Multiple diffraction theory, optical potentials, and an alternative approach to high-energy scattering by nuclei

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A formally exact perturbation expansion of the high-energy scattering amplitude by nuclei is developed, in which the leading term is given by the Glauber's multiple diffraction amplitude. The leading correction terms are examined in detail, which deal with the two approximations involved in the Glauber amplitude; (a) the linearization of the Green's function and (b) the fixed-scatterer assumption. Thus, the diffraction amplitude can be systematically improved rather simply so long as the scattering angle is not too large. It is stressed that, when the target profile function is replaced by the individual nucleon profiles, the correction to the linearization approximation is automatically included to all orders in the Glauber amplitude, insofar as the nonoverlapping interactions are concerned. The contents of the Glauber wave function are then analyzed to show explicitly that, within the approximations (a) and (b), the major part of the inelastic channel information, and thus of the optical potential, is already included in the theory. By comparing the diffraction theory with the optical potential approach, which is based on the multiple scattering theory and the conversion of the optical potential to a set of coupled equations, we exhibit a close connection between the two seemingly diverging approaches. As a result of this analysis, we derive a modified theory of high-energy scattering by nuclei, in which the elastic component is to be treated exactly without the linearization while the inelastic component is taken directly from the diffraction theory.

[NUCLEAR REACTIONS Systematic corrections to the high-energy nucleon-nucleus amplitude in the multiple diffraction theory of Glauber.]

I. INTRODUCTION

The multiple diffraction theory of Glauber[1] has been very successful in explaining the gross features of many high-energy hadron scatterings by target nuclei, often effective beyond the region of validity as deduced from the original derivation.[2] Therefore, many attempts have been made[3-13] in recent years to understand some of the reasons for its effectiveness, and to further improve the theory in order to extract more reliably the information on the target structure. The correlation effect among the target nucleons is often masked by the dominant forward peak in the cross section due to the purely geometrical effect, and thus manifests itself only appreciably at large momentum transfer region. However, this is precisely the region where the validity of the multiple diffraction theory (MDT) is most uncertain. It is the main purpose of this paper to examine the contents of the MDT, and develop a systematic perturbation expansion for *all* the correction terms. A formal expansion of the complete correction is given in Sec. II, with the Glauber amplitude as the leading term.

The MDT is derived essentially by taking into account the two assumptions: (a) the linearization of the Green's function which leads to a straight line trajectory for the projectile and makes the solution simple, and (b) the fixed-scatterer approximation which reduces the original many-particle scattering problem to a collision involving two particles at a time. The correction to the linearization approximation has been considered recently by Wallace[12] and we incorporate his result in our complete perturbation expansion. In addition to improving the large angle behavior of the cross section, this correction makes the additivity assumption of eikonal phases in the MDT inapplicable in the case of overlapping potentials, as suggested earlier by Feshbach.[14] We will also see that this is the possible reason for the breakdown of the MDT at large angles even if the two-particle scattering amplitude is used in the total amplitude. These problems are discussed in Sec. III.

We make the perturbation expansion of the correction to the Glauber amplitude complete by incorporating the effect of the motion of target particles during the scattering. Foldy and Walecka[15] examined the approximations involved in the derivation of the MDT and showed that the Glauber amplitude contains all the inelastic channel effects in the form of a closure. We analyze here directly the scattering wave function of the

MDT and show that it contains essentially the optical potential information in the approximations of (a) and (b) above. The result of this analysis is especially useful when we compare it with the optical potential approach, and also clarifies the correction terms we derive in Sec. IV for the motion of the target particles. The Glauber amplitude is deceptively simple, and yet contains much of the dynamical information between the projectile and target particles, in agreement with the analysis of Foldy and Walecka.

