c_2^{\eta'_\Delta} f_8(p_{KK}, \beta_{\phi_{2050}}, \beta_K, \beta_{\eta'_\Delta})e_2(p_{KK}, \beta_{\phi_{2050}}, \beta_K, \beta_K, \beta_{\eta'_\Delta})\}, \tag{C38}
\end{aligned}$$

$$\begin{aligned}
f_7(p, \beta_A, \beta_B) = & \frac{2\sqrt{2}p\beta_A^{3/2}}{3\sqrt{5}(\beta_B^2 + 2\beta_A^2)^{13/2}} [5(\beta_B^2 + 2\beta_A^2)^2(3\beta_B^6 - 17\beta_B^4\beta_A^2 + 16\beta_B^2\beta_A^4 + 12\beta_A^6) \\
& - 4\beta_A^2(\beta_B^2 + 2\beta_A^2)(-5\beta_B^4 + 9\beta_B^2\beta_A^2 + 10\beta_A^4)p^2 + 4\beta_A^4(\beta_B^2 + \beta_A^2)p^4], \tag{C39}
\end{aligned}$$

$$\begin{aligned}
f_8(p, \beta_A, \beta_B, \beta) = & \frac{4\sqrt{2}p\beta_A^{3/2}\beta_B^3(\beta_B^2 + \beta^2)}{9\sqrt{5}[\beta_B^4 + 2\beta^2\beta_A^2 + 2\beta_B^2(\beta^2 + \beta_A^2)]^{13/2}} \{5[\beta_B^4 + 2\beta_B^2\beta^2 - 2\beta_A^2(\beta_B^2 + \beta^2)][3\beta_B^4(\beta_B^2 + 2\beta^2) \\
& - 11\beta_B^2\beta_A^2 - 6(\beta_B^2 + \beta^2)\beta_A^4][\beta_B^4 + 2\beta^2\beta_A^2 + 2\beta_B^2(\beta^2 + \beta_A^2)]^2 \\
& + 4p^2(\beta_B^2 + \beta^2)^2\beta_A^2[5(\beta_B^6 + 2\beta_B^4\beta^2) - 9\beta_B^4\beta_A^2 - 10(\beta_B^2 + \beta^2)\beta_A^4][\beta_B^4 + 2\beta^2\beta_A^2 + 2\beta_B^2(\beta^2 + \beta_A^2)] \\
& + 4p^4(\beta_B^2 + \beta^2)^4\beta_A^4(\beta_B^2 + \beta_A^2)\}. \tag{C40}
\end{aligned}$$

$\phi(2050) \rightarrow K^+K^{*-}$  :

$$\begin{aligned}
C_{11}^{\phi(2050)KK^{*-}} = & (c_1^{\phi_{2050}} + c_2^{\phi_{2050}})\{f_9(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*})e_1(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}) \\
& + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{10}(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta})e_2(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) \\
& + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{10}(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta})e_2(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta})\}, \tag{C41}
\end{aligned}$$

$$\begin{aligned}
f_9(p, \beta_A, \beta_B, \beta_C) = & \frac{p\beta_B^{3/2}\beta_C^{3/2}\beta_A^{3/2}}{6\sqrt{10}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^{13/2}} \{20[6\beta_B^{10}\beta_C^{10} - 5\beta_B^8\beta_C^8(\beta_B^2 + \beta_C^2)\beta_A^2 \\
& - 20\beta_B^6\beta_C^6(\beta_B^2 + \beta_C^2)^2\beta_A^4 + 2\beta_B^4\beta_C^4(\beta_B^2 + \beta_C^2)^3\beta_A^6 + 14\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)^4\beta_A^8 + 3(\beta_B^2 + \beta_C^2)^5\beta_A^{10}] \\
& + 4p^2(\beta_B^2 + \beta_C^2)^2\beta_A^2[10\beta_B^6\beta_C^6 + \beta_B^4\beta_C^4(\beta_B^2 + \beta_C^2)\beta_A^2 - 14\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)^2\beta_A^4 \\
& - 5(\beta_B^2 + \beta_C^2)^3\beta_A^6] + p^4(\beta_B^2 + \beta_C^2)^4\beta_A^4[2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2)\beta_A^2]\}, \tag{C42}
\end{aligned}$$

$$\begin{aligned}
f_{10}(p, \beta_A, \beta_B, \beta_C, \beta) = & \frac{p\beta_B^{3/2}\beta_C^{3/2}\beta_A^{3/2}}{9\sqrt{10}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2) + 2(\beta_C^2 + \beta_A^2)\beta^2]^{13/2}} (20\{-\beta_C^2\beta_A^2 + \beta_B^2[\beta_C^2 - \beta_A^2] \\
& + 2[\beta_C^2 - \beta_A^2]\beta^2\}\{\beta_C^2\beta_A^2 + \beta_B^2[\beta_C^2 + \beta_A^2] + 2[\beta_C^2 + \beta_A^2]\beta^2\}^2\{\beta_B^4[6\beta_C^4 - 11\beta_C^2\beta_A^2 - 3\beta_A^4] \\
& - \beta_B^2[11\beta_C^4\beta_A^2 + 6\beta_C^2\beta_A^4 + 2\beta^2(-9\beta_C^4 + 11\beta_C^2\beta_A^2 + 6\beta_A^4)] \\
& - 3[4\beta_C^2\beta_A^4\beta^2 + 4\beta_A^4\beta^4 + \beta_C^4(\beta_A^4 - 4\beta^4)]\} + 4\beta_A^2\{\beta_B^2 + \beta_C^2 + 2\beta^2\}^2\{\beta_C^2\beta_A^2 + \beta_B^2[\beta_C^2 + \beta_A^2] \\
& + 2[\beta_C^2 + \beta_A^2]\beta^2\}p^2). \tag{C43}
\end{aligned}$$

$$\begin{aligned}
& + 2[\beta_C^2 + \beta_A^2]\beta^2\{\beta_B^4[10\beta_C^4 - 9\beta_C^2\beta_A^2 - 5\beta_A^4] - \beta_B^2[20\beta_A^4\beta^2 + \beta_C^4(9\beta_A^2 - 30\beta^2)] \\
& + 2\beta_C^2\beta_A^2(5\beta_A^2 + 9\beta^2)] - 5[4\beta_C^2\beta_A^4\beta^2 + 4\beta_A^4\beta^4 + \beta_C^4(\beta_A^4 - 4\beta^4)]\}p^2 \\
& + \beta_A^4\{\beta_B^2 + \beta_C^2 + 2\beta^2\}^4\{\beta_C^2\beta_A^2 + \beta_B^2[2\beta_C^2 + \beta_A^2] + 2[\beta_C^2 + \beta_A^2]\beta^2\}p^4). \tag{C43}
\end{aligned}$$

