PHYSICAL REVIEW C VOLUME 10, NUMBER 2 AUGUST 1974

Dynamic single-particle effects in fission

Y. Boneh
Nuclear Research Center, Negev, Beersheva, Israel, and Weizmann Institute of Science, Rehovot, Israel

Z. Fraenkel
Weizmann Institute of Science, Rehovot, Israel
(Received 20 March 1974)

We calculate for a model nucleus the single-particle excitations in the transition from the saddle point to the scission point in the fission process. The model nucleus consists of a square well potential of finite depth filled with noninteracting "protons" and "neutrons." At every stage of the transition between the saddle point and scission point the shape of the potential surface is equal to the surface of the fissioning nucleus as predicted by the liquid-drop model but the depth of the potential is held constant. The rate of change of the nuclear surface is assumed to be equal to that predicted by the dynamical liquid-drop-model calculations of Nix. It is found that for this time scale there is a small probability for the particles to be raised to levels above the potential well and thus be emitted from the nucleus. The calculated number of emitted neutrons and protons is in qualitative agreement with the experimental results for the emission of scission neutrons and protons. However, there is a large energy transfer from the collective to the single-particle degrees of freedom and hence the transition cannot be considered adiabatic for this time scale. The inclusion in the model of a residual interaction is expected to reduce both the number of particles emitted and the energy transfer from the collective to single-particle degrees of freedom, thus making the transition more nearly adiabatic.

I. INTRODUCTION

The analogy between the motion of an incompressible charged fluid under the influence of surface tension and nuclear fission is a basic feature of the liquid-drop model (LDM) of atomic nuclei.[1] The model is essentially a many-particle model, in which the particles have a mean free path much smaller than the total dimension of the system.[2] This seems to contradict an independent particle model (IPM) of fission in which each particle moves in an average potential well, nearly independent of the other nucleons. Yet since one of the basic features of LDM is the use of the over-all shape of the nucleus as a dynamical variable it is possible to relate the IPM to the LDM.

Using the shape of the nucleus to determine the boundaries of the average potential in which the particles move, Hill and Wheeler[3] showed that by an appropriate definition of the nuclear potential energy it is possible to reproduce approximately the LDM nuclear surface energy. In the "collective model" of fission of Hill and Wheeler the direct coupling of the particles with each other is neglected and the coupling between the particles and the collective motion of the nucleus is assumed to take place through the motion of the moving potential well. Consistency of the model requires that the single-particle excitation energy be small compared to the energy associated with the collective motion of the fissioning nucleus. It follows that the justification of the collective model relies on the assumption that the fission process is approximately adiabatic. [The requirement for near adiabaticity is manifestly called for in the dynamical calculations of fission in which one uses the cranking model for the calculations of the inertia tensor.[4]]

In the framework of the pure LDM the inertia tensor is obtained by treating the nucleus as a drop of homogenous incompressible nonviscous fluid.[5] Solving the hydrodynamical equations governing the motion of such a fluid, Nix[6] was able to describe the motion of the fissioning nucleus from the saddle point to the scission point. If this motion is nonadiabatic then the single-particle excitation energy should be treated as a viscosity term to be added to the classical hydrodynamic equations.

The emission of light particles in fission is probably closely connected with the nonadiabaticity of the transition from saddle shape to scission. It was first postulated by Halpern[7] that the fast potential change occurring in the neck region between the two fission fragments is responsible for the emission of light particles in fission. Hence the theoretical investigation of the emission of these particles may help to clarify to what extent the adiabatic approximation is justified when dealing with the nuclear motion towards the scission point.

In the present work a single-particle model is proposed for the calculation of the single-particle excitation energy and the emission probabilities of neutrons and protons in fission. We compare the results of the calculations with the experimentally measured number of particles emitted and discuss the probability of particle emission for the time scale predicted by nonviscous liquid-drop model calculations. Conversely, assuming the particles are emitted due to a fast potential change, the measured number of particles emitted determine the time scale needed for the nucleus to make the transition from saddle point to the scission point. For the same time scale, the over-all single-particle excitation energy may be compared with the collective kinetic energy and a quantitative measure of the adiabaticity of the fission process can be obtained.

