Nucleon-pair wave functions in a single-j shell

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(Received 10 March 2019; published 25 July 2019)

In this work we study approximate and exact solutions for nucleons in a single-j shell, from the perspective of nucleon-pair basis states, i.e., those coupled by pairs with good spins. We find that for four, five, and six particles in the $0h_{11/2}$ shell, a selected set of independent nucleon-pair basis states leads to approximate solutions of a realistic two-body interaction, without resorting to the diagonalization. We analytically show that for six particles in the $j = \frac{11}{2}$ shell, two nucleon-pair states with $J = 3$ and 11—which are coupled by three pairs of spin 0, 2, and 4 and by pairs of spin 0, 2, and 10, respectively—are eigenstates of *any* two-body interactions. In particular, we construct general analytic expressions for states of definite seniority numbers $\nu = 3, 4, 5$ in terms of nucleon-pair basis states, based on which we further derive exact wave functions for a few eigenstates of *any* two-body interactions in the midshells of $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$. The exact wave functions given here should be useful in interpreting electromagnetic moment and transition properties of corresponding nuclei.

DOI: 10.1103/PhysRevC.100.014318

I. INTRODUCTION

Atomic nuclei are quantum many-body systems composed of protons and neutrons. Complicated many-body wave functions are usually obtained by the diagonalization of the Hamiltonian matrix in the nuclear shell model [1,2]. Despite the complexities, simple patterns emerge in the low-lying structures [3], such as those of the nuclear pairing [4,5] reflecting the short-range and attractive nature of effective interactions between (valence) nucleons.

The seniority scheme [6] and its generalization [7,8] are very useful in interpreting regularities of semimagic nuclei, and have been receiving renewed interest in recent decades; see, e.g., [9–25]. The (generalized) seniority number ν refers to the number of nucleons which are not coupled to spin-zero pairs (i.e., S pairs) in a state. In the nucleon-pair approximation [26–28], unpaired nucleons are further coupled into pairs of nonzero good spins. From this perspective, some of the present authors have shown [29] that low-lying yrast states of even-even semimagic $N = 82$ isotones and Sn isotopes are well described by one-dimensional structures in terms of collective pairs of both zero and nonzero spins. Such one-dimensional structures coincide with the generalized seniority scheme for those of generalized seniority 0 and 2, but are unexpected for low-lying states with larger generalized seniority numbers.

In a single-j shell, the seniority is a quantum number which can be conveniently used for the classification of j^n configurations. In recent years, solvable eigenstates of systems with partial dynamical symmetry (suggesting the system has partial eigenstates keeping all quantum numbers) have been intensively studied [9–20]. If two-body interactions conserve the seniority, multiplicity-free states, i.e., those uniquely defined by J and ν, are solvable eigenstates. In Ref. [9] algebraic conditions of two-body interactions in the $j = \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$ shells for seniority conservation, and analytic expressions for eigenenergies of a few multiplicity-free states for n particles in the $j = \frac{9}{2}$ shell, were both derived. In Refs. [10,11] it was found that, surprisingly, two states of four particles in the $j = \frac{9}{2}$ shell with $J = 4$, $\nu = 4$ and with $J = 6$, $\nu = 4$, which are not multiplicity-free, are eigenstates of *any* two-body interactions. Analytic expressions for energies and wave functions of these two striking cases were obtained in Ref. [12], and analytic proofs that these two cases are eigenstates of *any* two-body interactions were given in Refs. [13,17]. In Ref. [15] it was shown that eigenstates of *any* two-body interactions have definite seniority quantum numbers, and are solvable states having their eigenenergies as linear combinations of two-body matrix elements with rational coefficients. Very recently, in Ref. [19] an approach using the m scheme and angular momentum projection technique, to derive solvable states, was developed, with which analytic expressions for eigenenergies of a number of solvable states in single-j shells up to $j = \frac{15}{2}$ were obtained.

In this paper we study approximate and exact solutions for nucleons in a single-j shell from the perspective of nucleon-pair basis states, i.e., those coupled by pairs of both zero and nonzero spins. The paper is organized as follows. In Sec. II we show that a selected set of independent nucleon-pair basis states, together with the Schmidt orthogonalization, leads

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to approximate solutions of a realistic two-body interaction, without resorting to the diagonalization. In Sec. III we analytically show that, for six particles in the $j = \frac{11}{2}$ shell, two nucleon-pair states with $J = 3$ and 11, both having seniority 4, are eigenstates of *any* two-body interactions. In Sec. IV, we construct general analytic expressions for states of seniority $\nu = 3, 4, 5$ in terms of nucleon-pair basis states, and derive exact wave functions for a few eigenstates of *any* two-body interactions in the midshells of $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$.

II. APPROXIMATE SOLUTIONS OF A REALISTIC TWO-BODY INTERACTION

In this section we shall show that a selected set of complete and independent nucleon-pair basis states leads to approximate solutions of a realistic two-body interaction. We exemplify this with cases of four, five, and six particles in the $0h_{11/2}$ shell. We take two-body matrix elements of Ref. [30], which were derived from the realistic CD-Bonn potential [31] using the G-matrix approach [32]. The shell-model calculations are realized using the nucleon-pair approximation [26–28] and adopting all possible pairs.

Let us first present the definitions of the nucleon-pair basis state and the shell-model Hamiltonian in a single-j shell. A pair basis state of $2N$ nucleons in a single-j shell is given by

$$(((A^{\dagger r_1} \times A^{\dagger r_2})^{J_2} \times A^{\dagger r_3})^{J_3} \cdots A^{\dagger r_N})^{J_N}|0\rangle. \tag{1}$$

Here the coupled pair with spin r is defined by

$$A_\mu^{\dagger r} = (a_j^\dagger \times a_j^\dagger)_\mu^r = \sum_{m_1 m_2} C_{jm_1 jm_2}^{r\mu} a_{jm_1}^\dagger a_{jm_2}^\dagger, \tag{2}$$

in which $C_{jm_1 jm_2}^{r\mu}$ is the Clebsch-Gordan coefficient. Note that the pair is not normalized. The shell-model Hamiltonian in a single-j shell is given by

$$H = \frac{1}{2}\sum_{JM} V_J A_M^{\dagger J} A_M^J = \frac{1}{2}\sum_J V_J \hat{J}[A^{\dagger J} \times \widetilde{A}^J]^{(0)}, \tag{3}$$

where $\hat{J} = \sqrt{2J+1}$, $A^{\dagger J}$ is the pair creation operator defined in Eq. (2), and $\widetilde{A}^J$ is the time reversal operator of the pair destruction. V_J is equal to the antisymmetrized and normalized two-body matrix element $\langle jjJ|V|jjJ\rangle$.

Now let us describe how we obtain our approximate solutions. We first define the energy of a coupled pair with spin r, denoted as $E_{\rm p}(r)$, to be equal to the corresponding two-body matrix element, i.e.,

$$E_{\rm p}(r) = \langle jjr|V|jjr\rangle. \tag{4}$$

We further define the unperturbed energy of a nucleon-pair basis state, coupled by N pairs with spin $r_1, r_2, \ldots, r_N$, respectively, as

$$E^{(0)} = \sum_i E_{\rm p}(r_i). \tag{5}$$

Then our procedure is as follows.

(1) We consider all combinations of pairs for a given J, and construct pair basis states in the form of Eq. (1) by coupling the pairs successively with $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_N$. Keeping those having nonvanishing overlaps with themselves, we obtain an overcomplete set.

(2) We put the above pair basis states in the overcomplete set in increasing order of $E^{(0)}$.

(3) From the above overcomplete set, we obtain a complete and independent set step by step, using the criterion for linear independence that the corresponding norm matrix has only nonzero eigenvalues.

(4) From the above complete and independent set of pair basis states, we use the Schmidt orthogonalization to obtain a complete and orthogonal set.

Below we shall show that this complete and orthogonal set provides us with approximate solutions of the shell-model Hamiltonian of Eq. (3).

We illustrate in Figs. 1 and 2 the accuracy of our approximate solutions for four, five, and six particles in the $0h_{11/2}$ shell. We put the approximate solutions (i.e., orthogonal basis states), in increasing order of the expectation energy (i.e., the diagonal matrix element, denoted as H_{ii}), and also put the eigenstates given by the diagonalization in increasing order of the eigenenergy (denoted as E_i). The approximate solutions and the eigenstates in such sequences are denoted as ϕ_1, ϕ_2, …, ϕ_D and as ψ_1, ψ_2, …, ψ_D, respectively, where D is the dimension of the space with a given J. In Fig. 1 we plot E_i versus H_{ii} for various J values. One sees that, for each J, E_i versus H_{ii} follows a very compact trajectory of $E_i = H_{ii}$. We also present the overlap $\langle\phi_i|\psi_i\rangle$ in Fig. 2, where one sees the overlaps are all remarkably close to 1. Thus the orthogonal basis states included in Figs. 1 and 2 are indeed approximate solutions of the shell-model Hamiltonian, with very good accuracy.

