

## Effects of quantum statistics near the critical point of nuclear matter

S. N. Fedotkin and A. G. Magner

*Institute for Nuclear Research NASU, 03680 Kiev, Ukraine*

M. I. Gorenstein

*Bogolyubov Institute for Theoretical Physics, 03143 Kiev, Ukraine*



(Received 11 March 2019; published 26 November 2019)

Effects of quantum statistics for nuclear matter equation of state are analyzed in terms of the recently proposed quantum van der Waals model. The system pressure is expanded over a small parameter  $\delta \propto n(mT)^{-3/2}[g(1 - bn)]^{-1}$ , where  $n$  and  $T$  are, respectively, the particle number density and temperature,  $m$  and  $g$  the particle mass and degeneracy factor. The parameter  $b$  corresponds to the van der Waals excluded volume. The corrections due to quantum statistics for the critical point values of  $T_c$ ,  $n_c$ , and the critical pressure  $P_c$  are found within the linear and quadratic orders over  $\delta$ . These approximate analytical results appear to be in a good agreement with exact numerical calculations in the quantum van der Waals model for interacting Fermi particles: the symmetric nuclear matter ( $g = 4$ ) and the pure neutron matter ( $g = 2$ ). They can be also applied to the system of interacting Bose particles like the matter composed of  $\alpha$  nuclei.

DOI: [10.1103/PhysRevC.100.054334](https://doi.org/10.1103/PhysRevC.100.054334)

### I. INTRODUCTION

A study of nuclear matter, an interacting system of protons and neutrons, has a long history; see, e.g., Refs. [1–11]. Realistic versions of the nuclear matter equation of state includes both the attractive and repulsive forces between protons and neutrons. A theoretical description of the thermodynamical behavior of this matter leads to the liquid-gas first-order phase transition, which ends at the critical point. Such a behavior is rather similar to that in the atomic gases and liquids. Experimentally, a presence of the liquid-gas phase transition in nuclear matter was reported and then analyzed in numerous papers (see, e.g., Refs. [12–17]).

In the present paper the recently proposed van der Waals equation of state with the quantum statistics [18] is used to describe the properties of nuclear matter. The aim of our consideration is to investigate the role and size of the effects of quantum statistics. Particularly, we investigate a dependence of the critical point parameters on the particle mass  $m$ , degeneracy factor  $g$ , and the van der Waals parameters  $a$  and  $b$ , which describe particle interactions for the systems of interacting nucleons and  $\alpha$  particles. Our analytical results are new and were not considered in the textbooks and previous publications. Our consideration will be restricted to small temperatures,  $T \lesssim 30$  MeV, and not too large particle densities. Within these restrictions, the number of nucleons becomes a conserved number, and the chemical potential of this system regulates the number density of nucleons. An extension to the fully relativistic hadron resonances gas formulation with van der Waals interactions between baryons and between antibaryons was considered in Ref. [19]. An application of this extended model to net baryon number fluctuations in relativistic nucleus-nucleus collisions was developed in Ref. [20]. We do not include the Coulomb forces and make no differentiation between protons and neutrons

(both these particles are named as nucleons). In addition, under these restrictions the nonrelativistic treatment becomes very accurate and is adopted in our studies.

The paper is organized as the following. In Sec. II we recall some results of the ideal Bose and Fermi gases. In Sec. III the quantum statistics effects near the critical point are studied for the symmetric nuclear matter. These results are then extended to the pure neutron matter and pure  $\alpha$  matter. Section IV summarizes the paper.

### II. IDEAL QUANTUM GASES

The pressure  $P(T, \mu)$  plays the role of the thermodynamical potential in the grand canonical ensemble (GCE) where temperature  $T$  and chemical potential  $\mu$  are independent variables. The particle number density  $n(T, \mu)$ , entropy density  $s(T, \mu)$ , and energy density  $\varepsilon(T, \mu)$  are given as

$$n = \left( \frac{\partial P}{\partial \mu} \right)_T, \quad s = \left( \frac{\partial P}{\partial T} \right)_\mu, \quad \varepsilon = Ts + \mu n - P. \quad (1)$$

In the thermodynamic limit  $V \rightarrow \infty$  considered in the present paper all intensive thermodynamical functions— $P$ ,  $n$ ,  $s$ , and  $\varepsilon$ —depend on  $T$  and  $\mu$ , rather than on the system volume  $V$ , see Ref. [21] for detailed discussions of finite-size effects in the ideal Bose gas. We start with the GCE expressions for the pressure  $P_{\text{id}}(T, \mu)$  and particle number density  $n_{\text{id}}(T, \mu)$  for the ideal nonrelativistic quantum gas [22,23],

$$P_{\text{id}} = \frac{g}{3} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \frac{p^2}{m} \left[ \exp \left( \frac{p^2}{2mT} - \frac{\mu}{T} \right) - \theta \right]^{-1}, \quad (2)$$

$$n_{\text{id}} = g \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \left[ \exp \left( \frac{p^2}{2mT} - \frac{\mu}{T} \right) - \theta \right]^{-1}, \quad (3)$$

where  $m$  and  $g$  are, respectively, the particle mass and degeneracy factor. The value of  $\theta = -1$  corresponds to the Fermi gas,  $\theta = 1$  to the Bose gas, and  $\theta = 0$  is the Boltzmann (classical) approximation when effects of the quantum statistics are neglected.<sup>1</sup>

