


Explicit derivation of the completeness condition in pseudoscalar meson photoproductionK. Nakayama *Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602, USA*

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By exploiting the underlying symmetries of the relative phases of the pseudoscalar meson photoproduction amplitude, we provide a consistent and explicit mathematical derivation of the completeness condition for the observables in this reaction. In particular, we determine all the possible sets of four double-spin observables that resolve the phase ambiguity of the amplitude in transversity basis up to an overall phase. The present work substantiates and corroborates the original findings of Chiang and Tabakin [*Phys. Rev. C* **55**, 2054 (1997)]. It is found, however, that the completeness condition of four double-spin observables to resolve the phase ambiguity holds only when the relative phases *do not* meet the condition of equal magnitudes. In situations where this condition occurs, it is shown that one needs extra chosen observables, resulting in the minimum number of observables required to resolve the phase ambiguity reaching up to eight, depending on the particular set of four double-spin observables considered. Furthermore, a way of gauging when the condition of equal magnitudes occurs is provided.

DOI: [10.1103/PhysRevC.100.035208](https://doi.org/10.1103/PhysRevC.100.035208)**I. INTRODUCTION**

The issue of model-independent determination of the pseudoscalar meson photoproduction amplitude has attracted much attention since the early stage of investigation of this reaction process. In particular, early papers on the minimum number of experimental observables required to determine the pseudoscalar meson photoproduction amplitude—the so-called *complete experiments*—have resulted in contradictory findings (for a brief account on these, see Ref. [1]). Barker, Donnachie, and Storrow [1] have cleared up this situation, by deriving the necessary and sufficient conditions for determining the full photoproduction amplitude up to discrete ambiguities. They also provided the rules for choosing further measurements to resolve these ambiguities. According to these authors, for a given kinematics (total energy of the system and meson production angle), one requires nine observables to determine the full reaction amplitude up to an arbitrary overall phase. Keaton and Workman [2], however, have realized that there are cases obeying the rules given in Ref. [1] that still leave unsolved ambiguities. Finally, Chiang and Tabakin [3], have shown that, instead of nine observables as claimed in Ref. [1], one requires a minimum of eight carefully chosen observables for a complete experiment. Apart from solving for the amplitude magnitudes and phases directly, Chiang and Tabakin [3] in their study also used a bilinear helicity product formulation to map an algebra of measurements over to the well-known algebra of the 4×4 gamma matrices. This latter method leads to an alternate proof that eight carefully chosen experiments suffice for determining the transversity amplitudes completely. The issue of complete experiments has been also discussed by Moravcsik [4] in the context of a general reaction process. There, a very similar approach to that of Ref. [3] is used for resolving the discrete phase ambiguities of the reaction amplitude with a geometrical interpretation.

Sandorf *et al.* [5] have concluded among other things that, while a mathematical solution to the problem of determining an amplitude free of ambiguities may require eight observables [3], experiments with realistically achievable uncertainties will require a significantly larger number of observables. Also, the Gent group has expended much effort along this line [6–8]. Recently, with the advances in experimental techniques, many spin observables in photoproduction reactions became possible to be measured, and this has attracted much interest in constraints on partial-wave analysis in the context of complete experiments [9–14]. Of particular interest in this connection is the issue of whether the baryon resonances can be extracted model independently or with minimal model inputs. Efforts in this direction are currently in progress [12–14].

In this work, we revisit the problem of complete experiments in pseudoscalar meson photoproduction from a mathematical point of view, i.e., under ideal experiments with zero uncertainties. Thus, it is most directly related to the work of Ref. [3]. We tackle this problem by solving for the amplitude magnitudes and phases directly, as has been done in Ref. [3]. In doing so, we shall reveal and exploit the underlying symmetries of the relative phases of the photoproduction amplitude, which allows a consistent and explicit mathematical derivation of the completeness condition for the observables covering all the relevant cases. The completeness condition of a set of four double-spin observables to resolve the phase ambiguity of the transversity amplitude is shown to hold, except in situations where the equal-relative-phase-magnitudes relation—as specified in Eq. (48) later in Sec. VI—occur. It will be shown that, when this situation occurs, one needs up to seven chosen double-spin observables, instead of four, to resolve the phase ambiguity. Furthermore, in the particular situation where the relative phases vanish, eight chosen double-spin observables are required to resolve the phase ambiguity.

The paper is organized as follows. In Sec. II, we introduce the notations used throughout this work and express the observables as bilinear combinations of the four basic transversity amplitudes. In addition, we group the observables and classify them in cases which are convenient for determining the possible sets of four observables that resolve the phase ambiguity. In Secs. III, IV, and V, we determine these sets of four double-spin observables, according to the classification introduced in Sec. II. There, we also consider the cases where the restriction on the relative phases for the completeness condition of the four observables is not satisfied. In Sec. VI, we discuss how to identify when this restriction is violated. Finally, a summary is given in Sec. VII.

II. NOTATIONS

The basic four independent amplitudes, M_j ($j = 1, \dots, 4$), that constitute the full pseudoscalar photoproduction amplitude can be expressed as

$$M_j = r_j e^{i\phi_j}, \quad \begin{cases} r_j = \text{magnitude,} \\ \phi_j = \text{phase.} \end{cases} \quad (1)$$

Then, following Ref. [3], the 16 nonredundant observables can be expressed in terms of these amplitudes M_j in transversity basis and grouped according to

$$\mathcal{S} = \begin{cases} d\sigma/d\Omega &= \frac{1}{2}[|r_1|^2 + |r_2|^2 + |r_3|^2 + |r_4|^2], \\ \Sigma &= \frac{1}{2}[|r_1|^2 + |r_2|^2 - |r_3|^2 - |r_4|^2], \\ T &= \frac{1}{2}[|r_1|^2 - |r_2|^2 - |r_3|^2 + |r_4|^2], \\ P &= \frac{1}{2}[-|r_1|^2 + |r_2|^2 - |r_3|^2 + |r_4|^2], \end{cases} \quad (2)$$

$$\mathcal{BT} = \begin{cases} O_{1+}^a &\equiv -G &= B_{13} \sin \phi_{13} + B_{24} \sin \phi_{24}, \\ O_{1-}^a &\equiv F &= B_{13} \sin \phi_{13} - B_{24} \sin \phi_{24}, \\ O_{2+}^a &\equiv E &= B_{13} \cos \phi_{13} + B_{24} \cos \phi_{24}, \\ O_{2-}^a &\equiv H &= B_{13} \cos \phi_{13} - B_{24} \cos \phi_{24}, \end{cases} \quad (3)$$

$$\mathcal{BR} = \begin{cases} O_{1+}^b &\equiv O_z &= B_{14} \sin \phi_{14} + B_{23} \sin \phi_{23}, \\ O_{1-}^b &\equiv -C_x &= B_{14} \sin \phi_{14} - B_{23} \sin \phi_{23}, \\ O_{2+}^b &\equiv -C_z &= B_{14} \cos \phi_{14} + B_{23} \cos \phi_{23}, \\ O_{2-}^b &\equiv -O_x &= B_{14} \cos \phi_{14} - B_{23} \cos \phi_{23}, \end{cases} \quad (4)$$

$$\mathcal{TR} = \begin{cases} O_{1+}^c &\equiv -L_x &= B_{12} \sin \phi_{12} + B_{34} \sin \phi_{34}, \\ O_{1-}^c &\equiv -T_z &= B_{12} \sin \phi_{12} - B_{34} \sin \phi_{34}, \\ O_{2+}^c &\equiv -L_z &= B_{12} \cos \phi_{12} + B_{34} \cos \phi_{34}, \\ O_{2-}^c &\equiv T_x &= B_{12} \cos \phi_{12} - B_{34} \cos \phi_{34}, \end{cases} \quad (5)$$

where

$$B_{ij} \equiv r_i r_j \quad \text{and} \quad \phi_{ij} \equiv \phi_i - \phi_j. \quad (6)$$

In the following we refer to ϕ_{ij} as the relative phase. The observables in \mathcal{S} include the unpolarized cross section,

$d\sigma/d\Omega$, and single-spin observables Σ (beam asymmetry), T (target asymmetry), and P (recoil asymmetry). It is clear from Eq. (2) that, together, they determine uniquely the magnitudes of the basic four amplitudes in transversity basis. Throughout this work, these four observables are assumed to be measured, so that the magnitudes of the basic transversity amplitudes are known. The remaining observables given in Eqs. (3), (4), and (5) are all double-spin observables, and some combinations of them will serve to determine the phases of the four transversity amplitudes up to an overall phase, i.e., the three relative phases ϕ_{ij} involved. We refer to the observables in each of \mathcal{BT} (beam-target asymmetry), \mathcal{BR} (beam-recoil asymmetry) and \mathcal{TR} (target-recoil asymmetry) as a group. We use $a = \mathcal{BT}$, $b = \mathcal{BR}$ and $c = \mathcal{TR}$.

In Ref. [3], the unnormalized spin asymmetries are denoted by $\check{\Omega}^\beta$, i.e., $\check{\Omega}^\beta \equiv (d\sigma/d\Omega)\Omega^\beta$, where Ω^β stands for a given spin asymmetry specified by the index β . Throughout this work, we simply use the same notation Ω^β for the unnormalized spin asymmetries $[(d\sigma/d\Omega)\Omega^\beta \rightarrow \Omega^\beta]$ to avoid overloading the notations. For example, Σ in Eq. (2) actually stands for $(d\sigma/d\Omega)\Sigma$, and so on.

From the above list of observables, one sees that all possible sets of four double-spin observables can be obtained by considering the following cases:

- (1) (2 + 2) case: two pairs of observables, each pair from distinct groups.
- (2) (2 + 1 + 1) case: a pair of observables from one group and two other observables, one from each of the remaining two groups.
- (3) (3 + 1) case: three observables from one group and one observable from another group.
- (4) 4 case: all four observables from one group.

In the following we shall consider each of the cases listed above.

III. PHASE FIXING FOR THE (2 + 2) CASE

We start by noticing that there are two basic types of combination of a pair of observables ($O_{n\nu}^m, O_{n'\nu'}^m$) in a given group, one type with $n = n'$ and the other with $n \neq n'$. Here, ($m = a, b, c$), ($n, n' = 1, 2$), and ($\nu, \nu' = \pm$). A pair of observables of the type (O_{n+}^m, O_{n-}^m) leads to a fourfold phase ambiguity, with twofold ambiguity in each of the relative phases involved, ϕ_{ij} and ϕ_{kl} . There are two distinct pairs of this type ($n = 1, 2$) in each group. On the other hand, a pair of observables of the type ($O_{1\nu}^m, O_{2\nu'}^m$), leads only to a 2-fold phase ambiguity. There are four distinct pairs of this type ($\nu, \nu' = \pm$) in each group.

To see the properties mentioned above, consider all the possible pairs one can form in a given group, say, group $a = \mathcal{BT}$. For the pair (O_{1+}^a, O_{1-}^a) = $(-G, F)$, we have from Eq. (3)

$$\begin{aligned} O_{1+}^a &= B_{13} \sin \phi_{13} + B_{24} \sin \phi_{24}, \\ O_{1-}^a &= B_{13} \sin \phi_{13} - B_{24} \sin \phi_{24}, \end{aligned} \quad (7)$$

which leads to

$$\begin{aligned}\sin \phi_{13} &= \frac{O_{1+}^a + O_{1-}^a}{2B_{13}} \implies \phi_{13} = \begin{cases} \alpha_{13}, \\ \pi - \alpha_{13}, \end{cases} \\ \sin \phi_{24} &= \frac{O_{1+}^a - O_{1-}^a}{2B_{24}} \implies \phi_{24} = \begin{cases} \alpha_{24}, \\ \pi - \alpha_{24}, \end{cases}\end{aligned}\quad (8)$$

where $-\pi/2 \leq \alpha_{13}, \alpha_{24} \leq +\pi/2$; α_{ij} 's are uniquely defined. In the following, we use the notation ϕ_{ij}^λ to designate

$$\phi_{ij}^+ = \alpha_{ij}, \quad \phi_{ij}^- = \pi - \alpha_{ij}. \quad (9)$$

Note that a (relative) phase is meaningful only modulo 2π .

