Density-dependent NN interaction from subsubleading chiral $3N$ forces: Intermediate-range contributions

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(Received 6 August 2019; published 13 January 2020)

From the subsubleading chiral three-nucleon forces [intermediate-range contributions, published in Phys. Rev. C **87**, 054007 (2013)] a density-dependent NN-interaction $V_{\rm med}$ is derived in isospin-symmetric nuclear matter. Following the division of the pertinent $3N$ diagrams into two-pion-one-pion exchange topology and ring topology, one evaluates for these all self-closings and concatenations of nucleon lines to an in-medium loop. In the case of the $2\pi 1\pi$-exchange topology, the momentum- and k_f-dependent potentials associated with the isospin operators (1 and $\vec\tau_1\cdot\vec\tau_2$) and five independent spin structures require at most one numerical integration. For the more challenging (concatenations of the) ring diagrams proportional to $c_{1,2,3,4}$, one ends up with regularized double-integrals $\int_0^\lambda dr\, r\int_0^{\pi/2}d\psi$ from which the λ^2 divergence has been subtracted and the logarithmic piece $\sim\ln(m_\pi/\lambda)$ is isolated. The derived semianalytical results are most helpful to implement the subsubleading chiral $3N$ forces into nuclear many-body calculations.

DOI: 10.1103/PhysRevC.101.014001

I. INTRODUCTION AND SUMMARY

It is well known that three-nucleon forces are an indispensable ingredient in accurate few-nucleon and nuclear structure calculations. Nowadays, chiral effective field theory is the appropriate tool to construct systematically the nuclear interactions in harmony with the symmetries of QCD. Three-nucleon forces appear first at N^{2}LO, where they consist of a zero-range contact term ($\sim c_E$), a midrange 1π-exchange component ($\sim c_D$), and a long-range 2π-exchange component ($\sim c_{1,3,4}$). The complete calculation of the chiral $3N$ forces to subleading order N^{3}LO [1,2] and even to subsubleading order N^{4}LO [3,4] has been achieved during the past decade by the Bochum-Bonn group. At present the focus lies on constructing $3N$ forces in chiral effective field theory with explicit $\Delta(1232)$ isobars, for which the longe-range 2π-exchange component has been derived recently in Ref. [5] at order N^{3}LO.

However, for the variety of existing many-body methods, that are commonly employed in calculations of nuclear matter or medium mass and heavy nuclei, it is technically very challenging to include the chiral three-nucleon forces directly. An alternative and approximate approach is to use instead a density-dependent two-nucleon interaction $V_{\rm med}$ that originates from the underlying $3N$ force. When restricting to on-shell scattering of two nucleons in isospin-symmetric spin-saturated nuclear matter, the resulting in-medium NN-potential $V_{\rm med}$ has the same isospin- and spin-structure as the free NN potential. The analytical expressions for $V_{\rm med}$ from the leading chiral $3N$ force at N^{2}LO (involving the parameters $c_{1,3,4}$, c_D, and c_E) have been presented in Ref. [6] and these have found many applications (e.g., to the thermodynamics of nuclear matter) in recent years [7–16]. But in order to perform nuclear many-body calculations that are consistent with their input at the two-body level, one needs also $V_{\rm med}$ derived from the subleading chiral $3N$ forces at order N^{3}LO. In two recent works this task has been completed for the short-range terms and relativistic $1/M$ corrections in Ref. [17], and for the long-range terms in Ref. [18]. In the latter case one is dealing with $3N$ diagrams which were divided in Ref. [1] into classes of 2π-exchange topology, $2\pi 1\pi$-exchange topology, and ring topology. For these topologies the self-closings of a nucleon line and the concatenations of any two nucleon lines to an in-medium loop had to be worked out to together with the summation/integration over the filled Fermi sea of density $\rho = 2k_f^3/3\pi^2$. The momentum- and k_f-dependent potentials associated with the isospin operators (1 and $\vec\tau_1\cdot\vec\tau_2$) and five independent spin structures $[1, \vec\sigma_1\cdot\vec\sigma_2, \vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q,\ i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,),\ \vec\sigma_1\cdot\vec p\,\vec\sigma_2\cdot\vec p+\vec\sigma_1\cdot\vec p\,'\vec\sigma_2\cdot\vec p\,']$ could all be expressed in terms of functions, which were either given in closed analytical form or required at most one numerical integration. In order to obtain for the (nonfactorizable) $3N$-ring diagrams such an expedient form it was crucial to invert the order the original loop integration and the added Fermi-sphere integral. Moreover, the method of dimensional regularization, as it was implicitly used in Ref. [1], could be recovered by subtracting asymptotic constants from the integrands in $\int_0^\infty dl\cdots$.

The purpose of the present paper is to extend the calculation of the in-medium NN-potential $V_{\rm med}$ to the subsubleading chiral $3N$ forces at order N^{4}LO. The long-range 2π-exchange component, symbolized by the left diagram in Fig. 1, has already been treated in Sec. IV of Ref. [18] through appropriate contributions to the two structure functions $\tilde g_+(q_2)$ and $\tilde h_-(q_2)$. As indicated by the notation, these structure functions are equal to f_π^2 times the isoscalar non-spin-flip and isovector spin-flip πN-scattering amplitudes at zero pion-energy $\omega = 0$ and squared momentum-transfer $t = -q_2^2$.

The present paper is organized as follows. Section II starts with the computation of $V_{\rm med}$ from the intermediate-range

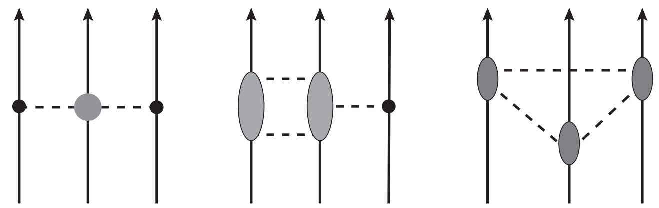

FIG. 1. 2π-exchange topology, $2\pi 1\pi$-exchange topology, and ring topology which comprise the long- and intermediate-range chiral $3N$ forces at subsubleading order N^{4}LO.

$2\pi 1\pi$-exchange component, symbolized by the middle diagram in Fig. 1. In comparison to Sec. III of Ref. [18] one encounters at N^{4}LO a richer spin- and momentum-dependence for this part of the chiral $3N$ force, and 12 instead of eight functions $f_j(q_1)$ are needed to represent all diagrams belonging to this topology. The contributions to $V_{\rm med}$ as they arise from self-closing, vertex-correction by 1π exchange, vertex-correction by 2π exchange, and double exchange are given by semianalytical expressions that comply with this extended structure. Note that Ref. [4] has concluded from a study of the $3N$ potential in coordinate space at the equilateral triangle configuration, that the N^{4}LO corrections to the intermediate-range topologies are numerically large and dominate in most cases over the nominally leading N^{3}LO terms. This feature could be traced back to the large coefficients $c_{2,3,4}$, which reflect the importance of the $\Delta(1232)$ isobar coupled to the πN system. At N^{4}LO the $3N$ diagrams belonging to the ring topology, symbolized by the right diagram in Fig. 1, fall into three classes according to their scaling with g_A^2. Section III is devoted to the simplest ring interaction proportional to $g_A^0 c_{1,2,3,4}$ and the contributions to $V_{\rm med}$ from self-closings and concatenations are given in three subsections. After angular integration the remaining double-integral $\int dl_0 dl$ is treated in polar coordinates and regularized by a (euclidean) cutoff λ. In this form the λ^2 divergence can be easily subtracted and the subsequent logarithmic piece $\sim \ln(m_\pi/\lambda)$ is isolated. A good check is provided by the fact that the total $\lambda^2 k_f^3$ divergence is of isoscalar central type and thus can be absorbed on the $3N$ short-distance parameter c_E. In Sec. IV the analogous calculations are carried out for the more involved ring interaction proportional to $g_A^2 c_{1,2,3,4}$. Finally, one considers in Sec. V the ring interaction proportional to $g_A^4 c_{1,2,3,4}$, which consists of a large number of terms with different isospin, spin, and momentum dependence. At that point one elaborates also a bit on euclidean loop integrals over four or three pion propagators. The self-closing contributions to $V_{\rm med}$ are given in closed analytical form in Sec. V A and one observes that these central, spin-spin, and tensor potentials linear in density ρ depend either on $c_2 + c_3$ or on c_1 and c_3. Concerning the contributions to $V_{\rm med}$ from concatenations, the pertinent expressions are presented in Sec. V B only for three selected pieces from the ring interaction $\sim g_A^4 c_{1,2,3,4}$. These give rise to isoscalar and isovector potentials accompanied by all five spin-structures. A complete list of the lengthy formulas for the remaining contributions to $V_{\rm med}$ from the concatenations of the $3N$-ring interaction $\sim g_A^4 c_{1,2,3,4}$ can be obtained from the author upon request.

In summary, after eventual partial-wave projection the presented results for $V_{\rm med}$ are suitable for an approximate implementation of the subsubleading chiral $3N$ forces of intermediate range into nuclear many-body calculations.

II. TWO-PION-ONE-PION EXCHANGE TOPOLOGY

The $2\pi 1\pi$-exchange $3N$ interaction arises from a large set of loop diagrams, and according to Eq. (3.1) in Ref. [4] it can be written in the general form

$$\begin{aligned} V_{3\rm N} = \frac{g_A^4}{256\pi f_\pi^6}\frac{\vec\sigma_3\cdot\vec q_3}{m_\pi^2+q_3^2}\Big\{&\vec\tau_1\cdot\vec\tau_3\Big[\vec\sigma_2\cdot\vec q_1\,\vec q_1\cdot\vec q_3\, f_1(q_1)+\vec\sigma_2\cdot\vec q_1\, f_2(q_1)+\vec\sigma_2\cdot\vec q_3\, f_3(q_1)\Big]\\ &+\vec\tau_2\cdot\vec\tau_3\Big[\vec\sigma_1\cdot\vec q_1\,\vec q_1\cdot\vec q_3\, f_4(q_1)+\vec\sigma_1\cdot\vec q_3\, f_5(q_1)+\vec\sigma_2\cdot\vec q_1\,\vec q_1\cdot\vec q_3\, f_6(q_1)+\vec\sigma_2\cdot\vec q_1\, f_7(q_1)+\vec\sigma_2\cdot\vec q_3\,\vec q_1\cdot\vec q_3\, f_8(q_1)+\vec\sigma_2\cdot\vec q_3\, f_9(q_1)\Big]\\ &+(\vec\tau_1\times\vec\tau_2)\cdot\vec\tau_3\Big[(\vec\sigma_1\times\vec\sigma_2)\cdot\vec q_1\Big(\vec q_1\cdot\vec q_3\, f_{10}(q_1)+f_{11}(q_1)\Big)+\vec\sigma_1\cdot(\vec q_1\times\vec q_3)\,\vec\sigma_2\cdot\vec q_1\, f_{12}(q_1)\Big]\Big\}, \end{aligned} \tag{1}$$

where $\vec q_j$ denotes the momentum transfer at nucleon $j \in \{1, 2, 3\}$, and $\vec q_1 + \vec q_2 + \vec q_3 = 0$ holds due to momentum conservation. Since a common prefactor $g_A^4/(256\pi f_\pi^6)$ has been pulled out in Eq. (1), the contributions to the reduced functions $f_j(s)$ at N^{3}LO read, according to Eq. (3.2) in Ref. [4],

$$f_1(s) = \frac{m_\pi}{s^2}\big(1-2g_A^2\big) - \frac{g_A^2 m_\pi}{4m_\pi^2+s^2} + \left[1+g_A^2+\frac{4m_\pi^2}{s^2}\big(2g_A^2-1\big)\right]A(s), \tag{2}$$