More recently, a slightly different approach to the high-energy scattering by nuclei has been developed.[16-18] The theory is based on the multiple scattering theory of Watson[19] and Kerman, McManus, and Thaler.[20] The optical potential for the elastic scattering, for example, is converted into a set of coupled equations, with the parameters in the set derived from the correlation functions for the target nucleons. In this way, the dominant approximations in the MDT are treated *separately*, by dealing with the approximation (b) first. The linearization approximation is then unnecessary because the resulting coupled set of equations can be solved by any number of accurate methods, numerically. These results are briefly summarized in Sec. V. For further refinements of the theory we refer to a recent study[18] where many of the approximations introduced for this approach are reviewed.

Both these approaches have been developed during the past years to a sufficient extent that a more systematic and complete analysis and comparison of the seemingly diverging points of view may be made. By noting the intrinsic advantages of both approaches, we derive in Sec. V an improved method for the high-energy scattering problem; it improves the elastic component of the scattering function by following the procedure of the optical potential approach, while the entire inelastic channel component is represented by the Glauber scattering function projected onto the inelastic channel space. Some refinements of the method, as well as the corrections to this amplitude, are also suggested.

II. EIKONAL PERTURBATION EXPANSION

We construct in this section an exact perturbation theory of high-energy scattering by a composite target in such a way that the leading amplitude contains the linearized propagator and fixed-scatterer approximation, as in the Glauber theory. More importantly, the accuracy of amplitude may be systematically improved by including additional terms in the expansion. Different ways of linearizing the Green's function will also be discussed.

Consider the elastic scattering of projectile particles by a composite target, described by

$$(H-E)\Psi(\vec{r},\vec{R})=0 \tag{2.1}$$

with

$$H=T_{\vec{R}}+H_T(\vec{r})+V(\vec{r},\vec{R}),\quad V=\sum_{i=1}^{N}V_i(\vec{r}-\vec{R}), \tag{2.2}$$

where $\vec{R}$ denotes the relative coordinate of the target center of mass and the projectile, $T_{\vec{R}}$ is the kinetic energy of the projectile, and H_T denotes the internal Hamiltonian of the target with N nucleons. The variable $\vec{r}$ denotes these internal variables collectively, as $\vec{r}=(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N)$. We have ($\hbar=M=c=1$)

$$H_T(\vec{r})=\sum_{i=1}^{N}(-\tfrac{1}{2}\vec{\nabla}_i^{\,2})+\sum_{i>j}v_{ij}\equiv T_{\vec{r}}+U(\vec{r}), \tag{2.3}$$

which generates the set of target states, properly antisymmetrized, as

$$H_T\psi_n(\vec{r})=E_n\psi_n(\vec{r}). \tag{2.4}$$

If we let

$$E=E_0+E_0'=E_n+E_n',$$
$$E_0'=\tfrac{1}{2}K^2,$$
$$T_{\vec{R}}=\tfrac{1}{2}\vec{P}^2=-\tfrac{1}{2}\nabla_{\vec{R}}^2,\quad \vec{P}=-i\vec{\nabla}_{\vec{R}},$$

then

$$H-E=\tfrac{1}{2}\vec{P}^2-\tfrac{1}{2}\vec{K}^2+V+(H_T-E_0). \tag{2.5}$$

For definiteness of discussion, we have chosen here the Schrödinger equation as the scattering equation; the necessary modifications to take into account the relativistic effect of the projectile can be incorporated later on following the standard procedure.[21] To simplify the formalism, we will neglect the exchange symmetry of the projectile with the target nucleons in case of nucleon scattering, and also the spin and isospin effects, as well as any production channels. Eventually these effects should be put in, as some of them are known to be important in the momentum transfer region of interest.