One easily sees that using our procedure the nucleon-pair basis state with the lowest $E^{(0)}$ is one of the approximate solutions. According to our calculation, in most cases, the nucleon-pair basis state with the lowest $E^{(0)}$ is also the approximate solution with the lowest expectation energy. This is consistent with our earlier work [29], where it was shown that most yrast states of $N = 82$ isotones and Sn isotopes can be well represented solely by one nucleon-pair basis state.

Next let us discuss the above approximate solutions from the perspective of seniority. It is well known that states with good seniority numbers are eigenstates of the monopole pairing interaction [33]. Nucleon-pair basis states in the form of Eq. (1) in general include components of different seniority numbers. In a nucleon-pair basis state with N' non-S pairs, the component of the largest seniority is that of $\nu = 2N'$ for an even-number system and that of $\nu = 2N' + 1$ for an odd-number system. This can be understood by noting that a seniority-ν state of an n-particle system ($n \geqslant \nu$) can be obtained by adding $\frac{n-\nu}{2}$ S pairs to a seniority-ν state of a ν-particle system [33]. Also note that, for a given J, the nucleon-pair basis state(s) having the smallest number of non-S pairs among all possible pair basis states necessarily has a definite seniority number, and the components of different seniority numbers are orthogonal to each other. Then we conclude that if we replace the second step in the previous procedure to obtain approximate solutions with

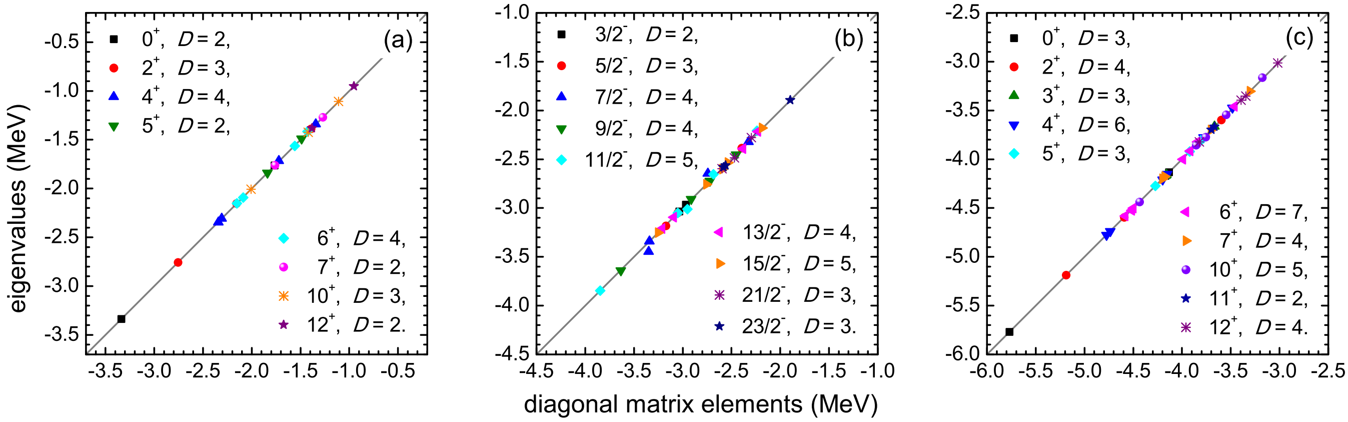

FIG. 1. Eigenvalues versus diagonal matrix elements of realistic shell-model Hamiltonian matrices, i.e., energies of exact solutions versus those of approximate ones, for (a) four particles, (b) five particles, and (c) six particles in the $0h_{11/2}$ shell. See text for further explanations.

(2′) We put the above pair basis states in the overcomplete set in increasing order of the number of non-S pairs

every basis state in the final complete and orthogonal set will have a definite seniority number.

According to our calculation, the above condition happens to be satisfied in our procedure to obtain the approximate solutions of four, five, and six particles in the $0h_{11/2}$ shell (except for the 4^+ and 6^+ cases of six particles in the $0h_{11/2}$ shell). Thus, the approximate solutions presented in this work have definite seniority numbers (for the two exceptional cases, we have verified that the approximate solutions also have definite seniority numbers). Then, the approximate decoupling, as shown in Figs. 1 and 2, indicates the realistic two-body interaction used in this work [30] approximately conserve the seniority. In Table I we present dimensions of subspaces with definite seniority numbers for a given J. As shown there, some subspaces of definite seniority numbers are one-dimensional, and thus approximate solutions included in them are multiplicity-free. These multiplicity-free states are necessarily the eigenstates if the two-body interaction conserves the seniority. Meanwhile, it is very interesting to note that a few subspaces are multidimensional, and thus approximate solutions included in them are not multiplicity-free. The origin of such approximate decoupling between states which are not multiplicity-free, needs further studies.

III. NUCLEON-PAIR STATES AS EXACT SOLUTIONS OF ANY TWO-BODY INTERACTIONS

In this section we shall show that for six particles in the $j = \frac{11}{2}$ shell, two nucleon-pair states with $J = 3$ and 11—which are coupled by three pairs of spin 0, 2, and 4 and by pairs of spin 0, 2, and 10, respectively, as shown below—are actually eigenstates of *any* two-body interactions:

$$((A^{\dagger(2)} \times A^{\dagger(4)})^{(3)} \times A^{\dagger(0)})^{(3)}|0\rangle, \tag{6}$$

$$((A^{\dagger(2)} \times A^{\dagger(10)})^{(11)} \times A^{\dagger(0)})^{(11)}|0\rangle. \tag{7}$$

Here we omit normalization factors for simplicity (they will be given later in Table III).

For an arbitrary J, the n-dimensional configuration space can be constructed with a set of complete and independent pair basis states in the form of Eq. (1), denoted as $|\beta_1\rangle, |\beta_2\rangle, \ldots, |\beta_n\rangle$. These pair basis states are in general nonorthogonal. Using the Schmidt orthogonalization, one can obtain the orthogonal (and non-normalized) basis states as follows, denoted by $|\gamma_1\rangle, |\gamma_2\rangle, \ldots, |\gamma_n\rangle$:

$$|\gamma_1\rangle = |\beta_1\rangle,$$
$$|\gamma_2\rangle = |\beta_2\rangle - \frac{\langle\beta_2|\gamma_1\rangle}{\langle\gamma_1|\gamma_1\rangle}|\gamma_1\rangle,$$

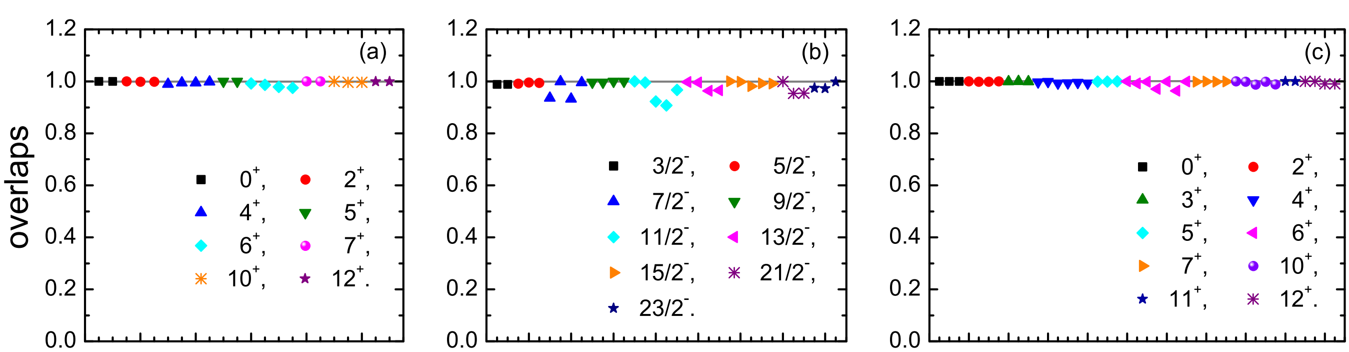

FIG. 2. Overlaps between approximate and exact solutions, namely $\langle\phi_i|\psi_i\rangle$ with $i = 1, 2, \ldots, D$, for (a) four particles, (b) five particles, and (c) six particles in the $0h_{11/2}$ shell. See text for further explanations.