II. MODEL

A. General

During the transition from the saddle point to the scission point the shape of the nucleus changes rapidly. The fast change in the shape of the nuclear potential may cause single-particle excitations in the nucleus. The amount of these single-particle excitations will depend on the transition time. For a very slow transition the motion is adiabatic, i.e., the wave function at any particular nuclear deformation is given by solutions of the static Schrödinger equation for this deformation.

Our model nucleus consists of a finite square-well potential whose surface in configuration space is equal to the surface of the nucleus as predicted by the LDM. The potential well is filled with Z "protons" and $A-Z$ "neutrons." The particles are assumed to have no residual interaction and no spin-orbit force.

The neglect of the spin-orbit force is mainly a matter of convenience. Its inclusion would not complicate our calculation in any essential way but would increase the computation time substantially. It is not believed to greatly affect our results and it was therefore neglected at this stage. The situation is quite different with respect to the residual interaction (e.g., pairing force). As discussed below, its inclusion will probably change the results of our calculation quite markedly. However it would also change the nature of our calculation from that of a single-particle Hamiltonian to that of a many-body Hamiltonian. The solution of this vastly more complicated problem was not attempted in the present calculation.

The depth of the nuclear potential is assumed to be constant throughout the transition from saddle point to scission. Its value for neutrons is chosen so that the neutron binding energy is equal to the experimental value for an (A, Z) nucleus in its ground state. Similarly the depth of the proton potential is given by the equation

$$V(\vec{r}) = V_C(\vec{r}) \qquad \text{outside the nucleus},$$

$$V(\vec{r}) = -(E_F + E_B) \qquad \text{inside the nucleus } (E_F, E_B > 0),$$

where E_F is the proton Fermi energy, E_B is the experimental proton ground-state binding energy, and $V_C(\vec{r})$ is the actual Coulomb potential for the given nuclear shape

$$V_C(\vec{r}) = \frac{Ze^2}{v} \int_v \frac{d^3\vec{r}'}{|\vec{r} - \vec{r}'|};$$

$\vec{r}$ is the radius vector and v is the nuclear volume. $V_C(\vec{r})$ changes therefore along the surface of the deformed nucleus and it also changes with the deformation of the nucleus during the fission process. The surface of the potential well approximately follows throughout the transition from saddle point to scission point the nuclear shape as obtained by the dynamical LDM calculations of Nix.[6] Figure 1 shows this shape for the saddle point, the scission point, and in an intermediate stage between saddle point and scission. The nuclear volume is rigorously conserved during the transition.

The time sequence (i.e., change of nuclear shape as a function of *time*) of the transition is treated in the present calculation as an independent function. The transition time predicted by the nonviscous LDM[6] as well as slower and faster transition times are investigated.

The first stage of the calculation consists of solving the static Schrödinger equation for the saddle-point configuration. We thus obtain the single-particle wave functions of the nucleons at this point, which is the starting point in our dynamical treatment of the fission process. For definiteness the nucleus is assumed to be in its ground state. (Our model can treat an excited nucleus at the saddle point equally well.)

In the second stage of the calculation the static solutions of the Schrödinger equation for the saddle point serve as initial conditions for calculating the time-dependent single-particle wave functions for

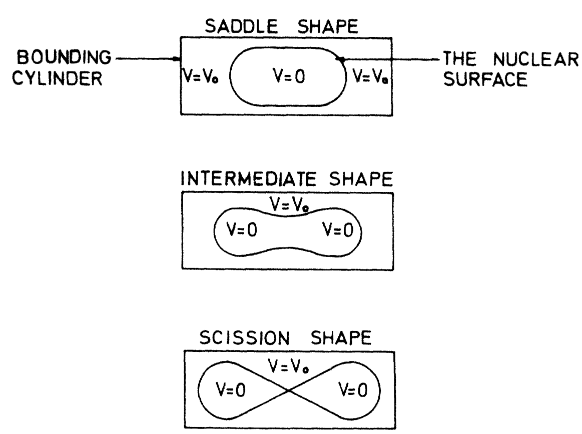

FIG. 1. The shape of the single-particle potential well at the saddle point, at an intermediate deformation, and at the scission point.

the transition from the saddle point to the scission point. This is done by solving the time-dependent Schrödinger equation for the transition.