The linearization of H and the corresponding Green's function $G\equiv(E^{(+)}-H)^{-1}$ follows closely the result for the potential scattering.[12] As has been noted by various authors, the procedure is far from being unique. To be able to compare different possibilities, we consider an identity

$$\begin{aligned}\vec{K}^2-\vec{P}^2&=\{\vec{K}^2-2\vec{K}_\alpha\cdot(\vec{P}-\vec{K}_\beta)-\vec{K}_\alpha^{\,2}\}\\&\quad+[2\vec{K}_\alpha\cdot(\vec{K}_\alpha-\vec{K}_\beta)-(\vec{P}-\vec{K}_\alpha)^2]\\&\equiv 2A-2C,\end{aligned} \tag{2.6}$$

where the individual A and C are yet to be specified, and where $\vec{K}_\alpha$ and $\vec{K}_\beta$ are arbitrary constant vectors. Various choices for these vectors will generate different starting approximations. [The only constraint on these vectors is to have A satisfy (2.11).] The kinematics of the elastic scattering is such that

$$K = |\vec{K}_i| = |\vec{K}_f|$$

and

$$\vec{q} \equiv \vec{K}_i - \vec{K}_f, \quad q = 2K \sin\tfrac{1}{2}\Theta \tag{2.7}$$

with

$$0 \leq q \leq 2K .$$

We also define the average momentum vector

$$\vec{K}_a \equiv \tfrac{1}{2}(\vec{K}_i + \vec{K}_f) \tag{2.8}$$

with the property

$$\vec{K}_a \cdot \vec{q} = 0, \quad K_a = K\cos\tfrac{1}{2}\Theta = K(1 - q^2/4K^2)$$

and

$$\vec{K}_0 \equiv K\hat{K}_a, \quad K_0 = |\vec{K}_0| = K , \tag{2.9}$$

which is in the $\vec{K}_a$ direction but has the magnitude K. Thus,

$$\vec{K}_i = \vec{K}_a + \tfrac{1}{2}\vec{q}, \quad \vec{K}_f = \vec{K}_a - \tfrac{1}{2}\vec{q}$$

with

$$\vec{K}_a \cdot (\vec{K}_i - \vec{K}_a) = 0, \quad \vec{K}_a \cdot (\vec{K}_f - \vec{K}_a) = 0 .$$

Now, we consider several typical choices for $\vec{K}_\alpha$ and $\vec{K}_\beta$. Firstly, we try (i) $\vec{K}_\alpha = \vec{K}_a = \vec{K}_\beta$: This gives immediately the result of Abarbanel and Itzykson[6] and Kujawski[9] as

$$\begin{aligned} A &= -\vec{K}_a \cdot (\vec{P} - \vec{K}_a), \\ C &= \tfrac{1}{2}[(\vec{P} - \vec{K}_a)^2 - \vec{K}^2 + \vec{K}_a^2] \\ &= \tfrac{1}{2}(\vec{P} - \vec{K}_f) \cdot (\vec{P} - \vec{K}_i) . \end{aligned} \tag{2.10}$$

The form of A is dictated by the requirement that

$$A e^{i\vec{K}_i \cdot \vec{R}} = 0 . \tag{2.11}$$

This condition on A is essential for a rigorous eikonal formulation of Ψ, in which the plane wave part dominates and C is treated as a perturbation. [The effect of (2.11) is clear in (2.16) and (2.23) below.] For the condition (2.11) to hold, $\hat{K}_\alpha = \hat{K}_a$ and $\vec{K}_\beta = \vec{K}_a$ are useful.

On the other hand, we can also try (ii) $\vec{K}_\alpha = \vec{K}_0$, $\vec{K}_\beta = \vec{K}_a$: With the requirement (2.11), we have

$$\begin{aligned} A &= -\vec{K}_0 \cdot (\vec{P} - \vec{K}_a) , \\ C &= \tfrac{1}{2}(\vec{P} - \vec{K}_0)^2 - \vec{K}_0 \cdot (\vec{K}_0 - \vec{K}_a). \end{aligned} \tag{2.12}$$

The choice (2.12) leads to the Glauber amplitude[1,2] in the lowest order.

Other possible choices for A are listed in Table I. The requirement (2.11) forces (iii) and (iv) to be equivalent to (i) and (ii), respectively, while the choices (v), (vii), and (vi), (viii) again become symmetric in $\vec{K}_i$ and $\vec{K}_f$ when (2.11) is imposed and reduce to (i) and (ii). On the other hand, the form (ix) is quite asymmetric in $\vec{K}_i$ and $\vec{K}_f$, and gives the basis for the original derivation[2] of the eikonal approximation for small-angle high-energy scattering.