TABLE I. Dimensions (denoted as D_i) of subspaces with definite seniority numbers (denoted as ν_i) for a given J, for four, five, and six particles in the $j = \frac{11}{2}$ shell.

	J	Subspace 1		Subspace 2		Subspace 3		Subspace 4	
		ν_1	D_1	ν_2	D_2	ν_3	D_3	ν_4	D_4
$(\frac{11}{2})^4$	0	0	1	2	0	4	1		
	2	0	0	2	1	4	2		
	4	0	0	2	1	4	3		
	5	0	0	2	0	4	2		
	6	0	0	2	1	4	3		
	7	0	0	2	0	4	2		
	10	0	0	2	1	4	2		
	12	0	0	2	0	4	2		
$(\frac{11}{2})^5$	3/2	1	0	3	1	5	1		
	5/2	1	0	3	1	5	2		
	7/2	1	0	3	1	5	3		
	9/2	1	0	3	2	5	2		
	11/2	1	1	3	1	5	3		
	13/2	1	0	3	1	5	3		
	15/2	1	0	3	2	5	3		
	21/2	1	0	3	1	5	2		
	23/2	1	0	3	1	5	2		
$(\frac{11}{2})^6$	0	0	1	2	0	4	1	6	1
	2	0	0	2	1	4	2	6	1
	3	0	0	2	0	4	1	6	2
	4	0	0	2	1	4	3	6	2
	5	0	0	2	0	4	2	6	1
	6	0	0	2	1	4	3	6	3
	7	0	0	2	0	4	2	6	2
	10	0	0	2	1	4	2	6	2
	11	0	0	2	0	4	1	6	1
	12	0	0	2	0	4	2	6	2

$$|\gamma_3\rangle = |\beta_3\rangle - \frac{\langle\beta_3|\gamma_1\rangle}{\langle\gamma_1|\gamma_1\rangle}|\gamma_1\rangle - \frac{\langle\beta_3|\gamma_2\rangle}{\langle\gamma_2|\gamma_2\rangle}|\gamma_2\rangle,$$
$$\vdots$$
$$|\gamma_n\rangle = |\beta_n\rangle - \frac{\langle\beta_n|\gamma_1\rangle}{\langle\gamma_1|\gamma_1\rangle}|\gamma_1\rangle - \frac{\langle\beta_n|\gamma_2\rangle}{\langle\gamma_2|\gamma_2\rangle}|\gamma_2\rangle - \cdots - \frac{\langle\beta_n|\gamma_{n-1}\rangle}{\langle\gamma_{n-1}|\gamma_{n-1}\rangle}|\gamma_{n-1}\rangle.$$

If $|\gamma_1\rangle$, namely $|\beta_1\rangle$, is the (non-normalized) eigenstate of *any* two-body interactions, matrix elements of the two-body operator $[A^{\dagger s} \times \tilde{A}^s]^{(0)}$ (with $s = 0, 2, \ldots, 2j-1$, respectively) between $|\gamma_1\rangle$ and $|\gamma_i\rangle$ (with $i = 2, 3, \ldots, n$, respectively) must vanish. This further gives the sufficient and necessary condition of $|\beta_1\rangle$ being the eigenstate of *any* two-body interactions, i.e.,

$$\frac{\langle\beta_1|[A^{\dagger s} \times \tilde{A}^s]^{(0)}|\beta_1\rangle}{\langle\beta_1|\beta_1\rangle} = \frac{\langle\beta_2|[A^{\dagger s} \times \tilde{A}^s]^{(0)}|\beta_1\rangle}{\langle\beta_2|\beta_1\rangle} = \cdots = \frac{\langle\beta_n|[A^{\dagger s} \times \tilde{A}^s]^{(0)}|\beta_1\rangle}{\langle\beta_n|\beta_1\rangle}, \quad (8)$$

for $s = 0, 2, \ldots, 2j-1$, respectively.

For six particles in the $j = \frac{11}{2}$ shell, the space of $J = 3$ is three-dimensional, constructed by the pair basis states as follows (the pair basis state $((A^{\dagger r_1} \times A^{\dagger r_2})^{J_2} \times A^{\dagger r_3})^J|0\rangle$ is denoted by $|r_1, r_2, r_3; J_2, J\rangle$).

$$J = 3 : |\beta_1\rangle = |2, 4, 0; 3, 3\rangle,$$
$$|\beta_2\rangle = |2, 2, 2; 2, 3\rangle,$$
$$|\beta_3\rangle = |2, 2, 4; 2, 3\rangle, \quad (9)$$

where $|\beta_1\rangle$ is the nucleon-pair state described by Eq. (6). The space of $J = 11$ is two-dimensional, constructed by the pair basis states as below.

$$J = 11 : |\beta_1\rangle = |2, 10, 0; 11, 11\rangle,$$
$$|\beta_2\rangle = |2, 2, 8; 4, 11\rangle, \quad (10)$$

where $|\beta_1\rangle$ is the nucleon-pair state described by Eq. (7). Below we shall show that Eq. (8) is exactly satisfied for both the case of $J = 3$ and that of $J = 11$, which suggests that the $|\beta_1\rangle$ state of $J = 3$ and that of $J = 11$ are the eigenstates of *any* two-body interactions. For the $J = 11$ case, the space is two-dimensional, and the orthogonal partner of the $|\beta_1\rangle$ state is the other eigenstate.

The general formulas to calculate matrix elements of two-body interactions in coupled pair basis states were given in

TABLE II. Exact matrix elements of the two-body operator $[A^{\dagger s} \times \tilde{A}^s]^{(0)}$ (with $s = 0, 2, \ldots, 2j-1$, respectively), as well as overlaps, between $|\beta_1\rangle$ and $|\beta_i\rangle$ (with $i = 1, \ldots, n$, respectively) for the case of $J = 3$ and that of $J = 11$. With these exact values, one sees Eq. (8) is exactly satisfied for both cases, indicating that the nucleon-pair basis state $|\beta_1\rangle$ of $J = 3$ and that of $J = 11$ are the eigenstates of *any* two-body interactions. See text for details.

	Overlap	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$
$J=3$							
(β_1, β_1)	$\frac{680}{429}$	$\frac{1360}{1287}$	$\frac{174080\sqrt{5}}{184041}$	$\frac{403580}{184041}$	$\frac{149780}{14157}\sqrt{\frac{1}{13}}$	$\frac{157580\sqrt{17}}{184041}$	$\frac{56060}{5577}\sqrt{\frac{1}{21}}$
(β_2, β_1)	$-\frac{81600}{7007}\sqrt{\frac{6}{143}}$	$-\frac{54400}{7007}\sqrt{\frac{6}{143}}$	$-\frac{6963200}{1002001}\sqrt{\frac{30}{143}}$	$-\frac{16143200}{1002001}\sqrt{\frac{6}{143}}$	$-\frac{5991200}{1002001}\sqrt{\frac{6}{11}}$	$-\frac{6303200}{1002001}\sqrt{\frac{102}{143}}$	$-\frac{6727200}{91091}\sqrt{\frac{2}{1001}}$
(β_3, β_1)	$\frac{333880}{21021}\sqrt{\frac{2}{143}}$	$\frac{667760}{63063}\sqrt{\frac{2}{143}}$	$\frac{85473280}{9018009}\sqrt{\frac{10}{143}}$	$\frac{198157780}{9018009}\sqrt{\frac{2}{143}}$	$\frac{73541980}{9018009}\sqrt{\frac{2}{11}}$	$\frac{77371780}{9018009}\sqrt{\frac{34}{143}}$	$\frac{27525460}{273273}\sqrt{\frac{2}{3003}}$
$J=11$							
(β_1, β_1)	$\frac{200}{231}$	$\frac{400}{693}$	$\frac{293600\sqrt{5}}{693693}$	$\frac{208900}{231231}$	$\frac{568300}{129591\sqrt{13}}$	$\frac{15714700}{1882881\sqrt{17}}$	$\frac{2534500}{323323\sqrt{21}}$
(β_2, β_1)	$\frac{1200}{77}\sqrt{\frac{2}{1729}}$	$\frac{800}{77}\sqrt{\frac{2}{1729}}$	$\frac{587200}{77077}\sqrt{\frac{10}{1729}}$	$\frac{1253400}{77077}\sqrt{\frac{2}{1729}}$	$\frac{1136600}{187187}\sqrt{\frac{2}{133}}$	$\frac{31429400}{209209}\sqrt{\frac{2}{29393}}$	$\frac{15207000}{2263261}\sqrt{\frac{6}{247}}$