Analogously, for the pair $(O_{2+}^a, O_{2-}^a) = (E, H)$, we have from Eq. (3)

$$\begin{aligned}O_{2+}^a &= B_{13} \cos \phi_{13} + B_{24} \cos \phi_{24}, \\ O_{2-}^a &= B_{13} \cos \phi_{13} - B_{24} \cos \phi_{24},\end{aligned}\quad (10)$$

which leads to the twofold ambiguity

$$\phi_{ij}^+ = \alpha_{ij}, \quad \phi_{ij}^- = -\alpha_{ij}, \quad (11)$$

where α_{ij} is uniquely defined with $0 \leq \alpha_{ij} \leq \pi$.

Next we consider the pair $(O_{1+}^a, O_{2-}^a) = (-G, H)$. From Eq. (3),

$$\begin{aligned}O_{1+}^a &= B_{13} \sin \phi_{13} + B_{24} \sin \phi_{24}, \\ O_{2-}^a &= B_{13} \cos \phi_{13} - B_{24} \cos \phi_{24}.\end{aligned}\quad (12)$$

We first combine the above two expressions into

$$O_{1+}^a{}^2 + O_{2-}^a{}^2 = B_{13}^2 + B_{24}^2 - 2B_{13}B_{24} \cos(\phi_{13} + \phi_{24}). \quad (13)$$

Now, we define angle $\zeta \equiv \zeta_{nv,n'v'}^m$ through¹

$$\cos \zeta \equiv \frac{O_{nv}^m}{N}, \quad \sin \zeta \equiv \frac{O_{n'v'}^m}{N}, \quad (14)$$

with $N \equiv N_{nv,n'v'}^m \equiv \sqrt{O_{nv}^m{}^2 + O_{n'v'}^m{}^2}$. In the following we simply use ζ and N to avoid the heavy notation, but it should be kept in mind that they depend on the given pair of observables. For the pair under consideration, we have

$$\cos \zeta \equiv \frac{O_{1+}^a}{N}, \quad \sin \zeta \equiv \frac{O_{2-}^a}{N}, \quad (15)$$

with $N \equiv \sqrt{O_{1+}^a{}^2 + O_{2-}^a{}^2}$.

Then, Eq. (12) can be expressed in terms of ζ as

$$\begin{aligned}N \cos \zeta &= B_{13} \sin \phi_{13} + B_{24} \sin \phi_{24}, \\ N \sin \zeta &= B_{13} \cos \phi_{13} - B_{24} \cos \phi_{24}.\end{aligned}\quad (16)$$

Multiplying the first equality in the above equation by $\sin \phi_{24}$ and the second one by $\cos \phi_{24}$ and subtracting the second from the first, we arrive at

$$\cos(\phi_{13} + \phi_{24}) = \frac{B_{24} + N \sin(\zeta - \phi_{24})}{B_{13}}. \quad (17)$$

¹ $\zeta_{nv,n'v'}^m$ has a geometrical interpretation as the polar angle of a vector in a two-dimensional coordinate system, where O_{nv}^m defines the x coordinate and $O_{n'v'}^m$, the y coordinate. This provides an intuitive understanding of the fact that such an angle, $\zeta_{nv,n'v'}^m$, can indeed always be found.

Inserting the above result into Eq. (13) yields

$$\sin(\zeta - \phi_{24}) = \frac{B_{13}^2 - B_{24}^2 - N^2}{2NB_{24}}, \quad (18)$$

leading to the following twofold ambiguity for ϕ_{24} :

$$\phi_{24} = \begin{cases} \zeta - \alpha_{24}, \\ \zeta - \pi + \alpha_{24}. \end{cases}\quad (19)$$

Analogously, from Eqs. (13) and (16), we find that

$$\sin(\zeta + \phi_{13}) = \frac{B_{13}^2 - B_{24}^2 + N^2}{2NB_{13}}, \quad (20)$$

leading to the twofold ambiguity

$$\phi_{13} = \begin{cases} -\zeta + \alpha_{13}, \\ -\zeta + \pi - \alpha_{13}. \end{cases}\quad (21)$$

Note that, in Eqs. (19) and (21), phases α_{24} and α_{13} are uniquely defined by

$$\begin{aligned}\sin(\alpha_{24}) &= \frac{B_{13}^2 - B_{24}^2 - N^2}{2NB_{24}}, \\ \sin(\alpha_{13}) &= \frac{B_{13}^2 - B_{24}^2 + N^2}{2NB_{13}},\end{aligned}\quad (22)$$

with $-\pi/2 \leq \alpha_{13}, \alpha_{24} \leq +\pi/2$.

Equations (19) and (21) show that ϕ_{13} and ϕ_{24} have a twofold ambiguity each. However, there is another constraint that $\cos(\phi_{13} + \phi_{24})$ is uniquely defined by Eq. (13). Then, first we note that the sum of ϕ_{13} and ϕ_{24} should be of the form $\phi_{13} + \phi_{24} = \pm\tilde{\alpha}$. Combining this with Eqs. (19) and (21) leads to the following possibilities for $\tilde{\alpha}$:

$$\tilde{\alpha} = \begin{cases} \lambda(\phi_{13}^\lambda + \phi_{24}^\lambda) = (\alpha_{13} - \alpha_{24}), \\ \lambda(\phi_{13}^\lambda + \phi_{24}^{\lambda'}) = (\alpha_{13} + \alpha_{24} - \pi), \end{cases}\quad (23)$$

where the notation introduced in Eq. (9) has been used. Here, $\lambda, \lambda' = \pm$ and $\lambda \neq \lambda'$.

Next, we calculate $\cos(\phi_{13} + \phi_{24}) = \cos(\pm\tilde{\alpha})$, with $\tilde{\alpha}$ given in Eq. (23). For $\tilde{\alpha} = \alpha_{13} - \alpha_{24}$, we obtain

$$\begin{aligned}\cos(\phi_{13} + \phi_{24}) &= \cos[\pm(\alpha_{13} - \alpha_{24})] \\ &= \cos \alpha_{13} \cos \alpha_{24} + \sin \alpha_{13} \sin \alpha_{24} \\ &= \sqrt{(1 - \sin^2 \alpha_{13})(1 - \sin^2 \alpha_{24})} \\ &\quad + \sin \alpha_{13} \sin \alpha_{24} \\ &= \frac{B_{13}^2 + B_{24}^2 - N^2}{2B_{13}B_{24}},\end{aligned}\quad (24)$$

where Eq. (22) has been used. This result coincides with Eq. (13). For $\tilde{\alpha} = \alpha_{13} + \alpha_{24} - \pi$, on the other hand, it is immediately seen that the result for $\cos(\phi_{13} + \phi_{24})$ does not agree with Eq. (13) since, in this case, apart from an overall sign, all that changes from the $\tilde{\alpha} = \alpha_{13} - \alpha_{24}$ case is the change in the sign of the term $\sin \alpha_{13} \sin \alpha_{24}$, which is nonzero in general, in Eq. (24).

Thus, we conclude that Eq. (13), together with Eqs. (19) and (21), leads to

$$\phi_{13} + \phi_{24} = \pm(\alpha_{13} - \alpha_{24}), \quad (25)$$

i.e., we end up with only twofold ambiguity for ϕ_{13} and ϕ_{24} , viz.,

$$\begin{cases} \phi_{13} = -\zeta + \alpha_{13}, \\ \phi_{24} = \zeta - \alpha_{24}, \end{cases} \quad \text{or} \quad \begin{cases} \phi_{13} = -\zeta - \alpha_{13} + \pi, \\ \phi_{24} = \zeta + \alpha_{24} - \pi. \end{cases} \quad (26)$$

For the pair $(O_{1-}^a, O_{2-}^a) = (F, H)$,

$$\begin{aligned} O_{1-}^a &= B_{13} \sin \phi_{13} - B_{24} \sin \phi_{24}, \\ O_{2-}^a &= B_{13} \cos \phi_{13} - B_{24} \cos \phi_{24}, \end{aligned} \quad (27)$$

the results can be readily obtained by simply changing the sign of ϕ_{24} everywhere in the results of the previous case of (O_{1+}^a, O_{2-}^a) . We obtain

$$\begin{cases} \phi_{13} = -\zeta + \alpha_{13}, \\ \phi_{24} = -\zeta + \alpha_{24}, \end{cases} \quad \text{or} \quad \begin{cases} \phi_{13} = -\zeta - \alpha_{13} + \pi, \\ \phi_{24} = -\zeta - \alpha_{24} + \pi. \end{cases} \quad (28)$$

For the pair $(O_{1-}^a, O_{2+}^a) = (F, E)$,

$$\begin{aligned} O_{1-}^a &= B_{13} \sin \phi_{13} - B_{24} \sin \phi_{24}, \\ O_{2+}^a &= B_{13} \cos \phi_{13} + B_{24} \cos \phi_{24}, \end{aligned} \quad (29)$$

the only change from the previous case of (O_{1+}^a, O_{2-}^a) , is in the sign of B_{24} . Thus, we can simply follow the steps of the derivation for the case of (O_{1+}^a, O_{2-}^a) , making there the replacement $B_{24} \rightarrow -B_{24}$. This leads to the change in the constraint given by Eq. (25) to

$$\phi_{13} + \phi_{24} = \pm(\alpha_{13} - \alpha_{24} + \pi). \quad (30)$$

Thus, we obtain the twofold ambiguity

$$\begin{cases} \phi_{13} = -\zeta - \alpha_{13} + \pi, \\ \phi_{24} = \zeta + \alpha_{24}, \end{cases} \quad \text{or} \quad \begin{cases} \phi_{13} = -\zeta + \alpha_{13}, \\ \phi_{24} = \zeta - \alpha_{24} + \pi. \end{cases} \quad (31)$$

For the pair $(O_{1+}^a, O_{2+}^a) = (-G, E)$,

$$\begin{aligned} O_{1+}^a &= B_{13} \sin \phi_{13} + B_{24} \sin \phi_{24}, \\ O_{2+}^a &= B_{13} \cos \phi_{13} + B_{24} \cos \phi_{24}, \end{aligned} \quad (32)$$

we simply flip the sign of ϕ_{24} in Eq. (31). We have

$$\begin{cases} \phi_{13} = -\zeta - \alpha_{13} + \pi, \\ \phi_{24} = -\zeta - \alpha_{24}, \end{cases} \quad \text{or} \quad \begin{cases} \phi_{13} = -\zeta + \alpha_{13}, \\ \phi_{24} = -\zeta + \alpha_{24} - \pi. \end{cases} \quad (33)$$

To avoid any confusion, we emphasize that, in all the cases discussed above, $(O_{1\pm}^a, O_{2\pm}^a)$ (with the signs \pm being independent), the phases α_{13} and α_{24} are uniquely defined and given by Eq. (22).