$$f_2(s) = f_7(s) = \big(4m_\pi^2+2s^2\big)A(s), \quad f_3(s) = \big[4\big(1-2g_A^2\big)m_\pi^2 + \big(1-3g_A^2\big)s^2\big]A(s), \tag{3}$$

$$f_5(s) = -s^2 f_4(s) = 2g_A^2 s^2 A(s), \quad f_{11}(s) = -\left(2m_\pi^2+\frac{s^2}{2}\right)A(s), \quad f_{6,8,9,10,12}(s) = 0 \tag{4}$$

with the heavy-baryon loop-function

$$A(s) = \frac{1}{2s}\arctan\frac{s}{2m_\pi}. \tag{5}$$

Likewise, one extracts from Eq. (3.3) in Ref. [4] the following contributions to the reduced functions $f_j(s)$ at N^{4}LO:

$$f_1(s) = \frac{16c_4}{3\pi}\left\{\left(4-g_A^{-2}\right)\frac{m_\pi^2}{s^2}+\left[\left(g_A^{-2}-4\right)\frac{m_\pi^2}{s^2}+\frac{1-g_A^{-2}}{2}-\frac{3m_\pi^2}{4m_\pi^2+s^2}\right]L(s)\right\}, \tag{6}$$

$$f_3(s) = \frac{16c_4}{3\pi}\left[\left(g_A^{-2}-1\right)m_\pi^2+\left(g_A^{-2}-4\right)s^2-\frac{12m_\pi^4}{4m_\pi^2+s^2}\right]L(s), \quad f_5(s) = -s^2 f_4(s) = \frac{16c_4}{\pi}s^2 L(s), \tag{7}$$

$$f_6(s) = \frac{8}{3\pi}\left\{(6c_1+c_2-3c_3)\frac{m_\pi^2}{s^2}+\left[(3c_3-6c_1-c_2)\frac{m_\pi^2}{s^2}+\frac{c_2}{2}+\frac{3(2c_1+c_3)m_\pi^2}{4m_\pi^2+s^2}\right]L(s)\right\}, \tag{8}$$

$$f_7(s) = \frac{8}{\pi g_A^2}\left\{\left[2\left(c_3+\frac{c_2}{3}-2c_1\right)m_\pi^2+\left(\frac{c_2}{6}+c_3\right)s^2\right]L(s)+\left((8\pi f_\pi)^2\bar{e}_{14}-\frac{5c_2}{18}-c_3\right)\frac{s^2}{2}\right\}, \tag{9}$$

$$f_9(s) = \frac{8}{\pi}\left[(8c_1-c_2-4c_3)m_\pi^2-(3c_2+13c_3)\frac{s^2}{4}-\frac{4(2c_1+c_3)m_\pi^4}{4m_\pi^2+s^2}\right]L(s), \tag{10}$$

$$f_{10}(s) = f_{12}(s) = \frac{4c_4}{\pi}L(s), \quad f_{2,8,11}(s) = 0 \tag{11}$$

with the frequently occurring logarithmic loop function

$$L(s) = \frac{\sqrt{4m_\pi^2+s^2}}{s}\ln\frac{s+\sqrt{4m_\pi^2+s^2}}{2m_\pi}. \tag{12}$$

Note that one has supplied in Eq. (9) through the last term proportional to $s^2/2$ that particular polynomial piece, which cannot be absorbed on the short-distance parameters c_D and c_E. The value of the low-energy constant $\bar{e}_{14}$ as extracted from πN scattering is $\bar{e}_{14} = 1.52\,\mathrm{GeV}^{-3}$ [3] or $\bar{e}_{14} = 1.18\,\mathrm{GeV}^{-3}$ [19]. One notices from Eqs. (4), (11) that there is yet no contribution to $f_8(s)$, but the corresponding structure $\vec{\sigma}_2\cdot\vec{q}_3\,\vec{q}_1\cdot\vec{q}_3$ will arise once explicit $\Delta(1232)$ isobars are considered in the derivation of the chiral $2\pi1\pi$-exchange $3N$ interaction.

A. Contributions to in-medium NN potential

Now one can turn to the contributions of the $2\pi1\pi$-exchange $3N$ interaction V_{3N} written in Eq. (1) to the in-medium NN-potential $V_{\rm med}$. Only the self-closing of nucleon line 1 gives a nonvanishing spin-isospin trace, and after relabeling $3\to1$ one obtains the contribution

$$V_{\rm med}^{(0)} = \frac{g_A^4 k_f^3 f_9(0)}{3(4\pi f_\pi^2)^3}\frac{\vec{\tau}_1\cdot\vec{\tau}_2}{m_\pi^2+q^2}\,\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q} = \frac{g_A^4 m_\pi^2 k_f^3}{24\pi^4 f_\pi^6}\frac{\vec{\tau}_1\cdot\vec{\tau}_2}{m_\pi^2+q^2}\,\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}\,(6c_1-c_2-5c_3), \tag{13}$$

which is of the form: 1π-exchange NN interaction times a factor linear in density $\rho = 2k_f^3/3\pi^2$. The last expression in Eq. (13) comes from evaluating $f_9(s)$ in Eq. (10) at $s=0$. In all forthcoming formulas for $V_{\rm med}$ one denotes by $\vec{q} = \vec{p}\,'-\vec{p}$ the momentum-transfer for the on-shell scattering process $N_1(\vec{p}\,)+N_2(-\vec{p}\,)\to N_1(\vec{p}\,')+N_2(-\vec{p}\,')$ in the nuclear matter rest frame. On the other hand the vertex corrections by 1π exchange, apparent in Eq. (1) through the second factor $\vec{\sigma}_3\cdot\vec{q}_3/(m_\pi^2+q_3^2)$, produce the contribution

$$\begin{aligned}
V_{\rm med}^{(1)} = \frac{g_A^4}{\left(8\pi f_\pi^2\right)^3}\Bigg\{&\left(2m_\pi^2\Gamma_0-\frac{4k_f^3}{3}\right)[\vec{\tau}_1\cdot\vec{\tau}_2 f_3(q)+3f_9(q)]-\left(2\Gamma_2+\frac{q^2}{2}\widetilde{\Gamma}_3\right)q^2[\vec{\tau}_1\cdot\vec{\tau}_2 f_1(q)+3f_6(q)]\\
&+\widetilde{\Gamma}_1 q^2[\vec{\tau}_1\cdot\vec{\tau}_2 f_2(q)+3f_7(q)]+\left(2k_f^3-3m_\pi^2\widetilde{\Gamma}_1\right)q^2 f_8(q)-3(\vec{\sigma}_1\cdot\vec{p}\,\vec{\sigma}_2\cdot\vec{p}+\vec{\sigma}_1\cdot\vec{p}\,'\,\vec{\sigma}_2\cdot\vec{p}\,')\widetilde{\Gamma}_3 f_5(q)\\
&+2\vec{\sigma}_1\cdot\vec{\sigma}_2\left[-3\Gamma_2 f_5(q)+\vec{\tau}_1\cdot\vec{\tau}_2\,q^2\left(2\Gamma_2[f_{10}(q)+f_{12}(q)]-\widetilde{\Gamma}_1 f_{11}(q)+\frac{q^2}{2}\widetilde{\Gamma}_3 f_{10}(q)\right)\right]\\
&+2\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}\left[\vec{\tau}_1\cdot\vec{\tau}_2\left(\widetilde{\Gamma}_1 f_{11}(q)-2\Gamma_2[f_{10}(q)+f_{12}(q)]-\frac{q^2}{2}\widetilde{\Gamma}_3 f_{10}(q)\right)-3\left(\Gamma_2+\frac{q^2}{4}\widetilde{\Gamma}_3\right)f_4(q)\right]\\
&+4\vec{\tau}_1\cdot\vec{\tau}_2\,\vec{\sigma}_1\cdot(\vec{q}\times\vec{p}\,)\,\vec{\sigma}_2\cdot(\vec{q}\times\vec{p}\,)\widetilde{\Gamma}_3 f_{12}(q)\;+i(\vec{\sigma}_1+\vec{\sigma}_2)\cdot(\vec{q}\times\vec{p}\,)\Bigg[\frac{3q^2}{2}\widetilde{\Gamma}_3 f_6(q)-3\widetilde{\Gamma}_1 f_7(q)\\
&+\vec{\tau}_1\cdot\vec{\tau}_2\left(\widetilde{\Gamma}_1[2f_{11}(q)-f_2(q)]+\frac{q^2}{2}\widetilde{\Gamma}_3[f_1(q)-2f_{10}(q)+2f_{12}(q)]\right)\Bigg]\Bigg\}
\end{aligned} \tag{14}$$

with the (p, k_f)-dependent functions Γ_0, $\widetilde{\Gamma}_1 = \Gamma_0 + \Gamma_1$, Γ_2 and $\widetilde{\Gamma}_3 = \Gamma_0 + 2\Gamma_1 + \Gamma_3$ defined in the Appendix of Ref. [17]. Moreover, the vertex corrections by 2π exchange, represented by the expression in curly brackets of Eq. (1), can be summarized as the 1π-exchange NN interaction times a (p, q, k_f)-dependent factor

$$V_{\rm med}^{(2)} = \frac{g_A^4}{(8\pi f_\pi^2)^3} \frac{\vec\tau_1\cdot\vec\tau_2}{m_\pi^2+q^2}\, \vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q\,\big[S_1(p,k_f) + q^2 S_2(p,k_f)\big]. \tag{15}$$

The two auxiliary functions $S_{1,2}(p, k_f)$ are computed as integrals over $f_j(s)$ in the following way:

$$S_1(p,k_f) = \int_{p-k_f}^{p+k_f} ds\,\frac{s}{p}\big[k_f^2-(p-s)^2\big]\bigg\{2s^2 f_{12}(s) - f_3(s) - f_5(s) - f_9(s) + \frac{1}{8p^2}\big[(p+s)^2-k_f^2\big][f_2(s)+f_7(s)-4f_{11}(s)]$$
$$+\frac{1}{24p^2}\big[k_f^2-(p-s)^2\big]\big(s^2+4sp+p^2-k_f^2\big)[4f_{10}(s)-2f_{12}(s)-f_1(s)-f_4(s)-f_6(s)]\bigg\}, \tag{16}$$

$$S_2(p,k_f) = \int_{p-k_f}^{p+k_f} ds\,\frac{s}{8p^3}\big[k_f^2-(p-s)^2\big]\big[(p+s)^2-k_f^2\big]\bigg\{f_8(s)$$
$$+\frac{1}{4p^2}\big(s^2+p^2-k_f^2\big)[4f_{10}(s)-2f_{12}(s)-f_1(s)-f_4(s)-f_6(s)]\bigg\}. \tag{17}$$