Refs. [26–28]. Based on these formulas, we first express the matrix element of the two-body operator $[A^{\dagger s} \times \tilde{A}^s]^{(0)}$ between coupled pair basis states of three pairs in a single-j shell in terms of three-pair overlaps, given in Eq. (A1) of Appendix A. Then we express in closed forms two specific three-pair overlaps: one overlap between the basis state of three non-S pairs and the basis state of two non-S pairs and one S pair, which is given in Eq. (A2) of Appendix A, and the other overlap between basis states both having two non-S pairs and one S pair, which is given in Eq. (A3). Using the above formulas we derive exact values for matrix elements, as well as overlaps, between $|\beta_1\rangle$ and $|\beta_i\rangle$ ($i = 1, \ldots, n$). These exact values are presented in Table II, where one sees Eq. (8) is exactly satisfied for both the case of $J = 3$ and that of $J = 11$, indicating the nucleon-pair basis state $|\beta_1\rangle$ of $J = 3$ and that of $J = 11$ (the factors for normalization will be given later in Table III) are the eigenstates of *any* two-body interactions. With the matrix elements and overlaps listed in Table II we also have the energies of these two eigenstates, given by

$$\begin{aligned} E_{J=3,\beta_1} = {} & \tfrac{1}{3}V_0 + \tfrac{640}{429}V_2 + \tfrac{1187}{572}V_4 + \tfrac{7489}{2244}V_6 \\ & + \tfrac{7879}{1716}V_8 + \tfrac{2803}{884}V_{10}, \end{aligned} \tag{11}$$

$$\begin{aligned} E_{J=11,\beta_1} = {} & \tfrac{1}{3}V_0 + \tfrac{3670}{3003}V_2 + \tfrac{6267}{4004}V_4 + \tfrac{5683}{2244}V_6 \\ & + \tfrac{157147}{32604}V_8 + \tfrac{76035}{16796}V_{10}. \end{aligned} \tag{12}$$

The above two eigenstates, which are nucleon-pair states, both have a definite seniority number $v = 4$, as components with smaller seniority numbers are not possible for $J = 3$ and 11. They actually belong to a special series [15,19] consisting of eigenstates of *any* two-body interactions in the midshells. In Ref. [19] analytic expressions for eigenenergies of such states are obtained. The eigenenergies given in Eqs. (11) and (12) of this work are the same as those derived in Ref. [19].

IV. STATES OF DEFINITE SENIORITY NUMBERS AND EXACT WAVE FUNCTIONS OF ANY TWO-BODY INTERACTIONS

In this section we shall construct analytic expressions for states of definite seniority numbers in terms of nucleon-pair basis states, and derive exact pair wave functions for a few eigenstates of *any* two-body interactions in the midshells of $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$.

The states with good seniority quantum numbers are eigenstates of the monopole pairing interaction [33],

$$P = \frac{2j+1}{2} A^{\dagger(0)} \tilde{A}^{(0)},$$

and the states with definite seniority numbers $v = 0,\ 1,\ 2$ can be easily constructed. For a system with an even particle number $2N$, the seniority-0 state with $J = 0$ is coupled by N spin-0 S pairs

$$\underbrace{A^{\dagger(0)} A^{\dagger(0)} \cdots A^{\dagger(0)}}_{N} |0\rangle, \tag{13}$$

and the seniority-2 states with $J = 2, \ldots, 2j-1$, respectively, are coupled by $(N-1)$ S pairs and one spin-J pair

$$\underbrace{A^{\dagger(0)} \cdots A^{\dagger(0)}}_{N-1} A^{\dagger J} |0\rangle. \tag{14}$$

For a system with an odd particle number $(2N+1)$, the seniority-1 state with $J = j$ is

$$\underbrace{A^{\dagger(0)} A^{\dagger(0)} \cdots A^{\dagger(0)}}_{N} a_j^{\dagger} |0\rangle. \tag{15}$$

The states with larger seniority numbers can be constructed numerically. With the procedure consisting of steps (1), (2′), (3) and (4) described in Sec. II, a complete and orthogonal set of states all having definite seniority numbers can be constructed. One sees that using this procedure, states of definite seniority numbers for a given J are constructed successively, from the smallest seniority to the largest one.

From another perspective, we shall construct as follows analytic expressions for states of definite seniority numbers $v = 3,\ 4,\ 5$, directly from one nucleon-pair basis state with

TABLE III. Exact pair wave functions (denoted by $\frac{1}{\mathcal{N}}|\alpha\rangle$ with $\mathcal{N}^2 = \langle\alpha|\alpha\rangle$) for eigenstates of *any* two-body interactions in the midshells of $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$. Due to the complexity, the normalization factors for the three $\nu = 6$ states in the $j = \frac{11}{2}$ shell are numerically calculated. For systems of $2N$ particles, we use $|r_1, r_2, \ldots, r_N; J_2, \ldots, J\rangle$ to denote $((A^{\dagger r_1} \times A^{\dagger r_2})^{J_2} \cdots \times A^{\dagger r_N})^J|0\rangle$; for systems of $(2N+1)$ particles, we use $|j, r_1, r_2, \ldots, r_N; J_1, J_2, \ldots, J\rangle$ to denote $(((a_j^\dagger \times A^{\dagger r_1})^{J_1} \times A^{\dagger r_2})^{J_2} \cdots \times A^{\dagger r_N})^J|0\rangle$. For these eigenstates, exact energies in the form of the linear combination of two-body matrix elements are given in Ref. [19]. The wave functions presented here should be useful in interpreting electromagnetic moment and transition properties of corresponding nuclei.

	J	ν	$\lvert\alpha\rangle$	$\mathcal{N}^2$
$(7/2)^4$	2	2	$\lvert 2, 0; 2\rangle$	2
	2	4	$\lvert 2, 2; 2\rangle - \frac{8\sqrt{6}}{21}\lvert 2, 0; 2\rangle$	$\frac{132}{49}$
	4	2	$\lvert 4, 0; 4\rangle$	2
	4	4	$\lvert 2, 2; 4\rangle + \frac{2\sqrt{110}}{21}\lvert 4, 0; 4\rangle$	$\frac{1300}{441}$
$(9/2)^5$	5/2	3	$\lvert\frac{9}{2}, 2, 0; \frac{5}{2}, \frac{5}{2}\rangle$	$\frac{4}{3}$
	5/2	5	$\lvert\frac{9}{2}, 2, 2; \frac{5}{2}, \frac{5}{2}\rangle + \frac{5\sqrt{231}}{132}\lvert\frac{9}{2}, 2, 0; \frac{5}{2}, \frac{5}{2}\rangle + \frac{\sqrt{455}}{28}\lvert\frac{9}{2}, 4, 0; \frac{5}{2}, \frac{5}{2}\rangle$	$\frac{2080}{693}$
	7/2	3	$\lvert\frac{9}{2}, 2, 0; \frac{7}{2}, \frac{7}{2}\rangle$	$\frac{416}{165}$
	7/2	5	$\lvert\frac{9}{2}, 2, 2; \frac{5}{2}, \frac{7}{2}\rangle + \frac{78\sqrt{105}}{1512}\lvert\frac{9}{2}, 4, 0; \frac{7}{2}, \frac{7}{2}\rangle$	$\frac{442}{693}$
	9/2	3	$\lvert\frac{9}{2}, 2, 0; \frac{9}{2}, \frac{9}{2}\rangle + \frac{\sqrt{5}}{4}\lvert\frac{9}{2}, 0, 0; \frac{9}{2}, \frac{9}{2}\rangle$	$\frac{26}{165}$
	11/2	3	$\lvert\frac{9}{2}, 2, 0; \frac{11}{2}, \frac{11}{2}\rangle$	$\frac{136}{165}$
	11/2	5	$\lvert\frac{9}{2}, 2, 2; \frac{7}{2}, \frac{11}{2}\rangle - \frac{\sqrt{91}}{22}\lvert\frac{9}{2}, 2, 0; \frac{11}{2}, \frac{11}{2}\rangle + \frac{13\sqrt{1890}}{1188}\lvert\frac{9}{2}, 4, 0; \frac{11}{2}, \frac{11}{2}\rangle$	$\frac{96824}{35937}$
	13/2	3	$\lvert\frac{9}{2}, 2, 0; \frac{13}{2}, \frac{13}{2}\rangle$	$\frac{16}{11}$
	13/2	5	$\lvert\frac{9}{2}, 2, 2; \frac{9}{2}, \frac{13}{2}\rangle + \frac{4\sqrt{5}}{11}\lvert\frac{9}{2}, 2, 0; \frac{13}{2}, \frac{13}{2}\rangle + \frac{5\sqrt{78}}{66}\lvert\frac{9}{2}, 4, 0; \frac{13}{2}, \frac{13}{2}\rangle$	$\frac{68}{1331}$
	15/2	3	$\lvert\frac{9}{2}, 4, 0; \frac{15}{2}, \frac{15}{2}\rangle$	$\frac{456}{715}$
	15/2	5	$\lvert\frac{9}{2}, 2, 2; \frac{11}{2}, \frac{15}{2}\rangle + \frac{\sqrt{13090}}{132}\lvert\frac{9}{2}, 4, 0; \frac{15}{2}, \frac{15}{2}\rangle$	$\frac{2125}{4719}$
	17/2	3	$\lvert\frac{9}{2}, 4, 0; \frac{17}{2}, \frac{17}{2}\rangle$	$\frac{200}{143}$
	17/2	5	$\lvert\frac{9}{2}, 2, 2; \frac{13}{2}, \frac{17}{2}\rangle + \frac{\sqrt{5005}}{66}\lvert\frac{9}{2}, 4, 0; \frac{17}{2}, \frac{17}{2}\rangle$	$\frac{266}{99}$
$(11/2)^6$	3	4	$\lvert 2, 4, 0; 3, 3\rangle$	$\frac{680}{429}$
	11	4	$\lvert 2, 10, 0; 11, 11\rangle$	$\frac{200}{231}$
	11	6	$\lvert 2, 2, 8; 4, 11\rangle - 18\sqrt{\frac{2}{1729}}\lvert 2, 10, 0; 11, 11\rangle$	2.596
	13	4	$\lvert 4, 10, 0; 13, 13\rangle$	$\frac{280}{429}$
	13	6	$\lvert 2, 2, 10; 4, 13\rangle + \frac{24}{7}\sqrt{\frac{15}{143}}\lvert 4, 10, 0; 13, 13\rangle$	0.729
	14	4	$\lvert 4, 10, 0; 14, 14\rangle$	$\frac{868}{429}$
	14	6	$\lvert 2, 2, 10; 4, 14\rangle + \frac{24}{7}\sqrt{\frac{15}{143}}\lvert 4, 10, 0; 14, 14\rangle$	4.426