From the preceding considerations in this section, we conclude that

- (i) Any pair of observables of the form (O_{1+}^m, O_{1-}^m) leads to a fourfold phase ambiguity of the form given by Eq. (9), while any pair of the form (O_{2+}^m, O_{2-}^m) leads to a fourfold ambiguity of the form given by Eq. (11). These result in [in view of the consistency relations given by Eq. (41) that shall be used later on to help

resolve the phase ambiguity]

$$(O_{1+}^a, O_{1-}^a) : \begin{cases} \phi_{13}^+ - \phi_{24}^+ = (\alpha_{13} - \alpha_{24}), \\ \phi_{13}^+ - \phi_{24}^- = [(\alpha_{13} + \alpha_{24}) - \pi], \\ \phi_{13}^- - \phi_{24}^+ = -[(\alpha_{13} + \alpha_{24}) - \pi], \\ \phi_{13}^- - \phi_{24}^- = -(\alpha_{13} - \alpha_{24}), \\ \phi_{13}^+ + \phi_{24}^+ = (\alpha_{13} + \alpha_{24}), \\ \phi_{13}^+ + \phi_{24}^- = (\alpha_{13} - \alpha_{24}) + \pi, \\ \phi_{13}^- + \phi_{24}^+ = -(\alpha_{13} - \alpha_{24}) + \pi, \\ \phi_{13}^- + \phi_{24}^- = -(\alpha_{13} + \alpha_{24}), \end{cases} \quad (34)$$

and

$$(O_{2+}^a, O_{2-}^a) : \begin{cases} \phi_{13}^+ - \phi_{24}^+ = (\alpha_{13} - \alpha_{24}), \\ \phi_{13}^+ - \phi_{24}^- = (\alpha_{13} + \alpha_{24}), \\ \phi_{13}^- - \phi_{24}^+ = -(\alpha_{13} + \alpha_{24}), \\ \phi_{13}^- - \phi_{24}^- = -(\alpha_{13} - \alpha_{24}), \\ \phi_{13}^+ + \phi_{24}^+ = (\alpha_{13} + \alpha_{24}), \\ \phi_{13}^+ + \phi_{24}^- = (\alpha_{13} - \alpha_{24}), \\ \phi_{13}^- + \phi_{24}^+ = -(\alpha_{13} - \alpha_{24}), \\ \phi_{13}^- + \phi_{24}^- = -(\alpha_{13} + \alpha_{24}), \end{cases} \quad (35)$$

- (ii) Any pair of observables of the form $(O_{1\pm}^m, O_{2\mp}^m) = (O_{1+}^m, O_{2-}^m)$ or (O_{1-}^m, O_{2+}^m) leads to a twofold ambiguity of the form given by Eqs. (26) and (31), while any pair of the form $(O_{1\nu}^m, O_{2\nu}^m)$ leads to a twofold ambiguity of the form given by Eqs. (28) and (33). These result in [recall that (relative) phases are modulo 2π]

$$(O_{1-}^a, O_{2-}^a) : \begin{cases} \phi_{13}^\lambda - \phi_{24}^\lambda = \lambda(\alpha_{13} - \alpha_{24}), \\ \phi_{13}^\lambda + \phi_{24}^\lambda = -2\zeta + \lambda(\alpha_{13} + \alpha_{24}), \end{cases} \quad (36)$$

$$(O_{1+}^a, O_{2-}^a) : \begin{cases} \phi_{13}^\lambda - \phi_{24}^\lambda = -2\zeta + \lambda(\alpha_{13} + \alpha_{24}), \\ \phi_{13}^\lambda + \phi_{24}^\lambda = \lambda(\alpha_{13} - \alpha_{24}), \end{cases} \quad (37)$$

with $\lambda = \pm$, and

$$(O_{1+}^a, O_{2+}^a) : \begin{cases} \phi_{13}^+ - \phi_{24}^- = (\alpha_{13} - \alpha_{24}) + \pi, \\ \phi_{13}^- - \phi_{24}^+ = -(\alpha_{13} - \alpha_{24}) + \pi, \\ \phi_{13}^+ + \phi_{24}^- = -2\zeta + (\alpha_{13} + \alpha_{24}) - \pi, \\ \phi_{13}^- + \phi_{24}^+ = -2\zeta - (\alpha_{13} + \alpha_{24}) + \pi, \end{cases} \quad (38)$$

$$(O_{1-}^a, O_{2+}^a) : \begin{cases} \phi_{13}^+ - \phi_{24}^- = -2\zeta + (\alpha_{13} + \alpha_{24}) - \pi, \\ \phi_{13}^- - \phi_{24}^+ = -2\zeta - (\alpha_{13} + \alpha_{24}) + \pi, \\ \phi_{13}^+ + \phi_{24}^- = -(\alpha_{13} - \alpha_{24}) + \pi, \\ \phi_{13}^- + \phi_{24}^+ = (\alpha_{13} - \alpha_{24}) + \pi. \end{cases} \quad (39)$$

From the results obtained above for the pairs of observables $(O_{n\nu}^a, O_{n'\nu'}^a)$ [$n, n' = 1, 2$ and $\nu, \nu' = \pm$ with $(n\nu) \neq (n'\nu')$] in group $a = BT$ [cf. Eq. (3)], it is straightforward to obtain the corresponding results for the pairs of observables in other two groups $b = BR$ and $c = TR$ [cf. Eqs. (4) and (5)]. All one has to do is to replace $(O_{n\nu}^a, O_{n'\nu'}^a)$ by $(O_{n\nu}^m, O_{n'\nu'}^m)$ ($m = b, c$) and

the relative phases ϕ_{13} and ϕ_{24} , respectively, by ϕ_{14} and ϕ_{23} in the case $m = b$ or by ϕ_{12} and ϕ_{34} in the case $m = c$.

The discrete ambiguities exhibited by the relative phases so far in this section [cf. Eqs. (34)–(39)] cannot be resolved without further constraint. This is provided by the property obeyed by the relative phases ($\phi_{ij} \equiv \phi_i - \phi_j$):²

$$\phi_{12} + \phi_{23} + \phi_{34} = \phi_{14}. \quad (40)$$

Here, it should be emphasized that this relation is satisfied up to addition of multiples of 2π , because phases are meaningful only modulo 2π . We refer to the above relation as the *consistency* relation, because it is going to be used to check on the “consistency” among the relative phases with discrete ambiguities, as we have shown in our considerations up to this point. As the reader shall see, the consistency relation allows one to resolve the discrete ambiguities for certain sets of four chosen observables. Equation (40) can be rewritten as

$$\phi_{24} - \phi_{13} = \phi_{34} - \phi_{12} \quad (a \longleftrightarrow c), \quad (41a)$$

$$\phi_{24} + \phi_{13} = \phi_{14} + \phi_{23} \quad (a \longleftrightarrow b), \quad (41b)$$

$$\phi_{34} + \phi_{12} = \phi_{14} - \phi_{23} \quad (c \longleftrightarrow b). \quad (41c)$$

The first relation in the above equations is used to relate the observables in group $a = \mathcal{BT}$ to those in group $c = \mathcal{TR}$, while the second relation connects the observables in group a to those in group $b = \mathcal{BR}$. The third relation connects the observables in group b to those in group c . Note that, apart from an irrelevant overall factor, Eq. (40) leads to a unique relation which connects the relative phases belonging to two specific groups of observables as exhibited in Eq. (41). Equation (41) has been also used by the authors of Refs. [3,4] in their analyses.

The logic for determining whether a given set of four observables can or cannot resolve the phase ambiguity is as follows. From the chosen set of four observables, using the appropriate consistency relation in Eq. (41), form all possible solutions due to the discrete ambiguities of the relative phases which, for the $(2+2)$ case, are given by Eqs. (34)–(39). Then, check if these solutions are linearly independent (nondegenerate) or dependent (degenerate). If there is no degeneracy in the possible solutions (i.e., they are *all* linearly independent), then, only one of them will be satisfied, in general, once the set of unique values of the phases α_{ij} and $\zeta (= \zeta_{nn',n''n'''})$ is provided by the measurements of the four observables in consideration.³ The precise relation of each α_{ij} to the corresponding ϕ_{ij} is known once the correct solution among the possible solutions is identified, thus resolving the ambiguity of ϕ_{ij} . Hence, this set of four observables resolves the phase ambiguity. If the degeneracy occurs among the

possible solutions, then, this set of observables cannot resolve the ambiguity. The logic just described applies to all cases (1)–(4) specified at the end of the previous section. Only the discrete ambiguities of the relative phases are case dependent, as shown later in Secs. IV and V.

It should be clear from the above consideration that, whether a set of four observables resolves the phase ambiguity or not rests on the linear independence of the possible solutions provided by the consistency relation [cf. Eqs. (41)] for that set of four observables.

We are now prepared to identify the possible sets of four double-spin observables that resolve the phase ambiguity of the transversity amplitude in the $(2+2)$ case defined in item (1) of the preceding section. There are three basic combinations of the pairs of observables to be considered:

- (aa) two pairs from item (i) above with $4 \times 4 = 16$ -fold phase ambiguity: (O_{n+}^m, O_{n-}^m) and $(O_{n'+}^{m'}, O_{n'-}^{m'})$ with $m \neq m'$.
- (bb) two pairs from item (ii) above with $2 \times 2 = 4$ -fold phase ambiguity: $(O_{1\nu}^m, O_{2\nu'}^m)$ and $(O_{1\mu}^{m'}, O_{2\mu'}^{m'})$ with $m \neq m'$.
- (ab) one pair from item (i) and one pair from item (ii) with $4 \times 2 = 8$ -fold phase ambiguity: (O_{n+}^m, O_{n-}^m) and $(O_{1\mu}^{m'}, O_{2\mu'}^{m'})$ with $m \neq m'$.

A. Case (aa)

First, consider case (aa). To be concrete, choose the set of pairs $[(O_{2+}^a, O_{2-}^a), (O_{2+}^c, O_{2-}^c)]$. From Eqs. (3) and (4), the observables in group a contain relative phases ϕ_{13} and ϕ_{24} , while those in group c contain relative phases ϕ_{12} and ϕ_{34} . Then, using Eq. (41a), we have

$$\phi_{13}^\lambda - \phi_{24}^{\lambda'} = \phi_{12}^{\lambda''} - \phi_{34}^{\lambda'''}, \quad (42)$$

where the indices on which these relative phases depend have been written explicitly. Inserting the corresponding fourfold phase ambiguity given by Eq. (35) into the above relation, we end up with 16 possible solutions:

$$\pm\alpha_{13} \pm \alpha_{24} = \pm\alpha_{12} \pm \alpha_{34}, \quad (43)$$

where all four signs \pm are independent. The 16 possible solutions given above are not all linearly independent. For example, consider the solution $\alpha_{13} + \alpha_{24} = \alpha_{12} + \alpha_{34}$ corresponding to $(\lambda, \lambda', \lambda'', \lambda''') = (+, -, +, -)$ in Eq. (42). This solution is degenerate, with the solution $-(\alpha_{13} + \alpha_{24}) = -(\alpha_{12} + \alpha_{34})$ corresponding to $(\lambda, \lambda', \lambda'', \lambda''') = (-, +, -, +)$. Hence, the phase ambiguity cannot be resolved in this case. It is also straightforward to see that none of the other combinations of the pairs of observables in case (aa) resolve the ambiguity. This includes the corresponding sets of pairs of observables from group a and group b and from b and c , in which cases we use the consistency relations given by Eqs. (41b) and (41c), respectively.

B. Case (bb)

For case (bb), we start by considering the set of two pairs $[(O_{1+}^a, O_{2-}^a), (O_{1-}^c, O_{2-}^c)]$. From Eqs. (3) and (4), the relative phases involved for this combination are (ϕ_{13}, ϕ_{24}) and

²Equation (40) may be seen as a direct consequence of the fact that a complex number can be represented by a vector in the complex plane and that the sum of all angles between neighboring vectors in a given set of vectors is 2π (or zero since phases are modulo 2π).

³Recall that the unpolarized cross section and single-spin observables are assumed to be measured. They fix the magnitudes of the four basic transversity amplitudes which enter in the determination of α_{ij} 's [cf. Eq. (22)].