Finally, there is the contribution $V_{\rm med}^{(3)}$ from the double exchange, which is separated into an isoscalar part

$$V_{\rm med}^{(3)} = \frac{3g_A^4}{(8\pi f_\pi^2)^3}\bigg[\vec\sigma_1\cdot\vec\sigma_2(2I_{2,2}-2I_{3,2}-H_{1,2}-\tilde I_{1,2}) + \vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q\left(\frac{H_{1,1}+\tilde I_{1,4}}{2} - I_{2,4} - I_{3,5}\right)$$
$$+(\vec\sigma_1\cdot\vec p\,\vec\sigma_2\cdot\vec p + \vec\sigma_1\cdot\vec p\,'\,\vec\sigma_2\cdot\vec p\,')\left(I_{2,3}-I_{3,3}-\frac{H_{1,3}+\tilde I_{1,3}}{2}\right)\bigg], \tag{18}$$

and an isovector part

$$V_{\rm med}^{(3)} = \frac{g_A^4\,\vec\tau_1\cdot\vec\tau_2}{(8\pi f_\pi^2)^3}\bigg\{2m_\pi^2 I_{5,0} - 2H_{5,0} + \frac{q^2}{2}(H_{4,1}+\tilde I_{4,4}) - p^2(H_{4,3}+\tilde I_{4,3}) - 3H_{4,2} - 3\tilde I_{4,2}$$
$$+ i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec p\times\vec q\,)\left[H_{10,1}+\tilde I_{10,1} - \frac{1}{2}(H_{4,1}+\tilde I_{4,1}) - 2I_{11,1}\right]$$
$$+\vec\sigma_1\cdot\vec\sigma_2\big[2I_{7,2} - H_{6,2} - \tilde I_{6,2} + H_{8,2} + \tilde I_{8,2} - 2I_{9,2} + 4H_{10,2} + 4\tilde I_{10,2} - 8I_{11,2} + 2H_{12,2}$$
$$-2\tilde I_{12,2} - 4m_\pi^2 I_{12,2} + 2p^2(H_{10,3}+\tilde I_{10,3}-2I_{11,3}) - q^2(H_{10,1}+\tilde I_{10,4}-2I_{11,4})\big]$$
$$+\vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q\left[\frac{H_{6,1}+\tilde I_{6,4}+\tilde I_{8,5}}{2} - I_{7,4} + H_{8,0} - H_{8,1} - I_{9,5} + H_{10,1} + \tilde I_{10,4}\right.$$
$$\left. -2I_{11,4} + H_{12,1} + \tilde I_{12,4} + 2m_\pi^2(2I_{12,4}+I_{12,5}-2I_{12,0})\right] + (\vec\sigma_1\cdot\vec p\,\vec\sigma_2\cdot\vec p + \vec\sigma_1\cdot\vec p\,'\,\vec\sigma_2\cdot\vec p\,')$$
$$\times\left[\frac{H_{8,3}-H_{6,3}+\tilde I_{8,3}-\tilde I_{6,3}}{2} + I_{7,3} - I_{9,3} - H_{10,3} - \tilde I_{10,3} + 2I_{11,3} + H_{12,3} - \tilde I_{12,3} - 2m_\pi^2 I_{12,3}\right]\bigg\}. \tag{19}$$

The double-indexed functions $H_{j,\nu}(p)$ are defined by

$$H_{j,0}(p) = \frac{1}{2p}\int_{p-k_f}^{p+k_f} ds\,s f_j(s)\big[k_f^2-(p-s)^2\big], \tag{20}$$

$$H_{j,1}(p) = \frac{1}{8p^3}\int_{p-k_f}^{p+k_f} ds\,s f_j(s)\big[k_f^2-(p-s)^2\big]\big[(p+s)^2-k_f^2\big], \tag{21}$$

$$H_{j,2}(p) = \frac{1}{48p^3}\int_{p-k_f}^{p+k_f} ds\,s f_j(s)\big[k_f^2-(p-s)^2\big]^2\big(s^2+4sp+p^2-k_f^2\big), \tag{22}$$

$$H_{j,3}(p) = \frac{1}{16p^5}\int_{p-k_f}^{p+k_f} ds\,s f_j(s)\big[k_f^2-(p-s)^2\big]\big[(p+s)^2-k_f^2\big]\big(p^2+s^2-k_f^2\big). \tag{23}$$

The other double-indexed functions $I_{j,\nu}(p,q)$ are defined by

$$I_{j,0}(p,q) = \frac{1}{2q}\int_{p-k_f}^{p+k_f} ds\, s f_j(s) \ln\frac{qX+2\sqrt{W}}{(2p+q)\big[m_\pi^2+(s-q)^2\big]}, \tag{24}$$

$$I_{j,1}(p,q) = \frac{1}{4p^2-q^2}\int_{p-k_f}^{p+k_f} ds\, s f_j(s)\bigg[\frac{p\big(s^2+m_\pi^2\big)-\sqrt{W}}{q^2}+\frac{p^2+k_f^2-s^2}{2p}\bigg], \tag{25}$$

$$\begin{aligned} I_{j,2}(p,q) = {}& \frac{1}{8q^2}\int_{p-k_f}^{p+k_f} ds\, s f_j(s)\bigg\{s\big(m_\pi^2+s^2+q^2\big)-p\bigg(m_\pi^2+s^2+\frac{3q^2}{4}\bigg)+\frac{p}{4p^2-q^2}\bigg(m_\pi^2+2k_f^2-s^2+\frac{q^2}{2}\bigg)^2 \\ & -\frac{X\sqrt{W}}{4p^2-q^2}-\frac{\big(k_f^2-s^2\big)^2}{p}-\frac{1}{2q}\big[m_\pi^2+(s+q)^2\big]\big[m_\pi^2+(s-q)^2\big]\ln\frac{qX+2\sqrt{W}}{(2p+q)\big[m_\pi^2+(s-q)^2\big]}\bigg\}, \end{aligned} \tag{26}$$

$$\begin{aligned} I_{j,3}(p,q) = {}& \frac{1}{(4p^2-q^2)^2}\int_{p-k_f}^{p+k_f} ds\, s f_j(s)\bigg\{\frac{X\sqrt{W}}{q^2}+\frac{q^2}{8p^3}\big(k_f^2-s^2\big)^2-\frac{3pq^2}{8}+\frac{p}{2}\big(2k_f^2-3m_\pi^2+p^2-s^2\big) \\ & +\frac{p}{q^2}\big(s^2+m_\pi^2\big)\big(2p^2+s^2-2k_f^2-m_\pi^2\big)+\frac{1}{4p}\big[s^2\big(2m_\pi^2+q^2-4s^2\big)+k_f^2\big(10s^2-2m_\pi^2-3q^2\big)-6k_f^4\big]\bigg\}, \end{aligned} \tag{27}$$

$$\begin{aligned} I_{j,4}(p,q) = {}& \frac{1}{4q^4}\int_{p-k_f}^{p+k_f} ds\, s f_j(s)\bigg\{\frac{X}{(4p^2-q^2)^2}\Big[\sqrt{W}(3q^2-4p^2)-8p^3X\Big] \\ & +\bigg[\frac{q^3}{2}+q\big(s^2-m_\pi^2\big)-\frac{3}{2q}\big(s^2+m_\pi^2\big)^2\bigg]\ln\frac{qX+2\sqrt{W}}{(2p+q)\big[m_\pi^2+(s-q)^2\big]} \\ & +\frac{p}{4p^2-q^2}\big[16k_f^4+8k_f^2\big(2m_\pi^2+q^2-2s^2\big)+3m_\pi^4+3s^4+2q^2\big(m_\pi^2-s^2\big)-10s^2m_\pi^2\big] \\ & +s\big(3s^2-q^2+3m_\pi^2\big)+2p^3-p\big(4k_f^2+4m_\pi^2+q^2\big)+\frac{2}{p}\big(k_f^2+q^2-s^2\big)\big(s^2-k_f^2\big)\bigg\}, \end{aligned} \tag{28}$$

$$I_{j,5}(p,q) = -I_{j,4}(p,q)+\frac{1}{2q^2}\int_{p-k_f}^{p+k_f} ds\, s f_j(s)\bigg[\frac{q^2-s^2-m_\pi^2}{q}\ln\frac{qX+2\sqrt{W}}{(2p+q)\big[m_\pi^2+(s-q)^2\big]}+\frac{k_f^2-(p-s)^2}{p}\bigg] \tag{29}$$

with the auxiliary polynomials

$$\begin{aligned} X &= m_\pi^2+2\big(k_f^2-p^2\big)+q^2-s^2, \\ W &= k_f^2q^4+p^2\big(m_\pi^2+s^2\big)^2+q^2\big[\big(k_f^2-p^2\big)^2+m_\pi^2\big(k_f^2+p^2\big)-s^2\big(k_f^2+p^2+m_\pi^2\big)\big]. \end{aligned} \tag{30}$$

Furthermore, the functions $\tilde{I}_{j,\nu}(p,q)$ with $j = 1, 4, 6, 8, 10, 12$ appearing in Eqs. (18), (19) are computed analogously by replacing in the integrand $f_j(s)$ by $\tilde{f}_j(s) = (s^2-m_\pi^2-q^2)f_j(s)$. The decomposition into $H_{j,\nu}$ and $I_{j,\nu}$ is obtained by canceling momentum factors against a pion propagator, while $\tilde{I}_{j,\nu}$ takes care of the s^2-dependent remainder terms.

III. RING INTERACTION PROPORTIONAL TO g_A^0

Next, one considers the 3*N*-ring interaction at N^{4}LO, which consists of three pieces with a different dependence on the axial-vector coupling constant: g_A^{2n}, $n = 0, 1, 2$. The g_A^0 part can be obtained directly from the well-known Feynman rules for the $\pi\pi NN$ Tomozawa-Weinberg vertex and the second-order $\pi\pi NN$-contact vertex proportional to $c_{1,2,3,4}$. Altogether, the 3*N*-ring interaction proportional to $g_A^0 c_{1,2,3,4}$ is given by a euclidean loop-integral of the form

$$V_{3\mathrm{N}} = -\frac{1}{f_\pi^6}\int_0^\infty dl_0\int\frac{d^3l_2}{(2\pi)^4}\frac{l_0^2}{\big(\bar{m}^2+l_1^2\big)\big(\bar{m}^2+l_2^2\big)\big(\bar{m}^2+l_3^2\big)}\bigg\{\vec{\tau}_2\cdot\vec{\tau}_3\big[2c_1m_\pi^2+(c_2+c_3)l_0^2+c_3\,\vec{l}_2\cdot\vec{l}_3\big]+\frac{c_4}{4}\vec{\tau}_1\cdot(\vec{\tau}_2\times\vec{\tau}_3)\,\vec{\sigma}_1\cdot(\vec{l}_3\times\vec{l}_2)\bigg\} \tag{31}$$

with $\bar{m} = \sqrt{m_\pi^2+l_0^2}$ and one has to set $\vec{l}_1 = \vec{l}_2-\vec{q}_3$ and $\vec{l}_3 = \vec{l}_2+\vec{q}_1$. Alternatively, one can take the (Fourier-transformed) coordinate-space potential in Eq. (4.8) of Ref. [4] and translate spatial gradients back to momentum factors $\vec{l}_{1,2,3}$. Note that the four-dimensional loop integral in Eq. (31) is quadratically divergent and therefore the 3*N*-ring interaction V_{3N} requires a regularization (e.g., by an ultraviolet cutoff) and a renormalization (by absorbing cutoff-dependent pieces on the 3*N* short-distance parameters c_E and $E_{1,\ldots,10}$ [20]).