$\phi(2050) \rightarrow K^{*+}K^{*-}$ :

$$\begin{aligned}
C_{12}^{\phi(2050)K^{*+}K^{*-}} & = 2(c_1^{\phi_{2050}} - c_2^{\phi_{2050}})\{-f_7(p_{K^*K^*}, \beta_{\phi_{2050}}, \beta_{K^*})e_1(p_{K^*K^*}, \beta_{\phi_{2050}}, \beta_{K^*}, \beta_{K^*}) \\
& + 2c_1^{\phi_{2050}}c_2^{\phi_{2050}}f_8(p_{K^*K^*}, \beta_{\phi_{2050}}, \beta_{K^*}, \beta_{\phi_{\Delta}})e_2(p_{K^*K^*}, \beta_{\phi_{2050}}, \beta_{K^*}, \beta_{\phi_{\Delta}}) \\
& + 2c_1^{\omega_{2050}}c_2^{\omega_{2050}}f_8(p_{K^*K^*}, \beta_{\phi_{2050}}, \beta_{K^*}, \beta_{\omega_{\Delta}})e_2(p_{K^*K^*}, \beta_{\phi_{2050}}, \beta_{K^*}, \beta_{\omega_{\Delta}})\}. \tag{C44}
\end{aligned}$$

$\phi(2050) \rightarrow K^+K_1^-(1270)$ :

$$\begin{aligned}
C_{01}^{\phi(2050)KK_1(1270)} & = -(c_1^{\phi_{2050}} + c_2^{\phi_{2050}})\{f_{11}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
& + c_1^{\eta_{\Delta}}c_2^{\eta_{\Delta}}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_{\Delta}}) \\
& + c_1^{\eta'_{\Delta}}c_2^{\eta'_{\Delta}}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_{\Delta}})\}\cos\theta \\
& + \frac{(c_1^{\phi_{2050}} - c_2^{\phi_{2050}})}{\sqrt{2}}\{2f_{11}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
& - c_1^{\phi_{\Delta}}c_2^{\phi_{\Delta}}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_{\Delta}}) \\
& - c_1^{\omega_{\Delta}}c_2^{\omega_{\Delta}}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_{\Delta}})\}\sin\theta, \tag{C45}
\end{aligned}$$

$$\begin{aligned}
C_{21}^{\phi(2050)KK_1(1270)} & = -(c_1^{\phi_{2050}} + c_2^{\phi_{2050}})\{f_{13}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
& + c_1^{\eta_{\Delta}}c_2^{\eta_{\Delta}}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_{\Delta}}) \\
& + c_1^{\eta'_{\Delta}}c_2^{\eta'_{\Delta}}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_{\Delta}})\}\cos\theta \\
& - \frac{(c_1^{\phi_{2050}} - c_2^{\phi_{2050}})}{\sqrt{2}}\{f_{13}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
& - 2c_1^{\phi_{\Delta}}c_2^{\phi_{\Delta}}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_{\Delta}}) \\
& - 2c_1^{\omega_{\Delta}}c_2^{\omega_{\Delta}}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_{\Delta}})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_{\Delta}})\}\sin\theta, \tag{C46}
\end{aligned}$$

$$\begin{aligned}
f_{11}(p, \beta_A, \beta_B, \beta_C) & = \frac{\beta_B^{7/2}\beta_C^{5/2}\beta_A^{3/2}}{12\sqrt{15}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^{15/2}}\{240\beta_A^2[11\beta_B^4\beta_C^4 - 14\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)\beta_A^2 \\
& + 3(\beta_B^2 + \beta_C^2)^2\beta_A^4][\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^3 + 20[6\beta_B^6\beta_C^6 - 33\beta_B^4\beta_C^4(\beta_B^2 + \beta_C^2)\beta_A^2 \\
& + 44\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)^2\beta_A^4 - (\beta_B^2 + \beta_C^2)^3\beta_A^6][\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^2p^2 \\
& + 4(\beta_B^2 + \beta_C^2)^2\beta_A^2[10\beta_B^6\beta_C^6 - 7\beta_B^4\beta_C^4(\beta_B^2 + \beta_C^2)\beta_A^2 - 23\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)^2\beta_A^4 \\
& - 6(\beta_B^2 + \beta_C^2)^3\beta_A^6]p^4 + [\beta_B^2 + \beta_C^2]^4\beta_A^4[2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2)\beta_A^2]p^6\}, \tag{C47}
\end{aligned}$$

$$\begin{aligned}
f_{12}(p, \beta_A, \beta_B, \beta_C, \beta) & = -\frac{\beta_B^{3/2}\beta_C^{5/2}\beta_A^{3/2}}{18\sqrt{15}[\beta_C^2(\beta_B^2 + 2\beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]^{15/2}}\{240\beta_B^2\beta_A^2[\beta_C^2(\beta_B^2 + 2\beta^2) \\
& + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]^3[11\beta_C^4(\beta_B^2 + 2\beta^2)^2 - 14\beta_C^2(\beta_B^2 + 2\beta^2)(\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2 \\
& + 3(\beta_B^2 + \beta_C^2 + 2\beta^2)^2\beta_A^4] + 20[\beta_C^2(\beta_B^2 + 2\beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]^2[6\beta_C^6(\beta_B^2 + \beta^2)(\beta_B^2 + 2\beta^2)^3 \\
& - 11\beta_C^4(\beta_B^2 + 2\beta^2)^2(3\beta_B^2 + 2\beta^2)(\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2 \\
& + 2\beta_C^2(\beta_B^2 + \beta_C^2 + 2\beta^2)^2(22\beta_B^4 + 41\beta_B^2\beta^2 - 6\beta^4)\beta_A^4 - (\beta_B^2 - 22\beta^2)(\beta_B^2 + \beta_C^2 + 2\beta^2)^3\beta_A^6]p^2 \\
& + 4[\beta_B^2 + \beta_C^2 + 2\beta^2]^2\beta_A^2[10\beta_C^6(\beta_B^2 + \beta^2)(\beta_B^2 + 2\beta^2)^3 - \beta_C^4(7\beta_B^2 - 2\beta^2)(\beta_B^2 + 2\beta^2)^2(\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2 \\
& - 6(\beta_B^2 + \beta_C^2 + 2\beta^2)^3\beta_A^6]p^4 + [\beta_B^2 + \beta_C^2 + 2\beta^2]^4\beta_A^4[2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]p^6\}.
\end{aligned}$$

$$\begin{aligned}
 & -\beta_C^2(\beta_B^2 + \beta_C^2 + 2\beta^2)^2(23\beta_B^4 + 72\beta_B^2\beta^2 + 52\beta^4)\beta_A^4 \\
 & -6(\beta_B^2 + \beta_C^2 + 2\beta^2)^3(\beta_B^2 + 3\beta^2)\beta_A^6 p^4 + [\beta_B^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2]^4\beta_A^4[2\beta_C^2 \\
 & \times (\beta_B^2 + \beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]p^6\}, \tag{C48}
 \end{aligned}$$