(ϕ_{12}, ϕ_{34}) . Then, inserting Eqs. (37) and (36) into Eq. (41a) yields the following four possible solutions:

$$\begin{aligned} -2\zeta + (\alpha_{13} + \alpha_{24}) &= (\alpha_{12} - \alpha_{34}), \\ -2\zeta + (\alpha_{13} + \alpha_{24}) &= -(\alpha_{12} - \alpha_{34}), \\ -2\zeta - (\alpha_{13} + \alpha_{24}) &= (\alpha_{12} - \alpha_{34}), \\ -2\zeta - (\alpha_{13} + \alpha_{24}) &= -(\alpha_{12} - \alpha_{34}). \end{aligned} \quad (44)$$

Since the above possible solutions are all linearly independent, there will be only one solution satisfied, in general, for the set of unique values of α_{13} , α_{24} , α_{12} , α_{34} and $\zeta (= \zeta_{1+,2-}^a)$, once they are extracted from the measurements of the four observables in question. The correct solution, then, will tell us the exact relation of each α_{ij} ($ij = 13, 24, 12, 34$) to the corresponding ϕ_{ij} , resolving the ambiguity of ϕ_{ij} . Hence this set of four observables will resolve the phase ambiguity.

Consider now the set of pairs $[(O_{1+}^a, O_{2-}^a), (O_{1+}^c, O_{2-}^c)]$. Again, with the help of Eq. (37), Eq. (41a) leads to

$$\begin{aligned} -2\zeta + (\alpha_{13} + \alpha_{24}) &= -2\zeta' + (\alpha_{12} + \alpha_{34}), \\ -2\zeta + (\alpha_{13} + \alpha_{24}) &= -2\zeta' - (\alpha_{12} + \alpha_{34}), \\ -2\zeta - (\alpha_{13} + \alpha_{24}) &= -2\zeta' + (\alpha_{12} + \alpha_{34}), \\ -2\zeta - (\alpha_{13} + \alpha_{24}) &= -2\zeta' - (\alpha_{12} + \alpha_{34}). \end{aligned} \quad (45)$$

$$\begin{aligned} (O_{1\pm}^m, O_{2\mp}^m) &= (O_{1+}^m, O_{2-}^m) \text{ or } (O_{1-}^m, O_{2+}^m) \longrightarrow \begin{cases} \phi_{ij} - \phi_{kl} \longrightarrow \zeta \text{ dependent,} \\ \phi_{ij} + \phi_{kl} \longrightarrow \zeta \text{ independent,} \end{cases} \\ (O_{1\pm}^m, O_{2\pm}^m) &= (O_{1+}^m, O_{2+}^m) \text{ or } (O_{1-}^m, O_{2-}^m) \longrightarrow \begin{cases} \phi_{ij} - \phi_{kl} \longrightarrow \zeta \text{ independent,} \\ \phi_{ij} + \phi_{kl} \longrightarrow \zeta \text{ dependent.} \end{cases} \end{aligned} \quad (47)$$

Then, from the pattern exhibited by the above three sets of four observables worked out explicitly and with the help of Eq. (47), we can easily determine those sets of two pairs of observables for case (bb) that cannot resolve the phase ambiguity. They are the sets which yield the phase relations in Eq. (41) being ζ independent. All the other sets do resolve the ambiguity. The results are displayed in Table I.

It should be noted, however, that there is a restriction to the fact that those sets of two pairs of observables can resolve the phase ambiguity. For example, for the set $[(O_{1+}^a, O_{2-}^a), (O_{1-}^c, O_{2-}^c)]$, from Eqs. (44) and (45), it is clear that when $\alpha_{13} = -\alpha_{24}$ and/or $\alpha_{12} = \alpha_{34}$ no ambiguity can be resolved since the possible solutions become degenerate. The same is true for the set $[(O_{1+}^a, O_{2-}^a), (O_{1+}^c, O_{2-}^c)]$ when $\alpha_{13} = -\alpha_{24}$ and/or $\alpha_{12} = -\alpha_{34}$. It is easy to see that, had we considered the set $[(O_{1+}^a, O_{2+}^a), (O_{1+}^c, O_{2+}^c)]$ instead, we would have found that when $\alpha_{13} = \alpha_{24}$ and/or $\alpha_{12} = \alpha_{34}$ no phase ambiguity can be resolved [cf. Eqs. (38) and (41a)]. Thus, in these situations, we need to measure one or two more extra observables to be able to resolve the phase ambiguity. For example, for the set of two pairs of observables $[(O_{1+}^a, O_{2-}^a), (O_{1-}^c, O_{2-}^c)]$, we require the extra observable O_{1-}^a to resolve the ambiguity in the case $\alpha_{13} = -\alpha_{24}$ and the extra observable O_{1+}^c in the case $\alpha_{12} = \alpha_{34}$. If $\alpha_{13} = -\alpha_{24}$ and $\alpha_{12} = \alpha_{34}$, simultaneously, then we require both extra

Note that ζ is distinct from ζ' [cf. Eq. (14)]. As in the previous case just discussed above, since the four possible solutions here are all linearly independent, the same reasoning as in the previous case applies, and we conclude that this set of four observables also resolves the phase ambiguity.

Now, take the set $[(O_{1-}^a, O_{2-}^a), (O_{1-}^c, O_{2-}^c)]$. In this case, we obtain the following results:

$$\begin{aligned} (\alpha_{13} - \alpha_{24}) &= (\alpha_{12} - \alpha_{34}), \\ (\alpha_{13} - \alpha_{24}) &= -(\alpha_{12} - \alpha_{34}), \\ -(\alpha_{13} - \alpha_{24}) &= (\alpha_{12} - \alpha_{34}), \\ -(\alpha_{13} - \alpha_{24}) &= -(\alpha_{12} - \alpha_{34}), \end{aligned} \quad (46)$$

and we see that this set of observables cannot resolve the phase ambiguity, since there are degenerate (or linearly dependent) solutions (first and fourth solutions and second and third solutions).

Now, from Eqs. (36)–(39), we note that the two relative phases, ϕ_{ij} and ϕ_{kl} , involved in a given pair of observables from the same group, have the following properties ($m = a, b, c$):

observables O_{1-}^a and O_{1+}^c . Note that O_{1+}^a differs by a sign of relative phase ϕ_{24} from O_{1-}^a . This later feature is true for all the observables of the form $O_{1\nu}^m$. Thus, for the sets of two pairs of the form $[(O_{1\pm}^a, O_{2\nu}^a), (O_{1\pm}^c, O_{2\nu}^c)]$, we need the extra observable $O_{1\mp}^a$ and/or $O_{1\mp}^c$ (here the \pm signs are not independent) to completely resolve the phase ambiguity, depending on whether $\alpha_{13} = \pm\alpha_{24}$ and/or $\alpha_{12} = \pm\alpha_{34}$. This means that we need a minimum of five or six chosen observables, instead of four, to resolve the phase ambiguity in these situations of equal magnitudes of the relative phases α_{ij} . It is straightforward to extended the above considerations to other sets of two pairs of observables involving groups a and b , and groups b and c . The results are given in Table I. Explicitly, the equal-relative-phase-magnitudes relations for the sets of two pairs of observables, in general, are

$$\begin{aligned} |\alpha_{13}| &= |\alpha_{24}| \text{ and/or } |\alpha_{12}| = |\alpha_{34}| \quad (a \longleftrightarrow c), \\ |\alpha_{13}| &= |\alpha_{24}| \text{ and/or } |\alpha_{14}| = |\alpha_{23}| \quad (a \longleftrightarrow b), \\ |\alpha_{12}| &= |\alpha_{34}| \text{ and/or } |\alpha_{14}| = |\alpha_{23}| \quad (c \longleftrightarrow b). \end{aligned} \quad (48)$$

Even with the additional observables as discussed above, the ambiguity still will not be resolved if $\alpha_{13} = \alpha_{24} = 0$ and/or $\alpha_{12} = \alpha_{34} = 0$. The only way to resolve the phase ambiguity in this case is to measure a set of eight chosen double-spin observables to determine both $\cos \phi_{ij}$ and $\sin \phi_{ij}$

TABLE I. Sets of two pairs of double-spin observables for case (bb) mentioned in the text. \checkmark = do resolve. \times = do not resolve. X = do not resolve. Observables indicated outside the parentheses are the additional ones required in case the equal-relative-phase-magnitudes condition, as given by Eq. (48), is met for the pairs of observables (in parentheses) that do resolve the phase ambiguity otherwise.

$(O_{1+}^b, O_{2+}^b), O_{1-}^b$ $(O_z, C_z), C_x$	$(O_{1+}^b, O_{2+}^b), O_{1-}^b$ $(O_z, O_x), C_x$	$(O_{1-}^b, O_{2+}^b), O_{1+}^b$ $(C_x, C_z), O_z$	$(O_{1-}^b, O_{2-}^b), O_{1+}^b$ $(C_x, O_x), O_z$	$(O_{1+}^c, O_{2+}^c), O_{1-}^c$ $(L_x, L_z), T_z$	$(O_{1+}^c, O_{2-}^c), O_{1-}^c$ $(L_x, T_x), T_z$	$(O_{1-}^c, O_{2+}^c), O_{1+}^c$ $(T_z, L_z), L_x$	$(O_{1-}^c, O_{2-}^c), O_{1+}^c$ $(T_x, T_z), L_x$
$(O_{1+}^g, O_{2+}^g), O_{1-}^g$ $(G, E), F$	\checkmark	\checkmark	\checkmark	\times	\checkmark	\checkmark	\times
$(O_{1+}^g, O_{2-}^g), O_{1-}^g$ $(G, H), F$	\checkmark	\times	\times	\checkmark	\checkmark	\checkmark	\checkmark
$(O_{1-}^g, O_{2+}^g), O_{1+}^g$ $(F, E), G$	\checkmark	\times	\times	\checkmark	\checkmark	\checkmark	\checkmark
$(O_{1-}^g, O_{2-}^g), O_{1+}^g$ $(F, H), F$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\times
$(O_{1+}^c, O_{2+}^c), O_{1-}^c$ $(L_x, L_z), T_z$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark
$(O_{1+}^c, O_{2-}^c), O_{1-}^c$ $(L_x, T_x), T_z$	\times	\checkmark	\checkmark	\times	\checkmark	\checkmark	\checkmark
$(O_{1-}^c, O_{2+}^c), O_{1+}^c$ $(T_z, L_z), L_x$	\times	\checkmark	\checkmark	\times	\checkmark	\checkmark	\checkmark
$(O_{1-}^c, O_{2-}^c), O_{1+}^c$ $(T_x, T_z), L_x$	\checkmark	\checkmark	\checkmark	\times	\checkmark	\checkmark	\checkmark

TABLE II. Sets of two pairs of double-spin observables for case (ab) mentioned in the text. \surd = do resolve. X = do not resolve. Observables indicated outside the parentheses are the additional ones required in case the equal-relative-phase-magnitudes condition, as given by Eq. (48), is met for the pairs of observables (in parentheses) that do resolve the phase ambiguity otherwise. The additional observable required is either one of the observables indicated for each pair, except for those indicated with **, which require two additional observables.

$(O_{1+}^a, O_{1-}^a), O_{2\pm}^a$ (G, F), E/H	$(O_{2+}^a, O_{2-}^a), O_{1\pm}^a$ (E, H), G/F	$(O_{1+}^b, O_{1-}^b), O_{2\pm}^b$ (O_z, C_x), C_z/O_x	$(O_{2+}^b, O_{2-}^b), O_{1\pm}^b$ (C_z, O_x), O_z/C_x	$(O_{1+}^c, O_{1-}^c), O_{2\pm}^c$ (L_x, T_z), L_z/T_x	$(O_{2+}^c, O_{2-}^c), O_{1\pm}^c$ (L_z, T_x), L_x/T_z
$(O_{1+}^a, O_{2+}^a), O_{1-}^a$ (G, E), F		\surd^{**}	\surd^{**}	X	X
$(O_{1+}^a, O_{2-}^a), O_{1-}^a$ (G, H), F		X	X	\surd	\surd
$(O_{1-}^a, O_{2+}^a), O_{1+}^a$ (F, E), G		X	X	\surd	\surd
$(O_{1-}^a, O_{2-}^a), O_{1+}^a$ (F, H), G		\surd^{**}	\surd^{**}	X	X
$(O_{1+}^b, O_{2+}^b), O_{1-}^b$ (O_z, C_z), C_x	\surd^{**}	\surd^{**}		X	X
$(O_{1+}^b, O_{2-}^b), O_{1-}^b$ (O_z, O_x), C_x	X	X		\surd^{**}	\surd^{**}
$(O_{1-}^b, O_{2+}^b), O_{1+}^b$ (C_x, C_z), O_z	X	X		\surd^{**}	\surd^{**}
$(O_{1-}^b, O_{2-}^b), O_{1+}^b$ (C_x, O_x), O_z	\surd^{**}	\surd^{**}		X	X
$(O_{1+}^c, O_{2+}^c), O_{1-}^c$ (L_x, L_z), T_z	X	X	\surd^{**}	\surd^{**}	
$(O_{1+}^c, O_{2-}^c), O_{1-}^c$ (L_x, T_x), T_z	\surd	\surd	X	X	
$(O_{1-}^c, O_{2+}^c), O_{1+}^c$ (T_z, L_z), L_x	\surd	\surd	X	X	
$(O_{1-}^c, O_{2-}^c), O_{1+}^c$ (T_z, T_x), L_x	X	X	\surd^{**}	\surd^{**}	

for all four relative phases ϕ_{ij} associated with the four basic photoproduction amplitudes.