TABLE I. Assignment of pion momenta, where $\vec{l}$ is unconstrained and $|\vec{l}_4| < k_f$ from a Fermi sphere.

concat.	N_3 on N_2	N_2 on N_3	N_3 on N_1	N_1 on N_3	N_1 on N_2	N_2 on N_1
$\vec{l}_1 =$	$\vec{l}_4+\vec{l}$	$-\vec{l}_4-\vec{l}$	$\vec{l}-\vec{p}\,'$	$\vec{p}-\vec{l}$	$\vec{l}-\vec{p}$	$\vec{p}\,'-\vec{l}$
$\vec{l}_2 =$	$\vec{l}-\vec{p}\,'$	$\vec{p}-\vec{l}$	$\vec{l}_4+\vec{l}$	$-\vec{l}_4-\vec{l}$	$\vec{l}-\vec{p}\,'$	$\vec{p}-\vec{l}$
$\vec{l}_3 =$	$\vec{l}-\vec{p}$	$\vec{p}\,'-\vec{l}$	$\vec{l}-\vec{p}$	$\vec{p}\,'-\vec{l}$	$\vec{l}_4+\vec{l}$	$-\vec{l}_4-\vec{l}$

A. In-medium *NN* potential from self-closing of nucleon lines

Only the self-closing of nucleon line 1 gives a nonvanishing spin-isospin trace, and after relabeling $3 \to 1$ one recognizes an isovector central potential $\sim k_f^3 \vec{\tau}_1\cdot\vec{\tau}_2$. Evaluating the pertinent loop integral in spherical coordinates $l_0 = r\cos\psi$, $l_2 = r\sin\psi$, $\hat{l}_2\cdot\hat{q} = \cos\theta$ and introducing a cutoff λ for the radial integral $\int_0^\lambda dr$, one gets the following contribution to the in-medium *NN* potential:

$$V_{\rm med}^{(0)} = \frac{k_f^3\,\vec{\tau}_1\cdot\vec{\tau}_2}{48\pi^4 f_\pi^6}\bigg\{\bigg[\bigg(2c_1-\frac{3c_2}{2}-3c_3\bigg)m_\pi^2-\bigg(\frac{c_2}{4}+\frac{c_3}{3}\bigg)q^2\bigg]\ln\frac{m_\pi}{\lambda}+\bigg(\frac{3c_2}{2}-2c_1+\frac{13c_3}{3}\bigg)\frac{m_\pi^2}{4} + \bigg(\frac{13c_2}{8}+\frac{11c_3}{3}\bigg)\frac{q^2}{12}+\bigg[\bigg(2c_1-c_2-\frac{7c_3}{3}\bigg)m_\pi^2-\bigg(\frac{c_2}{4}+\frac{c_3}{3}\bigg)q^2\bigg]L(q)\bigg\} \tag{32}$$

with the function $L(q)$ defined in Eq. (12). Note that the power divergence proportional to $\lambda^2 k_f^3$ has been dropped in Eq. (32), but it will be considered in the total balance at the end of this section. Note also that the coefficient $c_2/4 + c_3/3$ appears twice, such that the chiral limit $m_\pi \to 0$ of $V_{\rm med}^{(0)}$ exists.

B. In-medium *NN* potential from concatenations N_3 on N_2 and N_2 on N_3

Next, one has to work out for V_{3N} in Eq. (31) the six possible concatenations of two nucleon lines and their mirror graphs. The proper assignments of $\vec{l}_1$, $\vec{l}_2$, $\vec{l}_3$, with $\vec{l}$ the unconstrained loop-momentum and $\vec{l}_4$ from the interior of a Fermi-sphere $|\vec{l}_4| < k_f$, are given for each concatenation in Table I. The integral over the Fermi sphere and the angular part of the loop integral can always be solved analytically in terms of the following functions:

$$\bar{\Gamma}_0(l) = k_f - \bar{m}\bigg[\arctan\frac{k_f+l}{\bar{m}}+\arctan\frac{k_f-l}{\bar{m}}\bigg]+\frac{\bar{m}^2+k_f^2-l^2}{4l}\ln\frac{\bar{m}^2+(k_f+l)^2}{\bar{m}^2+(k_f-l)^2}, \tag{33}$$

$$\bar{\Gamma}_1(l) = \frac{k_f}{4l^2}\big(\bar{m}^2+k_f^2+l^2\big)-\frac{1}{16l^3}[\bar{m}^2+(k_f+l)^2][\bar{m}^2+(k_f-l)^2]\ln\frac{\bar{m}^2+(k_f+l)^2}{\bar{m}^2+(k_f-l)^2}, \tag{34}$$

$$\bar{\Gamma}_2(l) = \frac{k_f^3}{9}-\frac{\bar{m}^2}{3}\bar{\Gamma}_0(l)+\frac{1}{6}\big(k_f^2+\bar{m}^2-l^2\big)\bar{\Gamma}_1(l), \quad \bar{\Gamma}_3(l) = \frac{k_f^3}{3l^2}+\frac{l^2-\bar{m}^2-k_f^2}{2l^2}\bar{\Gamma}_1(l), \tag{35}$$

$$\Lambda(l) = \frac{1}{4p}\ln\frac{\bar{m}^2+(l+p)^2}{\bar{m}^2+(l-p)^2}, \quad \Omega(l) = \frac{1}{q\sqrt{B+q^2l^2}}\ln\frac{q\,l+\sqrt{B+q^2l^2}}{\sqrt{B}} \tag{36}$$

with the abbreviation $B = [\bar{m}^2+(l+p)^2][\bar{m}^2+(l-p)^2]$. For the remaining integration over dl_0dl one chooses polar coordinates $l_0 = r\cos\psi$, $l = r\sin\psi$ and sets a radial cutoff λ. By performing these calculational steps one obtains from the concatenations N_3 on N_2 and N_2 on N_3 an isoscalar central potential of the form

$$V_{\rm med}^{(1)} = \frac{3}{4\pi^5 f_\pi^6}\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi\bigg\{l_0^2 l\,\bar{\Gamma}_0(l)\bigg[c_3\Lambda(l)+\bigg(2c_1m_\pi^2+c_2l_0^2-\frac{c_3}{2}\big(2m_\pi^2+q^2\big)\bigg)\Omega(l)\bigg]-\frac{k_f^3}{6}(c_3+c_2\cos^2\psi)\sin^2 2\psi\bigg\}. \tag{37}$$

The purpose of the subtraction term at the end of the integrand is to remove a power divergence proportional to $\lambda^2 k_f^3$. After that the double integral in Eq. (37) has only a logarithmic dependence on the cutoff:

$$\frac{\pi k_f^3}{48}\bigg[c_2\bigg(3m_\pi^2+\frac{3k_f^2}{10}+\frac{p^2}{2}+\frac{q^2}{4}\bigg)-4c_1m_\pi^2+c_3\bigg(6m_\pi^2+\frac{2k_f^2}{5}+\frac{2p^2}{3}+q^2\bigg)\bigg]\ln\frac{m_\pi}{\lambda}, \tag{38}$$

and this detailed knowledge may be useful for numerical checks.

The last c_4 term in Eq. (31) produces in the same way a contribution to the isovector spin-orbit potential

$$V_{\rm med}^{(1)} = \frac{c_4 \vec\tau_1\cdot\vec\tau_2}{8\pi^5 f_\pi^6} i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,)\int_0^\lambda dr r\int_0^{\pi/2} d\psi\, \frac{l_0^2 l\,\bar\Gamma_0(l)}{4p^2-q^2}\bigg[\Lambda(l)+\bigg(p^2-l^2-\bar m^2-\frac{q^2}{2}\bigg)\Omega(l)\bigg] \tag{39}$$

with a large-λ behavior of the double integral: $-(\pi k_f^3/144)\ln(m_\pi/\lambda)$.

C. In-medium *NN* potential from remaining four concatenations

The other four concatenations, N_3 on N_1, N_1 on N_3, N_1 on N_2, and N_2 on N_1, applied to the $c_{1,2,3}$ term in Eq. (31) give rise to an isovector central potential of the form

$$V_{\rm med}^{(\rm cc)} = \frac{\vec\tau_1\cdot\vec\tau_2}{4\pi^5 f_\pi^6}\int_0^\lambda dr r\int_0^{\pi/2} d\psi\bigg\{l_0^2 l\bigg\{\big[4c_1m_\pi^2+2(c_2+c_3)l_0^2\big]\bar\Gamma_0(l)\Omega(l) + c_3\bar\Gamma_1(l)[\Lambda(l)+(l^2-\bar m^2-p^2)\Omega(l)]\bigg\} - \frac{k_f^3}{3}(c_3+c_2\cos^2\psi)\sin^2 2\psi\bigg\} \tag{40}$$

with a suitable subtraction term to have only a logarithmic λ dependence of the double integral:

$$\frac{\pi k_f^3}{24}\bigg[c_2\bigg(3m_\pi^2+\frac{3k_f^2}{10}+\frac{p^2}{2}+\frac{q^2}{4}\bigg)-4c_1m_\pi^2+c_3\bigg(6m_\pi^2+\frac{4k_f^2}{5}+\frac{4p^2+q^2}{3}\bigg)\bigg]\ln\frac{m_\pi}{\lambda}. \tag{41}$$

Under the same calculational treatment, the c_4 term in Eq. (31) produces a further contribution to the isovector spin-orbit potential

$$V_{\rm med}^{(\rm cc)} = \frac{c_4\,\vec\tau_1\cdot\vec\tau_2}{8\pi^5 f_\pi^6} i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,)\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\frac{l_0^2 l\,\bar\Gamma_1(l)}{4p^2-q^2}\big[(p^2+l^2+\bar m^2)\Omega(l)-\Lambda(l)\big] \tag{42}$$

with a large-λ behavior of the double integral: $-(\pi k_f^3/72)\ln(m_\pi/\lambda)$. Before closing this section, one takes a closer look at the balance of λ^2 divergences in the total sum $V_{\rm med}^{(0)}+V_{\rm med}^{(1)}+V_{\rm med}^{(\rm cc)}$. It reads

$$\frac{k_f^3\lambda^2}{192\pi^4 f_\pi^6}\bigg[-\vec\tau_1\cdot\vec\tau_2(c_2+2c_3)+3\bigg(\frac{c_2}{2}+c_3\bigg)+\vec\tau_1\cdot\vec\tau_2(c_2+2c_3)\bigg] = \frac{k_f^3\lambda^2}{128\pi^4 f_\pi^6}(c_2+2c_3), \tag{43}$$

such that the remaining isoscalar piece can be absorbed on the $3N$ short-distance parameter c_E. This perfect matching gives an a posteriori justification to drop or subtract the λ^2 divergences at any place. In the case of the pieces proportional to $\ln(m_\pi/\lambda)$ one can verify that these can be absorbed on the parameters $E_{1,\ldots,10}$ of the subleading $3N$-contact interaction [20] (see also Eq. (49) in Ref. [17]).

IV. RING INTERACTION PROPORTIONAL TO g_A^2

The $3N$-ring interaction proportional to $g_A^2 c_{1,2,3,4}$ can be inferred from the coordinate-space potential written in Eq. (4.7) of Ref. [4]. By exploiting the permutational symmetry (and parity invariance) one can obtain the following somewhat simpler form:

$$V_{\rm 3N} = -\frac{g_A^2}{f_\pi^6}\int_0^\infty dl_0\int\frac{d^3l_2}{(2\pi)^4}\,\frac{1}{(\bar m^2+l_1^2)(\bar m^2+l_2^2)(\bar m^2+l_3^2)}\bigg\{\vec\tau_2\cdot\vec\tau_3\,\vec l_1\cdot(\vec l_2+\vec l_3) \times\big[2c_1m_\pi^2+(c_2+c_3)l_0^2+c_3\,\vec l_2\cdot\vec l_3\big]+\frac{c_4}{2}\Big[\vec\tau_1\cdot(\vec\tau_2\times\vec\tau_3)\,\vec l_1\cdot\vec l_2\,\vec\sigma_2\cdot(\vec l_1\times\vec l_3)+\vec\tau_1\cdot(\vec\tau_2+\vec\tau_3) \times\Big(\bar m^2(\vec\sigma_2\times\vec l_3)\cdot(\vec\sigma_3\times\vec l_2)+\vec l_2\cdot\vec l_3\,\vec\sigma_2\cdot\vec l_1\,\vec\sigma_3\cdot\vec l_1+\vec l_1\cdot\vec l_2\,\vec l_1\cdot\vec l_3\,\vec\sigma_2\cdot\vec\sigma_3-2\vec l_1\cdot\vec l_3\,\vec\sigma_2\cdot\vec l_2\,\vec\sigma_3\cdot\vec l_1\Big)\Big]\bigg\}, \tag{44}$$

which involves only three different isospin-operators: $\vec\tau_2\cdot\vec\tau_3$, $\vec\tau_1\cdot(\vec\tau_2\times\vec\tau_3)$ and $\vec\tau_1\cdot(\vec\tau_2+\vec\tau_3)$. Again, $\bar m$ stands for $\bar m=\sqrt{m_\pi^2+l_0^2}$ and one has to set $\vec l_1=\vec l_2-\vec q_3$ and $\vec l_3=\vec l_2+\vec q_1$.