$$\begin{aligned}
 f_{13}(p, \beta_A, \beta_B, \beta_C) = & -\frac{\beta_B^{7/2}\beta_C^{5/2}\beta_A^{3/2}p^2}{60\sqrt{3}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^{15/2}}\{120\beta_B^{10}\beta_C^{10} - 420\beta_B^8\beta_C^8(\beta_B^2 + \beta_C^2)\beta_A^2 \\
 & - 752\beta_B^6\beta_C^6(\beta_B^2 + \beta_C^2)^2\beta_A^4 + 456\beta_B^4\beta_C^4(\beta_B^2 + \beta_C^2)^3\beta_A^6 + 888\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)^4\beta_A^8 \\
 & + 220(\beta_B^2 + \beta_C^2)^5\beta_A^{10} + 4(\beta_B^2 + \beta_C^2)^2\beta_A^2[10\beta_B^6\beta_C^6 - 7\beta_B^4\beta_C^4(\beta_B^2 + \beta_C^2)\beta_A^2 \\
 & - 26\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)^2\beta_A^4 - 9(\beta_B^2 + \beta_C^2)^3\beta_A^6]p^2 + (\beta_B^2 + \beta_C^2)^4\beta_A^4[2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2)\beta_A^2]p^4\}, \tag{C49}
 \end{aligned}$$

$$\begin{aligned}
 f_{14}(p, \beta_A, \beta_B, \beta_C, \beta) = & \frac{\beta_B^{3/2}\beta_C^{5/2}(\beta_B^2 + 2\beta^2)\beta_A^{3/2}p^2}{90\sqrt{3}[\beta_C^2(\beta_B^2 + 2\beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]^{15/2}}\{4[30\beta_C^{10}(\beta_B^2 + \beta^2)(\beta_B^2 + 2\beta^2)^4 \\
 & - 5\beta_C^8(\beta_B^2 + 2\beta^2)^3(\beta_B^2 + \beta_C^2 + 2\beta^2)(21\beta_B^2 + 10\beta^2)\beta_A^2 \\
 & - 4\beta_A^4\beta_C^6(\beta_B^2 + 2\beta^2)^2(\beta_B^2 + \beta_C^2 + 2\beta^2)^2(47\beta_B^2 + 55\beta^2) \\
 & + 6\beta_C^4(\beta_B^2 + \beta_C^2 + 2\beta^2)^3(19\beta_B^4 + 28\beta_B^2\beta^2 - 20\beta^4) \\
 & \times \beta_A^6 + 2\beta_C^2(\beta_B^2 + \beta_C^2 + 2\beta^2)^4(111\beta_B^2 + 95\beta^2)\beta_A^8 + 55(\beta_B^2 + \beta_C^2 + 2\beta^2)^5\beta_A^{10}] \\
 & + 4[\beta_B^2 + \beta_C^2 + 2\beta^2]^2\beta_A^2[10\beta_C^6(\beta_B^2 + \beta^2)(\beta_B^2 + 2\beta^2)^2 \\
 & - \beta_C^4(\beta_B^2 + \beta_C^2 + 2\beta^2)(7\beta_B^4 + 12\beta_B^2\beta^2 - 4\beta^4)\beta_A^2 - 26\beta_C^2(\beta_B^2 + \beta^2)(\beta_B^2 + \beta_C^2 + 2\beta^2)^2\beta_A^4 \\
 & - 9\beta_A^6(\beta_B^2 + \beta_C^2 + 2\beta^2)^3]p^2 + [\beta_B^2 + \beta_C^2 + 2\beta^2]^4\beta_A^4[2\beta_C^2(\beta_B^2 + \beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]p^4\}. \tag{C50}
 \end{aligned}$$

$\phi(2050) \rightarrow K^+K_1^-(1400)$ :

$$\begin{aligned}
 \mathcal{C}_{01}^{\phi(2050)KK_1(1400)} = & (c_1^{\phi_{2050}} + c_2^{\phi_{2050}})\{f_{11}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
 & + c_1^{\eta_\Delta}c_2^{\eta_\Delta}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta}) \\
 & + c_1^{\eta'_\Delta}c_2^{\eta'_\Delta}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})\}\sin\theta \\
 & + \frac{(c_1^{\phi_{2050}} - c_2^{\phi_{2050}})}{\sqrt{2}}\{2f_{11}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
 & - c_1^{\phi_\Delta}c_2^{\phi_\Delta}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta}) \\
 & - c_1^{\omega_\Delta}c_2^{\omega_\Delta}f_{12}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})\}\cos\theta, \tag{C51}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{C}_{21}^{\phi(2050)KK_1(1400)} = & (c_1^{\phi_{2050}} + c_2^{\phi_{2050}})\{f_{13}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
 & + c_1^{\eta_\Delta}c_2^{\eta_\Delta}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta}) \\
 & + c_1^{\eta'_\Delta}c_2^{\eta'_\Delta}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})\}\sin\theta \\
 & - \frac{(c_1^{\phi_{2050}} - c_2^{\phi_{2050}})}{\sqrt{2}}\{f_{13}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}) \\
 & - 2c_1^{\phi_\Delta}c_2^{\phi_\Delta}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta}) \\
 & - 2c_1^{\omega_\Delta}c_2^{\omega_\Delta}f_{14}(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})e_2(p_{KK_1}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})\}\cos\theta. \tag{C52}
 \end{aligned}$$

$\phi(2050) \rightarrow K^+K_0^{*-}(1430)$ :

$$\begin{aligned}
 \mathcal{C}_{20}^{\phi(2050)KK_0^*(1430)} = & (c_1^{\phi_{2050}} - c_2^{\phi_{2050}})\{c_1^{\phi_\Delta}c_2^{\phi_\Delta}f_{14}(p_{KK_0^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_0^*}, \beta_{\phi_\Delta})e_2(p_{KK_0^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_0^*}, \beta_{\phi_\Delta}) \\
 & + c_1^{\omega_\Delta}c_2^{\omega_\Delta}f_{14}(p_{KK_0^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_0^*}, \beta_{\omega_\Delta})e_2(p_{KK_0^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_0^*}, \beta_{\omega_\Delta})\}. \tag{C53}
 \end{aligned}$$

$\phi(2050) \rightarrow K^+K_2^{*-}(1430)$ :

$$\begin{aligned} \mathcal{C}_{22}^{\phi(2050)KK_2^{*}(1430)} &= \sqrt{3}(c_1^{\phi_{2050}} - c_2^{\phi_{2050}}) \{ f_{13}(p_{KK_2^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_2^*}) e_1(p_{KK_2^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_2^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{14}(p_{KK_2^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_2^*}, \beta_{\phi_\Delta}) e_2(p_{KK_2^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_2^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{14}(p_{KK_2^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_2^*}, \beta_{\omega_\Delta}) e_2(p_{KK_2^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_2^*}, \beta_{\omega_\Delta}) \}. \end{aligned} \quad (C54)$$