Equations (2) and (3) can be expressed in terms of the power series over fugacity  $z \equiv \exp(\mu/T)$  for  $z \leq 1$ :

$$P(T, z) \equiv \frac{gT}{\theta\Lambda^3} \text{Li}_{5/2}(\theta z) = \frac{gT}{\theta\Lambda^3} \sum_{k=1}^{\infty} \frac{(\theta z)^k}{k^{5/2}}, \quad (4)$$

$$n(T, z) \equiv \frac{g}{\theta\Lambda^3} \text{Li}_{3/2}(\theta z) = \frac{g}{\theta\Lambda^3} \sum_{k=1}^{\infty} \frac{(\theta z)^k}{k^{3/2}}, \quad (5)$$

where

$$\Lambda \equiv \hbar \sqrt{\frac{2\pi}{mT}} \quad (6)$$

is the de Broglie heat wavelength [23], and  $\text{Li}_\nu$  is the polylogarithmic function [24,25]. The values of  $\mu > 0$ , i.e.,  $z > 1$ , are forbidden in the ideal Bose gas. The point  $\mu = 0$  corresponds to an onset of the Bose-Einstein condensation in the system of bosons. For fermions, any values of  $\mu$  are possible, i.e., integrals (2) and (3) exist for  $\theta = -1$  at all real values of  $\mu$ . However, the power series (4) and (5) are convergent at  $z \leq 1$  only. For the Fermi statistics at  $z > 1$ , the integral representation of the corresponding polylogarithmic function can be used. Particularly, at  $z \rightarrow \infty$  one can use the asymptotic Sommerfeld expansion of the  $\text{Li}_\nu(-z)$  functions over  $1/\ln^2|z|$  [26].

Figure 1 shows lines of the constant values of fugacity  $z$  in the  $n$ - $T$  plane for the ideal Fermi gases [Fig. 1(a) and 1(b)], and Bose gas [Fig. 1(c)], see Eq. (5). Figure 1(a) corresponds to the isospin symmetric ideal nucleon gas (i.e., an equal number of protons and neutrons). We take  $m \cong 938$  MeV neglecting a small difference between proton and neutron masses. The degeneracy factor is then  $g = 4$ , which takes into account two spin and two isospin states of nucleon. The ideal neutron gas with  $g = 2$  is presented in Fig. 1(b), and the ideal Bose gas of  $\alpha$  nuclei ( $g = 1$  and  $m \cong 3727$  MeV) is shown in Fig. 1(c).

At  $z \ll 1$ , only one term  $k = 1$  is enough in Eqs. (4) and (5), which leads to the classical ideal gas relation

$$P = nT. \quad (7)$$

Note that the result (7) follows automatically from Eqs. (2) and (3) at  $\theta = 0$ . As seen from Fig. 1, the classical Boltzmann approximation at  $z \ll 1$  is valid for large  $T$  and/or small  $n$

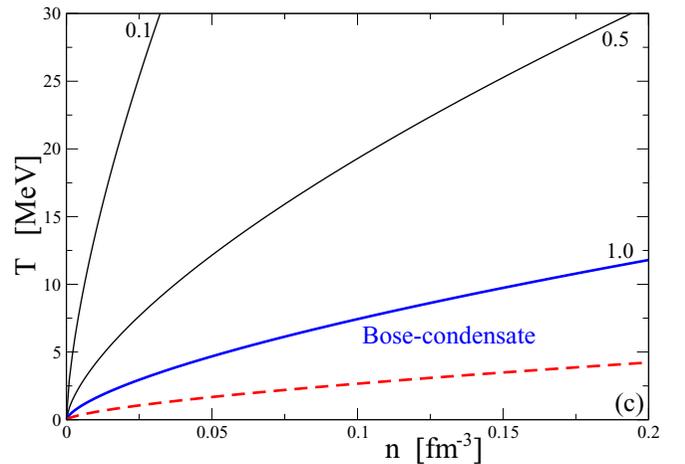
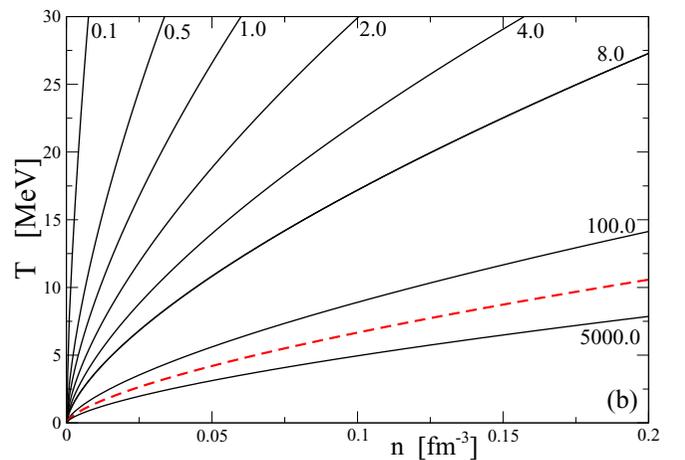
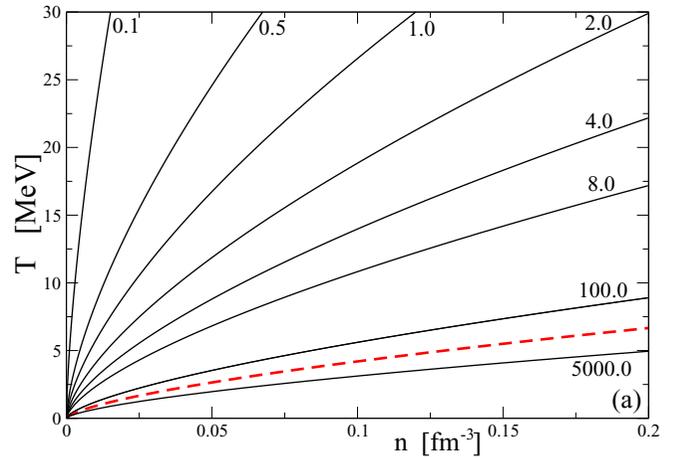