The third stage of the calculation consists of solving the static Schrödinger equation for the scission point, and obtaining the static single-particle wave functions for this configuration. The overlap integrals between these static solutions and the time-dependent wave functions yield the transition probabilities and hence the single-particle excitations during the saddle-to-scission transition.

In contrast to most calculations concerned with various single-particle aspects of the fission process, our calculation does not make use of the expansion of the single-particle wave functions into basis functions but solves the Schrödinger equation directly by a numerical method to be discussed below. The advantage of the present method is that it is applicable to a nucleus of arbitrary shape and the accuracy of the method is essentially independent of the deformation of the nucleus.

B. Single-particle eigenfunctions

The Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi = E\psi \qquad (1)$$

was solved numerically by transforming the elliptic differential equation to a system of difference equations. The method is quite similar to the work of Brandt and Kelson.[2] However, they used an infinite potential well for the nuclear potential.

Since the shapes (and thus the potentials) considered are axially symmetric Eq. (1) is essentially two dimensional. In order to obtain a numerical solution, cylindrical boundary conditions were imposed on the eigenfunction, i.e., the functions are assumed to vanish on a large cylinder which encloses the nucleus. The added cylindrical boundary condition has the effect of raising the single-particle energies and also changing the shape of the eigenfunctions. This effect can be made negligibly small by making the dimensions of the cylinder sufficiently large. In the Appendix the error due to the finite size of the cylinder is estimated.

The time-dependent equation for the transition from saddle to scission is

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V(\vec{x},t)\psi\,, \qquad (2)$$

which in this case is a complex two-dimensional elliptic equation. By separating ψ into its real and imaginary parts the complex equation is replaced by two coupled real differential equations,

$$\frac{\partial u}{\partial t} = \frac{-h}{2m}\nabla^2 v + \frac{V(\vec{x},t)\cdot v}{\hbar}\,,$$
$$\frac{\partial v}{\partial t} = \frac{\hbar}{2m}\nabla^2 u + \frac{V(\vec{x},t)\cdot u}{\hbar}\,. \qquad (3)$$

In order to solve numerically these equations, they are transformed into difference equations. The explicit form of the difference equations imposes restrictions on the value of Δt, the time step used in the numerical differentiation. It has to be chosen small enough in comparison with the square of the spatial grid dimensions to ensure the stability of the solution.[8] This difficulty was avoided by using an implicit form of the difference equations. In this scheme it was necessary to solve a large system of linear equations, repeating the procedure for each time step. Instead of using an iteration method to solve this system of equations, the alternating direction method[8] was used. In this method the solution of the two-dimensional equations is replaced by two successive solutions of much simpler one-dimensional equations.

The complex time-dependent Schrödinger equation was also solved numerically by Fuller[9] who used a simplified one-dimensional model to describe the fission process. In Fuller's work the explicit formulation was used and the difficulty of the instability of the solution was overcome by using a small enough time step.

C. Transition probabilities

Denote by $\phi_i(0)$ the eigenfunctions for the saddle shape. As mentioned above these functions serve as the initial conditions for calculating the time-dependent eigenfunctions in the transition to the scission point. Accordingly, we denote by $\phi_i(t)$ the functions which evolve in time from the functions $\phi_i(0)$. Denote by ψ_i the (static) eigenfunctions for the scission point, and the transition time from saddle to scission by T. Then

$$P_{ij} = |\langle\phi_i(T)|\psi_j\rangle|^2 = |c_{ij}|^2 \qquad (4)$$

is the probability of transition of a single particle which was in the ith state for the initial saddle-point configuration to the jth state at the time of scission. It was shown by Fuller[9] that the many-particle transition probability may be obtained by simply summing up the single-particle probabilities and that the summation does not violate the Pauli principle.

D. Emission probabilities

We assume that a particle is emitted from the nucleus if that particle makes a transition to level

j at the scission point and the energy E_j of this new level is greater than V, the depth of the potential well.

Levels of energy greater than V should be considered as an approximation to the true continuum states. These "quasibound" levels are the result of the cylindrical boundary conditions discussed before. The approximation of the continuum by discrete levels has been treated in connection with the Strutinsky procedure[10] and recently in the calculations of level densities.[11] Based on the result of these authors it is to be expected that as long as the particles are capable of making transitions only to levels slightly above the Fermi level the error introduced by replacing the continuum by a set of discrete levels is small.