$\phi(2050) \rightarrow K^+K^{*-}(1410)$ :

$$\begin{aligned} \mathcal{C}_{11}^{\phi(2050)KK^{*}(1410)} &= (c_1^{\phi_{2050}} + c_2^{\phi_{2050}}) \{ f_{15}(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}) e_1(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{16}(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) e_2(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{16}(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta}) e_2(p_{KK^*}, \beta_{\phi_{2050}}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta}) \}, \end{aligned} \quad (C55)$$

$$\begin{aligned} f_{15}(p, \beta_A, \beta_B, \beta_C) &= \frac{\beta_A^{3/2} \beta_B^{3/2} \beta_C^{3/2} p}{24\sqrt{15}[\beta_B^2 \beta_C^2 + \beta_A^2(\beta_B^2 + \beta_C^2)]^{17/2}} \{ 40[\beta_B^2 \beta_C^2 + \beta_A^2(\beta_B^2 + \beta_C^2)]^3 [18\beta_B^8 \beta_C^8 \\ &\quad + \beta_A^4 \beta_B^4 \beta_C^4 (275\beta_B^2 - 27\beta_C^2)(\beta_B^2 + \beta_C^2) - 11\beta_A^6 \beta_B^2 \beta_C^2 (\beta_B^2 - 3\beta_C^2)(\beta_B^2 + \beta_C^2)^2 \\ &\quad - 9\beta_A^8 (\beta_B^2 - \beta_C^2)(\beta_B^2 + \beta_C^2)^3 - 11\beta_A^2 \beta_B^6 \beta_C^6 (19\beta_B^2 + 3\beta_C^2)] - 4[\beta_B^2 \beta_C^2 + \beta_A^2(\beta_B^2 + \beta_C^2)]^2 [30\beta_B^{10} \beta_C^8 \\ &\quad + 2\beta_A^4 \beta_B^4 \beta_C^4 (265\beta_B^2 - 3\beta_C^2)(\beta_B^2 + \beta_C^2)^2 + 30\beta_A^8 (-\beta_B^2 + \beta_C^2)(\beta_B^2 + \beta_C^2)^4 \\ &\quad - 5\beta_A^2 \beta_B^6 \beta_C^6 (\beta_B^2 + \beta_C^2)(61\beta_B^2 + 12\beta_C^2) + \beta_A^6 \beta_B^2 \beta_C^2 (\beta_B^2 + \beta_C^2)^3 (79\beta_B^2 + 84\beta_C^2)] p^2 \\ &\quad - 2\beta_A^2 [\beta_B^2 + \beta_C^2]^2 [20\beta_B^{10} \beta_C^8 + 3\beta_A^8 (\beta_B^2 - \beta_C^2)(\beta_B^2 + \beta_C^2)^4 - 15\beta_A^4 \beta_B^4 \beta_C^4 (\beta_B^2 + \beta_C^2)^2 (5\beta_B^2 + \beta_C^2) \\ &\quad - 4\beta_A^6 \beta_B^2 \beta_C^2 (\beta_B^2 + \beta_C^2)^3 (4\beta_B^2 + 3\beta_C^2) - 6\beta_A^2 \beta_B^6 \beta_C^6 (6\beta_B^4 + 7\beta_B^2 \beta_C^2 + \beta_C^4)] p^4 \\ &\quad - \beta_A^4 \beta_B^4 \beta_C^4 [\beta_B^2 + \beta_C^2]^4 [2\beta_B^2 \beta_C^2 + \beta_A^2(\beta_B^2 + \beta_C^2)] p^6 \}, \end{aligned} \quad (C56)$$

$$\begin{aligned} f_{16}(p, \beta_A, \beta_B, \beta_C, \beta) &= -\frac{\beta_A^{3/2} \beta_B^{3/2} \beta_C^{3/2} p}{36\sqrt{15}[\beta_C^2(\beta_B^2 + 2\beta^2) + \beta_A^2(\beta_B^2 + \beta_C^2 + 2\beta^2)]^{17/2}} (40\{\beta_C^2[\beta_B^2 + 2\beta^2] \\ &\quad + \beta_A^2[\beta_B^2 + \beta_C^2 + 2\beta^2]\}^3 \{18\beta_C^8[\beta_B^2 + \beta^2][\beta_B^2 + 2\beta^2]^3 \\ &\quad - 9\beta_A^8[\beta_B^2 - \beta_C^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2]^3 + 11\beta_A^6 \beta_C^2[\beta_B^2 + \beta_C^2 + 2\beta^2]^2 [-\beta_B^4 + 24\beta^4 + \beta_B^2(3\beta_C^2 + 10\beta^2)] \\ &\quad - 11\beta_A^2 \beta_C^6[\beta_B^2 + 2\beta^2]^2 [19\beta_B^4 + 24\beta^4 + \beta_B^2(3\beta_C^2 + 50\beta^2)] \\ &\quad + \beta_A^4 \beta_C^4[\beta_B^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2] [275\beta_B^4 - 36\beta_C^2 \beta^2 + \beta_B^2(-27\beta_C^2 + 550\beta^2)] \} \\ &\quad - 4\{\beta_C^2[\beta_B^2 + 2\beta^2] + \beta_A^2[\beta_B^2 + \beta_C^2 + 2\beta^2]\}^2 \{30\beta_C^8[\beta_B^2 + \beta^2][\beta_B^2 + 2\beta^2]^4 \\ &\quad - 30\beta_A^8[\beta_B^2 - \beta_C^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2]^4 - 5\beta_A^2 \beta_C^6[\beta_B^2 + 2\beta^2]^2 [\beta_B^2 + \beta_C^2 + 2\beta^2] [61\beta_B^4 \\ &\quad + 4\beta^2(3\beta_C^2 + 25\beta^2) + 4\beta_B^2(3\beta_C^2 + 43\beta^2)] + \beta_A^6 \beta_C^2[\beta_B^2 + \beta_C^2 + 2\beta^2]^3 [79\beta_B^4 + 60\beta_C^2 \beta^2 \\ &\quad + 820\beta^4 + \beta_B^2(84\beta_C^2 + 568\beta^2)] + 2\beta_A^4 \beta_C^4[\beta_B^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2]^2 [265\beta_B^4 - 30\beta_C^2 \beta^2 + 190\beta^4 \\ &\quad + \beta_B^2(-3\beta_C^2 + 625\beta^2)] \} p^2 - 2\beta_A^2 \{\beta_B^2 + \beta_C^2 + 2\beta^2\}^2 \{\beta_C^2[\beta_B^2 + 2\beta^2] \\ &\quad + \beta_A^2[\beta_B^2 + \beta_C^2 + 2\beta^2]\} \{20\beta_C^6[\beta_B^2 + \beta^2][\beta_B^2 + 2\beta^2]^3 \\ &\quad + 3\beta_A^6[\beta_B^2 - \beta_C^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2]^3 - 2\beta_A^2 \beta_C^4[\beta_B^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2] [28\beta_B^4 + 3\beta_C^2 \beta^2 + 38\beta^4 \\ &\quad + 3\beta_B^2(\beta_C^2 + 25\beta^2)] - \beta_A^4 \beta_C^2[\beta_B^2 + \beta_C^2 + 2\beta^2]^2 [19\beta_B^4 + 9\beta_B^2(\beta_C^2 + 10\beta^2) \\ &\quad + 4\beta^2(3\beta_C^2 + 26\beta^2)] \} p^4 - \beta_A^4 \beta_C^2 \{\beta_B^2 + 2\beta^2\}^2 \{\beta_B^2 + \beta_C^2 + 2\beta^2\}^4 \{2\beta_C^2[\beta_B^2 + \beta^2] \\ &\quad + \beta_A^2[\beta_B^2 + \beta_C^2 + 2\beta^2]\} p^6). \end{aligned} \quad (C57)$$