FIG. 1. Lines of the fixed fugacities  $z = z(n, T)$  are shown for the ideal Fermi gas of nucleons (a) and neutrons (b), and the ideal Bose gas of  $\alpha$  particles (c). Dashed red lines correspond to  $s = 0$  for the entropy density (8) of the classical ideal gas. A blue line in (c) shows the largest value of  $z = 1$  in the ideal Bose gas. This is the line of the Bose-Einstein condensation, and the Bose condensate of  $\alpha$  particles exists below this line.

<sup>1</sup>The units with Boltzmann constant  $\kappa_B = 1$  are used. We keep the Planck constant in the formulas to illustrate the effects of quantum statistics, but put  $\hbar = h/2\pi = 1$  in all numerical calculations. For simplicity, we omitted here and below the subscript *id* for the ideal gas everywhere where it will not lead to a misunderstanding.

region of the  $n$ - $T$  plane. In fact, at very small  $n$ , one observes  $z < 1$  at small  $T$  too. However, the classical ideal gas equation (7) becomes wrong at small  $T$  and/or large  $n$ . The entropy

density of the classical ideal gas reads

$$s = \left( \frac{\partial P}{\partial T} \right)_\mu = n \left( \frac{5}{2} - \frac{\mu}{T} \right), \quad (8)$$

and it becomes negative at  $\mu/T > 5/2$ . The particle number density in the classical ideal gas at  $\mu/T = 5/2$  equals to

$$n = \frac{g}{\hbar^3} \left( \frac{mT}{2\pi} \right)^{3/2} \exp \left( \frac{5}{2} \right). \quad (9)$$

The relation (9) is shown by a dashed red line in Fig. 1. Under this line, the entropy density of the classical ideal gas becomes negative. This happens at small  $T$  and/or large  $n$  and means a contradiction with the third law of thermodynamics. Quantum statistics solves this problem: Eqs. (2) and (3) guarantee  $s \geq 0$  at all  $T$  and  $n$ , and  $s = 0$  at  $T = 0$ .

Inverting the  $z^k$  power series in Eq. (5), one finds the power expansion of  $z$  over the parameter  $\epsilon$  (see, e.g., Ref. [27]),

$$\epsilon \equiv -\frac{\theta n \Lambda^3}{4\sqrt{2}g} \equiv -\theta \frac{\hbar^3 \pi^{3/2} n}{2g(mT)^{3/2}}. \quad (10)$$

This expansion is inserted then to Eq. (4). At small  $|\epsilon| < 1$  the expansion of the pressure over the powers of  $\epsilon$  is rapidly convergent, and a few first terms give already a good approximation of the quantum statistics effects. Taking the three terms,  $k = 1, 2$ , and  $3$ , in Eqs. (4) and (5), one obtains a classical gas result (7) plus the corrections due to the effects of quantum statistics:

$$P(T, n) = nT[1 + \epsilon - c_2\epsilon^2 + O(\epsilon^3)], \quad (11)$$

where  $c_2 = 4[16/(9\sqrt{3}) - 1] \cong 0.106$ . For brevity, we call the linear and quadratic  $\epsilon$  terms in Eq. (11) as the first- and second-order quantum corrections.

Equation (11) demonstrates explicitly a deviation of the quantum ideal gas pressure from the classical ideal gas value (7): the Fermi statistics leads to an increasing of the classical pressure, while the Bose statistics to its decreasing. This is often interpreted [23] as the effective Fermi repulsion and Bose attraction between quantum particles.

### III. QUANTUM VAN DER WAALS MODEL

Recently, the van der Waals (vdW) equation of state was extended by taking into account the effects of quantum statistics in Ref. [18]. The pressure function of the quantum vdW (QvdW) model can be presented as [18]

$$P(T, n) = P_{\text{id}}[T, n_{\text{id}}(T, \mu^*)] - an^2, \quad (12)$$

$$n_{\text{id}}(T, \mu^*) = \frac{n}{1 - bn}, \quad (13)$$

where  $P_{\text{id}}$  and  $n_{\text{id}}$  are respectively given by Eqs. (2) and (3), and  $\mu^*$  is the solution of the transcendental equation

$$\mu^* = \mu - bP(T, \mu) - abn^2 + 2an,$$

see more details in Ref. [18]. Notice that for Bose particles, the restriction  $\mu^* \leq 0$  (or  $\mu^* \leq m$  for the relativistic chemical potential) should be satisfied. These restrictions correspond to those  $\mu \leq 0$  (or  $\mu \leq m$  in relativistic case) in the ideal Bose gas. The constants  $a > 0$  and  $b > 0$  are responsible

for, respectively, attractive and repulsive interactions between particles. The QvdW model given by Eqs. (12) and (13) satisfies the following conditions:

(i) In the Boltzmann approximation, i.e., at  $\theta = 0$  in Eqs. (2) and (3), the QvdW model (12) and (13) is reduced to the classical vdW model [23]

$$P = \frac{nT}{1 - nb} - an^2. \quad (14)$$

Note that the classical vdW model (14) is further reduced to the ideal classical gas (7) at  $a = 0$  and  $b = 0$ .