A. In-medium *NN* potential from self-closing of nucleon lines

Following the same procedure as in Sec. III A, one obtains from the self-closing of nucleon line 1 (providing a nonvanishing spin-isospin trace) a further contribution to the isovector central potential:

$$V_{\rm med}^{(0)} = \frac{g_A^2 k_f^3\,\vec\tau_1\cdot\vec\tau_2}{48\pi^4 f_\pi^6}\bigg\{\bigg[3m_\pi^2(4c_1-c_2-6c_3)-\frac{5q^2}{6}(c_2+4c_3)\bigg]\ln\frac{m_\pi}{\lambda}+\bigg(c_1+\frac{c_2}{12}+\frac{11c_3}{6}\bigg)m_\pi^2 +\bigg(\frac{41c_2}{8}+43c_3\bigg)\frac{q^2}{18}+\bigg[\frac{4m_\pi^2}{3}(6c_1-c_2-7c_3)m_\pi^2-\frac{5q^2}{6}(c_2+4c_3)+\frac{2m_\pi^2q^2}{4m_\pi^2+q^2}(2c_1-c_3)\bigg]L(q)\bigg\}, \tag{45}$$

where the λ^2 divergence has been dropped. Again, the appearance of the same coefficient c_2+4c_3 in the first and second line of Eq. (45) guarantees the chiral limit $m_\pi \to 0$ of $V_{\rm med}^{(0)}$.

B. In-medium *NN* potential from concatenations N_3 on N_2 and N_2 on N_3

The concatenations N_3 on N_2 and N_2 on N_3 give for the first term with isospin-factor $\vec\tau_2\cdot\vec\tau_3$ in Eq. (44) an isoscalar central potential of the form

$$V_{\rm med}^{(1)} = \frac{3g_A^2}{4\pi^5 f_\pi^6}\int_0^\lambda dr r\int_0^{\pi/2} d\psi\bigg\{l\bar\Gamma_1(l)\bigg\{\bigg[2c_1m_\pi^2+c_2l_0^2+c_3\bigg(r^2-2\bar m^2-p^2+\frac{q^2}{4p^2}(l^2+\bar m^2-p^2)\bigg)\bigg]\Lambda(l)$$
$$+\frac{c_3 l}{4p^2}(4p^2-q^2)+\bigg[2c_1m_\pi^2+c_2l_0^2-c_3\bigg(m_\pi^2+\frac{q^2}{2}\bigg)\bigg](l^2-\bar m^2-p^2)\Omega(l)\bigg\}-\frac{4k_f^3}{3}(c_3+c_2\cos^2\psi)\sin^4\psi\bigg\}, \quad (46)$$

where the subtraction term at the end of the second line removes the $\lambda^2k_f^3$ divergence. The remaining logarithmic dependence of the double integral on the cutoff λ is

$$\frac{\pi k_f^3}{12}\bigg[\frac{c_2}{4}\bigg(6m_\pi^2+k_f^2+\frac{5p^2}{3}+\frac{q^2}{6}\bigg)-6c_1m_\pi^2+c_3\bigg(9m_\pi^2+k_f^2+\frac{5p^2}{3}+q^2\bigg)\bigg]\ln\frac{m_\pi}{\lambda}. \quad (47)$$

The second term $\sim c_4\vec\tau_1\cdot(\vec\tau_2\times\vec\tau_3)$ in Eq. (44) produces an isovector spin-orbit potential of the form

$$V_{\rm med}^{(1)} = \frac{c_4g_A^2\vec\tau_1\cdot\vec\tau_2}{16\pi^5 f_\pi^6}i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,)\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\frac{l}{4p^2-q^2}\bigg\{2\bar\Gamma_2(l)\big[-2\Lambda(l)$$
$$+(2l^2+2\bar m^2-2p^2+q^2)\Omega(l)\big]+\bar\Gamma_3(l)(\bar m^2+p^2-l^2)\big[\Lambda(l)-(\bar m^2+l^2+p^2)\Omega(l)\big]\bigg\} \quad (48)$$

with a large-λ behavior of the double integral: $-(\pi k_f^3/18)\ln(m_\pi/\lambda)$. The third term proportional to $c_4\vec\tau_1\cdot(\vec\tau_2+\vec\tau_3)$ in Eq. (44) gives, on the one hand, rise to an isovector central potential of the form

$$V_{\rm med}^{(1)} = \frac{c_4g_A^2\vec\tau_1\cdot\vec\tau_2}{8\pi^5 f_\pi^6}\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\bigg\{2l\big[\bar m^2\bar\Gamma_0(l)+2\bar\Gamma_2(l)\big]\big[2\Lambda(l)-(2\bar m^2+q^2)\Omega(l)\big]$$
$$+l\Gamma_3(l)\bigg[\frac{l}{2}+(3l^2-\bar m^2-p^2)\Lambda(l)+\bigg(\frac{B}{2}-l^2(4\bar m^2+q^2)\bigg)\Omega(l)\bigg]-\frac{8k_f^3}{3}\sin^4\psi\bigg\}, \quad (49)$$

where the double integral has the large-λ behavior: $\pi k_f^3[m_\pi^2+k_f^2/10+p^2/6+7q^2/36]\ln(m_\pi/\lambda)$. On the other hand one gets a contribution to the isovector spin-orbit potential of the form

$$V_{\rm med}^{(1)} = \frac{c_4g_A^2\vec\tau_1\cdot\vec\tau_2}{8\pi^5 f_\pi^6}i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,)\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\frac{l}{4p^2-q^2}\bigg\{\big[\bar m^2\bar\Gamma_0(l)+2\bar\Gamma_2(l)\big]\big[2\Lambda(l)$$
$$+(2p^2-2l^2-2\bar m^2-q^2)\Omega(l)\big]+\bar\Gamma_3(l)(\bar m^2+p^2-l^2)\big[(\bar m^2+l^2+p^2)\Omega(l)-\Lambda(l)\big]\bigg\} \quad (50)$$

with a large-λ behavior of the double integral: $(\pi k_f^3/24)\ln(m_\pi/\lambda)$.

C. In-medium *NN* potential from remaining four concatenations

The other four concatenations, N_3 on N_1, N_1 on N_3, N_1 on N_2, and N_2 on N_1, applied the first term $\sim\vec\tau_2\cdot\vec\tau_3$ in Eq. (44) produce an isovector central potential of the form

$$V_{\rm med}^{(cc)} = \frac{g_A^2\vec\tau_1\cdot\vec\tau_2}{4\pi^5 f_\pi^6}\int_0^\lambda dr r\int_0^{\pi/2} d\psi\bigg\{l\big[2c_1m_\pi^2+(c_2+c_3)l_0^2\big]\bigg\{\bar\Gamma_0(l)\big[2\Lambda(l)-(2\bar m^2+q^2)\Omega(l)\big]$$
$$+\bar\Gamma_1(l)\big[\Lambda(l)+(l^2-p^2-\bar m^2)\Omega(l)\big]\bigg\}+c_3l\bigg\{\bar\Gamma_2(l)\big[2\Lambda(l)-(2\bar m^2+q^2)\Omega(l)\big]$$
$$+\bar\Gamma_3(l)\bigg[\frac{l}{2}+(l^2-p^2-\bar m^2)\Lambda(l)+\frac{1}{2}(l^2-p^2-\bar m^2)^2\Omega(l)\bigg]+\bar\Gamma_1(l)\bigg[\frac{l}{4p^2}(4p^2-q^2)$$
$$+\bigg(l^2-p^2-2\bar m^2-\frac{q^2}{4}+\frac{q^2}{4p^2}(l^2+\bar m^2)\bigg)\Lambda(l)+\bigg(\bar m^2+\frac{q^2}{2}\bigg)(\bar m^2+p^2-l^2)\Omega(l)\bigg]\bigg\}$$
$$-\frac{8k_f^3}{3}(c_3+c_2\cos^2\psi)\sin^4\psi\bigg\}, \quad (51)$$

where the double integral depends (after subtraction in the last line) logarithmically on the cutoff

$$\frac{\pi k_f^3}{4}\left[c_2\left(m_\pi^2+\frac{k_f^2}{10}+\frac{p^2}{6}+\frac{5q^2}{36}\right)-4c_1m_\pi^2+c_3\left(6m_\pi^2+\frac{11k_f^2}{15}+\frac{11p^2+5q^2}{9}\right)\right]\ln\frac{m_\pi}{\lambda}. \tag{52}$$

In the same way one obtains from the second term $\sim c_4\vec{\tau}_1\cdot(\vec{\tau}_2\times\vec{\tau}_3)$ in Eq. (44) a contribution to the isovector spin-orbit potential of the form

$$V_{\rm med}^{(\rm cc)}=\frac{c_4g_A^2\vec{\tau}_1\cdot\vec{\tau}_2}{8\pi^5f_\pi^6}i(\vec{\sigma}_1+\vec{\sigma}_2)\cdot(\vec{q}\times\vec{p}\,)\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi\,\frac{l\,\bar{\Gamma}_1(l)}{4p^2-q^2}\Big\{(l^2+p^2)\Lambda(l)-\big[(l^2-p^2)^2+q^2l^2+\bar{m}^2(l^2+p^2)\big]\Omega(l)\Big\} \tag{53}$$

with a large-λ behavior of the double integral: $-(\pi k_f^3/18)\ln(m_\pi/\lambda)$. Under the same calculational treatment the third term $\sim c_4\vec{\tau}_1\cdot(\vec{\tau}_2+\vec{\tau}_3)$ in Eq. (44) gives rise to three spin-dependent potentials with the common isospin-factor $3+\vec{\tau}_1\cdot\vec{\tau}_2$. The pertinent spin-spin potential has the form

$$\begin{aligned}V_{\rm med}^{(\rm cc)}=&\frac{c_4g_A^2\vec{\sigma}_1\cdot\vec{\sigma}_2}{16\pi^5f_\pi^6}(3+\vec{\tau}_1\cdot\vec{\tau}_2)\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi\bigg\{l\bar{\Gamma}_1(l)\bigg[\frac{l}{2p^2}(3p^2-l^2-\bar{m}^2-q^2)\\&+\left(l^2-\bar{m}^2-\frac{3p^2}{2}+(\bar{m}^2+l^2+q^2)\frac{\bar{m}^2+l^2}{2p^2}+\frac{q^2(4\bar{m}^2+4l^2+q^2)}{8p^2-2q^2}\right)\Lambda(l)\\&+\frac{q^2(\bar{m}^2+l^2+p^2)}{4p^2-q^2}(2p^2-2l^2-2\bar{m}^2-q^2)\Omega(l)\bigg]-\frac{16k_f^3}{9}\sin^4\psi\bigg\}\end{aligned} \tag{54}$$

with a large-λ behavior of the (subtracted) double integral: $(\pi k_f^3/9)[6m_\pi^2+k_f^2+2p^2+3q^2/4]\ln(m_\pi/\lambda)$. Note that the subtraction term acts only in the ${}^1S_0$ state with total isospin 1, therefore one can replace (of course only for this $\lambda^2k_f^3$ term) the operator $\vec{\sigma}_1\cdot\vec{\sigma}_2(3+\vec{\tau}_1\cdot\vec{\tau}_2)$ by $-3(3+\vec{\tau}_1\cdot\vec{\tau}_2)$. Next, there is a tensor-type potential of the form