$\phi(2050) \rightarrow K^+K^-(1460)$ :

$$\begin{aligned} \mathcal{C}_{10}^{\phi(2050)KK(1460)} = & -(c_1^{\phi_{2050}} - c_2^{\phi_{2050}}) \{ f_{15}(p_{KK_{1460}}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_{1460}}) e_1(p_{KK_{1460}}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_{1460}}) \\ & + c_1^{\eta_{\Delta}} c_2^{\eta_{\Delta}} f_{16}(p_{KK_{1460}}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_{1460}}, \beta_{\eta_{\Delta}}) e_2(p_{KK_{1460}}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_{1460}}, \beta_{\eta_{\Delta}}) \\ & + c_1^{\eta'_{\Delta}} c_2^{\eta'_{\Delta}} f_{16}(p_{KK_{1460}}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_{1460}}, \beta_{\eta'_{\Delta}}) e_2(p_{KK_{1460}}, \beta_{\phi_{2050}}, \beta_K, \beta_{K_{1460}}, \beta_{\eta'_{\Delta}}) \}. \end{aligned} \quad (C58)$$

$\phi(2050) \rightarrow \eta\phi$ :

$$\begin{aligned} \mathcal{C}_{11}^{\phi(2050)\eta\phi} = & 2 \{ (2c_1^{\eta} c_1^{\phi} c_1^{\phi_{2050}} + c_2^{\eta} c_2^{\phi} c_2^{\phi_{2050}}) f_9(p_{\eta\phi}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{\phi}) e_1(p_{\eta\phi}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{\phi}) \\ & - [2c_1^{\eta} c_1^{\phi} c_1^{\phi_{2050}} (c_1^{\phi_{\Delta}})^2 + c_2^{\eta} c_2^{\phi} c_2^{\phi_{2050}} (c_2^{\phi_{\Delta}})^2] f_{10}(p_{\eta\phi}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_{\Delta}}) e_2(p_{\eta\phi}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_{\Delta}}) \\ & - [2c_1^{\eta} c_1^{\phi} c_1^{\phi_{2050}} (c_1^{\omega_{\Delta}})^2 + c_2^{\eta} c_2^{\phi} c_2^{\phi_{2050}} (c_2^{\omega_{\Delta}})^2] f_{10}(p_{\eta\phi}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{\phi}, \beta_{\omega_{\Delta}}) e_2(p_{\eta\phi}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{\phi}, \beta_{\omega_{\Delta}}) \}. \end{aligned} \quad (C59)$$

$\phi(2050) \rightarrow \eta'\phi$ :

$$\begin{aligned} \mathcal{C}_{11}^{\phi(2050)\eta'\phi} = & 2 \{ (2c_1^{\eta'} c_1^{\phi} c_1^{\phi_{2050}} + c_2^{\eta'} c_2^{\phi} c_2^{\phi_{2050}}) f_9(p_{\eta'\phi}, \beta_{\phi_{2050}}, \beta_{\eta'}, \beta_{\phi}) e_1(p_{\eta'\phi}, \beta_{\phi_{2050}}, \beta_{\eta'}, \beta_{\phi}) \\ & - [2c_1^{\eta'} c_1^{\phi} c_1^{\phi_{2050}} (c_1^{\phi_{\Delta}})^2 + c_2^{\eta'} c_2^{\phi} c_2^{\phi_{2050}} (c_2^{\phi_{\Delta}})^2] f_{10}(p_{\eta'\phi}, \beta_{\phi_{2050}}, \beta_{\eta'}, \beta_{\phi}, \beta_{\phi_{\Delta}}) e_2(p_{\eta'\phi}, \beta_{\phi_{2050}}, \beta_{\eta'}, \beta_{\phi}, \beta_{\phi_{\Delta}}) \\ & - [2c_1^{\eta'} c_1^{\phi} c_1^{\phi_{2050}} (c_1^{\omega_{\Delta}})^2 + c_2^{\eta'} c_2^{\phi} c_2^{\phi_{2050}} (c_2^{\omega_{\Delta}})^2] f_{10}(p_{\eta'\phi}, \beta_{\phi_{2050}}, \beta_{\eta'}, \beta_{\phi}, \beta_{\omega_{\Delta}}) e_2(p_{\eta'\phi}, \beta_{\phi_{2050}}, \beta_{\eta'}, \beta_{\phi}, \beta_{\omega_{\Delta}}) \}. \end{aligned} \quad (C60)$$

$\phi(2050) \rightarrow \eta h_1(1380)$ :

$$\begin{aligned} \mathcal{C}_{01}^{\phi(2050)\eta h_1} = & 2 \{ -(2c_1^{h_1} c_1^{\eta} c_1^{\phi_{2050}} + c_2^{h_1} c_2^{\eta} c_2^{\phi_{2050}}) f_{11}(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}) e_1(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}) \\ & + [2c_1^{h_1} c_1^{\eta} (c_1^{\eta_{\Delta}})^2 c_1^{\phi_{2050}} + c_2^{h_1} c_2^{\eta} (c_2^{\eta_{\Delta}})^2 c_2^{\phi_{2050}}] f_{12}(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta_{\Delta}}) e_2(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta_{\Delta}}) \\ & + [2c_1^{h_1} c_1^{\eta} (c_1^{\eta'_{\Delta}})^2 c_1^{\phi_{2050}} + c_2^{h_1} c_2^{\eta} (c_2^{\eta'_{\Delta}})^2 c_2^{\phi_{2050}}] f_{12}(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta'_{\Delta}}) e_2(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta'_{\Delta}}) \}, \end{aligned} \quad (C61)$$

$$\begin{aligned} \mathcal{C}_{21}^{\phi(2050)\eta h_1} = & 2 \{ -(2c_1^{h_1} c_1^{\eta} c_1^{\phi_{2050}} + c_2^{h_1} c_2^{\eta} c_2^{\phi_{2050}}) f_{13}(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}) e_1(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}) \\ & + [2c_1^{h_1} c_1^{\eta} (c_1^{\eta_{\Delta}})^2 c_1^{\phi_{2050}} + c_2^{h_1} c_2^{\eta} (c_2^{\eta_{\Delta}})^2 c_2^{\phi_{2050}}] f_{14}(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta_{\Delta}}) e_2(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta_{\Delta}}) \\ & + [2c_1^{h_1} c_1^{\eta} (c_1^{\eta'_{\Delta}})^2 c_1^{\phi_{2050}} + c_2^{h_1} c_2^{\eta} (c_2^{\eta'_{\Delta}})^2 c_2^{\phi_{2050}}] f_{14}(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta'_{\Delta}}) e_2(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta}, \beta_{h_1}, \beta_{\eta'_{\Delta}}) \}. \end{aligned} \quad (C62)$$