(ii) At  $a = 0$  and  $b = 0$  the QvdW model (12) and (13) is transformed to the quantum ideal gas Eqs. (2) and (3).

(iii) The QvdW model (12) and (13), in contrast to its classical version (14), satisfies the third law of thermodynamics by having a non-negative entropy with  $s = 0$  at  $T = 0$ .

We fix the model parameters  $a$  and  $b$  using the ground-state properties of the symmetric nuclear matter (see, e.g., Ref. [28]): at  $T = 0$  and  $n = n_0 = 0.16 \text{ fm}^{-3}$ , one requires  $P = 0$  and the binding energy per nucleon  $\varepsilon(T = 0, n = n_0)/n_0 = -16 \text{ MeV}$ . With the steplike Fermi momentum distribution for nucleons the analytical expressions for the thermodynamical quantities at  $T = 0$  are obtained. One then finds from the above requirements:<sup>2</sup>

$$a = 329.8 \text{ MeV fm}^3, \quad b = 3.35 \text{ fm}^3. \quad (15)$$

These values are very close to those found in Ref. [18]. Small differences appear because of the nonrelativistic formulation used in the present studies.

In what follows, the first and second quantum correction of the QvdW model will be considered. This is analogous to that in Eq. (11) for the ideal quantum gas. Expanding  $P_{\text{id}}[T, n_{\text{id}}(T, \mu^*)]$  in Eq. (12) over the small parameter  $\delta$  (with  $\theta = -1$  for fermions and  $\theta = 1$  for bosons),

$$\delta = \frac{\epsilon}{1 - bn} = -\theta \frac{\hbar^3 \pi^{3/2} n}{2g(1 - bn)(mT)^{3/2}}, \quad (16)$$

one obtains

$$P(T, n) = \frac{nT}{1 - bn} [1 + \delta - c_2\delta^2 + O(\delta^3)] - an^2, \quad (17)$$

where  $c_2$  is the same small number coefficient as in Eq. (11). We proved that at small  $|\delta|$  the expansion of the pressure over powers of  $\delta$  becomes rapidly convergent to the exact results, and a few first terms give already a good approximation. Our Eq. (17), in contrast to Eq. (11) discussed in Refs. [23,27], takes into account the particle interaction effects. A new point of our consideration is the analytical estimates of the effects of quantum statistics in a system of interacting fermions and bosons. Similarly to the ideal gases, the quantum correction in Eq. (17) increases with the particle number density  $n$  and decreases with the system temperature  $T$ , particle mass  $m$ , and degeneracy factor  $g$ . A new feature of the quantum effects in the system of particle with the vdW interactions is the additional factor  $(1 - bn)^{-1}$  in the quantum correction  $\delta$ , i.e.,

<sup>2</sup>The multicomponent QvdW model with different  $a$  and  $b$  parameters for protons and neutrons was discussed in Refs. [29,32].

the quantum statistics effects becomes stronger due to the repulsive interactions between particles.

The vdW model, both in its classical form (14) and in its QvdW extension (12) and (13), describes the first-order liquid-gas phase transition. The critical point (CP) of this transition satisfies the following equations:

$$\left(\frac{\partial P}{\partial n}\right)_T = 0, \quad \left(\frac{\partial^2 P}{\partial n^2}\right)_T = 0. \quad (18)$$

Using Eq. (17) in the first approximation in  $\delta$ , one derives from Eq. (18) the system of two equations for the CP parameters  $n_c$  and  $T_c$  at the same first order:

$$\frac{T}{2an(1-nb)^2}[1+2\delta] = 1, \quad (19)$$

$$\frac{bT}{a(1-nb)^3} \left[ 1 + \frac{(1+2nb)}{bn} \delta \right] = 1. \quad (20)$$

Solving the system (19) and (20), one finds in the same first-order approximation over  $\delta$ :

$$\begin{aligned} T_c^{(1)} &\cong T_c^{(0)}(1-2\delta_0) \cong 19.0 \text{ MeV}, \\ n_c^{(1)} &\cong n_c^{(0)}(1-2\delta_0) \cong 0.065 \text{ fm}^{-3}. \end{aligned} \quad (21)$$

In Eq. (21),

$$\begin{aligned} T_c^{(0)} &= \frac{8a}{27b} \cong 29.2 \text{ MeV}, \\ n_c^{(0)} &= \frac{1}{3b} \cong 0.100 \text{ fm}^{-3}, \\ P_c^{(0)} &= \frac{a}{27b^2} \cong 1.09 \text{ MeV fm}^{-3} \end{aligned} \quad (22)$$

are the CP parameters of the classical vdW model, i.e., they are found from Eq. (18) for the equation (14). The parameter  $\delta_0$  in Eq. (21) is given by Eq. (16) calculated at the CP (22), i.e., at  $n = n_c^{(0)}$  and  $T = T_c^{(0)}$ . Substituting Eq. (21) for the critical temperature  $T_c^{(1)}$  and density  $n_c^{(1)}$  into Eq. (17) at the same first order, for the CP pressure  $P_c$  one obtains

$$P_c^{(1)} \cong 0.48 \text{ MeV fm}^{-3}. \quad (23)$$

The numerical calculations within the full QvdW model (12) and (13) give

$$\begin{aligned} T_c &\cong 19.7 \text{ MeV}, \\ n_c &\cong 0.072 \text{ fm}^{-3}, \\ P_c &\cong 0.52 \text{ MeV fm}^{-3}. \end{aligned} \quad (24)$$

These our results (24) appear to be essentially the same as those obtained in Ref. [18].