$[\frac{\nu}{2}]$ non-S pairs ($[\frac{\nu}{2}]$ denotes the largest integer not larger than $\frac{\nu}{2}$). First, we construct a seniority-ν state of a ν-particle system starting from a nucleon-pair basis state without S pairs. The key point we make use of is the property that such a seniority-ν state of a ν-particle system is a special eigenstate of the monopole pairing interaction, i.e., that with the eigenvalue of 0. The technique used here is that of calculating commutators between coupled clusters of fermion pairs [26–28]. In Appendix B we exemplify our method with the case of $\nu = 5$ and $J \neq j$. Second, we obtain the corresponding seniority-ν state of an n-particle system simply by adding $\frac{n-\nu}{2}$ S pairs.

Using the above procedure, for a system with an even particle number $2N$, we have the $\nu = 4$ states given by

$$\begin{cases} \underbrace{A^{\dagger(0)} \cdots A^{\dagger(0)}}_{N-2}[(A^{\dagger r} \times A^{\dagger r})^{(0)} + \frac{2\hat{r}}{2j-1}A^{\dagger(0)}A^{\dagger(0)}]|0\rangle & \text{if } J = 0, \\ \underbrace{A^{\dagger(0)} \cdots A^{\dagger(0)}}_{N-2}[(A^{\dagger r_1} \times A^{\dagger r_2})^J - \frac{4\hat{r}_1\hat{r}_2\hat{j}}{2j-3}\begin{Bmatrix} j & j & r_1 \\ r_2 & J & j \end{Bmatrix}A^{\dagger(0)}A^{\dagger J}]|0\rangle & \text{if } J = 2, \ldots, 2j-1, \\ \underbrace{A^{\dagger(0)} \cdots A^{\dagger(0)}}_{N-2}(A^{\dagger r_1} \times A^{\dagger r_2})^J|0\rangle & \text{otherwise.} \end{cases} \tag{16}$$

Here r, r_1, and r_2 are all not equal to 0. For a system with an odd particle number $(2N+1)$, we have the $\nu = 3$ states given by

$$\begin{cases} \underbrace{A^{\dagger(0)}A^{\dagger(0)}\cdots A^{\dagger(0)}}_{N-1}[(a_j^{\dagger}\times A^{\dagger r})^J + \frac{4\hat{r}}{4j-2}A^{\dagger(0)}a_j^{\dagger}]|0\rangle & \text{if } J=j, \\ \underbrace{A^{\dagger(0)}A^{\dagger(0)}\cdots A^{\dagger(0)}}_{N-1}(a_j^{\dagger}\times A^{\dagger r})^J|0\rangle & \text{otherwise}, \end{cases} \tag{17}$$

where r is not equal to 0. It is worthwhile to mention that, for example, for the $\nu = 4$ and $J = 0$ case, one knows in advance that the pair basis state $(A^{\dagger r} \times A^{\dagger r})^{(0)}$ includes the seniority-4 and seniority-0 components, and the seniority-0 component is $A^{\dagger(0)}A^{\dagger(0)}$, thus one can obtain the same expression for this case as in Eq. (16) by calculating corresponding overlaps and using the Schmidt orthogonalization. Similarly, the expression for the case of $\nu = 4$ and $J = 2, 4, \ldots, 2j-1$ as in Eq. (16) and that for the case of $\nu = 3$ and $J = j$ as in Eq. (17) can be also obtained by calculating corresponding overlaps and using the Schmidt orthogonalization.

The expressions for seniority-5 states are more complicated. For $J \neq j$, we have $\nu = 5$ states given by

$$\underbrace{A^{\dagger(0)}A^{\dagger(0)}\cdots A^{\dagger(0)}}_{N-2}\left\{[(a_j^{\dagger}\times A^{\dagger r_1})^{J_1}\times A^{\dagger r_2}]^J - \sum_{\{r_1'\}}[\lambda_1(r_1')]A^{\dagger(0)}(a_j^{\dagger}\times A^{\dagger r_1'})^J - \lambda_2 A^{\dagger(0)}(a_j^{\dagger}\times A^{\dagger r_1})^J - \lambda_3 A^{\dagger(0)}(a_j^{\dagger}\times A^{\dagger r_2})^J\right\}|0\rangle, \tag{18}$$

where

$$\begin{aligned} \lambda_1(r_1') &= (-)^{j+J}\frac{8\hat{j}\hat{r}_1\hat{r}_2\hat{r}_1'\hat{J}_1}{4j-10}\begin{Bmatrix} j & r_1 & J_1 \\ r_2 & J & r_1' \end{Bmatrix}\begin{Bmatrix} j & j & r_1 \\ r_2 & r_1' & j \end{Bmatrix}, \\ \lambda_2 &= (-)^{j+J_1+1}\frac{4\hat{j}\hat{r}_2\hat{J}_1}{4j-10}\begin{Bmatrix} j & r_1 & J_1 \\ J & r_2 & j \end{Bmatrix}, \\ \lambda_3 &= -\delta_{j,J_1}\frac{4\hat{r}_1}{4j-10}. \end{aligned} \tag{19}$$

Here r_1 and r_2 are not equal to 0; the new pair $A^{\dagger r_1'}$ is still described by Eq. (2), and r_1' is an even number satisfying the triangle conditions indicated by corresponding $6j$ symbols. For $J = j$, we have $\nu = 5$ states given by

$$\begin{aligned} \underbrace{A^{\dagger(0)}A^{\dagger(0)}\cdots A^{\dagger(0)}}_{N-2}\Bigg\{&[(a_j^{\dagger}\times A^{\dagger r_1})^{J_1}\times A^{\dagger r_2}]^j - \sum_{\{r_1'\}}(1-\delta_{r_1',0})[\lambda_1(r_1')]A^{\dagger(0)}(a_j^{\dagger}\times A^{\dagger r_1'})^j \\ &- \lambda_2 A^{\dagger(0)}(a_j^{\dagger}\times A^{\dagger r_1})^j - \lambda_3 A^{\dagger(0)}(a_j^{\dagger}\times A^{\dagger r_2})^j + \lambda_4 A^{\dagger(0)}A^{\dagger(0)}a_j^{\dagger}\Bigg\}|0\rangle, \end{aligned} \tag{20}$$

where $\lambda_1, \lambda_2, \lambda_3$ can be obtained by substituting $J = j$ in Eq. (19), and

$$\begin{aligned} \lambda_4 = \frac{4\hat{r}_1\hat{r}_2}{(4j-10)(2j-3)}\Bigg(&\hat{j}\hat{J}_1\sum_{\{r_1'\}}[2(1-\delta_{r_1',0})(2r_1'+1) + \frac{1}{2}\delta_{r_1',0}(2j+1)] \\ &\times\begin{Bmatrix} j & r_1 & J_1 \\ r_2 & j & r_1' \end{Bmatrix}\begin{Bmatrix} j & j & r_1 \\ r_2 & r_1' & j \end{Bmatrix} + (-)^{j+J_1}\hat{j}\hat{J}_1\begin{Bmatrix} j & r_1 & J_1 \\ j & r_2 & j \end{Bmatrix} + \delta_{j,J_1}\Bigg). \end{aligned} \tag{21}$$

