Comment on "Elimination of degeneracy in the γ-unstable Bohr Hamiltonian in the presence of an extended sextic potential"

R. Budaca [1,2] and P. Buganu[1]

[1] *"Horia Hulubei" National Institute for Physics and Nuclear Engineering,*
Strada Reactorului 30, RO-077125, POB-MG6 Bucharest-Măgurele, Romania
[2] *Academy of Romanian Scientists, 54 Splaiul Independenţei, RO-050094 Bucharest, Romania*

We argue that the analytical approach of [Phys. Rev. C **98**, 014312 (2018)] is incorrect. The authors obtain a solution for a sextic potential through the truncation of the power series of the wave-function ansatz. We show that the procedure should be equivalent to that of the quasiexactly solvable versions of the sextic potential used previously and that the quantization of the total energy is not performed properly. Additionally, we question the reliability of the transition probabilities calculated between states described by different collective potentials and, respectively, belonging to distinct Hilbert spaces.

The authors of Ref. [1] consider a γ-unstable version of the Bohr Hamiltonian with a β potential,

$$v(L;\beta) = \eta\frac{\hat{L}\hat{L}}{\beta^2} + \frac{\kappa^4}{\beta^2} + a\beta^2 + b\beta^4 + 4\beta^6. \quad (1)$$

The effect of the spin-dependent term was already studied in Ref. [2], therefore, the original contribution is related to the sextic potential. Making the correspondences,

$$\alpha' = \sqrt{\frac{9}{4} + \tau(\tau+3) + \eta L(L+1) + \kappa^4},$$
$$\beta' = \frac{b}{4}, \qquad \gamma' = \frac{1}{4}\left(\frac{b^2}{16} - a\right), \qquad \delta' = -\frac{\varepsilon}{2}, \quad (2)$$

and using successive function changes as well as a new variable $y = \beta^2$, the β equation can be brought to a form resembling the biconfluent Heun differential equation [3,4], whose canonical form is

$$yh''(y) + (1 + \alpha' - \beta' y - 2y^2)h'(y)$$
$$+ \left\{(\gamma' - \alpha' - 2)y - \tfrac{1}{2}[\delta' + \beta'(1 + \alpha')]\right\}h(y) = 0. \quad (3)$$

L and τ index the γ-angular states which obey the $SO(5)$ symmetry. For non-negative and integer α', its solution can be written as a series [5],

$$h(y) = \sum_{p=0}^{\infty} C_p x^p = \sum_{p=0}^{\infty} \frac{A_p}{(1+\alpha')_p p!} y^p, \quad (4)$$

where $(x)_p$ is a Pochhammer symbol and $A_0 = 1$. Plugging this series expansion in Eq. (3), one obtains a recurrence relation satisfied by the coefficients A_p,

$$A_{p+2} - A_{p+1}\left\{(p+1)\beta' + \tfrac{1}{2}[\delta' + \beta'(1+\alpha')]\right\}$$
$$+ A_p(\gamma' - 2 - \alpha' - 2p)(p+1)(p+\alpha'+1) = 0, \quad (5)$$

with the initial condition $A_{-1} = 0$. This equation shows that (4) becomes a polynomial of degree n if $\gamma' - 2 - \alpha' = 2n$

with $n = 0, 1, 2, \ldots$, and $A_{n+1} = 0$. Depending on the initial equation which is then brought to a form (3), the former condition can lead to the quantization of the energy [6]. In this case, however, it leads to a relation between the parameters of the collective potential and rotational quantum numbers,

$$a = \frac{b^2}{16} - 4\left[2n + 2 + \sqrt{\frac{9}{4} + \tau(\tau+3) + \eta L(L+1) + \kappa^4}\right]. \quad (6)$$