C. Case (ab)

We now turn attention to case (ab). In this case, it is straightforward to see that any pair of double-spin observables belonging to item (ii) that leads to the corresponding phase relations as given by Eq. (41) being ζ dependent, resolves the phase ambiguity, irrespective of the pair of observables belonging to item (i). Otherwise the phase ambiguity cannot be resolved. The results are displayed in Table II.

Analogous to the previous case (bb), here we have also the restriction of no equal relative-phase magnitudes, $|\alpha_{ij}|$'s, for the sets of two pairs of double-spin observables, as given in Table II, to be able to resolve the phase ambiguity. This case involves the pairs of observables (O_{n+}^m, O_{n-}^m) ($n = 1, 2$), in addition to those encountered in case (bb).

In the case of $[(O_{1+}^a, O_{1-}^a), (O_{1+}^c, O_{2+}^c)]$, e.g., from Eqs. (34) and (41a), the extra observable required to resolve the phase ambiguity is either O_{2+}^a or O_{2-}^a when $|\alpha_{13}| = |\alpha_{24}|$. Note that the relevant new pair of observables to help resolve the phase ambiguity here is either (O_{1+}^a, O_{2+}^a) or (O_{1-}^a, O_{2-}^a)

[cf. Eqs. (38) and (36)]. When $|\alpha_{12}| = |\alpha_{34}|$, the extra observable required is O_{1-}^c as in case (bb).

Now consider the set $[(O_{1+}^a, O_{1-}^a), (O_{1+}^b, O_{2+}^b)]$. In this case, from Eqs. (34) and (41b), it requires both O_{2+}^a and O_{2-}^a , in addition, to resolve the phase ambiguity when $|\alpha_{13}| = |\alpha_{24}|$. And, as above, extra observable O_{1-}^b when $|\alpha_{12}| = |\alpha_{34}|$.

The set $[(O_{1+}^c, O_{1-}^c), (O_{1+}^b, O_{2+}^b)]$, from Eqs. (34) and (41c), requires both O_{2+}^c and O_{2-}^c , in addition, to resolve the phase ambiguity when $|\alpha_{13}| = |\alpha_{24}|$, and O_{1-}^b , in addition, when $|\alpha_{12}| = |\alpha_{34}|$.

The two pairs of observables involving (O_{2+}^a, O_{2-}^a) , from Eqs. (35) and (41), always require both O_{1+}^a and O_{1-}^a , in addition, to resolve the phase ambiguity when $|\alpha_{13}| = |\alpha_{24}|$, irrespective of the other pair of observables from item (ii). The latter requires one extra observable when the corresponding relative phases have equal magnitudes.

We therefore see that in case (ab) the minimum number of double-spin observables required to resolve the phase ambiguity—when the magnitudes of the relative phases α_{ij} are equal—can be five, six or seven depending on the set of two pairs of observables that, otherwise, resolves the phase ambiguity. Based on the above considerations, the additional observables required to resolve the phase ambiguity are indicated in Table II.

IV. PHASE FIXING FOR THE (2 + 1 + 1) CASE

We start by considering two observables from a given group. For the sake of concreteness, consider the pair $(O_{1+}^a, O_{1-}^a) = (-G, F)$. This pair of observables was examined in the previous section, with the phase ambiguity given in Eqs. (8) and (9). Note that these two observables determine $\sin \phi_{13}$ and $\sin \phi_{24}$ [cf. Eq. (8)]:

$$\sin \phi_{13} = \frac{O_{1+}^a + O_{1-}^a}{2B_{13}}, \quad \sin \phi_{24} = \frac{O_{1+}^a - O_{1-}^a}{2B_{24}}. \quad (49)$$

Appropriate combination of ϕ_{24}^λ and $\phi_{13}^{\lambda'}$ results in [cf. Eq. (34)]

$$(O_{1+}^a, O_{1-}^a) : \begin{cases} \phi_{24}^+ - \phi_{13}^+ = (\alpha_{24} - \alpha_{13}), \\ \phi_{24}^+ - \phi_{13}^- = (\alpha_{24} + \alpha_{13}) - \pi, \\ \phi_{24}^- - \phi_{13}^+ = -(\alpha_{24} + \alpha_{13}) + \pi, \\ \phi_{24}^- - \phi_{13}^- = -(\alpha_{24} - \alpha_{13}). \end{cases} \quad (50)$$

Now we consider two observables from the remaining two groups, $b = \mathcal{BR}$ and $c = \mathcal{TR}$. For a given observable in one of these two groups, say $c = \mathcal{TR}$, there will be four possible combinations of the pairs of observables one can form involving another observable from group $b = \mathcal{BR}$ [cf. Eqs. (4) and (5)]. For example, for the observable O_{1+}^c , we have the combinations (O_{1-}^b, O_{1+}^c) , (O_{1+}^b, O_{1+}^c) , (O_{2-}^b, O_{1+}^c) , and (O_{2+}^b, O_{1+}^c) .

A. $(O_{1\pm}^b, O_{1\pm}^c)$

We start by considering the pair $(O_{1-}^b, O_{1+}^c) = (-C_x, -L_x)$. From Eqs. (4) and (5),

$$\begin{aligned} O_{1-}^b &= B_{14} \sin \phi_{14} - B_{23} \sin \phi_{23}, \\ O_{1+}^c &= B_{12} \sin \phi_{12} + B_{34} \sin \phi_{34}. \end{aligned} \quad (51)$$

Expressing ϕ_{14} and ϕ_{23} as

$$\begin{aligned} \phi_{14} &= \phi_{24} + \phi_{12}, \\ \phi_{23} &= \phi_{13} - \phi_{12}, \end{aligned} \quad (52)$$

we have

$$O_{1-}^b = A_c \sin \phi_{12} + A_s \cos \phi_{12}, \quad (53)$$

with

$$\begin{aligned} A_c &\equiv B_{14} \cos \phi_{24} + B_{23} \cos \phi_{13}, \\ A_s &\equiv B_{14} \sin \phi_{24} - B_{23} \sin \phi_{13}. \end{aligned} \quad (54)$$

Using $\cos \phi_{ij} = \pm \sqrt{1 - \sin^2 \phi_{ij}}$, we solve Eq. (53) for $\sin \phi_{12}$ to obtain

$$\sin \phi_{12} = \frac{A_c O_{1-}^b \pm A_s \sqrt{D^2 - (O_{1-}^b)^2}}{D^2}, \quad (55)$$

with

$$D^2 \equiv A_c^2 + A_s^2 = B_{14}^2 + B_{23}^2 + 2B_{14}B_{23} \cos(\phi_{24} + \phi_{13}). \quad (56)$$

We now note that, while A_s is uniquely determined [cf. Eq. (49)], A_c has a fourfold ambiguity because knowing only

$\sin \phi_{ij}$ implies that $\cos \phi_{ij}$ is known up to a sign. In particular, according to the notation of (9),

$$\text{knowing } \sin \phi_{ij}^\lambda \implies \cos \phi_{ij}^\lambda = \lambda \cos \alpha_{ij}. \quad (57)$$

Since A_c depends on $\cos \phi_{24}^\lambda$ and $\cos \phi_{13}^{\lambda'}$ [cf. Eq. (54)], we introduce the notations $A_c^{\lambda\lambda'}$ and $D^{\lambda\lambda'2}$, such that

$$\begin{aligned} A_c^{\lambda\lambda'} &= B_{14} \cos \phi_{24}^\lambda + B_{23} \cos \phi_{13}^{\lambda'}, \\ D^{\lambda\lambda'2} &= B_{14}^2 + B_{23}^2 + 2B_{14}B_{23} \cos(\phi_{24}^\lambda + \phi_{13}^{\lambda'}). \end{aligned} \quad (58)$$

and, from Eq. (55), we see that ϕ_{12} , in turn, depends on λ and λ' , i.e.,

$$\sin \phi_{12}^{\lambda\lambda'}(\eta) = \frac{A_c^{\lambda\lambda'} O_{1-}^b + \eta A_s \sqrt{D^{\lambda\lambda'2} - (O_{1-}^b)^2}}{D^{\lambda\lambda'2}}, \quad (59)$$

where η takes the values ± 1 .

Due to Eq. (57), it is clear that

$$\begin{aligned} A_c^{++} &= -A_c^{--} \quad \text{and} \quad A_c^{+-} = -A_c^{-+}, \\ D^{++2} &= D^{--2} \quad \text{and} \quad D^{+-2} = D^{-+2}. \end{aligned} \quad (60)$$

Then, we have

$$\begin{aligned} \sin \phi_{12}^{++}(\eta) &= \frac{A_c^{++} O_{1-}^b + \eta A_s \sqrt{D^{++2} - (O_{1-}^b)^2}}{D^{++2}}, \\ \sin \phi_{12}^{+-}(\eta) &= \frac{A_c^{+-} O_{1-}^b + \eta A_s \sqrt{D^{+-2} - (O_{1-}^b)^2}}{D^{+-2}}, \\ \sin \phi_{12}^{-+}(\eta) &= \frac{-A_c^{-+} O_{1-}^b + \eta A_s \sqrt{D^{-+2} - (O_{1-}^b)^2}}{D^{-+2}}, \\ \sin \phi_{12}^{--}(\eta) &= \frac{-A_c^{--} O_{1-}^b + \eta A_s \sqrt{D^{--2} - (O_{1-}^b)^2}}{D^{--2}}. \end{aligned} \quad (61)$$

From the above results, we see that there are, in general, eight possible $\sin \phi_{12}^{\lambda\lambda'}(\eta)$'s (recall that λ , λ' , and η take two possible values each), and each of them leads to a twofold ambiguity

$$\phi_{12}^{\lambda\lambda'}(\eta) = \begin{cases} \alpha_{12}^{\lambda\lambda'}(\eta), \\ \pi - \alpha_{12}^{\lambda\lambda'}(\eta). \end{cases} \quad (62)$$

An inspection of Eq. (61) reveals that

$$\begin{aligned} \sin \phi_{12}^{++}(\pm) &= -\sin \phi_{12}^{--}(\mp), \\ \sin \phi_{12}^{+-}(\pm) &= -\sin \phi_{12}^{-+}(\mp), \end{aligned} \quad (63)$$

and, consequently,

$$\alpha_{12}^{++}(\pm) = -\alpha_{12}^{--}(\mp) \quad \text{and} \quad \alpha_{12}^{+-}(\pm) = -\alpha_{12}^{-+}(\mp). \quad (64)$$

Note that since all $\sin \phi_{12}^{\lambda\lambda'}(\eta)$'s are distinct from each other, so are $\alpha_{12}^{\lambda\lambda'}(\eta)$'s.