A summary of the results for the CP parameters is presented in Table I. A difference of the results for the classical vdW model (22) and QvdW model (24) demonstrates a role of the effects of Fermi statistics at the CP of the symmetric nuclear matter. The size of these effects appears to be rather significant. On the other hand, it is remarkable that the first-order correction (21) and (23) reproduce these effects of quantum statistics with a high accuracy. The second-order correction in the expansion (17) leads to an improvement of the CP parameters,  $T_c^{(2)} = 19.7 \text{ MeV}$ ,  $n_c^{(2)} = 0.072 \text{ fm}^{-3}$  and

TABLE I. Results for the CP parameters of the symmetric nuclear matter ( $g = 4$ ,  $m = 938 \text{ MeV}$ ).

Critical points	Eq. (22)	First-order correction Eqs. (21) and (23)	QvdW Eq. (24)
$T_c$ (MeV)	29.2	19.0	19.7
$n_c$ (fm <sup>-3</sup> )	0.100	0.065	0.072
$P_c$ (MeV fm <sup>-3</sup> )	1.09	0.48	0.52

$P_c^{(2)} = 0.53 \text{ MeV fm}^{-3}$ , which become closer to the numerical results (24) of the full QvdW model. The contribution of the second order in  $\delta$  is much smaller than the first-order correction in Eq. (21), that shows a fast convergence in  $\delta$  by a few first terms. Therefore, high-order corrections due to the quantum statistics effects can be neglected.

The vdW model gives an example of the equation of state for the system of interacting nucleons in terms of the model parameters  $a$  and  $b$ . Many other examples were recently considered in Ref. [30]. All models investigated in that paper have rather different high-order virial-expansion coefficients. However, if the parameters of these different models are fixed by a requirement to reproduce properties of the nuclear ground state, the obtained values of  $T_c$  and  $n_c$  appear to be quite similar. For example, different  $T_c$  values come to the narrow region  $T_c = 18 \pm 2 \text{ MeV}$  for the symmetric nuclear matter. The effects of Fermi statistics leads to much stronger changes of the  $T_c$  values: about 10 MeV in the nucleon matter and 20 MeV in the neutron matter. Note that our consideration made for the QvdW model can be straightforwardly extended to other types of interparticle interaction.

The vdW model does not take into account very accurately the hard-sphere repulsion effects at high particle number density. On the other hand, near the CP, one finds  $n_c \cong 1/3b$ , where  $b = 4v_0$  is four times larger than the hard-core particle volume  $v_0$ . Thus,  $n_c$  is about ten times smaller than the hard-packing limit. At these conditions, the excluded volume procedure used in the vdW equation of state is justified.

As stated in the paper, our analysis can be applied beyond the vdW model. In fact, similar estimates of the quantum statistic effects can be straightforwardly done also for the mean-field models. Concerning these models see, e.g., Ref. [31] and references therein.

The CP parameters of the classical vdW model (22) depend on the interaction parameters  $a$  and  $b$ , and they are not sensitive to the values of particle mass  $m$  and degeneracy factor  $g$ . The effects of quantum statistics change this conclusion. The parameter  $\delta$  given by Eq. (16) is proportional to  $m^{-3/2}g^{-1}$ , i.e., the effects of quantum statistics become smaller for heavier

TABLE II. Results for the CP parameters for the neutron matter ( $g = 2$ ,  $m = 938 \text{ MeV}$ ).

Critical points	Eq. (22)	First-order correction	QvdW
$T_c$ (MeV)	29.2	8.7	10.8
$n_c$ (fm <sup>-3</sup> )	0.100	0.030	0.051
$P_c$ (MeV fm <sup>-3</sup> )	1.09	0.13	0.20