The total number of particles emitted from level i is thus given by

$$N_i = 2\sum_{j=k}^{\infty} P_{ij},$$

where the summation over j extends over all unbound levels. The factor of 2 is due to the spin degeneracy. For protons, k is the lowest level above the Coulomb barrier.

The possibility of particle emission after the scission stage is not treated here. The assumption that the particle is emitted from the nucleus only upon attaining an energy higher than the total potential depth is also justified for protons, since, as seen below, the effect of tunneling through the Coulomb barrier is negligible. Another effect which must be considered for both protons and neutrons is the reabsorption of the emitted particle by the moving fragments.

E. Single-particle excitation energy

The total single-particle energy at the saddle point is

$$E_0 = 2\sum_{i=1}^{n} E_i' .$$

(E_i' is the single-particle energy of level i at the *saddle* configuration.) The summation extends over all occupied levels.

The expectation value of the energy of a particle which makes a transition from level i in the saddle configuration is

$$\langle E\rangle_i = \sum_{j=1}^{\infty} P_{ij} E_j ,$$

where E_j is the single-particle energy of level j at the scission configuration. The total single-particle energy at scission is thus

$$E_s = 2\sum_{i=1}^{n}\sum_{j=1}^{\infty} P_{ij} E_j .$$

Denote by E_{s_0} the total single-particle energy for the scission ground state, namely, for the case of the lowest possible levels being occupied at the scission point

$$E_{s_0} = 2\sum_{j=1}^{n} E_j .$$

Clearly $E_s \geq E_{s_0}$ and it is possible to define an excitation energy E_{ex} given by $E_{ex} = E_s - E_{s_0}$. E_{ex} measures in general the nonadiabaticity of the fission process. It is to be expected that the shorter the time scale for the transition from saddle point to scission the greater the value of E_{ex}.

Due to the cylindrical symmetry assumed for the nuclear shape the magnetic quantum number for the *total* wave function and parity (in case of symmetric fission only) are conserved. In addition since residual interactions are not incorporated into the present model, the *single-particle* magnetic quantum number and parity are also conserved and can be used to classify energy levels. The conservation of the single-particle quantum numbers results in nuclear "excitation" *even for the case of an adiabatic transition*[12] since levels belonging to different quantum numbers may cross and hence even if the lowest possible levels were occupied at the saddle point, the nucleus may be in an "excited state" at the scission point. This "adiabatic excitation" must be subtracted from the over-all nuclear excitation E_{ex} in order to obtain the real single-particle energy gain.

Denote by E_a the total single-particle energy at scission for the *adiabatic* case $E_a = 2\sum_{i=1}^{n} E_i$. The summation extends over all levels which *were occupied in the saddle configuration*. The adiabatic excitation is $(E_a - E_{s_0})$ and the "viscous" or nonadiabatic excitation is given by $E_v = E_s - E_a$. For the adiabatic case $E_v = 0$.

It should be emphasized that the adiabatic excitation $(E_a - E_{s_0})$ in our model is the result of the conservation of the single-particle quantum numbers. However for finite transition times a significant part of this excitation may remain even upon the addition of an appropriate residual interaction because of "slippage."[4] This question and its possible connection with the dependence of $(E_a - E_{s_0})$ on the nuclear shape and asymmetric fission are discussed in Ref. 12.

F. Choice of physical parameters

Our calculations were made for $^{237}_{93}$Np since it is the heaviest nucleus for which there are results of dynamical LDM calculations.[6] Figure 1 shows the saddle-point and scission shapes of this nucleus. The volume of both shapes is equal to $v = \frac{4}{3}\pi r_0^3 A$ with $r_0 = 1.2249\times10^{-13}$ cm.[6] The value of the potential depth for neutrons was chosen so that the binding energy corresponds to the experimental value $E_B = 6.95$ MeV.[13] In the same way the proton potential was chosen to yield the experimental binding energy ($E_B = 5.4$ MeV[13]). Based on the above, one obtains for the nuclear potential of the neutrons $V = -46$ MeV and for total potential of the protons $V = -37$ MeV inside the nucleus and $V = V_c(\vec{r})$ outside.