$$\begin{aligned}V_{\rm med}^{(\rm cc)}=&\frac{c_4g_A^2}{8\pi^5f_\pi^6}(3+\vec{\tau}_1\cdot\vec{\tau}_2)(\vec{\sigma}_1\cdot\vec{p}\,\vec{\sigma}_2\cdot\vec{p}+\vec{\sigma}_1\cdot\vec{p}\,'\vec{\sigma}_2\cdot\vec{p}\,')\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi\,\frac{l\bar{\Gamma}_1(l)}{4p^2-q^2}\\&\times\bigg\{\frac{l}{2p^2}\bigg[3\bar{m}^2+3l^2-p^2+\frac{5q^2}{4}-\frac{3q^2}{4p^2}(\bar{m}^2+l^2)\bigg]+\bigg[\frac{p^2}{2}+l^2-\bar{m}^2-\frac{5q^2}{8}\\&-\frac{6(\bar{m}^2+l^2)^2+q^2(\bar{m}^2+3l^2)}{4p^2}+\frac{3q^2}{8p^4}(\bar{m}^2+l^2)^2-\frac{q^2(4\bar{m}^2+4l^2+q^2)}{4p^2-q^2}\bigg]\Lambda(l)\\&+\frac{q^2}{4p^2-q^2}\big[4(\bar{m}^2+l^2)^2+4p^2(\bar{m}^2-l^2)+q^2(\bar{m}^2+3l^2+p^2)\big]\Omega(l)\bigg\}\end{aligned} \tag{55}$$

with a large-λ behavior of the double-integral: $-(\pi k_f^3/36)\ln(m_\pi/\lambda)$. Finally, one gets an ordinary tensor potential which has the form

$$\begin{aligned}V_{\rm med}^{(\rm cc)}=&\frac{c_4g_A^2}{8\pi^5f_\pi^6}(3+\vec{\tau}_1\cdot\vec{\tau}_2)\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q}\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi\,\frac{l\bar{\Gamma}_1(l)}{4p^2-q^2}\bigg\{\bigg[\frac{q^2(4\bar{m}^2+4l^2+q^2)}{8p^2-2q^2}+\frac{\bar{m}^2+l^2}{2p^2}(q^2-4p^2)\bigg]\Lambda(l)\\&+\frac{l}{2p^2}(2p^2-q^2)+\bigg[3(\bar{m}^2+l^2)^2+p^2(4\bar{m}^2+4l^2+p^2)+q^2(\bar{m}^2+2l^2+p^2)-\frac{8p^2}{4p^2-q^2}(\bar{m}^2+l^2+p^2)^2\bigg]\Omega(l)\bigg\}\end{aligned} \tag{56}$$

with a large-λ behavior of the double integral: $-(\pi k_f^3/24)\ln(m_\pi/\lambda)$. One likes to consider the balance of λ^2 divergences in the total sum $V_{\rm med}^{(0)}+V_{\rm med}^{(1)}+V_{\rm med}^{(\rm cc)}$. With the equivalent form of the piece from Eq. (54), the balance reads

$$\begin{aligned}&\frac{g_A^2k_f^3\lambda^2}{96\pi^4f_\pi^6}\bigg\{-\vec{\tau}_1\cdot\vec{\tau}_2(c_2+6c_3)+\bigg[3\left(\frac{c_2}{2}+3c_3\right)+3c_4\vec{\tau}_1\cdot\vec{\tau}_2\bigg]+\Big[\vec{\tau}_1\cdot\vec{\tau}_2(c_2+6c_3)-3c_4(3+\vec{\tau}_1\cdot\vec{\tau}_2)\Big]\bigg\}\\&=\frac{g_A^2k_f^3\lambda^2}{64\pi^4f_\pi^6}(c_2+6c_3-6c_4),\end{aligned} \tag{57}$$

and one observes that the remaining isoscalar piece can again be absorbed on the $3N$ short-distance parameter c_E.

V. RING INTERACTION PROPORTIONAL TO g_A^4

The $3N$-ring interaction proportional to $g_A^4 c_{1,2,3,4}$ can be inferred from the coordinate-space potential written in Eq. (4.6) of Ref. [4]. In momentum space this part of V_{3N} at N^{4}LO is given by a euclidean loop integral over three pion propagators (one of them squared) times a long series of terms with different spin, isospin, and momentum dependence, which reads

$$
\begin{aligned}
V_{3N} = {}& \frac{g_A^4}{f_\pi^6}\int_0^\infty dl_0 \int \frac{d^3l_2}{(2\pi)^4}\frac{1}{(\bar m^2+l_1^2)(\bar m^2+l_2^2)^2(\bar m^2+l_3^2)} \\
&\times\Big\{\bar m^2\Big[(\vec\sigma_1\times\vec l_3)\cdot(\vec\sigma_3\times\vec l_1)\big[2l_0^2(c_2+c_3)\vec\tau_1\cdot\vec\tau_3-6c_1m_\pi^2+\vec l_1\cdot\vec l_3\big(c_4(\vec\tau_1+\vec\tau_3)\cdot\vec\tau_2-3c_3\big)\big] \\
&+2(\vec\sigma_1\times\vec l_2)\cdot(\vec\sigma_2\times\vec l_1)\big[2l_0^2(c_2+c_3)\vec\tau_1\cdot\vec\tau_2-6c_1m_\pi^2+\vec l_1\cdot\vec l_2\big(c_4(\vec\tau_1+\vec\tau_2)\cdot\vec\tau_3-3c_3\big)\big]\Big] \\
&+\frac{c_4}{2}\vec l_1\cdot\vec l_2\big[2\vec l_1\cdot\vec l_3\,\vec\sigma_1\cdot(\vec l_3\times\vec l_2)-\vec l_2\cdot\vec l_3\,\vec\sigma_2\cdot(\vec l_3\times\vec l_1)\big]\vec\tau_1\cdot(\vec\tau_2\times\vec\tau_3)+\vec l_1\cdot\vec l_2\,\vec l_1\cdot\vec l_3\,\vec l_2\cdot\vec l_3 \\
&\times\big[2c_3\vec\tau_1\cdot(2\vec\tau_2+\vec\tau_3)+2\vec\sigma_2\cdot\vec\sigma_3\big(c_4\vec\tau_1\cdot(\vec\tau_2+\vec\tau_3)-3c_3\big)+\vec\sigma_1\cdot\vec\sigma_3\big(c_4(\vec\tau_1+\vec\tau_3)\cdot\vec\tau_2-3c_3\big)\big] \\
&+2\vec l_1\cdot\vec l_2\,\vec l_1\cdot\vec l_3\big[(c_2+c_3)l_0^2(2\vec\sigma_2\cdot\vec\sigma_3\,\vec\tau_2\cdot\vec\tau_3-3)-6c_1m_\pi^2\vec\sigma_2\cdot\vec\sigma_3+\vec\sigma_1\cdot\vec l_3\,\vec\sigma_2\cdot\vec l_2\big(3c_3-c_4(\vec\tau_1+\vec\tau_2)\cdot\vec\tau_3\big)\big] \\
&+\vec l_1\cdot\vec l_2\,\vec l_2\cdot\vec l_3\big[4c_1m_\pi^2\vec\tau_1\cdot\vec\tau_3-3l_0^2(c_2+c_3)+2\vec\sigma_1\cdot\vec\sigma_3\big(l_0^2(c_2+c_3)\vec\tau_1\cdot\vec\tau_3-3c_1m_\pi^2\big)+2\vec\sigma_1\cdot\vec l_1\,\vec\sigma_2\cdot\vec l_3 \\
&\times\big(3c_3-c_4(\vec\tau_1+\vec\tau_2)\cdot\vec\tau_3\big)\big]+2\vec l_1\cdot\vec l_3\,\vec l_2\cdot\vec l_3\big[4c_1m_\pi^2\vec\tau_1\cdot\vec\tau_2+\vec\sigma_1\cdot\vec l_1\,\vec\sigma_3\cdot\vec l_2\big(3c_3-c_4(\vec\tau_1+\vec\tau_3)\cdot\vec\tau_2\big)\big] \\
&+(\vec l_1\cdot\vec l_3)^2\vec\sigma_1\cdot\vec l_2\,\vec\sigma_3\cdot\vec l_2\big(c_4(\vec\tau_1+\vec\tau_3)\cdot\vec\tau_2-3c_3\big)+2(\vec l_1\cdot\vec l_2)^2\vec\sigma_1\cdot\vec l_3\,\vec\sigma_2\cdot\vec l_3\big(c_4(\vec\tau_1+\vec\tau_2)\cdot\vec\tau_3-3c_3\big) \\
&+4\big(\vec l_2\cdot\vec l_3\,\vec\sigma_1\cdot\vec l_1\,\vec\sigma_2\cdot\vec l_3+\vec l_1\cdot\vec l_3\,\vec\sigma_1\cdot\vec l_3\,\vec\sigma_2\cdot\vec l_2-\vec l_1\cdot\vec l_2\,\vec\sigma_1\cdot\vec l_3\,\vec\sigma_2\cdot\vec l_3\big)\big(3c_1m_\pi^2-l_0^2(c_2+c_3)\vec\tau_1\cdot\vec\tau_2\big) \\
&+2\big(2\vec l_2\cdot\vec l_3\,\vec\sigma_1\cdot\vec l_1-\vec l_1\cdot\vec l_3\,\vec\sigma_1\cdot\vec l_2\big)\vec\sigma_3\cdot\vec l_2\big(3c_1m_\pi^2-l_0^2(c_2+c_3)\vec\tau_1\cdot\vec\tau_3\big)\Big\}
\end{aligned}
\tag{58}
$$

with $\bar m = \sqrt{m_\pi^2+l_0^2}$ and one has to set $\vec l_1 = \vec l_2 - \vec q_3$ and $\vec l_3 = \vec l_2 + \vec q_1$. Without the prefactor g_A^4/f_π^6, the first line in Eq. (58) defines a euclidean three-point function $\tilde J(q_1, q_2, q_3)$ that is symmetric under $q_1 \leftrightarrow q_3$. By applying the Cutkosky cutting rule to the first and third pion-propagator, one can easily compute its imaginary part $\mathrm{Im}\tilde J(q_1, \mu, q_3)$ as a 2π-phase space integral over a squared pion propagator, and obtains the following spectral representation:

$$
\tilde J(q_1,q_2,q_3) = \frac{1}{16\pi^2}\int_{2m_\pi}^\infty d\mu\,\frac{\mu^2\sqrt{\mu^2-4m_\pi^2}}{\mu^2+q_2^2}\big[(\mu q_1 q_3)^2+m_\pi^2 G\big]^{-1}
\tag{59}
$$

with the abbreviation $G = [\mu^2 + (q_1 + q_3)^2][\mu^2 + (q_1 - q_3)^2]$. By a partial-fraction decomposition of the two denominators in Eq. (59) one is able to find an analytical solution of the spectral integral in terms of the even loop function $L(s) = L(-s)$, defined in Eq. (12). The final result for $\tilde J(q_1, q_2, q_3)$ reads

$$
\tilde J(q_1,q_2,q_3) = \frac{1}{16\pi^2}\left\{\frac{b_+^2[L(q_2)-L(b_+)]}{(q_2^2-b_+^2)C}+\frac{b_-^2[L(b_-)-L(q_2)]}{(q_2^2-b_-^2)C}\right\}
\tag{60}
$$

with the auxiliary variables $b_\pm = (q_1\sqrt{4m_\pi^2+q_3^2} \pm q_3\sqrt{4m_\pi^2+q_1^2}\,)/(2m_\pi)$ and the combination $C = q_1 q_3\sqrt{(4m_\pi^2+q_1^2)(4m_\pi^2+q_3^2)}$. Likewise, the (bare) euclidean loop integral over three pion propagators in the first line of Eq. (44) defines a totally symmetric three-point function $J(q_1, q_2, q_3)$. It possesses a more involved spectral representation

$$
J(q_1,q_2,q_3) = \frac{1}{16\pi^2}\int_{2m_\pi}^\infty d\mu\,\frac{\mu}{(\mu^2+q_2^2)\sqrt{G}}\ln\frac{\mu(\mu^2+q_1^2+q_3^2)+\sqrt{(\mu^2-4m_\pi^2)G}}{\mu(\mu^2+q_1^2+q_3^2)-\sqrt{(\mu^2-4m_\pi^2)G}},
\tag{61}
$$

which does not allow for a solution in terms of elementary functions.

