Connected kernel methods in nuclear reactions. II. Rearrangement reactions and the distorted-wave series

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The newly developed connected kernel equations for N-body scattering are applied to the study of rearrangement reactions. An integral equation for the transition operator is obtained which has a connected kernel and whose inhomogeneous terms contain the standard distorted-wave Born approximation (DWBA) and a two-step part. The two-step part is similar to the expression usually used for the calculation of two-step reactions but with a nonstandard prescription for the driving potentials. The difference comes about because of the off-shell transformations used to simplify the original connected kernel equations. A multistep DWBA series is generated and the conditions for its convergence are discussed.

I. INTRODUCTION

Connected kernel equations (CKE) have recently been derived for the scattering operators in the N-body problem.[1-3] These equations eliminate the disconnectedness problems of N-body scattering theory[4] in a manner analogous to the Faddeev-Lovelace treatment of the three-body problem.[5,6] It has been noted that some rearrangement transitions can be expected to have significant two-step components.[7-9] However, in the usual formulation, the distorted-wave Born approximation (DWBA) is not the inhomogeneous term of a connected kernel integral equation.[10] The calculation of a second order correction is therefore somewhat problematical.[11]

In this paper we derive a linear integral equation with connected kernel whose inhomogeneous term contains the DWBA and a two-step term. This equation may be iterated to give a multi-step DWBA series which permits rearrangement in the intermediate steps. The propagator of an intermediate state is a Green function which includes the effect of the optical potential in that intermediate state and a projection operator on two cluster states. The effect of the intermediate breakup states must be included in the effective transition operators and treated properly as a three- (or more) body problem.[12] All the transition operators must be chosen consistently. The transition operators for the two-step process, when approximated in lowest order, are similar to those used previously,[13] but differ in detail. An ambiguity still remains because of the possibility of off-shell transformations of the full transition matrix which results in the mixing of terms of different orders in the DWBA series.

In Sec. II, we recall the CKEs and the resulting coupled channel equation for the effective transition operator for rearrangement is derived and the multi-step DWBA series generated. The conditions for the convergence of this series and the effects of off-shell ambiguities are also discussed in this section. In Sec. III the equations are used to determine the transition operators in two-step calculations and these are compared with the usual prescriptions. Conclusions are given in Sec. IV.

II. MULTI-STEP DWBA SERIES

The connected kernel equations obtained in Refs. 1–3 are for the transition operator

$$U_{\mu\nu}^{(-)} = V^{\nu} + V^{\mu} G V^{\nu}, \tag{1}$$

where V^{ν} is the residual interaction in the channel ν, and G is the full Green function. We have used the post form of the amplitude

$$\langle \Phi_{\mu} \mid U_{\mu\nu}^{(-)} \mid \Phi_{\nu} \rangle = \langle \Psi_{\mu}^{(-)} \mid V^{\nu} \mid \Phi_{\nu} \rangle . \tag{2}$$

The prior form can be used equally well. The CKE for $U_{\mu\nu}$ is

$$U_{\mu\nu} = V_{\nu}^{\mu} + \sum_{\sigma} W_{\mu}^{\sigma} G_0 U_{\sigma\nu} . \tag{3}$$

We use Greek indices to label two cluster channels. The potential part of the channel Hamiltonian H_{ν} is written V_{ν}. This includes the interactions between all pairs having both particles in the same cluster. If a pair has both particles in the same cluster we describe such interactions as being *internal* to the channel. If a pair has one particle in each of the two clusters of the channel we refer to the interaction of that pair as being *external* to the channel. The operator V_{ν}^{μ} is the

sum of those interactions which are both internal to the channel ν and external to the channel μ. Note that V_ν^ν is identically zero. The sum is over two cluster channels, σ, and the operator W_μ^σ is the sum of all σ-connected diagrams not ending in an interaction internal to channel μ. The $U_{\mu\nu}$ agrees with $U_{\mu\nu}^{(-)}$ on the energy shell, but an off-shell transformation has been performed to simplify Eq. (3).