The nuclear shape during the transition from saddle to scission was obtained by interpolation between the initial and final configurations while keeping the volume constant. The nuclear potential depth, V, was not changed from saddle to scission, but for protons the Coulomb potential naturally varied with the nuclear shape.

The parametrization used to describe the nuclear surface is that used by Nix,[6] namely the nuclear surface is described by three smoothly joined quadratic surfaces of revolution. For shapes with reflection symmetry, the generator functions for the surface are in cylindrical coordinates:

$$\rho^2(z,t) = \begin{cases} a_2^2(t) - b_2^2(t)[z - l_2(t)]^2 & z_2(t) \le z\,, \\ a_3^2(t) + b_3^2(t)z^2 & 0 \le z \le z_2(t)\,. \end{cases} \tag{5}$$

Six parameters are needed to parametrize the surface (of symmetric shapes). Two of these (e.g., l_2 and z_2) are eliminated by the requirement that the functions and their first derivatives are continuous at the point $z = z_2$. Denote by a_{i2}, a_{i3}, b_{i2}, and b_{i3}, the four parameters needed to describe the saddle shape and by a_{f2}, a_{f3}, b_{f2}, and b_{f3} the parameters for scission. The time-dependent parameters for the transition stage were obtained by:

$$a_2^2(t) = a_{i2}^2 + (a_{f2}^2 - a_{i2}^2)\beta(t)\,,$$

$$b_2^2(t) = b_{i2}^2 + (b_{f2}^2 - b_{i2}^2)\beta(t)\,.$$

The six parameters (including l_2 and z_2) were divided by a scale factor[6] in order to insure volume conservation. $\beta(t)$ is a continuous function of time, $\beta(t) = 0$ for $t = 0$, and $\beta(t) = 1$ for $t = T$, the time of transition from saddle point to scission.

The time sequence $\beta(t)$ may be so chosen that the function $\rho(z,t)$ will reproduce very closely the liquid-drop-model time sequence.[6] Yet we found it more convenient to carry out most of the calculations using the function $\beta_F(t)$ proposed by Fuller[9]

$$\beta_F(t) = \tfrac{1}{2}\left\{1 + \tfrac{1}{2}\left[3\left(\frac{t}{\tau} - 1\right) - \left(\frac{t}{\tau} - 1\right)^3\right]\right\} \tag{6}$$

$\beta_F(t) = 0$ for $t = 0$ and $\beta_F(2\tau) = 1$. This function has the advantage of having zero slope not only at $t = 0$ but also at $t = 2\tau$. Due to the peculiar shape of the function, τ rather than 2τ serves a measure for the time interval of the transition to scission.[9] The transition time as calculated in Ref. 6 for a nonviscous fluid is 3.10^{-21} sec for an initial excitation (in the fission mode) of 1 MeV at the saddle configuration.

III. RESULTS

In Table I we show the number of emitted neutrons N_i and the excitation energy E_v for neutrons. The numbers are grouped according to the magnetic quantum number m and parity (symmetric fission is considered). The "effective" transition time τ is 3×10^{-21} sec. In the table we also show the total single-particle energies for the saddle and scission configurations, E_0 and $E_s(2\tau)$, respectively, as well as the total energy E_a at scission for an adiabatic transition.

The total (even plus odd parity) neutron energy for an adiabatic transition is $\sum E_a = 3742.1$ MeV. This number is to be compared with the total neutron *ground-state* energy at scission $\sum E_{s_0} = 3700.5$ MeV. Hence the total adiabatic excitation energy of the neutrons at scission is $\sum E_a - \sum E_{s_0} = 41.6$ MeV. [Note that the ground-state energy for neutrons at the scission point is higher than that at the saddle point. The lower *total* (proton plus neutron) ground-state energy at the scission point is due to the reduction of the Coulomb energy rather than that of the nuclear energy.]

The large excitation energy for the $m = 1$, odd-parity group results from the unusual behavior of the sixth level, the highest occupied level for $m = 1$, odd parity. This level almost crosses levels lying above it and a particle occupying this level has an appreciable probability to make a transition to these higher levels[14] thus gaining substantial energy. In Table II the excitation energies and the number of emitted particles are shown as a function of the effective transition time τ for neutrons and protons. In view of the fact that the height of the Coulomb barrier is not constant along the nuclear surface we show the number of protons emitted to levels above the *lowest* value of the Coulomb barrier ($V_{\min}$) and above the *highest* value of the Coulomb barrier ($V_{\max}$).