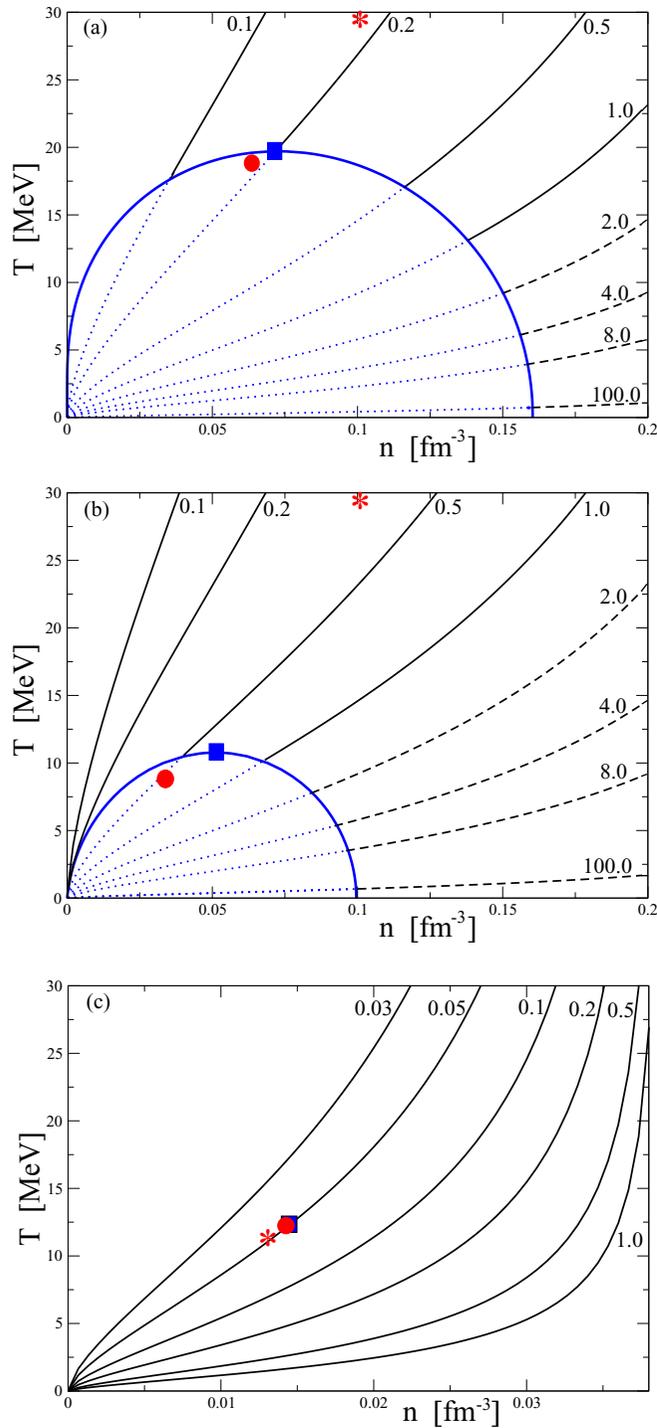


FIG. 2. Lines of the constant values of the parameter  $\delta$  [Eq. (16)] in the  $T$ - $n$  plane for the (a) symmetric nuclear matter, (b) neutron matter, and (c)  $|\delta|$  for  $\alpha$  matter. The boundaries of the liquid-gas mixed phase regions are shown in (a) and (b) by solid blue lines. The red close dots, blue squares, and asterisks in (a)–(c) show the CP parameters in the, respectively, QvdW model within the first-order correction, the full QvdW model, and classical vdW model.

particles (e.g., for the light nuclei admixture in the nuclear matter) or for large values of degeneracy factor  $g$ . Particularly, for  $g \gg 1$  the quantum statistics effects become negligible,

TABLE III. Results for the CP parameters for the  $\alpha$  particle matter ( $g = 1$ ,  $m = 3727$  MeV).

Critical points	Eq. (22)	First-order correction	QvdW
$T_c$ (MeV)	10.14	11.17	11.20
$n_c$ ( $\text{fm}^{-3}$ )	0.0117	0.0129	0.0130
$P_c$ ( $\text{MeV fm}^{-3}$ )	0.0443	0.0538	0.0540

and the QvdW model (12), (13) is reduced at  $T > 0$  to the classical vdW model (14). These results are in agreement with the numerical calculations in the recent paper [32].

The effects of quantum statistics become larger in the neutron matter within the QvdW model with the same  $a$  and  $b$  parameters, but  $g = 2$  instead of  $g = 4$  for the symmetric nuclear matter. This leads to essentially larger effects of the quantum statistics and stronger changes of the CP parameters. These results are summarized<sup>3</sup> in Table II.

In Fig. 2 the expansion parameter  $\delta$  entered to Eq. (17) is shown in the  $n$ - $T$  plane in the QvdW model for the symmetric nuclear matter [Fig. 2(a)] and pure neutron matter [Fig. 2(b)], and the parameter  $-\delta$  for the pure  $\alpha$  matter [Fig. 2(c)]. For the symmetric nuclear matter in Fig. 2(a) the values of  $\delta$  in a vicinity of the CP is rather small,  $\delta \approx 0.2$ . This explains a good accuracy of the first-order corrections of the quantum statistics effects.

For the neutron matter shown in Fig. 2(b) the parameter of the quantum correction is  $\delta \approx 0.5$  near the CP. The correction due to the quantum statistics becomes indeed larger than that for the symmetric nuclear matter. This also explains why an agreement of the first-order quantum results for the CP parameters with their exact QvdW numerical values is worse than in the case of symmetric nuclear matter. The second-order correction in Eq. (17) leads to the critical values  $T_c^{(2)} = 10.6$  MeV,  $n_c^{(2)} = 0.052$   $\text{fm}^{-3}$ , and  $P_c^{(2)} = 0.20$   $\text{MeV fm}^{-3}$ . This improves significantly an agreement with results of the full QvdW model calculations presented in Table II.

The calculations for interacting  $\alpha$  particles within the QvdW model are shown in Fig. 2(c) and Table III. For illustrative purposes we take the vdW parameters  $a = 976.7$   $\text{MeV fm}^3$  and  $b = 28.55$   $\text{fm}^3$  for interacting  $\alpha$  particles, obtained from the critical point values  $T_c$  and  $n_c$  estimated in Ref. [33]. One finds a rather small value  $|\delta| \approx 0.05$  for the  $\alpha$  matter in a vicinity of the CP,  $\delta < 0$  [see Fig. 2(c)]. Note that the Bose and Fermi statistics lead to the same absolute values of the first order quantum correction (16), but with the different signs. Thus, the quantum Bose effects change the critical temperature  $T_c$  and density  $n_c$  into the opposite direction, i.e.,  $T_c$  and  $n_c$  are moved to larger values due to the Bose effects in comparison with their classical vdW values. As seen from Fig. 2(c) and Table III, there is about a 10% increase of the  $T_c^{(0)}$  and  $n_c^{(0)}$  values due to the Bose statistics correction.