The CKEs may be separated into coupled channel form by a method similar to the one used by Grassberger and Sandhas to put the three-body equations into optical potential form.[14] If one makes a spectral representation of the kernel W_μ^σ and separates out those cuts corresponding to two bound clusters in the channel σ, one obtains coupled channel equations for the transition amplitudes, and equations for the optical potentials and effective transition operators appearing in the coupled channel equations.[2,12] The coupled channel equations obtained are

$$U_{\mu\nu} = U_{\mu\nu}^{(1)} + \sum_\sigma U_{\mu\sigma}^{(1)} \Gamma_\sigma U_{\sigma\nu}, \tag{4}$$

where Γ_σ is the renormalised two-cluster propagator

$$\Gamma_\sigma = \sum_{\epsilon_\sigma^1 \epsilon_\sigma^2} \int d\vec{p} \, | \, \epsilon_\sigma^1 \epsilon_\sigma^2 \vec{p} \rangle \frac{1}{E - \epsilon_\sigma^1 - \epsilon_\sigma^2 - p^2/2\mu_\sigma + i\eta} \times \langle \epsilon_\sigma^1 \epsilon_\sigma^2 \vec{p} \, | \, . \tag{5}$$

The exact intrinsic energies of the two clusters in the channel σ are ϵ_σ^1 and ϵ_σ^2, and their relative momentum is $\vec{p}$. We have written the equations in the c.m. frame so μ_σ is the reduced mass in the channel σ. Note that Γ_σ contains a projection operator on the bound states in the channel σ. The operator $U^{(1)}$ satisfies an equation of the form

$$U_{\mu\nu}^{(1)} = V_\nu^\mu + \sum_\sigma (W_\mu^\sigma G_0 - V_\sigma^\mu \Gamma_\sigma) U_{\sigma\nu}^{(1)} . \tag{6}$$

Equations (4) and (6) are a separation of Eq. (3) into two parts. Since Eq. (3) is an operator equation it involves all types of states including states with three or more clusters. The separation above isolates the part of the equation describing two-cluster scattering [Eq. (4)] from the part describing scattering of the three or more clusters [Eq. (6)]. The Eqs. (4) are coupled channel equations with rearrangement so we may interpret the diagonal elements of $U^{(1)}$ as generalized optical potentials and the off diagonal elements as transition operators. Equation (6) is a CKE for the effective interactions to be used in Eq. (4). Approximation methods for dealing with this equation will be discussed in a future paper.[12] For this work we only investigate the implications of the existence of Eq. (4). It is enough here to remark that in "lowest order" (the precise meaning of this will be specified in Ref. 12) Eq. (6) leads to a standard impulse approximation for the optical potentials and to the transition operators usually used in the DWBA.

We introduce the notation

$$U_{\mu\mu}^{(1)} = \mathfrak{v}_\mu , \qquad U_{\mu\nu}^{(1)} = v_{\mu\nu} . \tag{7}$$

In the lowest order the transition operators for rearrangement $v_{\mu\nu}$ are given by V_ν^μ plus corrections arising from continuum intermediate states.

With the definitions of Eq. (7), Eq. (4) may be written

$$U = \mathfrak{v} + v + \mathfrak{v}\Gamma U + v\Gamma U, \tag{8}$$

where we have used matrix form and suppressed the indices. If we now solve the part containing the optical potential (third term on the right), we obtain

$$U = \frac{1}{1 - \mathfrak{v}\Gamma} \mathfrak{v} + \frac{1}{1 - \mathfrak{v}\Gamma} v(1 + \Gamma U) . \tag{9}$$

We can easily convince ourselves that

$$U = U^{(1)} + U\Gamma U^{(1)} \tag{10}$$

by iterating both Eqs. (4) and (10) and noting that they agree term by term. Therefore we also have

$$U = \mathfrak{v} \frac{1}{1 - \Gamma\mathfrak{v}} + (1 + U\Gamma) v \frac{1}{1 - \Gamma\mathfrak{v}} . \tag{11}$$

Substituting Eq. (11) into Eq. (9) we obtain

$$U = \frac{1}{1 - \mathfrak{v}\Gamma} \mathfrak{v} + \frac{1}{1 - \mathfrak{v}\Gamma} [v + v\Gamma v + v\Gamma U\Gamma v] \frac{1}{1 - \Gamma\mathfrak{v}} . \tag{12}$$