For protons we considered the effect of tunneling through the potential barrier. The integral $I(t) = \int_{\text{out}} |\phi_i(\vec{r},t)|^2 d^3\vec{r}$ was performed over the volume *outside* the nuclear boundary. This integral was calculated for all the pertinent wave functions ϕ_i. The potential at the scission point $V(\vec{r}, 2\tau)$ was held constant and the development in time of $\phi_i(\vec{r},t)$ was calculated from $t = 2\tau$ to $t = 10\tau$. To a very good approximation $I(10\tau) = I(2\tau)$, ($\tau = 1.10^{-21}$ sec). It follows that for the time scale considered here the effect of tunneling is negligible.

TABLE I. The total neutron energy E_0 for the saddle configuration; the total neutron energy at the scission configuration for an *adiabatic transition*, E_a; the total neutron energy for the scission configuration for a finite transition time, $E_s(2\tau)$; the total "viscous" excitation energy $E_v(2\tau) = E_s(2\tau) - E_a$; and the total number of emitted neutrons $N_i(2\tau)$. The values are given separately for each parity and magnetic quantum number. All values were calculated for $\tau = 3\times10^{-21}$ sec. The energies are measured from the bottom of the potential well.

Magnetic quantum number m	Number of particles	Total energy at saddle configuration E_0 (MeV)	Total energy at scission for adiabatic transition E_a (MeV)	Total energy at scission $E_s(2\tau)$ (MeV)	Total "viscous" excitation energy $E_v(2\tau)$ (MeV)	Number of neutrons emitted N_i
			Even parity			
0	2× 9	368.7	364.0	367.0	3.0	0.002
1	2×14	675.5	729.6	732.4	2.8	0.211
2	2× 8	398.4	449.2	450.0	0.8	0.043
3	2× 6	355.2	411.2	411.2	~0	0.035
4	2× 3	209.7	237.6	237.6	~0	0.014
Total	80	2007.5	2191.6	2198.2	6.6	0.31
			Odd parity			
0	2× 8	346.4	294.5	299.1	4.6	~0
1	2×12	601.3	571.2	588.4	17.2	~0
2	2× 6	290.1	294.2	294.2	~0	~0
3	2× 4	231.2	243.2	243.2	~0	~0
4	2× 2	140.4	147.4	147.4	~0	~0
Total	64	1609.4	1550.5	1572.3	21.8	0

IV. DISCUSSION

The experimental value for the number of the "scission" neutrons emitted in the spontaneous fission of ^{252}Cf is estimated to be 0.25–0.35 neutrons per fission[15,16] whereas the number of protons emitted is 5×10^{-5} protons per fission.[17] We assume that these so-called scission particles are emitted during the transition from saddle point to the scission point.[9] The experimental numbers are therefore to be compared with our calculated results (Table II). It is seen that for the transition time $\tau = 6\times10^{-21}$ sec the number of neutrons and protons emitted are of the same order of magnitude as the experimental values. For the same transition time, the total excitation energy E_v is approximately 35 MeV. This number is to be compared with the difference in the potential energy between the saddle point and the scission point which is 40 MeV according to liquid-drop-model calculations.[6] The excitation energy is therefore evidently very high since it is close to the potential energy available for the transition.

The number of particles emitted as given in Tables I and II was calculated by assuming that once a particle gains an energy higher than the potential barrier the particle is emitted irrespective of the spatial distribution of its wave function. In order to estimate the effect of the reabsorption of the particles by the fragments we consider the probability of a particle which was in the ith state at the saddle point to be outside the nuclear volume at the moment of scission. This probability may be written in the form

$$P_{\text{out}} = \int_{\text{out}} \left| \sum_{j=0}^{k-1} c_{ij}\psi_j \right|^2 d^3\vec{r} + \int_{\text{out}} \left| \sum_{j=k}^{\infty} c_{ij}\psi_j \right|^2 d^3\vec{r} + \text{interference term}, \tag{7}$$

where the expansion coefficients c_{ij} are defined by Eq. (4), k is the first unbound level, and the integration is carried out over the space outside the nucleus at scission. The first term of Eq. (7) is related to the (small) probability of *bound* particles to be outside the nucleus. The second term gives

TABLE II. Number of neutrons and protons emitted and the excitation energies E_v as a function of τ.