This relation differs from that obtained in Ref. [1], which seems to be obtained by considering $n + 1$ instead of n. Let us turn now to the second condition, where A_{n+1} is actually a polynomial of degree $n + 1$ in A_1 defined by the compatibility condition of the system of $n + 1$ equations for A_n coefficients. For exemplification, let us write explicitly the first two conditions (5),

$$-\tfrac{1}{2}[\delta' + \beta'(1+\alpha')]A_0 + A_1 = 0,$$
$$2n(1+\alpha')A_0 - \left\{\beta' + \tfrac{1}{2}[\delta' + \beta'(1+\alpha')]\right\}A_1 + A_2 = 0. \quad (7)$$

From the first equation, we obtain that $A_1 = \tfrac{1}{2}[\delta' + \beta'(1 + \alpha')]$. As A_1 contains the energy through δ', solving the secular equation $A_{n+1} = 0$ will provide us with the eigenvalues ε^k indexed by the solution's order $k = \overline{1, n+1}$. At this point, it is important to note that n is not a quantum number but rather an integer parameter which sets the truncation of the polynomial solutions as well as the parameters of the solvable potential through (6). When $n = 0$, only the first condition remains together with $A_1 = 0$. It can be easily checked that now all coefficients A_p with $p > 0$ vanish, and the series (4) is truncated. It also leads to $\delta' + \beta'(1 + \alpha') = 0$, and consequently,

$$\varepsilon^0 = \frac{b}{2}\left[1 + \sqrt{\frac{9}{4} + \tau(\tau+3) + \eta L(L+1) + \kappa^4}\right], \quad (8)$$

where b is expressed through other parameters and rotational quantum numbers as in (6) with $n = 0$.

The above equation resembles the energy obtained in Ref. [1] but is quite different. Also, it is valid only for $n = 0$ truncation. Moreover, there is just a single solution for the energy. Therefore, one cannot describe β excited bands with only $n = 0$. For this purpose, we must take, at least, $n = 1$, whose condition $A_2 = 0$ will provide two energy solutions,

$$\varepsilon_\pm^1 = \frac{b}{2}\left[2 + \alpha' \pm \sqrt{1 + \frac{128(1 + \alpha')}{b^2}}\right], \qquad (9)$$

as roots of the compatibility condition of the system of equations for A_0 and A_1. The lowest energy corresponds to the ground band level, whereas the highest corresponds to the β band state. Thus, different $K = 0$ states of the same τ and L are distinguished by the order of the solution and not the truncation degree n. This method was used to describe collective excitations with a quasiexactly solvable sextic potential in Refs. [7–11]. The truncation of the power-series expansion of ansatz (4) is the essence of quasiexact solvability of a model [12]. The connection between polynomial solutions of Heun equations of various types and quasiexact solvability is well known [13–15].

In the applications of the quasiexactly solvable sextic potential to various collective conditions [7–11], the authors went to great effort for assuring that the potential to be state independent. This was performed by a seesaw adjustment of the truncation order and the rotational quantum numbers. In the commented paper, the authors ignored this aspect. Such that, states with different L and τ quantum numbers will be actually described with different potentials. As can be seen from (6), if one considers a as a constant parameter, then b will depend on L and τ or vice versa. Therefore, either the coefficient of β^2 or β^4 must be an operator in the γ-angular variables, whose eigenvalues are the same $SO(5)$ spherical harmonics determined for the centrifugal term. The last part is not true because of the rational form of the expression (6). This problem can be analyzed also from another point of view by considering that the potential depends on energy as in Refs. [16–18]. The consequences of state dependence of the potential include the violation of the continuity equation, the loss of completeness property, and the noncommutativity of the original coordinates and momenta [19]. The conservation of the norm in time can be recovered by a change in the scalar product by employing a correction to the integration metric. Such a modification, however, reinstates the completeness condition only in some particular cases of energy dependence. Therefore, even with such an amendment the calculation of averages and transition matrix elements within such a theory is not reliable. In Ref. [1], neither of the presented approaches were considered for the analytical formalism.

In conclusion, we have proved that the quantization condition for the energy from Ref. [1] is incorrect on many levels, and it should be similar to the exactly solvable approaches [7–11]. The formalism of Ref. [1] also suffers from inconsistencies stemming from quantum theory. All these issues were perpetuated from the previous papers [20,21].

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