We note that the first term, $(1 - \mathfrak{v}\Gamma)^{-1}\mathfrak{v}$ is simply the scattering matrix in the presence of the optical potentials alone. We therefore define

$$U^{\text{opt}} = \frac{1}{1 - \mathfrak{v}\Gamma} \mathfrak{v} . \tag{13}$$

One must be slightly careful of the notation here. We are using the notation U for a scattering matrix and $U^{(1)}$ or $\mathfrak{v}$ to refer to an optical potential. U^{opt} is a scattering matrix and should not be confused with an optical potential. The inverse operators on either side of the second term of Eq. (12) are simply the optical potential distortion operators. We define

$$\Omega_\alpha^{(\pm)} = [1 - \Gamma_\alpha (E + i\eta) \mathfrak{v}_\alpha]^{-1} = 1 + \frac{1}{\Gamma_\alpha^{-1} - \mathfrak{v}_\alpha \pm i\eta} \mathfrak{v}_\alpha . \tag{14}$$

With these definitions we may write Eq. (12) as

$$U = U^{\text{opt}} + \Omega^{(-)\dagger}[v + v\Gamma v + v\Gamma U\Gamma v]\Omega^{(+)} . \quad (15)$$

We now procede to define an equation for the operator in brackets in the above equation. This operator is the exact effective interaction operator to be used in a distorted wave calculation. Defining

$$U = U^{\text{opt}} + \Omega^{(-)\dagger}\mathcal{T}\Omega^{(+)} \quad (16)$$

straightforward manipulations yield the equation

$$\mathcal{T} = v + v\mathcal{G}v + v\mathcal{G}\mathcal{T}\mathcal{G}v , \quad (17)$$

where the propagators $\mathcal{G}$ are the two-cluster propagators now including the distortions due to the optical potentials:

$$\begin{aligned}\mathcal{G} &= \Gamma + \Gamma\mathcal{U}\mathcal{G} \\ &= (1 - \Gamma\mathcal{U})^{-1}\Gamma .\end{aligned} \quad (18)$$

Equation (17) is our main result. The series generated by iterating this equation has as its first term the distorted-wave Born approximation. The second term is a two-step process and so forth. Note that the solution of Eq. (17) will be identical to the solution of the equation

$$\mathcal{T} = v + v\mathcal{G}\mathcal{T} . \quad (19)$$

This can easily be seen by comparing the iterated forms of Eqs. (17) and (19). The divergences in previous DWBA series arise from building bound states of non-interacting subsystems.[10,15] In this formulation, the presence of the exact renormalized bound state projection operators followed by transition operators eliminates the possibility of having non-interacting subsystems. The two-cluster part of the problem has been treated as a two-body problem, while the three-cluster part (corresponding to breakup states) enters only in the equations for the transition operators and is there treated as a three-body problem.[12] There is therefore no ambiguity in whether or not the breakup terms should be included in the intermediate states as there is in previous theories. In order to maintain proper connectivity, they must not be included. Equation (19) may be used to study the convergence of the iterated series by examining the eigenvalues of the operator $v\mathcal{G}$. The equation can be seen to be an ordinary inhomogeneous integral equation of the second kind. The convergence of the equation will therefore be determined entirely by the eigenvalues of the operator $v\mathcal{G}$. If this operator has any eigenvalues outside the unit circle the iterated series will diverge. If all its eigenvalues have norm less than one the DWBA series will converge. The transition operators entering in the multi-step processes are now uniquely determined by Eqs. (6), (7), and (17). There is no remaining post-prior ambiguity.[13] Since we have a complete prescription, i.e., one which generates terms of all orders, one must maintain consistency over all. An ambiguity does still remain, however. The operator $U_{\mu\nu}$ is required only on the energy shell. One may therefore modify it off the energy shell as one pleases. These modifications are discussed in some detail in Ref. 12. The result found there is that one can transform the operator Γ by an operator which is equal to 1 on the energy shell but may take any value off shell. The kernel of Eq. (6) is changed by such a transformation and the optical potentials and transition operators of Eq. (7) modified thereby. This makes no difference if an exact calculation is performed, but, for example, if the series obtained by iterating Eq. (17) is truncated, one will obtain different results corresponding to different off-shell continuations. Through modifications of the operators v and $\mathcal{U}$ (and therefore $\mathcal{G}$), the kernel of Eq. (17) will be changed and so will the convergence rate of the series. We note that there are now two series involved: that used for calculating the optical potentials and transition operators [the iteration of Eq. (16)], and that used for calculating the effective interaction for use in a DWBA [iteration of Eq. (17)]. It is not clear that one can optimize the convergence of both series simultaneously. This interesting question will require further study.