A. In-medium NN potential from self-closing of nucleon lines

In this subsection the in-medium NN-potential $V_{\rm med}^{(0)}$ is computed as it arises from the self-closing of nucleon lines for the (extremely lengthy) $3N$-ring interaction written in Eq. (58). One gets a nonvanishing spin-isospin trace from closing N_1, N_2, and N_3, respectively. After performing the angular and radial integrals the summed contributions are sorted according to their two-body spin and isospin operators. The complete list of contributions to $V_{\rm med}^{(0)}$ consists of an isoscalar central potential

$$
V_{\rm med}^{(0)} = \frac{g_A^4(c_2+c_3)k_f^3}{96\pi^4 f_\pi^6}\left\{-\left[\frac{135m_\pi^2}{2}+\frac{53q^2}{4}\right]\ln\frac{m_\pi}{\lambda}-\frac{73m_\pi^2}{16}+\frac{371q^2}{96}+\left[\frac{12m_\pi^4}{4m_\pi^2+q^2}-35m_\pi^2-\frac{53q^2}{4}\right]L(q)\right\},
\tag{62}
$$

an isovector central potential

$$V_{\rm med}^{(0)} = \frac{g_A^4 k_f^3\,\vec\tau_1\cdot\vec\tau_2}{96\pi^4 f_\pi^6}\bigg\{15\bigg[(7c_3-4c_1)m_\pi^2+\frac{23c_3q^2}{18}\bigg]\ln\frac{m_\pi}{\lambda}+\bigg(\frac{389c_3}{24}-27c_1\bigg)m_\pi^2+\frac{8m_\pi^4(2c_1-c_3)}{4m_\pi^2+q^2}$$
$$-\frac{553c_3q^2}{144}+\bigg[4m_\pi^2\bigg(\frac{29c_3}{3}-11c_1\bigg)+\frac{115c_3q^2}{6}+\frac{8m_\pi^4(4c_1+3c_3)}{4m_\pi^2+q^2}+\frac{32m_\pi^6(2c_1-c_3)}{(4m_\pi^2+q^2)^2}\bigg]L(q)\bigg\}, \quad (63)$$

an isoscalar spin-spin potential

$$V_{\rm med}^{(0)} = \frac{g_A^4 k_f^3\,\vec\sigma_1\cdot\vec\sigma_2}{96\pi^4 f_\pi^6}\bigg\{\big[18(4c_1-5c_3)m_\pi^2-11c_3q^2\big]\ln\frac{m_\pi}{\lambda}+\bigg(18c_1-\frac{35c_3}{2}\bigg)m_\pi^2+\frac{c_3q^2}{3}$$
$$+\bigg[(48c_1-26c_3)m_\pi^2-11c_3q^2+\frac{24m_\pi^4(c_3-2c_1)}{4m_\pi^2+q^2}\bigg]L(q)\bigg\}, \quad (64)$$

an isovector spin-spin potential

$$V_{\rm med}^{(0)} = \frac{g_A^4(c_2+c_3)k_f^3}{48\pi^4 f_\pi^6}\vec\sigma_1\cdot\vec\sigma_2\,\vec\tau_1\cdot\vec\tau_2\bigg\{\big(6m_\pi^2+q^2\big)\ln\frac{m_\pi}{\lambda}+\frac{m_\pi^2}{2}-\frac{q^2}{6}+\big(2m_\pi^2+q^2\big)L(q)\bigg\}, \quad (65)$$

an isoscalar tensor potential

$$V_{\rm med}^{(0)} = \frac{g_A^4 k_f^3}{48\pi^4 f_\pi^6}\vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q\,\bigg\{4c_3\ln\frac{m_\pi}{\lambda}+\frac{61c_3}{48}+\frac{2m_\pi^2}{q^2}(3c_1+c_3)+\bigg[4c_3-\frac{2m_\pi^2}{q^2}(3c_1+c_3)+\frac{3m_\pi^2(c_3-2c_1)}{4m_\pi^2+q^2}\bigg]L(q)\bigg\}, \quad (66)$$

and an isovector tensor potential

$$V_{\rm med}^{(0)} = \frac{g_A^4(c_2+c_3)k_f^3}{72\pi^4 f_\pi^6}\vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q\;\vec\tau_1\cdot\vec\tau_2\bigg\{-\ln\frac{m_\pi}{\lambda}-\frac{7}{48}+\frac{m_\pi^2}{q^2}-\bigg(1+\frac{m_\pi^2}{q^2}\bigg)L(q)\bigg\}. \quad (67)$$

Note that c_4 has dropped out and the dependence on the other three low-energy constants $c_{1,2,3}$ is well structured. The isoscalar central and isovector spin-dependent potentials are solely proportional to the sum c_2+c_3, whereas the other potentials depend separately on c_1 and c_3. The total λ^2 divergence behind the central and spin-spin potentials written in Eqs. (62)–(65) is for S waves, where the replacements $\vec\sigma_1\cdot\vec\sigma_2\,\vec\tau_1\cdot\vec\tau_2\to-3$ and $\vec\sigma_1\cdot\vec\sigma_2\to-2-\vec\tau_1\cdot\vec\tau_2$ apply, equivalent to $g_A^4k_f^3\lambda^2[83c_3/2-77c_2/2+75c_3\,\vec\tau_1\cdot\vec\tau_2]/(256\pi^4f_\pi^6)$. In combination with the λ^2 divergences from concatenations $V_{\rm med}^{(2)}+V_{\rm med}^{(cc)}$ for all interaction terms in Eq. (58) (several examples are given in the next subsection) it reduces to an isoscalar component only

$$\frac{g_A^4k_f^3\lambda^2}{(4\pi)^4f_\pi^6}\bigg\{\frac{83c_3}{2}-\frac{77c_2}{2}+75c_3\,\vec\tau_1\cdot\vec\tau_2+(1+2)\bigg[\frac{13c_2}{12}-\frac{277c_3}{12}+40c_4-25c_3\,\vec\tau_1\cdot\vec\tau_2\bigg]\bigg\}$$
$$=\frac{3g_A^4k_f^3\lambda^2}{(4\pi)^4f_\pi^6}\bigg[40c_4-\frac{1}{4}(47c_2+37c_3)\bigg], \quad (68)$$

that can be absorbed on the $3N$ short-distance parameter c_E. This property serves as a good check on the present calculation.

B. In-medium *NN* potential from concatenations for three selected terms

The $3N$-ring interaction V_{3N} written in Eq. (58) consists of a very large number of terms. In this paper the contributions to $V_{\rm med}$ from concatenations of two nucleon lines are considered only for three selected terms. The analogous formulas for all the other terms can be obtained from the author upon request.

1. Term proportional to $c_4\vec\tau_1\cdot(\vec\tau_2\times\vec\tau_3)$ in the fourth line of Eq. (58)

It gives for the concatenations N_3 on N_1 and N_1 on N_3 an isovector spin-orbit potential of the form

$$V_{\rm med}^{(2)} = \frac{c_4 g_A^4\,\vec\tau_1\cdot\vec\tau_2}{16\pi^5 f_\pi^6}i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,)\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi\,\frac{l}{4p^2-q^2}\bigg\{\bar\gamma_2(l)\bigg[\frac{l}{p^2}(4p^2-q^2)$$
$$+\bigg(4p^2+\frac{q^2}{p^2}(\bar m^2+l^2)-4l^2-8\bar m^2-3q^2\bigg)\Lambda(l)+(2\bar m^2+q^2)(2\bar m^2+2l^2-2p^2+q^2)\Omega(l)\bigg]$$

$$+\bar\gamma_3(l)\bigg[\frac{l}{4p^2}(4p^2-q^2)(\bar m^2+l^2-p^2)+\bigg(\frac{q^2}{4p^2}(\bar m^2+l^2)^2+2(p^2-l^2)(\bar m^2+l^2+p^2)$$
$$+\frac{q^2}{4}(2\bar m^2-2l^2+p^2)\bigg)\Lambda(l)+(l^2-\bar m^2-p^2)\bigg(B+\frac{q^2}{2}(\bar m^2+3l^2+p^2)\bigg)\Omega(l)\bigg]\bigg\} \tag{69}$$

with a large-λ behavior of the double integral: $(5\pi k_f^3/72)\ln(m_\pi/\lambda)$. The new functions $\bar\gamma_{2,3}(l)$ appearing in Eq. (69) are $\bar\gamma_{2,3}(l)=-\partial\bar\Gamma_{2,3}(l)/\partial\bar m^2$ with $\bar\Gamma_{2,3}(l)$ given in Eq. (35). The other four concatenations produce also an isovector spin-orbit potential of the form

$$V_{\rm med}^{(\rm cc)}=\frac{c_4g_A^4\vec\tau_1\cdot\vec\tau_2}{16\pi^5f_\pi^6}i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,)\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi\,\bigg\{l\bar\Gamma_2(l)\bigg\{\bigg[\frac{1}{p^2}+\frac{4}{4p^2-q^2}\bigg]\Lambda(l)$$
$$-\frac{l}{B}\bigg[\frac{\bar m^2+l^2}{p^2}-1+\frac{2\bar m^2+q^2}{B+q^2l^2}(\bar m^2+p^2-l^2)\bigg]+\bigg[2-\frac{4(\bar m^2+l^2+p^2)}{4p^2-q^2}+\frac{2\bar m^2+q^2}{B+q^2l^2}(l^2-\bar m^2-p^2)\bigg]\Omega(l)\bigg\}$$
$$+\frac{l}{2}\bar\Gamma_3(l)\bigg\{\frac{l(\bar m^2+p^2-l^2)}{B+q^2l^2}-\frac{l}{p^2}+\bigg(\frac{\bar m^2+l^2-p^2}{p^2}-\frac{4\bar m^2+2q^2}{4p^2-q^2}\bigg)\Lambda(l)$$
$$+\bigg[\frac{\bar m^2+l^2+p^2}{4p^2-q^2}(4\bar m^2+8p^2)-\bar m^2-3l^2-p^2+\frac{q^2l^2(l^2-\bar m^2-p^2)}{B+q^2l^2}\bigg]\Omega(l)\bigg\}\bigg\} \tag{70}$$

with a large-λ behavior of the double integral: $(5\pi k_f^3/36)\ln(m_\pi/\lambda)$.