$\phi_1(1850) \rightarrow K^{*+}K^{*-}$ :

$$\begin{aligned} \mathcal{C}_{12}^{\phi_1 K^{*+} K^{*-}} = & (c_1^{\phi_1} - c_2^{\phi_1}) \{ f_{17}(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}) e_1(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{K^{*-}}) \\ & + c_1^{\phi_{\Delta}} c_2^{\phi_{\Delta}} f_{18}(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{\phi_{\Delta}}) e_2(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{K^{*-}}, \beta_{\phi_{\Delta}}) \\ & + c_1^{\omega_{\Delta}} c_2^{\omega_{\Delta}} f_{18}(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{\omega_{\Delta}}) e_2(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{K^{*-}}, \beta_{\omega_{\Delta}}) \}, \end{aligned} \quad (C63)$$

$$\begin{aligned} \mathcal{C}_{32}^{\phi_1 K^{*+} K^{*-}} = & (c_1^{\phi_1} - c_2^{\phi_1}) \{ f_{19}(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}) e_1(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{K^{*-}}) \\ & + c_1^{\phi_{\Delta}} c_2^{\phi_{\Delta}} f_{20}(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{\phi_{\Delta}}) e_2(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{K^{*-}}, \beta_{\phi_{\Delta}}) \\ & + c_1^{\omega_{\Delta}} c_2^{\omega_{\Delta}} f_{20}(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{\omega_{\Delta}}) e_2(p_{K^{*+}K^{*-}}, \beta_{\phi_1}, \beta_{K^{*+}}, \beta_{K^{*-}}, \beta_{\omega_{\Delta}}) \}, \end{aligned} \quad (C64)$$

$$f_{17}(p, \beta_A, \beta_B) = \frac{16\sqrt{2}\beta_A^{7/2} p [-5(2\beta_A^2\beta_B^2 + \beta_B^4) + (\beta_A^2 + \beta_B^2)p^2]}{15\sqrt{5}(2\beta_A^2 + \beta_B^2)^{9/2}}, \quad (C65)$$

$$f_{18}(p, \beta_A, \beta_B, \beta) = \frac{64\sqrt{2}\beta_A^{7/2}\beta_B^3(\beta^2 + \beta_B^2)p [5[2b^2\beta_A^2\beta_B^4 + 2(\beta^2 + \beta_A^2)\beta_B^6 + \beta_B^8] - [\beta^2 + \beta_B^2]^2[\beta_A^2 + \beta_B^2]p^2]}{45\sqrt{5}[2b^2\beta_A^2 + 2(\beta^2 + \beta_A^2)\beta_B^2 + \beta_B^4]^{9/2}}, \quad (C66)$$

$$f_{19}(p, \beta_A, \beta_B) = -\frac{32\beta_A^{7/2}(\beta_B^2 + \beta_A^2)p^3}{5\sqrt{35}(\beta_B^2 + 2\beta_A^2)^{9/2}}, \quad (C67)$$

$$f_{20}(p, \beta_A, \beta_B, \beta) = \frac{128\beta_B^3\beta_A^{7/2}(\beta_B^2 + \beta_A^2)(\beta_B^2 + \beta^2)^3 p^3}{15\sqrt{35}[\beta_B^4 + 2\beta_A^2\beta^2 + 2\beta_B^2(\beta_A^2 + \beta^2)]^{9/2}}. \quad (C68)$$



$\phi_2(1850) \rightarrow K^+K^-$ :

$$\begin{aligned} C_{30}^{\phi_2 KK} &= \frac{5\sqrt{2}}{12} (c_1^{\phi_2} - c_2^{\phi_2}) \{c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{20}(p_{KK}, \beta_{\phi_2}, \beta_K, \beta_{\phi_\Delta}) e_2(p_{KK}, \beta_{\phi_2}, \beta_K, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{20}(p_{KK}, \beta_{\phi_2}, \beta_K, \beta_{\omega_\Delta}) e_2(p_{KK}, \beta_{\phi_2}, \beta_K, \beta_{\omega_\Delta})\}. \end{aligned} \quad (C69)$$

$\phi_2(1850) \rightarrow K^+K^{*-}$ :

$$\begin{aligned} C_{11}^{\phi_2 KK^*} &= (c_1^{\phi_2} + c_2^{\phi_2}) \{f_{21}(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}) e_1(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{22}(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) e_2(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{22}(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta}) e_2(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta})\}, \end{aligned} \quad (C70)$$

$$\begin{aligned} C_{31}^{\phi_2 KK^*} &= (c_1^{\phi_2} + c_2^{\phi_2}) \{f_{23}(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}) e_1(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{24}(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) e_2(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{24}(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta}) e_2(p_{KK^*}, \beta_{\phi_2}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta})\}, \end{aligned} \quad (C71)$$

$$\begin{aligned} f_{21}(p, \beta_A, \beta_B, \beta_C) &= \frac{2\beta_B^{3/2} \beta_C^{3/2} (\beta_B^2 + \beta_C^2) \beta_A^{7/2} p}{5[\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2)]^{9/2}} \{20[\beta_B^2 \beta_C^4 \beta_A^2 + \beta_B^4 \beta_C^2 (\beta_C^2 + \beta_A^2)] \\ &\quad - [\beta_B^2 + \beta_C^2][2\beta_B^2 \beta_C^2 + (\beta_B^2 + \beta_C^2) \beta_A^2] p^2\}, \end{aligned} \quad (C72)$$

$$\begin{aligned} f_{22}(p, \beta_A, \beta_B, \beta_C, \beta) &= -\frac{4\beta_B^{3/2} \beta_C^{3/2} \beta_A^{7/2} (\beta_B^2 + \beta_C^2 + 2\beta^2) p}{45[\beta_C^2 (\beta_B^2 + 2\beta^2) + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2\beta^2)]^{9/2}} \{20\beta_B^2 \beta_C^2 [\beta_C^2 (\beta_B^2 + 2\beta^2) \\ &\quad + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2\beta^2)] - [\beta_B^2 + \beta_C^2 + 2\beta^2][2\beta_C^2 (\beta_B^2 + \beta^2) + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2\beta^2)] p^2\}, \end{aligned} \quad (C73)$$

$$\begin{aligned} f_{23}(p, \beta_A, \beta_B, \beta_C) &= -\frac{2\sqrt{\frac{2}{7}} \beta_B^{3/2} \beta_C^{3/2} (\beta_B^2 + \beta_C^2)^2 \beta_A^{7/2} [2\beta_B^2 \beta_C^2 + (\beta_B^2 + \beta_C^2) \beta_A^2] p^3}{15[\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2)]^{9/2}}, \end{aligned} \quad (C74)$$

$$\begin{aligned} f_{24}(p, \beta_A, \beta_B, \beta_C, \beta) &= -\frac{4\sqrt{\frac{2}{7}} \beta_B^{3/2} \beta_C^{3/2} \beta_A^{7/2} (\beta_B^2 + \beta_C^2 + 2\beta^2)^2 [2\beta_C^2 (\beta_B^2 + \beta^2) + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2\beta^2)] p^3}{15[\beta_C^2 (\beta_B^2 + 2\beta^2) + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2\beta^2)]^{9/2}}. \end{aligned} \quad (C75)$$