Transition time τ (sec)	Number of particles: Neutrons	Protons V_{max}	Protons V_{min}	Excitation energy E_v (MeV): Neutrons	Protons
1.5×10^{-21}	2.09	0.02	0.09	44.4	30.5
3×10^{-21}	0.31	9×10^{-4}	0.01	28.4	26.7
6×10^{-21}	0.10	1×10^{-4}	7×10^{-4}	22.8	12.3

approximately (except for the interference term) the probability of *unbound* nucleons to be outside the nuclear volume at the moment of scission. It is this term which is of interest in the present discussion. Calculations show that the interference term is small. Hence we may consider the second term of Eq. (7) to be a lower limit for the emission probability of the particle. In Table III the number of neutrons emitted according to the last assumption is shown. It is seen from the table that only for the time scale 1.5×10^{-21} sec does the calculated number agree with the experimental results. Since the number of protons emitted is smaller by three orders of magnitude, we could not calculate this number for protons.

The transition probabilities for neutrons were calculated for several levels near the Fermi energy using the Landau-Zener approximation.[18] The results[19] agree only qualitatively with the transition probabilities obtained by the present numerical method. This is to be expected since the conditions justifying the use of the Landau-Zener approximation[14] are only partially fulfilled for the level structure and time scale considered here.

As already mentioned a spin-orbit force was not included in the model. The inclusion of a spin-orbit term would affect to some extent the position of the energy levels and hence would also change the number of emitted particles and the excitation energy. Yet the addition of the spin-orbit force would not change the essential features of our model since this force does not change significantly with deformation.[2]

The inclusion of residual interactions will prevent single-particle excitations for infinite transition times from saddle to scission. In addition, the residual interaction will increase the minimum distance between the neighboring levels and so reduce the transition probability between these levels. Hence the number of particles emitted and the excitation energies shown in Table II will be smaller if a residual interaction is incorporated into the model.

Despite the deficiencies of our model, the emission probabilities calculated here seem to support the assumption that neutrons and protons are emitted as the result of the nonadiabaticity of the nuclear transition from saddle point to scission point.

TABLE III. Number of neutrons emitted based on the integral of the wave function outside the nucleus.

τ (sec)	Number of neutrons
1.5×10^{-21}	0.44
3.0×10^{-21}	0.06
$6\ \ \times10^{-21}$	~0

The number of protons and neutrons emitted is in reasonable agreement with the experimental values for the time scale predicted by the liquid-drop model. The results also indicate that some internal nuclear excitation arises as a result of the rapid nuclear transition from saddle to scission. Hence the time interval predicted for this transition by the liquid-drop model without the inclusion of a viscosity term is probably too short. In order to give a better estimate for the transition time both spin-orbit and residual interactions must be added to the model proposed here.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the very many useful discussions with I. Kelson, M. W. Kirson, J. R. Nix, Z. Paltiel, H. C. Pauli, H. A. Weidenmuller, and L. Wilets which have greatly helped us in the various aspects of this work. We wish to thank in particular T. Ledergerber for his very helpful suggestions.

APPENDIX. NUMERICAL ACCURACY

The accuracy of the calculations was checked separately for the static and the time-dependent calculations. The results given here were obtained by using a grid with 40×30 mesh points in the z and ρ directions, respectively. For symmetric fission it is possible, by using appropriate boundary conditions, to treat only one half of the nuclear shape and due to the cylindircal symmetry only the upper right hand part of the (ρ, z) plane need be considered. The dimensions of the cylinder enclosing the nucleus was taken to be $z_0=20\times10^{-13}$ cm and $\rho_0=7\times10^{-13}$ cm in comparison with $z_{max}=18.82\times10^{-13}$ and $\rho_{max}=5.43\times10^{-13}$ cm, the maximum (half) length and the maximum radius of the nucleus considered.