Thus we are essentially down to the two choices, and, in most of the following discussion, we specifically deal with the form (2.12), although th identical formalism can be developed with (2.10). The linearization of $E - H$ is therefore given by the separation

$$E - H = (A - V) - (C + D) , \tag{2.13}$$

where

$$D \equiv H_T - E_0 ,$$

and we treat the last terms $(C + D)$ as a perturbation. With the definition

$$G_{\text{eik}} \equiv (A + i\epsilon - V)^{-1} = -\frac{i}{K}\,\delta(\vec{B} - \vec{B}')\theta(z - z')\,e^{-i\phi + i\phi'} , \tag{2.14}$$

where

$$\phi = \vec{K}_0 \cdot \vec{R} + \chi_+ , \quad \chi_+ \equiv -\frac{1}{K}\int_{-\infty}^{z} V\,dz' ,$$

we have

$$\begin{aligned} G = (E + i\epsilon - H)^{-1} &\equiv G^{(+)} \\ &= G_{\text{eik}} + G_{\text{eik}}\,(C + D)G \\ &= G_{\text{eik}} + G_{\text{eik}}\,(C + D)G_{\text{eik}} + \cdots . \end{aligned} \tag{2.15}$$

The scattering function with the incoming wave

TABLE I. Different choices for the linearized operator A, and their corrections C. The case (i) gives the Abarbanel-Itzykson amplitude as its leading term, and the choice (ii) is for the Glauber amplitude. All the other possibilities, except (ix), are shown to reduce to the first two cases when the condition (2.11) is imposed.

	$\vec{K}_\alpha$	$\vec{K}_\beta$	A	C
(i)	$\vec{K}_a$	$\vec{K}_a$	$-\vec{K}_a \cdot (\vec{P} - \vec{K}_a)$	$\frac{1}{2}(\vec{P} - \vec{K}_a)^2 + \frac{1}{2}(K_a^2 - K^2)$
(ii)	$\vec{K}_0$	$\vec{K}_a$	$-\vec{K}_0 \cdot (\vec{P} - \vec{K}_a)$	$\frac{1}{2}(\vec{P} - \vec{K}_0)^2 - K^2 + KK_a$
(iii)	$\vec{K}_a$	$\vec{K}_0$	(i)	(i)
(iv)	$\vec{K}_0$	$\vec{K}_0$	(ii)	(ii)
(v)	$\vec{K}_a$	$\vec{K}_i$	(i)	(i)
(vi)	$\vec{K}_0$	$\vec{K}_i$	(ii)	(ii)
(vii)	$\vec{K}_a$	$\vec{K}_f$	(i)	(i)
(viii)	$\vec{K}_0$	$\vec{K}_f$	(ii)	(ii)
(ix)	$\vec{K}_i$	$\vec{K}_i$	$-\vec{K}_i \cdot (\vec{P} - \vec{K}_i)$	$\frac{1}{2}(\vec{P} - \vec{K}_i)^2$

in the initial channel i is then given by the series

$$\Psi_i = \Psi_{i,0} + G^{(+)} V \Psi_{i,0}$$
$$= \Psi_{i,0} + G_{\rm eik} V \Psi_{i,0} + G_{\rm eik} (C+D) G_{\rm eik} V \Psi_{i,0} + \cdots , \tag{2.16}$$

where

$$\Psi_{i,0} = \psi_0(\vec{r}) e^{i\vec{K}_i \cdot \vec{R}} ,$$

and the elastic amplitude by

$$\mathfrak{F}_{fi} = (\Psi_{f,0} | V | \Psi_i) \equiv (\Psi_{f,0} | \mathcal{T} | \Psi_{i,0}) , \tag{2.17}$$

where

$$\mathcal{T} = (V + V G_{\rm eik} V) + V G_{\rm eik} (C+D) G_{\rm eik} V + \cdots$$
$$\equiv \mathcal{T}^G + \mathcal{T}^{(1)} + \mathcal{T}^{(2)} + \cdots . \tag{2.18}$$