$\phi_2(1850) \rightarrow K^{*+}K^{*-}$ :

$$\begin{aligned} C_{12}^{\phi_2 K^* K^*} &= -\sqrt{10} (c_1^{\phi_2} - c_2^{\phi_2}) \{f_{17}(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}) e_1(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{18}(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{\phi_\Delta}) e_2(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{18}(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{\omega_\Delta}) e_2(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{K^*}, \beta_{\omega_\Delta})\}. \end{aligned} \quad (C76)$$

$$\begin{aligned} C_{32}^{\phi_2 K^* K^*} &= -\sqrt{\frac{5}{3}} (c_1^{\phi_2} - c_2^{\phi_2}) \{f_{19}(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}) e_1(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{20}(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{\phi_\Delta}) e_2(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{20}(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{\omega_\Delta}) e_2(p_{K^* K^*}, \beta_{\phi_2}, \beta_{K^*}, \beta_{K^*}, \beta_{\omega_\Delta})\}. \end{aligned} \quad (C77)$$

$\phi_2(1850) \rightarrow \eta\phi$ :

$$\begin{aligned} C_{11}^{\phi_2 \eta\phi} &= -2 \{ (2c_1^\eta c_1^\phi c_1^{\phi_2} + c_2^\eta c_2^\phi c_2^{\phi_2}) f_{21}(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi) e_1(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi) \\ &\quad + [2c_1^\eta c_1^\phi c_1^{\phi_2} (c_1^{\phi_\Delta})^2 + c_2^\eta c_2^\phi c_2^{\phi_2} (c_2^{\phi_\Delta})^2] f_{22}(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta}) e_2(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta}) \\ &\quad + [2c_1^\eta c_1^\phi c_1^{\phi_2} (c_1^{\omega_\Delta})^2 + c_2^\eta c_2^\phi c_2^{\phi_2} (c_2^{\omega_\Delta})^2] f_{22}(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta}) e_2(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta}) \}, \end{aligned} \quad (C78)$$

$$\begin{aligned} C_{31}^{\phi_2 \eta\phi} &= -2 \{ (2c_1^\eta c_1^\phi c_1^{\phi_2} + c_2^\eta c_2^\phi c_2^{\phi_2}) f_{23}(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi) e_1(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi) \\ &\quad + [2c_1^\eta c_1^\phi c_1^{\phi_2} (c_1^{\phi_\Delta})^2 + c_2^\eta c_2^\phi c_2^{\phi_2} (c_2^{\phi_\Delta})^2] f_{24}(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta}) e_2(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta}) \\ &\quad + [2c_1^\eta c_1^\phi c_1^{\phi_2} (c_1^{\omega_\Delta})^2 + c_2^\eta c_2^\phi c_2^{\phi_2} (c_2^{\omega_\Delta})^2] f_{24}(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta}) e_2(p_{\eta\phi}, \beta_{\phi_2}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta}) \}. \end{aligned} \quad (C79)$$

$\phi_3(1850) \rightarrow K^+K^-$ :

$$\begin{aligned} C_{30}^{\phi_3 KK} &= -\frac{5}{4}(c_1^{\phi_3} - c_2^{\phi_3})\{2f_{19}(p_{KK}, \beta_{\phi_3}, \beta_K)e_1(p_{KK}, \beta_{\phi_3}, \beta_K, \beta_K) \\ &\quad + c_1^{\eta_\Delta} c_2^{\eta_\Delta} f_{20}(p_{KK}, \beta_{\phi_3}, \beta_K, \beta_{\eta_\Delta})e_2(p_{KK}, \beta_{\phi_3}, \beta_K, \beta_K, \beta_{\eta_\Delta}) \\ &\quad + c_1^{\eta'_\Delta} c_2^{\eta'_\Delta} f_{20}(p_{KK}, \beta_{\phi_3}, \beta_K, \beta_{\eta'_\Delta})e_2(p_{KK}, \beta_{\phi_3}, \beta_K, \beta_K, \beta_{\eta'_\Delta})\}. \end{aligned} \quad (C80)$$

$\phi_3(1850) \rightarrow K^+K^{*-}$ :

$$\begin{aligned} C_{31}^{\phi_3 K K^*} &= -\sqrt{\frac{5}{6}}(c_1^{\phi_3} + c_2^{\phi_3})\{3f_{23}(p_{KK^*}, \beta_{\phi_3}, \beta_K, \beta_{K^*})e_1(p_{KK^*}, \beta_{\phi_3}, \beta_K, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{24}(p_{KK^*}, \beta_{\phi_3}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta})e_2(p_{KK^*}, \beta_{\phi_3}, \beta_K, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{24}(p_{KK^*}, \beta_{\phi_3}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta})e_2(p_{KK^*}, \beta_{\phi_3}, \beta_K, \beta_{K^*}, \beta_{\omega_\Delta})\}. \end{aligned} \quad (C81)$$

$\phi_3(1850) \rightarrow K^{*+}K^{*-}$ :

$$\begin{aligned} C_{12}^{\phi_3 K^* K^*} &= 6\sqrt{\frac{5}{3}}(c_1^{\phi_3} - c_2^{\phi_3})\{f_{17}(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*})e_1(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{18}(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{\phi_\Delta})e_2(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{18}(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{\omega_\Delta})e_2(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{K^*}, \beta_{\omega_\Delta})\}, \end{aligned} \quad (C82)$$

$$\begin{aligned} C_{32}^{\phi_3 K^* K^*} &= \sqrt{\frac{5}{3}}(c_1^{\phi_3} - c_2^{\phi_3})\{f_{19}(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*})e_1(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{K^*}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{20}(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{\phi_\Delta})e_2(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{K^*}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{20}(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{\omega_\Delta})e_2(p_{K^*K^*}, \beta_{\phi_3}, \beta_{K^*}, \beta_{K^*}, \beta_{\omega_\Delta})\}. \end{aligned} \quad (C83)$$