Now, taking the equation for O_{1+}^c in (51) and solving for $\sin \phi_{34}$ yields

$$\sin \phi_{34}^{\lambda\lambda'}(\eta) = \frac{O_{1+}^c - B_{12} \sin \phi_{12}^{\lambda\lambda'}(\eta)}{B_{34}}, \quad (65)$$

where we have displayed all the indices of the relative phases ϕ_{12} and ϕ_{34} explicitly. The above result leads to the twofold

ambiguity

$$\phi_{34}^{\lambda\lambda'}(\eta) = \begin{cases} \alpha_{34}^{\lambda\lambda'}(\eta), \\ \pi - \alpha_{34}^{\lambda\lambda'}(\eta), \end{cases} \quad (66)$$

with all eight $\alpha_{34}^{\lambda\lambda'}(\eta)$'s being distinct from each other to the extent that $\sin \phi_{12}^{\lambda\lambda'}(\eta)$'s are. However, $\alpha_{34}^{\lambda\lambda'}(\eta)$ lacks the symmetry exhibited by $\alpha_{12}^{\lambda\lambda'}(\eta)$ in Eq. (64), i.e., $\alpha_{34}^{\lambda\lambda'}(\eta)$'s are not related to each other in general.

Appropriate combinations of the relative phases $\phi_{34}^{\lambda\lambda'}(\eta)$ and $\phi_{12}^{\lambda\lambda'}(\eta)$ involved in each pair contain, in general, a four-fold ambiguity of the form given by

$$\phi_{34}^{\lambda\lambda'}(\eta) - \phi_{12}^{\lambda\lambda'}(\eta) = \begin{cases} (\alpha_{34}^{\lambda\lambda'}(\eta) - \alpha_{12}^{\lambda\lambda'}(\eta)), \\ (\alpha_{34}^{\lambda\lambda'}(\eta) + \alpha_{12}^{\lambda\lambda'}(\eta)) - \pi, \\ -(\alpha_{34}^{\lambda\lambda'}(\eta) + \alpha_{12}^{\lambda\lambda'}(\eta)) + \pi, \\ -(\alpha_{34}^{\lambda\lambda'}(\eta) - \alpha_{12}^{\lambda\lambda'}(\eta)), \end{cases} \quad (67)$$

for a given set of $\{\lambda, \lambda', \eta\}$ (note that λ, λ' , and η take two possible values each).

At this stage, in analogy to what we have done in the $(2+2)$ case in the previous section, we invoke the consistency relation (40) reexpressed as [cf. Eq. (41a)]

$$\phi_{24}^{\lambda} - \phi_{13}^{\lambda'} = \phi_{34}^{\lambda\lambda'}(\eta) - \phi_{12}^{\lambda\lambda'}(\eta). \quad (68)$$

Inserting Eq. (67) into the above equation, we arrive at the possible solutions

$$\phi_{24}^{\lambda} - \phi_{13}^{\lambda'} = \begin{cases} (\alpha_{34}^{\lambda\lambda'}(\eta) - \alpha_{12}^{\lambda\lambda'}(\eta)), \\ (\alpha_{34}^{\lambda\lambda'}(\eta) + \alpha_{12}^{\lambda\lambda'}(\eta)) - \pi, \\ -(\alpha_{34}^{\lambda\lambda'}(\eta) + \alpha_{12}^{\lambda\lambda'}(\eta)) + \pi, \\ -(\alpha_{34}^{\lambda\lambda'}(\eta) - \alpha_{12}^{\lambda\lambda'}(\eta)), \end{cases} \quad (69)$$

for a given set of $\{\lambda, \lambda', \eta\}$. The left-hand side of the above equation is given by Eq. (50). Since λ, λ' , and η take two possible values each, we have $2 \times 2 = 4$ distinct combinations on the left-hand side of the above equation [cf. Eq. (50)] and, on the right-hand side, we have $4 \times 2 = 8$ distinct combinations. This is a total of $4 \times 8 = 32$ possible solutions. It happens that these 32 solutions are all linearly independent, i.e., there are no degenerate solutions among them. This follows from the fact that all $\sin \phi_{34}^{\lambda\lambda'}(\eta)$'s—and, in turn, all $\alpha_{34}^{\lambda\lambda'}(\eta)$'s—are distinct from each other as pointed out previously [see below Eq. (66)]. Thus, once the unique values of α_{13}, α_{24} , and the associated $\alpha_{12}^{\lambda\lambda'}(\eta)$ and $\alpha_{34}^{\lambda\lambda'}(\eta)$ are provided by the measurements of the observables $[(O_{1+}^a, O_{1+}^a), (O_{1-}^b, O_{1+}^c)]$, there will be only one solution satisfying the consistency relation (68). Therefore, we conclude that this set of observables will resolve the phase ambiguity.

It is clear that the preceding results for the pair of observables (O_{1-}^b, O_{1+}^c) actually hold for any of the pairs $(O_{1\pm}^b, O_{1\pm}^c)$, with the signs \pm being independent, since the only difference is the sign change of B_{23} and/or B_{34} according to the particular combination of the observables in the pair considered. These sign changes do not affect any of the properties exhibited by the phases $\alpha_{12}^{\lambda\lambda'}(\eta)$ and $\alpha_{34}^{\lambda\lambda'}(\eta)$. Thus, any one of the pairs of observables $(O_{1\pm}^b, O_{1\pm}^c)$, together with the pair (O_{1+}^a, O_{1-}^a) , can resolve the phase ambiguity of the transversity amplitude.

B. $(O_{2\pm}^b, O_{1\pm}^c)$

We now consider the pair $(O_{2-}^b, O_{1+}^c) = (-O_x, -L_x)$,

$$O_{2-}^b = B_{14} \cos \phi_{14} - B_{23} \cos \phi_{23},$$

$$O_{1+}^c = B_{12} \sin \phi_{12} + B_{34} \sin \phi_{34}. \quad (70)$$

In this case, inserting Eq. (52) into the expression for O_{2-}^b in the above equation, yields

$$O_{2-}^b = A_c \cos \phi_{12} - A_s \sin \phi_{12}, \quad (71)$$

with

$$A_c \equiv B_{14} \cos \phi_{24} - B_{23} \cos \phi_{13},$$

$$A_s \equiv B_{14} \sin \phi_{24} + B_{23} \sin \phi_{13}. \quad (72)$$

Solving Eq. (71) for $\sin \phi_{12}$, we have

$$\sin \phi_{12} = \frac{-O_{2-}^b A_s \pm A_c \sqrt{D^2 - (O_{2-}^b)^2}}{D^2}, \quad (73)$$

where

$$D^2 \equiv A_c^2 + A_s^2 = B_{14}^2 + B_{23}^2 - 2B_{14}B_{23} \cos(\phi_{24} + \phi_{13}). \quad (74)$$

Using the same notation introduced in Eq. (58), we write Eq. (73) as

$$\sin \phi_{12}^{\lambda\lambda'}(\eta) = \frac{-O_{2-}^b A_s + \eta A_c \sqrt{D^{\lambda\lambda'2} - (O_{2-}^b)^2}}{D^{\lambda\lambda'2}}. \quad (75)$$

Noticing that both $A_c^{\lambda\lambda'}$ and $D^{\lambda\lambda'2}$ here have the same symmetry as in Eq. (60), we can verify in this case that

$$\sin \phi_{12}^{++}(\pm) = \sin \phi_{12}^{--}(\mp),$$

$$\sin \phi_{12}^{+-}(\pm) = \sin \phi_{12}^{-+}(\mp), \quad (76)$$

and, consequently,

$$\alpha_{12}^{++}(\pm) = \alpha_{12}^{--}(\mp) \quad \text{and} \quad \alpha_{12}^{+-}(\pm) = \alpha_{12}^{-+}(\mp). \quad (77)$$

Also note that, for a given set of $\{\lambda, \lambda', \eta\}$, Eq. (75) leads to a twofold phase ambiguity as given by Eq. (62).

Solving now the equation for O_{1+}^c in (70) for $\sin \phi_{34}$, we have

$$\sin \phi_{34}^{\lambda\lambda'}(\eta) = \frac{O_{1+}^c - B_{12} \sin \phi_{12}^{\lambda\lambda'}(\eta)}{B_{34}}, \quad (78)$$

leading to a twofold phase ambiguity as given by Eq. (66). Here we note that, unlike in the case of the pair of observables (O_{1-}^b, O_{1+}^c) , where $\sin \phi_{34}^{\lambda\lambda'}(\eta)$ has no symmetry, this quantity given by Eq. (78) above exhibits the following symmetry:

$$\sin \phi_{34}^{++}(\pm) = \sin \phi_{34}^{--}(\mp),$$

$$\sin \phi_{34}^{+-}(\pm) = \sin \phi_{34}^{-+}(\mp), \quad (79)$$

where Eq. (76) has been used. Consequently,

$$\alpha_{34}^{++}(\pm) = \alpha_{34}^{--}(\mp) \quad \text{and} \quad \alpha_{34}^{+-}(\pm) = \alpha_{34}^{-+}(\mp). \quad (80)$$

The relative phases $\alpha_{12}^{\lambda\lambda'}(\eta)$ and $\alpha_{34}^{\lambda\lambda'}(\eta)$ derived here, with the symmetry properties given by Eqs. (77) and (80), should obey Eq. (69). It happens that the set of pairs $[(O_{2-}^b, O_{1+}^c), (O_{1+}^a, O_{1-}^a)]$ cannot resolve the phase ambiguity. To see this, it suffices to consider the following two particular

solutions from Eq. (69):

$$\begin{aligned}\phi_{24}^+ - \phi_{13}^+ &= \alpha_{34}^{++}(+) - \alpha_{12}^{++}(+) \implies (\alpha_{24} - \alpha_{13}) = [\alpha_{34}^{++}(+) - \alpha_{12}^{++}(+)], \\ \phi_{24}^- - \phi_{13}^- &= \alpha_{34}^{--}(-) - \alpha_{12}^{--}(-) \implies -(\alpha_{24} - \alpha_{13}) = -[\alpha_{34}^{++}(+) - \alpha_{12}^{++}(+)],\end{aligned}\quad (81)$$

where we have made use of Eqs. (50), (77), and (80). This shows that these solutions are linearly dependent (degenerate) and, consequently, the set of observables in consideration cannot resolve the phase ambiguity. Degeneracy of the solutions involving $\alpha_{ij}^{+-}(\pm)$ and $\alpha_{ij}^{-+}(\mp)$ also occurs.

The above consideration shows that any of the pairs of observables ($O_{2\pm}^b, O_{1\pm}^c$), together with the pair (O_{1+}^a, O_{1-}^a), cannot resolve the phase ambiguity of the transversity amplitude.

C. ($O_{2\pm}^b, O_{2\pm}^c$)

For (O_{2-}^b, O_{2+}^c) = ($-O_x, -L_z$),

$$\begin{aligned}O_{2-}^b &= B_{14} \cos \phi_{14} - B_{23} \cos \phi_{23}, \\ O_{2+}^c &= B_{12} \cos \phi_{12} + B_{34} \cos \phi_{34},\end{aligned}\quad (82)$$

proceeding analogously to the case of (O_{1-}^b, O_{1+}^c), we have

$$\cos \phi_{12}^{\lambda\lambda'}(\eta) = \frac{A_c^{\lambda\lambda'} O_{2-}^b + \eta A_s \sqrt{D^{\lambda\lambda'2} - (O_{2-}^b)^2}}{D^{\lambda\lambda'2}},\quad (83)$$

where

$$\begin{aligned}A_s &= B_{14} \sin \phi_{24} + B_{23} \sin \phi_{13}, \\ A_c^{\lambda\lambda'} &= B_{14} \cos \phi_{24}^\lambda - B_{23} \cos \phi_{13}^{\lambda'}, \\ D^{\lambda\lambda'2} &= B_{14}^2 + B_{23}^2 - 2B_{14}B_{23} \cos(\phi_{24}^\lambda + \phi_{13}^{\lambda'}).\end{aligned}\quad (84)$$

(O_{1+}^a, O_{1-}^a) and ($O_{n\pm}^b, O_{n\pm}^c$) ($n = 1, 2$) \rightarrow do resolve the ambiguity,

(O_{1+}^a, O_{1-}^a) and ($O_{2\pm}^b, O_{1\pm}^c$) \rightarrow do not resolve the ambiguity. (88)

In the above relations, the \pm signs are independent.