<sup>3</sup>The properties of the pure neutron matter appear to be very sensitive to the values of the vdW parameters  $a$  and  $b$  for neutrons. This is discussed in Ref. [32].

At small  $T$  and/or large  $n$ , the parameter  $\delta \propto n(1 - bn)^{-1}T^{-3/2}$  becomes large. In this region of the phase diagram, the QvdW model should be treated within the full quantum statistical formulation (12) and (13). In the  $n$ - $T$  region with  $\delta > 1$  shown in Figs. 2(a) and 2(b) by the dashed lines the first-order quantum approximation loses their meaning.

The mixed gas-liquid phases of the symmetric nuclear matter and pure neutron matter correspond to the  $n$ - $T$  regions under the blue solid lines presented in Figs. 2(a) and 2(b), respectively. The physical states inside the mixed phase correspond to the equilibrium of the gas and liquid components with equal values of  $T$ ,  $\mu$ , and  $P$ . However, these components have different particle number densities,  $n_{\text{gas}} < n_{\text{liq}}$ . They become very different,  $n_{\text{gas}} \ll n_{\text{liq}}$ , at small temperatures. In this case, the effects of quantum statistics are small for the gas component ( $\delta \ll 1$ ) but are rather large ( $\delta \gtrsim 1$ ) for the liquid one. For the Bose  $\alpha$  particle system large quantum effects mean a possibility of the Bose-Einstein condensation. Thus, both effects—the first-order phase transition and a formation of the Bose condensate—should be treated simultaneously (see, e.g., Ref. [33]). These questions are, however, outside of the scope of the present paper.

Rough estimates give  $|\delta_0| \ll 1$  in most cases of the CP for different atomic gases [34] and, thus, small effects of quantum statistics. This happens, despite of much smaller values of the atomic critical temperature  $T_A \ll T_c$  in comparison with the nuclear matter  $T_c$  values. The small effects of quantum statistics in the atomic systems in their gas and liquid states is a consequence of very small atomic densities as compared to the nuclear ones,  $n_A \ll n_c$ . One exception from these arguments is the atomic system of He-4. The experimental value for its CP parameters [34,35],  $T_c(\text{He}) \cong 5.2 \text{ K}^0$  and  $n_c(\text{He}) \cong 10^{22} \text{ cm}^{-3}$ , lead to the estimate  $\delta_0(\text{He}) \approx -0.1$ . Therefore, our consideration within the first-order quantum correction over  $\delta$  looks rather suitable to the analysis of the quantum statistics effects near the CP of this atomic system.

#### IV. SUMMARY

The QvdW equation of state has been used to study the quantum statistics effects in a vicinity of the critical point of

nuclear matter. To obtain the analytical expressions, the first-order quantum statistics correction over the small parameter  $\delta$  is considered. An explicit dependence on the system parameters is demonstrated. Particularly, the CP position depends on the particle mass  $m$  and degeneracy factor  $g$ . Such a dependence is absent within the classical vdW model. The quantum corrections to the CP parameters of the symmetric nuclear matter appear to be quite significant. For example, the value of  $T_c^{(0)} = 29.2 \text{ MeV}$  in the classical vdW model decreases to the value  $T_c^{(1)} = 19.0 \text{ MeV}$ . On the other hand, this approximate analytical result within the first-order quantum correction is already close to the numerical value of  $T_c = 19.7 \text{ MeV}$  obtained by the numerical calculations within the full QvdW model. The quantum correction of the CP parameters becomes even larger for the pure neutron matter. In this case, the classical value of  $T_c^{(0)} = 29.2 \text{ MeV}$  decreases to  $T_c^{(1)} = 8.7 \text{ MeV}$ , which is still close to the numerical value of the full QvdW model value  $T_c = 10.8 \text{ MeV}$ , and the second-order corrections improves this agreement.

Our consideration is straightforwardly extended to the system of interacting bosons. The quantum corrections have different signs for fermions and bosons. An example of the pure  $\alpha$  matter has been considered. The CP temperature  $T_c^{(0)} = 11.2 \text{ MeV}$  for the classical vdW model of  $\alpha$  particles increases by about of 10% to  $T_c^{(1)} = 12.3 \text{ MeV}$  for  $\alpha$  matter within the first-order approximation in the QvdW model.

#### ACKNOWLEDGMENTS

We thank D. V. Anchishkin, B. I. Lev, A. Motornenko, R. V. Poberezhnyuk, A. I. Sanzhur, and V. Vovchenko, for many fruitful discussions. The work of S.N.F. and A.G.M. on the project “Nuclear matter in extreme conditions” was supported by the Program “Fundamental researches in high energy physics and nuclear physics (international collaboration)” at the Department of Nuclear Physics and Energy of the National Academy of Sciences of Ukraine. The work of M.I.G. was supported by the Program of Fundamental Research of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine.