2. *The term proportional to $\vec l_1\cdot\vec l_2\,\vec l_1\cdot\vec l_3\,\vec l_2\cdot\vec l_3$ multiplied by the fifth line in Eq. (58)*

It gives for the concatenations N_3 on N_1 and N_3 on N_1 a combination of central and spin-spin potentials of the form

$$V_{\rm med}^{(2)}=\frac{g_A^4}{8\pi^5f_\pi^6}\bigg[\frac{3c_3}{2}+3(c_3-c_4)\vec\sigma_1\cdot\vec\sigma_2-(2c_3+3c_4)\vec\tau_1\cdot\vec\tau_2-c_4\vec\sigma_1\cdot\vec\sigma_2\,\vec\tau_1\cdot\vec\tau_2\bigg]\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi$$
$$\times\bigg\{l\bar\gamma_2(l)\bigg[\frac{l}{p^2}(4p^2-q^2)+\bigg(\frac{q^2}{p^2}(\bar m^2+l^2)-8\bar m^2-3q^2\bigg)\Lambda(l)+(2\bar m^2+q^2)^2\,\Omega(l)\bigg]$$
$$+l\bar\gamma_3(l)\bigg[l(3l^2-p^2-2\bar m^2)+\frac{lq^2}{2p^2}(\bar m^2-l^2)+\bigg(p^2-l^2+3\bar m^2+\frac{q^2}{2}-\frac{q^2}{2p^2}(\bar m^2+l^2)\bigg)$$
$$\times(\bar m^2+p^2-l^2)\Lambda(l)-\bigg(\bar m^2+\frac{q^2}{2}\bigg)(\bar m^2+p^2-l^2)^2\,\Omega(l)\bigg]-\frac{8k_f^3}{3}\sin^8\psi\bigg\} \tag{71}$$

with a large-λ behavior of the double-integral: $(\pi k_f^3/24)[35m_\pi^2+18k_f^2/5+6p^2+43q^2/12]\ln(m_\pi/\lambda)$. The other four concatenations produce the same combination of central and spin-spin potentials, which takes the form

$$V_{\rm med}^{(\rm cc)}=\frac{g_A^4}{8\pi^5f_\pi^6}\bigg[\frac{3c_3}{2}+3(c_3-c_4)\vec\sigma_1\cdot\vec\sigma_2-(2c_3+3c_4)\vec\tau_1\cdot\vec\tau_2-c_4\vec\sigma_1\cdot\vec\sigma_2\,\vec\tau_1\cdot\vec\tau_2\bigg]\int_0^\lambda dr\,r\int_0^{\pi/2}d\psi$$
$$\times\bigg\{l\bar\Gamma_2(l)\bigg[\frac{l}{B}\bigg(\frac{q^2}{p^2}(\bar m^2+l^2)-8\bar m^2-3q^2+\frac{\bar m^2+l^2+p^2}{B+q^2l^2}(2\bar m^2+q^2)^2\bigg)$$
$$+\bigg(8-\frac{q^2}{p^2}\bigg)\Lambda(l)+\bigg(\frac{2\bar m^2+q^2}{B+q^2l^2}(\bar m^2+l^2+p^2)-4\bigg)(2\bar m^2+q^2)\Omega(l)\bigg]$$
$$+l\bar\Gamma_3(l)\bigg[\frac{l}{B}\bigg(3(p^2-l^2)^2+\bar m^2(4\bar m^2-l^2+7p^2)+\frac{l^2(2\bar m^2+q^2)}{B+q^2l^2}(\bar m^2+l^2+p^2)\bigg(2\bar m^2+\frac{q^2}{2}\bigg)$$
$$-\frac{\bar m^2q^2}{p^2}(\bar m^2+l^2)-\frac{q^2}{2}(3\bar m^2+l^2+p^2)\bigg)+\bigg(4l^2-4p^2-6\bar m^2+\frac{\bar m^2q^2}{p^2}\bigg)\Lambda(l)$$
$$+\bigg(3\bar m^2-l^2+p^2+q^2+\frac{2\bar m^2+q^2}{2(B+q^2l^2)}(l^4-(\bar m^2+p^2)^2)\bigg)(\bar m^2+p^2-l^2)\Omega(l)\bigg]-\frac{16k_f^3}{3}\sin^8\psi\bigg\} \tag{72}$$

with a large-λ behavior of the double integral: $(\pi k_f^3/12)[35m_\pi^2+79k_f^2/20+79p^2/12+3q^2]\ln(m_\pi/\lambda)$.

3. The spin- and isospin-dependent term in the last line of Eq. (58)

It gives for the concatenations N_3 on N_1 and N_3 on N_1 an isoscalar central potential

$$V_{\rm med}^{(2)} = \frac{3g_A^4}{4\pi^5 f_\pi^6}\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\bigg\{\big[(c_2+c_3)l_0^2 - c_1 m_\pi^2\big]\Big[l\bar\gamma_2(l)\Big((2\bar m^2+q^2)\Omega(l)-2\Lambda(l)\Big)$$

$$+\,l\bar\gamma_3(l)\Big(l-2(\bar m^2+p^2)\Lambda(l)+(l^2(q^2-2\bar m^2)+B)\Omega(l)\Big)\Big] - \frac{4k_f^3}{3}(c_2+c_3)\sin^6\psi\cos^2\psi\bigg\} \tag{73}$$

with a large-λ behavior of the double integral

$$\frac{\pi k_f^3}{48}\bigg[10c_1m_\pi^2 + (c_2+c_3)\bigg(5m_\pi^2+\frac{9k_f^2}{5}+3p^2-\frac{7q^2}{12}\bigg)\bigg]\ln\frac{m_\pi}{\lambda}, \tag{74}$$

and an isoscalar spin-orbit potential

$$V_{\rm med}^{(2)} = \frac{3g_A^4}{2\pi^5 f_\pi^6} i(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec p\,)\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\big[(c_2+c_3)l_0^2 - c_1m_\pi^2\big]\frac{l}{4p^2-q^2}\bigg\{\bar\gamma_2(l)\big[-2\Lambda(l)$$

$$+(2\bar m^2+q^2+2l^2-2p^2)\Omega(l)-2\Lambda(l)\big] + \frac{\bar\gamma_2(l)}{2}(\bar m^2+p^2-l^2)\big[\Lambda(l)-(\bar m^2+p^2+l^2)\Omega(l)\big]\bigg\} \tag{75}$$

with a large-λ behavior of the double integral: $-(\pi k_f^3/576)(c_2+c_3)\ln(m_\pi/\lambda)$. The other four concatenations produce a spin-spin potential

$$V_{\rm med}^{(\rm cc)} = \frac{g_A^4\vec\sigma_1\cdot\vec\sigma_2}{2\pi^5 f_\pi^6}\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\bigg\{\big[(c_2+c_3)l_0^2\vec\tau_1\cdot\vec\tau_2 - 3c_1m_\pi^2\big]\frac{l\bar\Gamma_1(l)}{4p^2-q^2}\bigg[\bigg(3\bar m^2+2l^2+2p^2+\frac{q^2}{2}\bigg)\Lambda(l)$$

$$-\bigg(2B+(\bar m^2+p^2+l^2)\bigg(\bar m^2+\frac{q^2}{2}\bigg)+2q^2l^2\bigg)\Omega(l)\bigg] - \frac{4k_f^3}{9}(c_2+c_3)\vec\tau_1\cdot\vec\tau_2\sin^6\psi\cos^2\psi\bigg\} \tag{76}$$

with a large-λ behavior of the double integral

$$\frac{\pi k_f^3}{24}\bigg[5c_1m_\pi^2+(c_2+c_3)\vec\tau_1\cdot\vec\tau_2\bigg(\frac{5m_\pi^2}{6}+\frac{11}{120}(k_f^2+p^2)+\frac{q^2}{5}\bigg)\bigg]\ln\frac{m_\pi}{\lambda}, \tag{77}$$

an ordinary tensor potential

$$V_{\rm med}^{(\rm cc)} = \frac{g_A^4}{2\pi^5 f_\pi^6}\vec\sigma_1\cdot\vec q\,\vec\sigma_2\cdot\vec q\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\big[(c_2+c_3)l_0^2\vec\tau_1\cdot\vec\tau_2 - 3c_1m_\pi^2\big]\frac{l\bar\Gamma_1(l)}{4p^2-q^2}$$

$$\times\bigg\{\frac{l}{B}\bigg[\frac{2p^2-\bar m^2-q^2}{B+q^2l^2}\bigg((\bar m^2+p^2)^2+l^2\bigg(\bar m^2-p^2+\frac{q^2}{2}\bigg)\bigg)+\frac{\bar m^2+q^2}{2}-l^2$$

$$-\frac{1}{q^2}(2l^4+l^2(5\bar m^2-6p^2)+3\bar m^4+5\bar m^2p^2+4p^4)\bigg]+\bigg[\frac{6\bar m^2+4l^2+2q^2}{4p^2-q^2}-\frac12\bigg]\Lambda(l)$$

$$+\bigg[\frac{2p^2-\bar m^2-q^2}{B+q^2l^2}\bigg((\bar m^2+p^2)^2+l^2\bigg(\bar m^2-p^2+\frac{q^2}{2}\bigg)\bigg)-\frac{2(\bar m^2+p^2+l^2)}{4p^2-q^2}(3\bar m^2+2l^2+q^2)$$

$$+\frac{1}{q^2}(2l^4+l^2(5\bar m^2-6p^2)+3\bar m^4+5\bar m^2p^2+4p^4)+2\bar m^2+\frac12(11l^2-7p^2+3q^2)\bigg]\Omega(l)\bigg\} \tag{78}$$

with a large-λ behavior of the double integral: $(\pi k_f^3/160)(c_2+c_3)\vec\tau_1\cdot\vec\tau_2\ln(m_\pi/\lambda)$, and a tensor-type potential

$$V_{\rm med}^{(\rm cc)} = \frac{g_A^4}{2\pi^5 f_\pi^6}(\vec\sigma_1\cdot\vec p\,\vec\sigma_2\cdot\vec p+\vec\sigma_1\cdot\vec p\,'\vec\sigma_2\cdot\vec p\,')\int_0^\lambda dr r\int_0^{\pi/2} d\psi\,\big[(c_2+c_3)l_0^2\vec\tau_1\cdot\vec\tau_2 - 3c_1m_\pi^2\big]\frac{l\bar\Gamma_1(l)}{4p^2-q^2}$$

$$\times\bigg\{\frac{l}{B}\bigg[\frac{2\bar m^2+q^2}{B+q^2l^2}\bigg((\bar m^2+p^2)^2+l^2\bigg(\bar m^2-p^2+\frac{q^2}{2}\bigg)\bigg)+\frac{3\bar m^2-q^2}{2}-l^2+\frac{1}{2p^2}(2l^4+5l^2\bar m^2+3\bar m^4)\bigg]$$

$$-\bigg[\frac{4}{4p^2-q^2}(3\bar m^2+2l^2+q^2)+1+\frac{3\bar m^2+2l^2}{2p^2}\bigg]\Lambda(l)+\bigg[\frac{4(\bar m^2+p^2+l^2)}{4p^2-q^2}(3\bar m^2+2l^2+q^2)$$

$$+\frac{2\bar m^2+q^2}{B+q^2l^2}\bigg((\bar m^2+p^2)^2+l^2\bigg(\bar m^2-p^2+\frac{q^2}{2}\bigg)\bigg)+p^2-5l^2-3\bar m^2-\frac{3q^2}{2}\bigg]\Omega(l)\bigg\} \tag{79}$$

with a large-λ behavior of the double integral: $(11\pi k_f^3/2880)(c_2+c_3)\vec\tau_1\cdot\vec\tau_2\ln(m_\pi/\lambda)$.

VI. SUMMARY AND OUTLOOK

In this work the density-dependent in-medium NN-interaction $V_{\rm med}$ has been derived from the subsubleading chiral $3N$ forces. This is necessary since for the intermediate-range topologies ($2\pi 1\pi$ exchange and ring diagrams), the N^{4}LO corrections of Ref. [4] dominate in most cases over the nominally leading N^{3}LO terms. The loop integrals representing the $3N$-ring interaction proportional to $c_{1,2,3,4}$ have been regularized by a (euclidean) cutoff λ and each contribution to $V_{\rm med}$ has been presented such that the absorption of [λ^2 and $\ln(m_\pi/\lambda)$] divergences on the $3N$ short-distance parameters becomes obvious. In the next step, partial-wave matrix elements of $V_{\rm med}$ will be calculated numerically [21] in order to study quantitatively the effects of the subleading [17,18] as well as subsubleading chiral $3N$ forces. At the same time the construction of $3N$ forces in chiral effective field theory with explicit $\Delta(1232)$ isobars by the Bochum group should be accompanied by a calculation of the corresponding density-dependent NN-potential $V_{\rm med}$. On the other hand, neutron matter calculations with (sub)subleading chiral $3N$ forces require the ρ_n-dependent NN interaction in pure neutron matter.

ACKNOWLEDGMENTS

I thank H. Krebs for detailed information on the chiral three-nucleon forces at N^{4}LO and for providing me some unpublished results. This work has been supported in part by DFG and NSFC (CRC110).

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