For the $\nu = 5$ case, it is not feasible to obtain general expressions as in Eqs. (18)–(21), using the Schmidt orthogonalization. We have verified the above Eqs. (16)–(21) numerically. Note that the above states of definite seniority numbers are not normalized, and the normalization factors can be obtained by calculating corresponding overlaps.

As shown in Sec. III, two nucleon-pair states of six particles in the $j = \frac{11}{2}$ shell are the eigenstates of *any* two-body interactions. These two states having seniority 4 belong to a special series [15,19] consisting of eigenstates of *any* two-body interactions, which are in the midshells and multiplicity-free, i.e., uniquely defined by J and ν. As explained in Refs. [15,19], the emergence of such a series is related to the fact that nondiagonal elements of the Hamiltonian matrix between two states with seniority numbers differing by 2, vanish in the midshells. In Ref. [19] exact eigenenergies in the form of the linear combination of two-body matrix elements are derived for a number of states of this series in the midshells of $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \frac{15}{2}$.

Below we shall derive exact wave functions for a few states of this series in terms of nucleon-pair basis states. Note that an eigenstate of this series is uniquely defined by J and ν, thus a wave function of such J and ν values is necessarily the eigenfunction of *any* two-body interactions. We derive and present in Table III such eigenfunctions of *any* two-body interactions in the midshells of $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$, denoted as $\frac{1}{\mathcal{N}}|\alpha\rangle$ with $\mathcal{N}^2 = \langle\alpha|\alpha\rangle$. For the states with definite seniority numbers $\nu = 2, 3, 4, 5$, the non-normalized wave function $|\alpha\rangle$ is constructed starting from a pair basis state of $[\frac{\nu}{2}]$ non-S pairs and using the analytic expressions given in Eqs. (14), (16)–(21), respectively. For the three $\nu = 6$ states of six particles in the $j = \frac{11}{2}$ shell, the non-normalized eigenfunction $|\alpha\rangle$ is constructed by using the Schmidt orthogonalization, with exact overlaps which are calculated using Eqs. (A2) and (A3) of Appendix A. The normalization factors $\mathcal{N}$ are given by $\mathcal{N}^2 = \langle\alpha|\alpha\rangle$. Due to the complexity, the normalization factors for the three $\nu = 6$ states of six particles in the $j = \frac{11}{2}$ shell are numerically calculated.

As shown in Table III, the structures of these eigenstates of *any* two-body interactions, in terms of coupled nucleon-pair basis states, are very compact. Note that in Table III the pair basis states included in the same pair wave function are nonorthogonal. It is also worthwhile to mention that, for example, for the $J = \frac{5}{2}$ and $\nu = 5$ case of five particles in the $j = \frac{9}{2}$ shell, the expression of $|\alpha\rangle$ is derived starting from the pair basis state $|\frac{9}{2}, 2, 2; \frac{5}{2}, \frac{5}{2}\rangle$ and using Eqs. (18) and (19); one easily sees the second and third pair basis states in the $|\alpha\rangle$, i.e., $|\frac{9}{2}, 2, 0; \frac{5}{2}, \frac{5}{2}\rangle$ and $|\frac{9}{2}, 4, 0; \frac{5}{2}, \frac{5}{2}\rangle$, both have $\nu = 3$; as the $J = \frac{5}{2}$ and $\nu = 3$ subspace of five particles in the $j = \frac{9}{2}$ shell is one-dimensional, these two pair basis states are different from each other only by a factor, i.e., $|\frac{9}{2}, 4, 0; \frac{5}{2}, \frac{5}{2}\rangle = \sqrt{\frac{39}{55}}|\frac{9}{2}, 2, 0; \frac{5}{2}, \frac{5}{2}\rangle$. Similarly, in the $|\alpha\rangle$ of $J = \frac{11}{2}$ and $\nu = 5$, $|\frac{9}{2}, 4, 0; \frac{11}{2}, \frac{11}{2}\rangle = \sqrt{\frac{30}{13}}|\frac{9}{2}, 2, 0; \frac{11}{2}, \frac{11}{2}\rangle$; in the $|\alpha\rangle$ of $J = \frac{13}{2}$ and $\nu = 5$, $|\frac{9}{2}, 4, 0; \frac{13}{2}, \frac{13}{2}\rangle = 3\sqrt{\frac{3}{130}}|\frac{9}{2}, 2, 0; \frac{13}{2}, \frac{13}{2}\rangle$. One can transform the exact wave functions of Table III into those of the m scheme by using the Clebsch-Gordan coefficients. Note that, as shown in Eq. (2), coupled pairs used in this work are not normalized.

As exemplified in, e.g., Refs. [18,20], seniority quantum numbers are very useful in interpreting electromagnetic properties of semimagic nuclei with valence particles or holes dominantly occupying a high-j shell. It will be interesting to study electromagnetic moment and transition properties of semimagic nuclei—such as Ca isotopes around ^{44}Ca with neutrons in the $0f_{7/2}$ shell, Rh isotopes with proton holes in the $0g_{9/2}$ shell, Sn isotopes around ^{126}Sn with neutron holes in the $0h_{11/2}$ shell, and Pb isotopes around ^{213}Pb with neutrons in the $1g_{9/2}$ shell—using the exact wave functions given in Table III which have definite seniority quantum numbers and are eigenfunctions of *any* two-body interactions.

V. SUMMARY

In this work we study approximate and exact solutions for nucleons in a single-j shell from the perspective of nucleon-pair basis states, i.e., those coupled by pairs with both zero and nonzero good spins [26–28]. We find that for four, five, and six particles in the $0h_{11/2}$ shell, a selected set of independent nucleon-pair basis states leads to approximate solutions of a realistic two-body interaction, without resorting to the diagonalization.

We analytically show that for six particles in the $j = \frac{11}{2}$ shell, two nucleon-pair states with $J = 3$ and 11—which are coupled by three pairs of spin 0, 2, and 4 and by pairs of spin 0, 2, and 10, respectively—are eigenstates of *any* two-body interactions. We also present exact energies of these two eigenstates, which are the same as corresponding results of Ref. [19].

In particular, we construct general analytic expressions for states of definite seniority numbers $\nu = 3, 4, 5$ in terms of nucleon-pair basis states. Based on these expressions, we further derive exact wave functions for a few eigenstates of *any* two-body interactions in the midshells of $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}$. Exact energies of these eigenstates were derived recently in Ref. [19]. The exact wave functions obtained in this work should be useful in interpreting electromagnetic moment and transition properties of corresponding nuclei.

ACKNOWLEDGMENTS

We thank the National Natural Science Foundation of China (Grants No. 11875134, No. 11605122, No. 11775081, and No. 11675101), the Natural Science Foundation of Shanghai (Grant No. 17ZR1408900), the Program of Shanghai Academic/Technology Research Leader (Grant No. 16XD1401600), and the National Key R&D Program of China (Grant No. 2018YFA0404403) for financial support.