$\phi_3(1850) \rightarrow K^+K_1^-(1270)$ :

$$\begin{aligned} C_{21}^{\phi_3 K K_1(1270)} &= (c_1^{\phi_3} + c_2^{\phi_3})\{f_{25}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}) \\ &\quad + c_1^{\eta_\Delta} c_2^{\eta_\Delta} f_{26}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta}) \\ &\quad + c_1^{\eta'_\Delta} c_2^{\eta'_\Delta} f_{26}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})\} \cos \theta \\ &\quad + (c_1^{\phi_3} - c_2^{\phi_3})\{f_{27}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}) \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{28}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta}) \\ &\quad + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{28}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})\} \sin \theta, \end{aligned} \quad (C84)$$

$$\begin{aligned} C_{41}^{\phi_3 K K_1(1270)} &= (c_1^{\phi_3} + c_2^{\phi_3})\{f_{29}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}) \\ &\quad + c_1^{\eta_\Delta} c_2^{\eta_\Delta} f_{30}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta_\Delta}) \\ &\quad + c_1^{\eta'_\Delta} c_2^{\eta'_\Delta} f_{30}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\eta'_\Delta})\} \cos \theta \\ &\quad + (c_1^{\phi_3} - c_2^{\phi_3})\sqrt{2}\left\{-\frac{1}{2}f_{29}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1})e_1(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}) \right. \\ &\quad + c_1^{\phi_\Delta} c_2^{\phi_\Delta} f_{30}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\phi_\Delta}) \\ &\quad \left. + c_1^{\omega_\Delta} c_2^{\omega_\Delta} f_{30}(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})e_2(p_{KK_1}, \beta_{\phi_3}, \beta_K, \beta_{K_1}, \beta_{\omega_\Delta})\right\} \sin \theta, \end{aligned} \quad (C85)$$

$$\begin{aligned} f_{25}(p, \beta_A, \beta_B, \beta_C) &= \frac{\sqrt{2}\beta_B^{7/2}\beta_C^{5/2}(\beta_B^2 + \beta_C^2)\beta_A^{7/2}p^2}{7\sqrt{15}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^{11/2}}\{28[4\beta_B^4\beta_C^4 + 5\beta_B^2\beta_C^2(\beta_B^2 + \beta_C^2)\beta_A^2 + (\beta_B^2 + \beta_C^2)^2\beta_A^4] \\ &\quad - 3[\beta_B^2 + \beta_C^2][2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2)\beta_A^2]p^2\}, \end{aligned} \quad (C86)$$

$$f_{26}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{2\sqrt{2}\beta_B^{3/2}\beta_C^{5/2}(\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^{7/2}p^2}{21\sqrt{15}[\beta_C^2(\beta_B^2 + 2\beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]^{11/2}} \{28[4\beta_C^4(\beta_B^2 + \beta^2)(\beta_B^2 + 2\beta^2)^2 + \beta_C^2(\beta_B^2 + \beta_C^2 + 2\beta^2)(5\beta_B^4 + 18\beta_B^2\beta^2 + 16\beta^4)\beta_A^2 + (\beta_B^2 + \beta_C^2 + 2\beta^2)^2(\beta_B^2 + 4\beta^2)\beta_A^4] - 3[\beta_B^2 + 2\beta^2][\beta_B^2 + \beta_C^2 + 2\beta^2][2\beta_C^2(\beta_B^2 + \beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]p^2\}, \quad (C87)$$

$$f_{27}(p, \beta_A, \beta_B, \beta_C) = \frac{4\beta_B^{7/2}\beta_C^{5/2}(\beta_B^2 + \beta_C^2)\beta_A^{7/2}p^2}{7\sqrt{15}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^{11/2}} \{28[\beta_B^2\beta_C^4\beta_A^2 + \beta_B^4\beta_C^2(\beta_C^2 + \beta_A^2)] - [\beta_B^2 + \beta_C^2][2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2)\beta_A^2]p^2\}, \quad (C88)$$

$$f_{28}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{2\beta_B^{3/2}\beta_C^{5/2}\beta_A^{7/2}(\beta_B^2 + \beta_C^2 + 2\beta^2)^2 p^2}{21\sqrt{15}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2) + 2(\beta_C^2 + \beta_A^2)\beta^2]^{11/2}} \{28\beta_B^2\beta_A^2[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2) + 2(\beta_C^2 + \beta_A^2)\beta^2] + [\beta_B^2 + 2\beta^2][\beta_C^2\beta_A^2 + \beta_B^2(2\beta_C^2 + \beta_A^2) + 2(\beta_C^2 + \beta_A^2)\beta^2]p^2\}, \quad (C89)$$

$$f_{29}(p, \beta_A, \beta_B, \beta_C) = \frac{\sqrt{2}\beta_B^{7/2}\beta_C^{5/2}(\beta_B^2 + \beta_C^2)^2\beta_A^{7/2}[2\beta_B^2\beta_C^2 + (\beta_B^2 + \beta_C^2)\beta_A^2]p^4}{21\sqrt{5}[\beta_C^2\beta_A^2 + \beta_B^2(\beta_C^2 + \beta_A^2)]^{11/2}}, \quad (C90)$$

$$f_{30}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{2\sqrt{2}\beta_B^{3/2}\beta_C^{5/2}(\beta_B^2 + 2\beta^2)(\beta_B^2 + \beta_C^2 + 2\beta^2)^2\beta_A^{7/2}p^4}{63\sqrt{5}[\beta_C^2(\beta_B^2 + 2\beta^2) + (\beta_B^2 + \beta_C^2 + 2\beta^2)\beta_A^2]^{11/2}}. \quad (C91)$$

$\phi_3(1850) \rightarrow \eta\phi$ :

$$C_{31}^{\phi_3\eta\phi} = \sqrt{\frac{10}{3}} \{3(2c_1^\eta c_1^\phi c_1^{\phi_3} + c_2^\eta c_2^\phi c_2^{\phi_3})f_{23}(p_{\eta\phi}, \beta_{\phi_3}, \beta_\eta, \beta_\phi)e_1(p_{\eta\phi}, \beta_{\phi_3}, \beta_\eta, \beta_\phi) + [2c_1^\eta c_1^\phi c_1^{\phi_3} (c_1^{\phi_\Delta})^2 + c_2^\eta c_2^\phi c_2^{\phi_3} (c_2^{\phi_\Delta})^2]f_{24}(p_{\eta\phi}, \beta_{\phi_3}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta})e_2(p_{\eta\phi}, \beta_{\phi_3}, \beta_\eta, \beta_\phi, \beta_{\phi_\Delta}) + [2c_1^\eta c_1^\phi c_1^{\phi_3} (c_1^{\omega_\Delta})^2 + c_2^\eta c_2^\phi c_2^{\phi_3} (c_2^{\omega_\Delta})^2]f_{24}(p_{\eta\phi}, \beta_{\phi_3}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta})e_2(p_{\eta\phi}, \beta_{\phi_3}, \beta_\eta, \beta_\phi, \beta_{\omega_\Delta})\}. \quad (C92)$$

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