III. TWO-STEP APPROXIMATION

Let us now consider the two-step approximation to Eq. (17) and determine its implications for actual calculations. To the second order, taking matrix elements in the channel wave functions Φ_μ and Φ_ν, and considering one intermediate channel λ, we have

$$\langle\Phi_\mu \mid U_{\mu\nu} \mid \Phi_\nu\rangle \simeq \langle\chi_\mu^{(-)} \mid [v_{\mu\nu} + v_{\mu\lambda}\mathcal{G}_\lambda v_{\lambda\nu}] \mid \chi_\nu^{(+)}\rangle , \quad (20)$$

where

$$\mid \chi_\mu^{(\pm)}\rangle = \Omega_\mu^{(\pm)} \mid \Phi_\mu\rangle . \quad (21)$$

This is similar in form to the starting point of most two-step calculations but it is by no means identical. The usual starting point for the analysis of a two-step process is the equation[13]

$$\langle\Phi_\mu \mid U_{\mu\nu}^{(-)} \mid \Phi_\nu\rangle = \langle\Phi_\mu \mid \Omega_\mu^{(-)\dagger}\mathcal{T}_{\mu\nu}^{(-)}\Omega_\nu^{(+)} \mid \Phi_\nu\rangle \quad (22)$$

with

$$\begin{aligned}\mathcal{T}_{\mu\nu}^{(-)} &= \overline{V}^\nu + \overline{V}^\mu G \overline{V}^\nu , \\ \overline{V}^\alpha &= V^\alpha - \mathcal{U}^\alpha .\end{aligned} \quad (23)$$

One obtains the effect of a two-step process by ap-

proximating the G by $\mathcal{G}_\lambda$ thus giving

$$\langle \Phi_\mu | \, U^{(-)}_{\mu\nu} \, | \Phi_\nu \rangle \simeq \langle \chi^{(-)}_\mu | \, [\bar{V}^\nu + \bar{V}^\mu \mathcal{G}_\lambda \bar{V}^\nu] \, | \chi^{(+)}_\nu \rangle \, . \tag{24}$$

As we have written it above, the projection operator on the bound states of the channel λ has been included in the approximation for G, but there is no compelling theoretical reason for doing so. In the CKE method, the projection operator is required in order to arrive at the coupled channel equations. The corrections coming from the breakup states in the channel λ go into the expression for the effective transition operators as an additive contribution. When the multi-step reaction we are considering is the successive transfer of single nucleons, the success of the DWBA using the simple approximation

$$v_{\mu\nu} \approx V^\mu_\nu \tag{25}$$

suggests that the truncation of the breakup states in the CKE approach should be a good one. [For example, in a (d, p) reaction, this gives simply $v_{\mu\nu} \approx V_{n-p}$].

Now let us compare the potentials appearing in Eqs. (24) and (20). As an example, consider the double stripping mechanism proceeding through the deuteron channel. Explicitly, we take the channel identifications $\mu = p + (A+2)$, $\lambda = d + (A+1)$, and $\nu = t + A$. We label the first transferred neutron as n_1, the second as n_2. We then obtain for the potentials in Eq. (20)

$$\begin{aligned} v_{\mu\nu} &= V^p_t = V_{p-n_1} + V_{p-n_2}, \\ v_{\lambda\nu} &= V^d_t = V_{d-n_1}, \\ v_{\mu\lambda} &= V^p_d = V_{p-n_2}, \end{aligned} \tag{26}$$

and for the potentials in Eq. (24)

$$\begin{aligned} \bar{V}^\nu &= V_{t-(A-2)} - \mathcal{U}_t = V_{d-(A-2)} + V_{n_1-(A-2)} - \mathcal{U}_t, \\ \bar{V}^\mu &= V_{p-A} - \mathcal{U}_p \approx V_{p-n_2}. \end{aligned} \tag{27}$$

As a second example, consider the stripping-pick-up mechanism for the $A(h, t)$ reaction. Explicitly, we take channel $\nu = h + A$, $\lambda = \nu + (A-1)$, and $\mu = t + A'$. With these identifications we get

$$\begin{aligned} v_{\mu\nu} &= V^t_h = V_{p-d} + V_{n-C}, \\ v_{\mu\lambda} &= V^t_\alpha = V_{p-t}, \\ v_{\lambda\nu} &= V^\alpha_h = V_{n-C}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \bar{V}^\nu &= V_{h-A} - \mathcal{U}_h \approx V_{n-h}, \\ \bar{V}^\mu &= V_{t-A'} - \mathcal{U}_t \approx V_{p-t}. \end{aligned} \tag{29}$$