D. (O_{2+}^a, O_{2-}^a)

We now turn attention to the case of the pair of observables from group a being (O_{2+}^a, O_{2-}^a) = (E, H),

$$\begin{aligned}O_{2+}^a &= B_{13} \cos \phi_{13} + B_{24} \cos \phi_{24}, \\ O_{2-}^a &= B_{13} \cos \phi_{13} - B_{24} \cos \phi_{24}.\end{aligned}\quad (89)$$

The difference from the previous case of (O_{1+}^a, O_{1-}^a) is that (O_{2+}^a, O_{2-}^a) determines $\cos \phi_{24}$ and $\cos \phi_{13}$ uniquely, instead of $\sin \phi_{24}$ and $\sin \phi_{13}$. This implies that, for the pair (O_{1-}^b, O_{1+}^c), the quantity A_c defined in Eq. (54) becomes uniquely determined, while A_s will have a fourfold ambiguity and the quantity D^2 in Eq. (56) depends on $(\lambda\lambda')$, but remains unchanged otherwise, viz.,

$$\begin{aligned}A_c &= B_{14} \cos \phi_{24} + B_{23} \cos \phi_{13}, \\ A_s^{\lambda\lambda'} &= B_{14} \sin \phi_{24}^\lambda - B_{23} \sin \phi_{13}^{\lambda'}, \\ D^{\lambda\lambda'2} &= B_{14}^2 + B_{23}^2 + 2B_{14}B_{23} \cos(\phi_{24}^\lambda + \phi_{13}^{\lambda'}).\end{aligned}\quad (90)$$

It is clear that $\cos \phi_{12}^{\lambda\lambda'}(\eta)$ above exhibits the symmetry

$$\begin{aligned}\cos \phi_{12}^{++}(\pm) &= -\cos \phi_{12}^{--}(\mp), \\ \cos \phi_{12}^{+-}(\pm) &= -\cos \phi_{12}^{-+}(\mp),\end{aligned}\quad (85)$$

and, consequently,

$$\alpha_{12}^{++}(\pm) = \pi + \alpha_{12}^{--}(\mp) \quad \text{and} \quad \alpha_{12}^{+-}(\pm) = \pi + \alpha_{12}^{-+}(\mp).\quad (86)$$

Now, solving the equation for O_{2+}^c in (82) for $\cos \phi_{23}$ yields

$$\cos \phi_{34}^{\lambda\lambda'}(\eta) = \frac{O_{2+}^c - B_{12} \cos \phi_{12}^{\lambda\lambda'}(\eta)}{B_{34}},\quad (87)$$

which reveals that all eight possible values of it are distinct. Consequently, all $\alpha_{34}^{\lambda\lambda'}(\eta)$'s are distinct, resulting in linear independence of all possible solutions from the consistency relation (41a). Then, it follows that any pair of observables of the form ($O_{2\pm}^b, O_{2\pm}^c$) together with the pair (O_{1+}^a, O_{1-}^a) can resolve the phase ambiguity.

Summarizing the results obtained in this section so far, we have

Then, Eq. (59) changes to

$$\sin \phi_{12}^{\lambda\lambda'}(\eta) = \frac{A_c O_{1-}^b + \eta A_s^{\lambda\lambda'} \sqrt{D^{\lambda\lambda'2} - (O_{1-}^b)^2}}{D^{\lambda\lambda'2}}.\quad (91)$$

Analogously, for the pair (O_{2-}^b, O_{1+}^c), Eq. (75) changes to

$$\sin \phi_{12}^{\lambda\lambda'}(\eta) = \frac{-O_{2-}^b A_s^{\lambda\lambda'} + \eta A_c \sqrt{D^{\lambda\lambda'2} - (O_{2-}^b)^2}}{D^{\lambda\lambda'2}}.\quad (92)$$

In the above equation $A_c, A_c^{\lambda\lambda'}$ and $D^{\lambda\lambda'2}$ are given by Eq. (90) except for the change in the sign of B_{23} .

It then, follows that the symmetry properties of $\sin \phi_{12}^{\lambda\lambda'}(\eta)$ given in the above two equations have interchanged from the corresponding quantities in the case of (O_{1+}^a, O_{1-}^a). This, in turn, interchanges the property of $\alpha_{34}^{\lambda\lambda'}(\eta)$. One can now see that the role of ($O_{1\pm}^b, O_{1\pm}^c$) and ($O_{2\pm}^b, O_{1\pm}^c$) interchanges

in Eq. (88), i.e.,

$$\begin{aligned} (O_{2+}^a, O_{2-}^a) \text{ and } (O_{n\pm}^b, O_{n\pm}^c) \text{ (} n = 1, 2) &\rightarrow \text{ do not resolve the ambiguity,} \\ (O_{2+}^a, O_{2-}^a) \text{ and } (O_{2\pm}^b, O_{1\pm}^c) &\rightarrow \text{ do resolve the ambiguity.} \end{aligned} \quad (93)$$

E. $(O_{1\pm}^a, O_{2\pm}^a)$

In the case of $(O_{1\pm}^a, O_{2\mp}^a)$ (here the signs \pm are not independent), we note that $\phi_{24}^\lambda - \phi_{13}^\lambda$ is ζ dependent [cf. Eqs. (37) and (39)]. Therefore, in this case, the phase ambiguity will be resolved because the possible solutions in Eq. (69) will all be linearly independent. For the case of $(O_{1\pm}^a, O_{2\pm}^a)$ (not independent \pm signs), however, $\phi_{24}^\lambda - \phi_{13}^\lambda$ is ζ independent [cf. Eqs. (36) and (38)] and the above argument valid for $(O_{1\pm}^a, O_{2\mp}^a)$ does not apply. However, it happens that the relative phases ϕ_{24} and ϕ_{13} in the $(O_{1\pm}^a, O_{2\pm}^a)$ case are given by [cf. Eqs. (28) and (33)]

$$\begin{cases} \phi_{13} = -\zeta + \alpha_{13}, \\ \phi_{24} = -\zeta + \alpha_{24} - \delta_+ \pi \end{cases} \quad (94)$$

or

$$\begin{cases} \phi_{13} = -\zeta - \alpha_{13} + \pi, \\ \phi_{24} = -\zeta - \alpha_{24} + \delta_- \pi \end{cases} \quad (95)$$

with twofold ambiguity. $\delta_+ = 1$ and $\delta_- = 0$ for (O_{1+}^a, O_{2+}^a) and $\delta_+ = 0$ and $\delta_- = 1$ for (O_{1-}^a, O_{2-}^a) . It is then easy to see that all $\cos \phi_{ij}$ ($ij = 24, 34$) are distinct from each other. The same is true for $\sin \phi_{ij}$. This implies that the quantities A_c and A_s entering into Eqs. (59) and (75) have all distinct values, in general, as can be seen from their definitions in Eqs. (54) and (72) for the cases (O_{1-}^b, O_{1+}^c) and (O_{2-}^b, O_{1+}^c) , respectively. Hence, all the phases $\alpha_{12}^{\lambda\lambda'}(\eta)$ and $\alpha_{34}^{\lambda\lambda'}(\eta)$ entering into Eq. (69) assume distinct values in general, resulting in linearly independent possible solutions. Consequently, the phase ambiguity can be resolved with the pairs $(O_{1\pm}^a, O_{2\pm}^a)$ as well.

We conclude that any pair of the form $(O_{1\pm}^a, O_{2\pm}^a)$, together with any pair of the form $(O_{1\pm}^b, O_{1\pm}^c)$ or $(O_{2\pm}^b, O_{1\pm}^c)$, will resolve the phase ambiguity. Here all the signs \pm are independent.

This completes the analysis of all possible $(2+1+1)$ cases. Collecting the results for all the possibilities, the following sets of four observables will resolve the phase ambiguity in the $(2+1+1)$ case:

- (i) (O_{1+}^a, O_{1-}^a) and $[(O_{1\pm}^b, O_{1\pm}^c) \text{ or } (O_{2\pm}^b, O_{2\pm}^c)]$.
- (ii) $(O_{1\pm}^a, O_{2\pm}^a)$ and $[(O_{1\pm}^b, O_{1\pm}^c) \text{ or } (O_{2\pm}^b, O_{2\pm}^c) \text{ or } (O_{2\pm}^b, O_{1\pm}^c)]$.
- (iii) (O_{2+}^a, O_{2-}^a) and $(O_{2\pm}^b, O_{1\pm}^c)$.

with any permutation of a, b, c . Here, the \pm signs are all independent. The results are displayed in Table III for the case $[2(a) + 1(b) + 1(c)]$. Other combinations can be obtained by an appropriate permutation of a, b, c .

As in the $(2+2)$ case discussed in preceding Sec. III, here one has also the restriction of no equal-relative-phase-magnitudes in order to enable the sets of two pairs of observables, as given in Table III, to resolve the phase ambiguity.

Analogous considerations for the $(2+2)$ case allows one to identify the additional observables required to resolve the phase ambiguity when this restriction is not met. They are indicated also in Table III for the case $[2(a) + 1(b) + 1(c)]$.

V. PHASE FIXING FOR THE $(3+1)$ AND 4 CASES

It is straightforward to show that no sets of observables with the $(3+1)$ or 4 cases can resolve the phase ambiguity.

Consider the $(3+1)$ case of three observables from, say, group $a = BT$ and one from group $b = BR$. Then, from Eqs. (3) and (4), one has the following possible sets of four observables: $[(O_{nv}^a, O_{n'v'}^a), (O_{n''v''}^a, O_{n'''v'''}^a)]$, with $[n, n', n'', n''' = 1, 2; v, v', v'', v''' = \pm]$ and $(n, v) \neq (n', v')$ and $(n'', v'') \neq (n, v), (n', v')$. For concreteness, consider the set $[(O_{1+}^a, O_{1-}^a), (O_{2+}^a, O_{1+}^b)]$. The pair of observables (O_{1+}^a, O_{1-}^a) determines $\sin \phi_{13}$ and $\sin \phi_{24}$ uniquely, yielding the twofold ambiguity for each of the relative phases ϕ_{13} and ϕ_{24} as given by Eq. (8). This, then, leads to the following four possible expressions for the observable O_{2+}^a :

$$\begin{aligned} O_{2+}^a &= B_{13} \cos \phi_{13} + B_{24} \cos \phi_{24} \\ &= \begin{cases} B_{13} \cos \alpha_{13} + B_{24} \cos \alpha_{24}, \\ B_{13} \cos \alpha_{13} - B_{24} \cos \alpha_{24}, \\ -(B_{13} \cos \alpha_{13} + B_{24} \cos \alpha_{24}), \\ -(B_{13} \cos \alpha_{13} - B_{24} \cos \alpha_{24}), \end{cases} \end{aligned} \quad (96)$$

where Eq. (57) has been used. Since these expressions are all linearly independent, only one of them will be satisfied—except perhaps for a few special cases—once O_{2+}^a is measured. That is, O_{2+}^a should in principle be able to resolve the discrete ambiguities of ϕ_{13} and ϕ_{24} . The remaining observable O_{1+}^b ,

$$O_{1+}^b = B_{14} \sin \phi_{14} + B_{23} \sin \phi_{23}, \quad (97)$$

however, can determine neither ϕ_{14} nor ϕ_{23} , one of which is needed, in addition to ϕ_{13} and ϕ_{24} , for resolving the phase ambiguity of the transversity amplitude up to an arbitrary phase. The analogous reasoning applies to all other sets of four observables in the $(3+1)$ case. The reader may convince himself/herself that none of these sets are capable of resolving the phase ambiguity.