The error in the calculated energies is estimated to be less than 1%. This estimate was obtained by comparing the known energies for a spherical potential well *without* cylindrical boundary conditions with the ones calculated by the grid method including cylindrical boundary conditions. Using this method we obtained at once an estimate for the error due to both the finite size of the outer cylinder and the finite number of mesh points used in the calculations. The effect of the large cylinder enclosing the nucleus was also checked by repeating our calculations using a cylinder whose dimensions were $z_0=23\times10^{-13}$ cm and $\rho_0=8\times10^{-13}$ cm. The results were essentially the same as those obtained for the dimensions used in the rest of the calculations.

The time steps used were chosen according to the transition time considered. For the transition time $\tau = 3\times10^{-21}$ sec the time step was taken to be $\Delta t = 3.0\times10^{-24}$ sec. The error was estimated by calculating the norm $\langle \phi(\vec{r}, t)|\phi(\vec{r}, t)\rangle$ which should be equal to unity. It was found that the error was smaller than 0.1%, namely $\langle \phi(\vec{r}, t)|\phi(\vec{r}, t)\rangle = 1.0 \pm (0.5\times10^{-4})$. For the other time steps used, the error did not exceed 0.3%.

The conservation of the volume of an *infinite* potential well does not guarantee the conservation of the nuclear density. Thus when changing the shape of the potential well an appropriate correction must be introduced to the energies calculated. We calculated directly the nuclear density for the saddle shape

$$\rho(\vec{r}) = 2\sum_{i=1}^{n} |\phi_i(\vec{r}, 0)|^2 ,$$

the scission shape

$$\rho_{sci}(\vec{r}) = 2\sum_{j=1}^{n} |\psi_j(\vec{r})|^2 ,$$

and the density,

$$\rho(\vec{r}, 2\tau) = 2\sum_{i=1}^{n} |\phi_i(\vec{r}, 2\tau)|^2 .$$

It was found that the nuclear density remained constant to a very good approximation for the three cases considered. Thus in our case it is not necessary to correct the energies calculated upon varying the shape of the potential well. The fact that the nuclear density follows the shape of the potential also serves as an indication of the consistency of the model.

[1] N. Bohr and J. A. Wheeler, Phys. Rev. 56, 426 (1939).
[2] A. Brandt and I. Kelson, Phys. Rev. 183, 1025 (1969).
[3] D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953).
[4] J. J. Griffin, in *Proceedings of the Second International Atomic Energy Symposium on Physics and Chemistry of Fission, Vienna, Austria, 1969* (International Atomic Energy Agency, Vienna, 1969).
[5] J. R. Nix and W. J. Swiatecki, Nucl. Phys. 71 (1965).
[6] J. R. Nix, Nucl. Phys. A130, 241 (1969).
[7] I. Halpern, in *Proceedings of the Symposium on the Physics and Chemistry of Fission, Salzburg, 1965* (International Atomic Energy Agency, Vienna, Austria, 1965).
[8] R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial Value Problems* (Interscience, New York, 1967), 2nd ed.
[9] R. W. Fuller, Phys. Rev. 126, 684 (1962).
[10] C. K. Ross and R. K. Bhaduri, Nucl. Phys. A188, 566 (1972).
[11] U. Mosel, to be published.
[12] Y. Boneh, Z. Fraenkel, and Z. Paltiel, in Proceedings of the Third Symposium on Physics and Chemistry of Fission, Rochester, New York, 1973 (to be published), IAEA/SM-174/22.
[13] M. Hillman, Brookhaven National Laboratory Report No. BNL 846, 1964 (unpublished).
[14] L. Wilets, *Theories of Nuclear Fission* (Clarendon, Oxford, 1964).
[15] H. R. Bowman, S. G. Thompson, J. C. D. Milton, and W. J. Swiatecki, Phys. Rev. 126, 2120 (1962).
[16] E. Cheifetz, and Z. Fraenkel, Phys. Rev. Lett. 21, 36 (1968).
[17] S. W. Cosper, J. Cerny, and R. C. Gatti, Phys. Rev. 154, 1193 (1967).
[18] C. Zener, Proc. Roy. Soc. Lond. A137, 696 (1932).
[19] Z. Paltiel, M. Sc. thesis, Weizmann Institute of Science, 1973 (unpublished).