We have used C to indicate the A-1 particle core which contains neither the transferred neutron nor proton. We note a number of differences between the results obtained beginning with Eq. (20) and those obtained beginning with Eq. (24). First we observe that the transition operators in the two-step process depend on the intermediate channel in our formalism but not in the iterated Green function formalism. The one-step terms are essentially the same up to the post-prior ambiguity. In the two-step term we note that we have two "short-range" interactions in the (t,p) example (i.e., an interaction of the transferred particle with the projectile) while in the (h, t) example we have one short-range interaction and one "long-range" interaction (i.e., an interaction of the transferred particle with the nuclear core). In the iterated Green function method this is reversed. Although this difference resembles the replacement of a post DWBA by a prior DWBA[13] this is not done arbitrarily but is in fact required by our formalism.

The difference in the two prescriptions arises from the fact that an off-shell transformation has been performed on the operator $U^{(-)}_{\mu\nu}$ in order that the connected kernel equation it satisfies have a simple Born term and in order that the DWBA be recovered in a straightforward manner from these equations. This transformation is described in Refs. 2 and 3.

The effect of the transformation is quite striking. Compare, for example, the Eq. (1) defining $U^{(-)}$, and Eq. (3) for U. When expanded in a perturbation series we see that no diagrams in the expansion for $U^{(-)}_{\mu\nu}$ begin with an interaction internal to the channel ν. When Eq. (3) is iterated, however, we see that all its terms begin with an interaction internal to the channel ν.

Since the second order terms include propagation off the energy shell, it is important to consider which off-shell extension one is using and the effect the choice of off-shell extension has on the convergence of a DWBA series. The CKE method leads to a DWBA series which will not be forced to diverge by the presence of disconnected graphs, in contrast to the series obtained from iterating a resolvent equation for a many-body Green function.[15] This suggests that the CKE method may be a more reliable basis for calculating higher order corrections than the iterated Green function method. It should be noted, however, that even a connected kernel integral equation does not guarantee that the iterated series will converge. Such divergences in connected kernel equations have been found in the three-body problem, both in studies of the multiple scattering series[16] and of the optical potential.[17] But such divergences can only be studied in the framework of a well-behaved integral equation, such as the CKE.

IV. SUMMARY AND CONCLUSIONS

The coupled channel equations arising from the connected kernel equations for the N-body two-cluster scattering amplitude has been written in distorted wave form and an equation obtained for the effective transition operator for rearrangement processes. This operator yields the exact rearrangement amplitudes when its matrix element is taken between distorted waves. The equation for it is an inhomogeneous linear integral equation and the inhomogeneity contains two terms, the first of which yields (in lowest order) the usual DWBA. The second term describes rearrangement two-step processes. The equation may be iterated to yield a multi-step DWBA series. The intermediate state propagators in this expansion contain optical distortions and projection operators on the two-body bound states. This projection operator must be present to avoid divergence difficulties. The coupled channel equations can only handle two-cluster channels properly. Three-cluster channels must be dealt with by writing Faddeev-like three-body equations for the effective operators appearing in the coupled channel equations.

The DWBA series obtained here is applied to two standard two-step examples, (t, d, p) and (h, α, t). The resulting prescriptions for the transition operators differ somewhat from the standard prescriptions. This is because the scattering operator used, $U_{\mu\nu}$, differs off the energy shell from the usually employed operator $U^{(-)}_{\mu\nu}$. This off-shell transformation was chosen in order to have the scattering operator satisfy a reasonably simple connected kernel equation and one which would reduce to the standard prescription for the DWBA when the appropriate assumptions are made.

The formulation presented not only allows one to establish prescriptions for two-step processes but to determine those operators controlling the convergence rate of the entire series. The ambiguities arising from the off-shell indeterminacy may also be studied. The connected kernel framework therefore provides us with a precise means of formulating and studying these questions which are not treatable with previous theories. In this formalism it is clear, for example, that the convergence rate of the series used to calculate an optical potential and the series used to calculate a rearrangement process in a distorted wave matrix element using those optical potentials, are not controlled by the same operator. Therefore, if one chooses an off-shell continuation for the purpose of optimizing the convergence rate of one series, the other may suffer.

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