- 
- [1] B. K. Jennings, S. Das Gupta, and N. Mobed, *Phys. Rev. C* **25**, 278 (1982).
  - [2] G. Röpke, L. Münchow, and H. Schulz, *Nucl. Phys. A* **379**, 536 (1982).
  - [3] G. Fái and J. Randrup, *Nucl. Phys. A* **381**, 557 (1982).
  - [4] T. Biro, H. W. Barz, B. Lukacs, and J. Zimanyi, *Phys. Rev. C* **27**, 2695 (1983).
  - [5] L. P. Csernai, H. Stöcker, P. R. Subramanian, G. Buchwald, G. Graebner, A. Rosenhauer, J. A. Maruhn, and W. Greiner, *Phys. Rev. C* **28**, 2001 (1983).
  - [6] L. P. Csernai and J. I. Kapusta, *Phys. Rep.* **131**, 223 (1986).
  - [7] B. D. Serot and J. D. Walecka, *Adv. Nucl. Phys.* **16**, 1 (1986).
  - [8] J. Zimanyi and S. A. Moszkowski, *Phys. Rev. C* **42**, 1416 (1990).
  - [9] R. Brockmann and R. Machleidt, *Phys. Rev. C* **42**, 1965 (1990).
  - [10] A. S. Hirsch, A. Bujak, J. E. Finn, L. J. Gutay, R. W. Minich, N. T. Porile, R. P. Scharenberg, B. C. Stringfellow, and F. Turkot, *Nucl. Phys. A* **606**, 508 (1996).
  - [11] M. Bender, P. H. Heenen, and P. G. Reinhard, *Rev. Mod. Phys.* **75**, 121 (2003).
  - [12] J. E. Finn, S. Agarwal, A. Bujak, J. Chuang, L. J. Gutay, A. S. Hirsch, R. W. Minich, N. T. Porile, R. P. Scharenberg, B. C. Stringfellow, and F. Turkot, *Phys. Rev. Lett.* **49**, 1321 (1982).
  - [13] R. W. Minich, S. Agarwal, A. Bujak, J. Chuang, J. E. Finn, L. J. Gutay, A. S. Hirsch, N. T. Porile, R. P. Scharenberg, B. C. Stringfellow, and F. Turkot, *Phys. Lett. B* **118**, 458 (1982).
  - [14] A. S. Hirsch, A. Bujak, J. E. Finn, L. J. Gutay, R. W. Minich, N. T. Porile, R. P. Scharenberg, B. C. Stringfellow, and F. Turkot, *Phys. Rev. C* **29**, 508 (1984).

- [15] J. Pochodzalla, T. Möhlenkamp, T. Rubehn, A. Schüttauf, A. Wörner, E. Zude, M. Begemann-Blaich, Th. Blaich, H. Emling, A. Ferrero, C. Gross, G. Immé, I. Iori, G. J. Kunde, W. D. Kunze, V. Lindenstruth, U. Lynen, A. Moroni, W. F. J. Müller, B. Ocker *et al.*, *Phys. Rev. Lett.* **76**, 2823 (1996).
- [16] J. B. Natowitz, K. Hagel, Y. Ma, M. Murray, L. Qin, R. Wada, and J. Wang, *Phys. Rev. Lett.* **89**, 212701 (2002).
- [17] V. A. Karnaukhov, H. Oeschler, S. P. Avdeyev, E. V. Duginova, V. K. Rodionov, A. Budzanowski, W. Karcz, O. V. Bochkarev, E. A. Kuzmin, L. V. Chulkov, E. Norbeck, and A. S. Botvina, *Phys. Rev. C* **67**, 011601(R) (2003).
- [18] V. Vovchenko, D. Anchishkin, and M. Gorenstein, *Phys. Rev. C* **91**, 064314 (2015).
- [19] V. Vovchenko, M. I. Gorenstein, and H. Stoecker, *Phys. Rev. Lett.* **118**, 182301 (2017).
- [20] V. Vovchenko, L. Jiang, M. I. Gorenstein, and H. Stoecker, *Phys. Rev. C* **98**, 024910 (2018).
- [21] V. V. Begun and M. I. Gorenstein, *Phys. Rev. C* **77**, 064903 (2008).
- [22] W. Greiner, L. Neise, and H. Stöcker, *Thermodynamics and Statistical Mechanics* (Springer-Verlag, New York, 1995).
- [23] L. D. Landau and E. M. Lifshitz, *Statistical Physics, Course of Theoretical Physics* (Pergamon, Oxford, 1975), Vol. 5.
- [24] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, 4th ed. (Fizmatlit, Moscow, 1963).
- [25] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Nauka, Moscow, 1986).
- [26] M. Brack, C. Guet, and H.-B. Håkansson, *Phys. Rep.* **123**, 275 (1985).
- [27] R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley, New York, 1975), Vol. 1.
- [28] H. A. Bethe, *Ann. Rev. Nucl. Part. Sci.* **21**, 93 (1971).
- [29] V. Vovchenko, A. Motornenko, P. Alba, M. I. Gorenstein, L. M. Satarov, and H. Stoecker, *Phys. Rev. C* **96**, 045202 (2017).
- [30] V. Vovchenko, *Phys. Rev. C* **96**, 015206 (2017).
- [31] D. Anchishkin and V. Vovchenko, *J. Phys. G* **42**, 105102 (2015).
- [32] R. V. Poberezhnyuk, V. Vovchenko, M. I. Gorenstein, and H. Stoecker, *Phys. Rev. C* **99**, 024907 (2019).
- [33] L. M. Satarov, M. I. Gorenstein, A. Motornenko, V. Vovchenko, I. N. Mishustin, and H. Stoecker, *J. Phys. G: Nucl. Part. Phys.* **44**, 125102 (2017).
- [34] K. A. Kobe and R. E. Lynn, *Chem. Rev.* **52**, 117 (1953).
- [35] R. D. McCarty, *J. Phys. Chem. Ref. Data* **2**, 923 (1973).