The differential cross section is

$$\frac{d\sigma_{\rm el}}{d\Omega} = \frac{1}{4\pi^2} | \mathfrak{F}_{fi} |^2 , \tag{2.19}$$

with

$$\mathfrak{F}_{fi} = \mathfrak{F}_{fi}^G + \mathfrak{F}_{fi}^{C1} + \mathfrak{F}_{fi}^{D1} + \cdots ,$$

where

$$\mathfrak{F}_{fi}^G = (\Psi_{f,0} | V + V G_{\rm eik} V | \Psi_{i,0}) \tag{2.20}$$

and

$$\mathfrak{F}_{fi}^{C1} = (\Psi_{f,0} | V G_{\rm eik} C G_{\rm eik} V | \Psi_{i,0}) , \tag{2.21}$$

$$\mathfrak{F}_{fi}^{D1} = (\Psi_{f,0} | V G_{\rm eik} D G_{\rm eik} V | \Psi_{i,0}) . \tag{2.22}$$

The amplitudes $\mathfrak{F}_{fi}^{C1}$ and $\mathfrak{F}_{fi}^{D1}$ are the leading corrections to $\mathfrak{F}_{fi}^G$ corresponding to the linearization and fixed-scatterer approximations, and will be discussed in great detail in Secs. III and IV, respectively. In this section, we briefly consider $\mathfrak{F}_{fi}^G$, mainly to define notations. Define

$$\Psi_i^{\rm eik} = (1 + G_{\rm eik} V) \Psi_{i,0}$$

or

$$(A - V) \Psi_i^{\rm eik} = (i\vec{K}_0 \cdot \vec{\nabla}_{\vec{R}} + K K_a - V) \Psi_i^{\rm eik} = 0 . \tag{2.23}$$

The solution of (2.23) is

$$\Psi_i^{\rm eik} = \psi_0(\vec{r}) e^{i\vec{K}_i \cdot \vec{R}} \Phi_G^{(+)}(\vec{r}, \vec{R}) , \tag{2.24}$$

where

$$\Phi_G^{(+)} = e^{i\chi_+} , \quad \chi_+ = -\frac{1}{K} \int_{-\infty}^{z} V(\vec{r}, \vec{R}') dz' . \tag{2.25}$$

In (2.25), the z axis is taken along the average momentum direction $\hat{K}_a$, and the dz' integration in χ_+ is carried out with the $\vec{r}$ variable held fixed as parameters. In an analogous way, we may also define

$$\Psi_f^{\rm eik} = \psi_0(\vec{r}) e^{i\vec{K}_f \cdot \vec{R}} \Phi_G^{(-)*} \tag{2.26}$$

and

$$\Phi_G^{(-)} = e^{i\chi_-} , \quad \chi_- = -\frac{1}{K} \int_{z}^{\infty} V(\vec{r}, \vec{R}') dz' .$$

Thus,

$$\mathfrak{F}_{fi}^G = \int d\vec{r} | \psi_0(\vec{r}) |^2 \int d\vec{R}\, e^{i\vec{q} \cdot \vec{R}} V(\vec{r}, \vec{R}) e^{i\chi_+(\vec{R}, \vec{r})}$$
$$= iK \int d\vec{r} | \psi_0(\vec{r}) |^2 \int d\vec{B}\, e^{i\vec{q} \cdot \vec{B}} (e^{i\chi_0} - 1) , \tag{2.27}$$

where

$$\chi_0 \equiv \chi_+ + \chi_-$$
$$= -\frac{1}{K} \int_{-\infty}^{\infty} V(\vec{r}, \vec{R}) dz = \chi_0(\vec{B}, \vec{b}) \tag{2.28}$$

and

$$\vec{R} = \vec{B} + \hat{K}_a z ,$$
$$\vec{b} = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_N) , \quad \vec{r}_i = \vec{b}_i + z_i \hat{K}_a .$$