In the case of four observables from one given group (4) case, say, $[(O_{1+}^a, O_{1-}^a), (O_{2+}^a, O_{2-}^a)]$, it is clear from Eq. (3) that they determine the relative phases ϕ_{13} and ϕ_{24} uniquely, but no information about a third relative phase is available for resolving the phase ambiguity.

VI. IDENTIFYING WHEN THE EQUAL-RELATIVE-PHASE-MAGNITUDES CONDITION OCCURS

As seen in Secs. III and IV, the completeness condition for a set of four double-spin observables to resolve the phase

TABLE III. Sets of two pairs of double-spin observables for case $[2(a) + 1(b) + 1(c)]$. Other combinations can be obtained by appropriate permutations of the indices a, b, c . \checkmark = do resolve. X = do not resolve. Observables indicated outside the parentheses are the additional ones required in case the equal-relative-phase-magnitudes condition, as given by Eq. (48), is met for the pairs of observables (in parentheses) that do resolve the phase ambiguity otherwise. The additional observable required is either one of the observables indicated for each pair, except for those marked with **, which require any two additional observables from those indicated.

	$(O_{1+}^a, O_{1-}^a), O_{2\pm}^a$ (G, F), E/H	$(O_{1+}^a, O_{2+}^a), O_{1-}^a$ (G, E), F	$(O_{1+}^a, O_{2-}^a), O_{1-}^a$ (G, H), F	$(O_{1-}^a, O_{2+}^a), O_{1+}^a$ (F, E), G	$(O_{1-}^a, O_{2-}^a), O_{1+}^a$ (F, H), G	$(O_{2+}^a, O_{2-}^a), O_{1\pm}^a$ (E, H), G/F
$O_{1-}^b/O_{1+}^c, (O_{1+}^b, O_{1+}^c)$ $C_x/T_z, (O_z, L_x)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	X
$O_{1-}^b/O_{1+}^c, (O_{1+}^b, O_{1-}^c)$ $C_x/L_x, (O_z, T_z)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	X
$O_{1-}^b, (O_{1+}^b, O_{2+}^c)$ $C_x, (O_z, L_z)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1-}^b, (O_{1+}^b, O_{2-}^c)$ $C_x, (O_z, T_x)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1+}^b/O_{1-}^c, (O_{1-}^b, O_{1+}^c)$ $O_z/T_z, (C_x, L_x)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	X
$O_{1+}^b/O_{1+}^c, (O_{1-}^b, O_{1-}^c)$ $O_z/L_x, (C_x, T_z)$	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark	X
$O_{1+}^b, (O_{1-}^b, O_{2+}^c)$ $O_z, (C_x, L_z)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1+}^b, (O_{1-}^b, O_{2-}^c)$ $O_z, (C_x, T_x)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1-}^c, (O_{2+}^b, O_{1+}^c)$ $T_z, (C_z, L_x)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1+}^c, (O_{2+}^b, O_{1-}^c)$ $L_x, (C_z, T_z)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1\pm}^b/O_{1\pm}^c, (O_{2+}^b, O_{2+}^c)$ (C_z, L_z)	** \checkmark	** \checkmark	\checkmark	\checkmark	\checkmark	X
$O_{1\pm}^b/O_{1\pm}^c, (O_{2+}^b, O_{2-}^c)$ (C_z, T_x)	** \checkmark	** \checkmark	\checkmark	\checkmark	\checkmark	X
$O_{1-}^c, (O_{2-}^b, O_{1+}^c)$ $T_z, (O_x, L_x)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1+}^c, (O_{2-}^b, O_{1-}^c)$ $L_x, (O_x, T_z)$	X	\checkmark	\checkmark	\checkmark	\checkmark	\checkmark **
$O_{1\pm}^b/O_{1\pm}^c, (O_{2-}^b, O_{2+}^c)$ (O_x, L_z)	** \checkmark	** \checkmark	\checkmark	\checkmark	\checkmark	X
$O_{1\pm}^b/O_{1\pm}^c, (O_{2-}^b, O_{2-}^c)$ (O_x, T_x)	** \checkmark	** \checkmark	\checkmark	\checkmark	\checkmark	X

ambiguity of the transversity amplitude holds, provided the equal-relative-phase-magnitudes relation [cf. Eq. (48)] is not met. This restriction would not cause a significant problem if this is a rarely occurring situation. However, we find no reason *a priori* to expect that this is indeed a rare case. This forces one to verify if the no equal-relative-phase-magnitudes condition is met for each kinematics (total energy of the system and meson production angle) where the four double-spin observables are measured, for the completeness argument that only four carefully selected double-spin observables are needed. Can

one know when the equal-magnitudes relation is realized? The answer to this question is yes, as we show in the following.

To be concrete, consider the pair of observables of the form $(O_{n\pm}^a, O_{n\mp}^a)$ ($n = 1, 2$), from Eqs. (7), (8), and (10). When the corresponding phases satisfy $\alpha_{13} = \pm\alpha_{24}$, these observables obey the relation

$$B_{13}(O_{n\pm}^a - O_{n\mp}^a) = \pm B_{24}(O_{n\pm}^a + O_{n\mp}^a). \quad (98)$$

Hence, by measuring the cross section and single-spin observables (which determine B_{13} and B_{24}) and the double-spin

observables in the above equation, we will be able to gauge if the equal-magnitudes relation, $|\alpha_{13}| = |\alpha_{24}|$, is met. Note that in the particular case of $\alpha_{13} = \alpha_{24} = 0$, we have

$$O_{1+}^a = O_{1-}^a = 0 \quad \text{and} \quad \frac{O_{2-}^a}{O_{2+}^a} = \frac{B_{13} - B_{24}}{B_{13} + B_{24}}. \quad (99)$$

For the pair of observables of the form $(O_{1\pm}^a, O_{2\pm}^a)$ (\pm signs are independent), from Eq. (22), when $\alpha_{13} = \pm\alpha_{24}$, we have

$$O_{1\pm}^a{}^2 + O_{2\pm}^a{}^2 = (B_{13} \mp B_{24})^2. \quad (100)$$

Note that the \pm sign on the right-hand side of the above equation goes with the \pm sign of α_{24} . In the particular case of $\alpha_{13} = \alpha_{24} = 0$, we have

$$O_{1\pm}^a = O_{2\pm}^a = 0 \quad \text{and} \quad B_{13} = \pm B_{24}. \quad (101)$$

For the pair (O_{1-}^b, O_{1+}^c) , when $\alpha_{12}^{\lambda\lambda'}(\eta) = \pm\alpha_{34}^{\lambda\lambda'}(\eta)$, we have, from Eqs. (59) and (65),

$$\frac{A_c^{\lambda\lambda'} O_{1-}^b + \eta A_s \sqrt{D^{\lambda\lambda'2} - (O_{1-}^b)^2}}{D^{\lambda\lambda'2}} = \frac{O_{1+}^c}{B_{12} \pm B_{34}}, \quad (102)$$

where $A_c^{\lambda\lambda'}$, A_s , and $D^{\lambda\lambda'2}$ are given by Eqs. (54) and (59). In the particular case of $\alpha_{12}^{\lambda\lambda'}(\eta) = \alpha_{34}^{\lambda\lambda'}(\eta) = 0$, we have

$$O_{1+}^c = 0 \quad \text{and} \quad |O_{1-}^b| = |A_s|, \quad (103)$$

where Eq. (56) has been used too. Equations (102) and (103) hold for all the pairs of observables of the form $(O_{n\pm}^b, O_{n\pm}^c)$ ($n = 1, 2$ and \pm signs are independent) with the appropriate signs of B_{23} and B_{34} in $A_c^{\lambda\lambda'}$, A_s , and $D^{\lambda\lambda'2}$, and also of B_{12} and B_{34} .

Analogously, for the pair (O_{2-}^b, O_{1+}^c) , from Eqs. (75) and (78), we obtain when $\alpha_{12}^{\lambda\lambda'}(\eta) = \pm\alpha_{34}^{\lambda\lambda'}(\eta)$

$$\frac{-A_s^{\lambda\lambda'} O_{2-}^b + \eta A_c \sqrt{D^{\lambda\lambda'2} - (O_{2-}^b)^2}}{D^{\lambda\lambda'2}} = \frac{O_{1+}^c}{B_{12} \pm B_{34}}, \quad (104)$$

where $A_s^{\lambda\lambda'}$, A_c , and $D^{\lambda\lambda'2}$ are given by Eqs. (72) and (74). In the particular case of $\alpha_{12}^{\lambda\lambda'}(\eta) = \alpha_{34}^{\lambda\lambda'}(\eta) = 0$, we have

$$O_{1+}^c = 0 \quad \text{and} \quad |O_{2-}^b| = |A_c|. \quad (105)$$

Equations (104) and (105) hold for all the pairs of observables of the form $(O_{2\pm}^b, O_{1\pm}^c)$ (\pm signs are independent) with the appropriate signs of B_{23} and B_{34} in $A_c^{\lambda\lambda'}$, A_s , and $D^{\lambda\lambda'2}$, and also of B_{12} and B_{34} .

Equations (98)–(105) enable one to gauge when the equal-relative-phase-magnitudes relation is met for any of the sets of two pairs of observables as listed in Table I, II, and III, which, otherwise, can resolve the phase ambiguity.

VII. SUMMARY

By revealing and exploiting the underlying symmetries of the relative phases of the pseudoscalar photoproduction amplitude, we have provided a consistent and explicit mathematical derivation of the completeness condition for the observables in this reaction covering all the relevant cases. In particular, we have determined all the possible sets of four observables that resolve the phase ambiguity of the

transversity amplitude up to an overall phase. The present work substantiates and corroborates the original findings of Ref. [3]. However, the completeness condition of a set of four double-spin observables to resolve the phase ambiguity holds only if the relative phases do not have equal magnitudes as specified in Eq. (48). In situations where the equal-magnitudes condition occurs, we have shown that one or two or even three extra chosen observables are required, depending on the particular set of two pairs of observables considered, as given in Tables I, II, and III, resulting in five or six or seven as the minimum number of chosen double-spin observables required to resolve the phase ambiguity. In the particular case of vanishing relative phases, one needs eight chosen observables to resolve the phase ambiguity. This results in a minimum of up to twelve chosen observables to determine the amplitude up to an overall phase: four, to determine the magnitudes of the basic four transversity amplitudes that comprise the full photoproduction amplitude, and up to eight more to resolve the phase ambiguity depending on the particular set of four double-spin observables.

To apply the argument of the completeness condition of a set of four double-spin observables to resolve the phase ambiguity of the photoproduction amplitude, we need to make sure that the restriction of no equal-relative-phase-magnitudes, as specified in Eq. (48), is satisfied. We have shown that it is possible to gauge whether this restriction is satisfied or not for each kinematics where the set of four double-spin observables is measured, because these observables obey the well defined relationships that are unique to the case of equal-relative-phase-magnitudes, as seen in Sec. VI.

We also remark that quantum mechanics does not allow one to determine the overall phase of the reaction amplitude from experiment. For this, some physics input is required. This fact must have a strong impact on partial-wave analysis in the context of complete experiments for extracting the baryon resonances since, if the overall phase of the amplitude is unknown, the corresponding partial-wave amplitude is an ill defined quantity. The issues related to the unknown overall phase have been discussed earlier by several authors. In particular, Omelaenko [15] mentioned the overall phase problem for photoproduction in the summary section of his paper on discrete ambiguities in truncated partial-wave analysis. In the classic review paper by Bowcock and Burkhardt [16], this problem is discussed as well. Dean and Lee [17] also investigated this problem mainly for the formalism of πN scattering. Two recent publications [18,19] treat the same problem, but mostly in the simpler context of spinless particle scattering.

Finally, the present type of analysis may be applied to other reaction processes where the interest in determining the complete experiments exists.

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