APPENDIX A: MATRIX ELEMENTS BETWEEN COUPLED PAIR BASIS STATES OF THREE PAIRS IN A SINGLE-j SHELL

In this Appendix we describe how we derive exact values for matrix elements of the two-body operator $[A^{\dagger s} \times \tilde{A}^{s}]^{(0)}$ between coupled pair basis states of three pairs in a single-j shell [the pair basis state $((A^{\dagger r_1} \times A^{\dagger r_2})^{J_2} \times A^{\dagger r_3})^{J}|0\rangle$ is denoted by $|r_1, r_2, r_3; J_2, J\rangle$ here]. The general formulas to calculate matrix elements of two-body interactions in coupled pair basis states were given in Refs. [26–28]. Based on these formulas, we first express the matrix element of the two-body operator $[A^{\dagger s} \times \tilde{A}^{s}]^{(0)}$ in terms of three-pair overlaps, and then obtain closed-form expressions for two specific three-pair overlaps, i.e., that between the basis state of three non-S pairs and the basis state of two non-S pairs and one S pair, and that between basis states both having two non-S pairs and one S pair. The matrix element of the two-body operator $[A^{\dagger s} \times \tilde{A}^s]^{(0)}$ is given by

$$\begin{aligned}
&\langle r_1, r_2, r_3; J_2, J|[A^{\dagger s} \times \tilde{A}^s]^{(0)}|s_1, s_2, s_3; J_2', J\rangle \\
&= \frac{2}{\hat{s}}(\delta_{r_3,s} + \delta_{r_2,s} + \delta_{r_1,s})\langle r_1, r_2, r_3; J_2, J|s_1, s_2, s_3; J_2', J\rangle + \sum_{\{r_2'\},t_2} 8\hat{r}_2\hat{r}_3\hat{J}_2\hat{t}_2\hat{r}_2'\langle r_1, r_2', s; t_2, J|s_1, s_2, s_3; J_2', J\rangle \sum_t (-)^{J_2+J+t+t_2+1} \\
&\quad \times (2t+1)\begin{Bmatrix} J_2 & t & t_2 \\ s & J & r_3 \end{Bmatrix}\begin{Bmatrix} r_1 & r_2 & J_2 \\ t & t_2 & r_2' \end{Bmatrix}\begin{Bmatrix} j & j & r_2 \\ t & r_2' & j \end{Bmatrix}\begin{Bmatrix} j & j & r_3 \\ s & t & j \end{Bmatrix} \\
&\quad + \sum_{\{r_1'\},t_2} 8\hat{r}_1\hat{r}_3\hat{J}_2\hat{t}_2\hat{r}_1'\langle r_1', r_2, s; t_2, J|s_1, s_2, s_3; J_2', J\rangle \sum_t (-)^{J+t+1} \\
&\quad \times (2t+1)\begin{Bmatrix} J_2 & t & t_2 \\ s & J & r_3 \end{Bmatrix}\begin{Bmatrix} r_1 & r_2 & J_2 \\ t_2 & t & r_1' \end{Bmatrix}\begin{Bmatrix} j & j & r_1 \\ t & r_1' & j \end{Bmatrix}\begin{Bmatrix} j & j & r_3 \\ s & t & j \end{Bmatrix} \\
&\quad + \sum_{\{r_1'\}} 8\hat{r}_1\hat{r}_2\hat{r}_1'\langle r_1', s, r_3; J_2, J|s_1, s_2, s_3; J_2', J\rangle \sum_t (-)^{J_2+1}(2t+1)\begin{Bmatrix} r_1 & t & r_1' \\ s & J_2 & r_2 \end{Bmatrix}\begin{Bmatrix} j & j & r_1 \\ t & r_1' & j \end{Bmatrix}\begin{Bmatrix} j & j & r_2 \\ s & t & j \end{Bmatrix}. \qquad \text{(A1)}
\end{aligned}$$

Note that r_1, r_2, r_3 and s_1, s_2, s_3 can be zero or nonzero. In the summations r_1', r_2' are even numbers, and r_1', r_2', t, t_2 must satisfy the triangle conditions indicated by corresponding $6j$ symbols. The new pairs, i.e., the spin-r_1' and spin-r_2' pairs, are still described by Eq. (2). The overlap between one basis state of three non-S pairs and one basis state of two non-S pairs and one S pair is given by

$$\begin{aligned}
&\langle r_1, r_2, r_3; J_2, J|s_1, s_2, 0; J, J\rangle \\
&= \sum_{\{r_2'\}} (-)^J \frac{32\hat{r}_2\hat{r}_3\hat{r}_2'\hat{J}_2}{\hat{j}}\begin{Bmatrix} r_1 & r_2 & J_2 \\ r_3 & J & r_2' \end{Bmatrix}\begin{Bmatrix} j & j & r_2 \\ r_3 & r_2' & j \end{Bmatrix}\left(\delta_{r_1,s_1}\delta_{r_2',s_2} + (-)^J\delta_{r_1,s_2}\delta_{r_2',s_1} - 4\hat{r}_1\hat{r}_2'\hat{s}_1\hat{s}_2\begin{Bmatrix} j & j & r_1 \\ j & j & r_2' \\ s_1 & s_2 & J \end{Bmatrix}\right) \\
&\quad + \sum_{\{r_1'\}} (-)^{J_2} \frac{32\hat{r}_1\hat{r}_3\hat{r}_1'\hat{J}_2}{\hat{j}}\begin{Bmatrix} r_1 & r_2 & J_2 \\ J & r_3 & r_1' \end{Bmatrix}\begin{Bmatrix} j & j & r_1 \\ r_3 & r_1' & j \end{Bmatrix}\left(\delta_{r_1',s_1}\delta_{r_2,s_2} + (-)^J\delta_{r_1',s_2}\delta_{r_2,s_1} - 4\hat{r}_1'\hat{r}_2\hat{s}_1\hat{s}_2\begin{Bmatrix} j & j & r_1' \\ j & j & r_2 \\ s_1 & s_2 & J \end{Bmatrix}\right) \\
&\quad + [1 + (-)^{J_2}]\frac{16\hat{r}_1\hat{r}_2}{\hat{j}}\begin{Bmatrix} j & j & r_1 \\ r_2 & J_2 & j \end{Bmatrix}\left(\delta_{J_2,s_1}\delta_{r_3,s_2} + (-)^J\delta_{J_2,s_2}\delta_{r_3,s_1} - 4\hat{J}_2\hat{r}_3\hat{s}_1\hat{s}_2\begin{Bmatrix} j & j & J_2 \\ j & j & r_3 \\ s_1 & s_2 & J \end{Bmatrix}\right), \qquad \text{(A2)}
\end{aligned}$$

Here r_1, r_2, and r_3, as well as s_1 and s_2, are all not equal to 0. In the summations r_1', r_2' are even numbers satisfying the triangle conditions indicated by corresponding $6j$ symbols. The overlap between basis states both having two non-S pairs and one S pair is given by

$$\langle r_1, r_2, 0; J, J|s_1, s_2, 0; J, J\rangle = \left(8 - \frac{64}{2j+1}\right)\left(\delta_{r_1,s_1}\delta_{r_2,s_2} + (-)^J\delta_{r_1,s_2}\delta_{r_2,s_1} - 4\hat{r}_1\hat{r}_2\hat{s}_1\hat{s}_2\begin{Bmatrix} j & j & r_1 \\ j & j & r_2 \\ s_1 & s_2 & J \end{Bmatrix}\right), \qquad \text{(A3)}$$

where r_1 and r_2, as well as s_1 and s_2, are all not equal to 0. Using the above formulas we obtain exact values presented in Table II, for matrix elements of the two-body operator $[A^{\dagger s} \times \tilde{A}^s]^{(0)}$, as well as overlaps, between the three-pair basis states.

APPENDIX B: CONSTRUCTION OF A SENIORITY-ν STATE OF A ν-PARTICLE SYSTEM WITH $\nu = 5$ AND $J \neq j$

In this Appendix we describe how we construct a seniority-ν state of a ν-particle system. We denote such states as $|\alpha\rangle = \alpha^\dagger|0\rangle$. Because $|\alpha\rangle$ manifests itself as the eigenfunction of the monopole pairing interaction with a special eigenvalue of 0, the commutator between $A^{\dagger(0)}\tilde{A}^{(0)}$ (proportional to the operator of the monopole pairing interaction) and the creation operator of $|\alpha\rangle$ must satisfy

$$[A^{\dagger(0)}\tilde{A}^{(0)}, \alpha^\dagger]|0\rangle = A^{\dagger(0)}\tilde{A}^{(0)}\alpha^\dagger|0\rangle = 0. \qquad \text{(B1)}$$

We then construct the expression of $|\alpha\rangle$ satisfying Eq. (B1), using the technique of calculating commutators between coupled clusters of fermion pairs [26–28].