The multiple diffraction theory of Glauber goes one step further and expresses the nucleus profile function by a product of individual nucleon profile functions as

$$\Gamma \equiv 1 - e^{i\chi_0} = 1 - \prod_{i=1}^{N} [1 - \Gamma_i(\vec{B}, \vec{b}_i)] , \tag{2.29}$$

where

$$\Gamma_i = 1 - e^{i\chi_0(i)} \equiv \frac{1}{2\pi i K} \int d\vec{q}_\perp e^{-i\vec{q} \cdot (\vec{B} - \vec{b}_i)} F_i(\vec{q}, \vec{K}) . \tag{2.30}$$

In (2.30), F_i is the projectile-free nucleon amplitude, where

$$\chi_0 = \sum_{i=1}^{N} \chi_0(i) , \tag{2.31}$$

which follows the form of V given by (2.2). Thus, $\mathfrak{F}_{fi}^G$ becomes

$$\mathfrak{F}_{fi}^G = -iK \int d^2B\, e^{i\vec{q} \cdot \vec{B}} \langle \psi_0 | \Gamma | \psi_0 \rangle$$
$$= -iK \int d^2B\, e^{i\vec{q} \cdot \vec{B}} \left\langle \psi_0 \middle| 1 - \prod_{i=1}^{N} (1 - \Gamma_i) \middle| \psi_0 \right\rangle , \tag{2.32}$$

which is the multiple diffraction amplitude of Glauber.[1] We note that in the final form (2.32), Γ_i could be a more general profile function than that defined by (2.30). In fact, the identification (2.29) between Γ and χ_0 is a rather arbitrary one, and we will show in the next section that, under certain conditions (of nonoverlapping interactions), (2.32) may have more general validity than the derivation presented above.[13,14]

III. CORRECTIONS TO THE LINEARIZED GREEN'S FUNCTION

One of the two major corrections to the Glauber amplitude $\mathfrak{F}_{fi}^{G}$ is concerned with the linearization of the Green's function, i.e., $G \to G_{\text{eik}}$, and we consider the correction term $\mathfrak{F}_{fi}^{G1}$ of (2.21) in detail in this section. Obviously, the similar correction appears also in the pure potential scattering, and we thus follow closely the result of a recent study of this problem by Wallace.[12,13] In this case of potential scattering, the leading correction analogous to $\mathfrak{F}_{fi}^{G1}$ is seen to greatly improve the behavior of the amplitude up to $q \lesssim K$, corresponding to the center of mass scattering angle $\Theta \lesssim \frac{1}{3}\pi$. For angles larger than this, however, the expansion in C as we have done in (2.15), with of course $D=0$, is much more slowly convergent. We will therefore consider alternate treatments of the correction term, specifically for the region $q \gtrsim K$. Finally, we investigate under what conditions the substitution of the form (2.32) would have a more general validity than the form involving χ_0 alone.

(a) Following rather closely the result of Ref. 12 in this part of the discussion, we have, with the choice (ii) of (2.12) for A and C,

$$C = \tfrac{1}{2}(\vec{P}-\vec{K}_0)^2 - K^2 + KK_a = \tfrac{1}{2}(\vec{P}-\vec{K}_f)\cdot(\vec{P}-\vec{K}_i) + \lambda A \,, \tag{3.1}$$

where

$$\lambda = 1-\cos\tfrac{1}{2}\Theta \,, \qquad A = -\vec{K}_0\cdot(\vec{P}-\vec{K}_a) \,.$$

Thus, using the result of Sec. II, (2.24) and (2.26),

$$\mathfrak{F}_{fi}^{G1} = \int d\vec{R} \int d\vec{r}\,\psi_0^*(\vec{r}) e^{-i\vec{K}_f\cdot\vec{R}} (e^{i\chi_-}-1)[\tfrac{1}{2}(\vec{P}-\vec{K}_f)\cdot(\vec{P}-\vec{K}_i)+\lambda A](e^{i\chi_+}-1)e^{i\vec{K}_i\cdot\vec{R}}\psi_0(\vec{r}) \,. \tag{3.2}$$