We exemplify our method with the case of $\nu = 5$ and $J \neq j$. We start from a nucleon-pair basis state without S pairs, denoted as

$$|\beta\rangle = \beta^\dagger|0\rangle = ((a_j^\dagger \times A^{\dagger r_1})^{J_1} \times A^{\dagger r_2})^J|0\rangle, \qquad \text{(B2)}$$

where r_1 and r_2 are not equal to 0, and $J \neq j$. The commutator between $A^{\dagger(0)}\tilde{A}^{(0)}$ and the creation operator of $|\beta\rangle$ is given by

$$[A^{\dagger(0)}\tilde{A}^{(0)},\ ((a_j^\dagger \times A^{\dagger r_1})^{J_1} \times A^{\dagger r_2})^J]|0\rangle = \left\{ \sum_{\{r_1'\}} [c_1(r_1')] A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r_1'})^J + c_2 A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r_1})^J + c_3 A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r_2})^J \right\} |0\rangle, \quad \text{(B3)}$$

where

$$c_1(r_1') = (-)^{j+J} \frac{8\hat{r}_1\hat{r}_2\hat{r}_1'\hat{J}_1}{\hat{j}} \begin{Bmatrix} j & r_1 & J_1 \\ r_2 & J & r_1' \end{Bmatrix} \begin{Bmatrix} j & j & r_1 \\ r_2 & r_1' & j \end{Bmatrix},$$

$$c_2 = (-)^{j+J_1+1} \frac{4\hat{r}_2\hat{J}_1}{\hat{j}} \begin{Bmatrix} j & r_1 & J_1 \\ J & r_2 & j \end{Bmatrix}, \quad \text{(B4)}$$

$$c_3 = -\delta_{J_1,j} \frac{4\hat{r}_1}{2j+1},$$

and r_1' is an even number satisfying the triangle conditions indicated by corresponding $6j$ symbols. As $J \neq j$ in this case, $r_1' \neq 0$. Then one sees $[A^{\dagger(0)}\tilde{A}^{(0)}, \beta^\dagger]|0\rangle$ is equal to a linear combination of pair basis states all having one S pair, denoted as

$$|\beta'\rangle = \beta'^\dagger|0\rangle = A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r})^J|0\rangle, \quad \text{(B5)}$$

with $r \neq 0$ and $J \neq j$. We next obtain the commutator between $A^{\dagger(0)}\tilde{A}^{(0)}$ and the creation operator of $|\beta'\rangle$,

$$[A^{\dagger(0)}\tilde{A}^{(0)},\ A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r})^J]|0\rangle = d A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r})^J|0\rangle, \quad d = \frac{4j-10}{2j+1}. \quad \text{(B6)}$$

One sees that $[A^{\dagger(0)}\tilde{A}^{(0)}, \beta'^\dagger]|0\rangle$ is proportional to $|\beta'\rangle$ itself. This is because $A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r})^J|0\rangle$ with $r \neq 0$ and $J \neq j$ has definite seniority number $\nu = 3$ and is an eigenstate of the monopole pairing interaction.

We now construct the $|\alpha\rangle$ of $\nu = 5$ and $J \neq j$ as follows, using the above Eqs. (B3), (B4), and (B6):

$$\left\{ [(a_j^\dagger \times A^{\dagger r_1})^{J_1} \times A^{\dagger r_2}]^J - \sum_{\{r_1'\}} [\lambda_1(r_1')] A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r_1'})^J - \lambda_2 A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r_1})^J - \lambda_3 A^{\dagger(0)}(a_j^\dagger \times A^{\dagger r_2})^J \right\} |0\rangle, \quad \text{(B7)}$$

where λ_i is given by c_i/d,

$$\lambda_1(r_1') = \frac{c_1(r_1')}{d} = (-)^{j+J} \frac{8\hat{j}\hat{r}_1\hat{r}_2\hat{r}_1'\hat{J}_1}{4j-10} \begin{Bmatrix} j & r_1 & J_1 \\ r_2 & J & r_1' \end{Bmatrix} \begin{Bmatrix} j & j & r_1 \\ r_2 & r_1' & j \end{Bmatrix},$$

$$\lambda_2 = \frac{c_2}{d} = (-)^{j+J_1+1} \frac{4\hat{j}\hat{r}_2\hat{J}_1}{4j-10} \begin{Bmatrix} j & r_1 & J_1 \\ J & r_2 & j \end{Bmatrix},$$

$$\lambda_3 = \frac{c_3}{d} = -\delta_{j,J_1} \frac{4\hat{r}_1}{4j-10}. \quad \text{(B8)}$$

One sees that, with the $|\alpha\rangle$ given by Eqs. (B7) and (B8), Eq. (B1) is exactly satisfied. Thus this $|\alpha\rangle$ is the (non-normalized) eigenfunction of the monopole pairing interaction, which has a special eigenvalue of 0.

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[1] M. Mayer, Phys. Rev. **75**, 1969 (1949).
[2] O. Haxel, J. H. D. Jensen, and H. E. Suess, Phys. Rev. **75**, 1766 (1949).
[3] R. F. Casten, *Nuclear Structure From a Simple Perspective* (Oxford University Press, Oxford, 1990).
[4] D. J. Dean and M. Hjorth-Jensen, Rev. Mod. Phys. **75**, 607 (2003).
[5] Y. M. Zhao and A. Arima, Phys. Rep. **545**, 1 (2014).
[6] G. Racah, Phys. Rev. **62**, 438 (1942); **63**, 367 (1943).
[7] I. Talmi, Nucl. Phys. **172**, 1 (1971).
[8] S. Shlomo and I. Talmi, Nucl. Phys. A **198**, 81 (1972).
[9] D. J. Rowe and G. Rosensteel, Phys. Rev. Lett. **87**, 172501 (2001).
[10] A. Escuderos and L. Zamick, Phys. Rev. C **73**, 044302 (2006).
[11] L. Zamick, Phys. Rev. C **75**, 064305 (2007).
[12] P. Van Isacker and S. Heinze, Phys. Rev. Lett. **100**, 052501 (2008).
[13] L. Zamick and P. Van Isacker, Phys. Rev. C **78**, 044327 (2008).
[14] C. Qi, X. B. Wang, Z. X. Xu, R. J. Liotta, R. Wyss, and F. R. Xu, Phys. Rev. C **82**, 014304 (2010).
[15] I. Talmi, Nucl. Phys. A **846**, 31 (2010).
[16] C. Qi, Phys. Rev. C **83**, 014307 (2011).
[17] C. Qi, Z. X. Xu, and R. J. Liotta, Nucl. Phys. A **884-885**, 21 (2012).
[18] C. Qi, Phys. Lett. B **773**, 616 (2017).
[19] Y. B. Qian and C. Qi, Phys. Rev. C **98**, 061303(R) (2018).
[20] A. I. Morales, G. Benzoni, H. Watanabe *et al.*, Phys. Lett. B **781**, 706 (2018).
[21] I. O. Morales, P. Van Isacker, and I. Talmi, Phys. Lett. B **703**, 606 (2011).

[22] Y. Lei, Z. Y. Xu, Y. M. Zhao, S. Pittel, and A. Arima, Phys. Rev. C **83**, 024302 (2011).
[23] M. A. Caprio, F. Q. Luo, K. Cai, V. Hellemans, and Ch. Constantinou, Phys. Rev. C **85**, 034324 (2012); M. A. Caprio, F. Q. Luo, K. Cai, Ch. Constantinou, and V. Hellemans, J. Phys. G **39**, 105108 (2012).
[24] L. Y. Jia and C. Qi, Phys. Rev. C **94**, 044312 (2016).
[25] G. J. Fu, L. Y. Jia, Y. M. Zhao, and A. Arima, Phys. Rev. C **96**, 044306 (2017).
[26] J. Q. Chen, B. Q. Chen, and A. Klein, Nucl. Phys. A **554**, 61 (1993); J. Q. Chen, *ibid.* **562**, 218 (1993).
[27] J. Q. Chen, Nucl. Phys. A **626**, 686 (1997).
[28] Y. M. Zhao, N. Yoshinaga, S. Yamaji, J. Q. Chen, and A. Arima, Phys. Rev. C **62**, 014304 (2000).
[29] Y. Y. Cheng, Y. M. Zhao, and A. Arima, Phys. Rev. C **94**, 024307 (2016); Y. Y. Cheng, C. Qi, Y. M. Zhao, and A. Arima, *ibid.* **94**, 024321 (2016).
[30] C. Qi and Z. X. Xu, Phys. Rev. C **86**, 044323 (2012).
[31] R. Machleidt, Phys. Rev. C **63**, 024001 (2001).
[32] M. Hjorth-Jensen, T. T. S. Kuo, and E. Osnes, Phys. Rep. **261**, 125 (1995).
[33] I. Talmi, *Simple Models of Complex Nuclei* (Harwood, New York, 1993).