To reduce (3.2) further, the following relations are useful:

$$\begin{aligned}
&(\vec{P}-\vec{K}_i)e^{i\vec{K}_i\cdot\vec{R}}(e^{i\chi_+}-1) = e^{i\chi_+}e^{i\vec{K}_i\cdot\vec{R}}(\vec{\nabla}_{\vec{R}}\chi_+),\\
&(\vec{P}^*-\vec{K}_f)e^{-i\vec{K}_f\cdot\vec{R}}(e^{i\chi_-}-1) = -e^{i\chi_-}e^{-i\vec{K}_f\cdot\vec{R}}(\vec{\nabla}_{\vec{R}}\chi_-),\\
&Ae^{i\vec{K}_i\cdot\vec{R}}(e^{i\chi_+}-1) = -e^{i\vec{K}_i\cdot\vec{R}}e^{i\chi_+}\vec{K}_0\cdot(\vec{\nabla}_{\vec{R}}\chi_+) \,,
\end{aligned} \tag{3.3}$$

and also

$$\vec{\nabla}_{\vec{R}}\chi_+ = -\frac{V}{K}\hat{z} - \frac{1}{K}\int_{-\infty}^{z}(\vec{\nabla}_{\vec{b}}V)dz' \,, \qquad \vec{\nabla}_{\vec{R}}\chi_- = -\frac{V}{K}\hat{z} - \frac{1}{K}\int_{z}^{\infty}(\vec{\nabla}_{\vec{b}}V)dz' \,. \tag{3.4}$$

(3.2) may now be rewritten as

$$\begin{aligned}
\mathfrak{F}_{fi}^{G1} &= \int d\vec{r}\,|\psi_0|^2 \int d\vec{B}\,e^{i\vec{q}\cdot\vec{B}} \int_{-\infty}^{\infty} dz\,\{\lambda(e^{i\chi_0}-e^{i\chi_+})(-\vec{K}_0\cdot\vec{\nabla}_{\vec{R}}\chi_+) - \tfrac{1}{2}(\vec{\nabla}_{\vec{R}}\chi_-)\cdot(\vec{\nabla}_{\vec{R}}\chi_+)e^{i\chi_0}\}\\
&= \int d\vec{r}\,|\psi_0|^2 \int d\vec{B}\,e^{i\vec{q}\cdot\vec{B}}e^{i\chi_0(\vec{B},\vec{b})}\\
&\quad\times\left\{-i\lambda K(1-i\chi_0-e^{-i\chi_0}) - \frac{1}{2K^2}\frac{\partial}{\partial B}\int_{-\infty}^{\infty} dz(zV)\left[\int_{-\infty}^{z}\frac{\partial V}{\partial B}dz' - \int_{z}^{\infty}\frac{\partial V}{\partial B}dz'\right]\right\}
\end{aligned} \tag{3.5}$$

The last expression in (3.5) was obtained by an integration by parts. For the potential scattering,[12] the second term in the curly brackets above can further be reduced in case when the potential $V(\vec{R})$ is independent of the aximuthal angle φ. This is, however, not in general the case with the composite particle scattering, because V now depends in a complicated way on the target variable $\vec{r}$ and $\vec{R}$ (on $\vec{R}-\vec{r}$).

If we assume that $V(\vec{r},\vec{R})$ is still nearly φ independent,[13] with

$$[L_\varphi, V] \approx 0 \,, \tag{3.6}$$

where L_φ is the rotation operator around the z axis; then (3.5) immediately reduces to a simpler form:

$$\mathfrak{F}_{fi}^{G1} \approx \int d\vec{r}\,|\psi_0|^2 \int d\vec{B}\,e^{i\vec{q}\cdot\vec{B}}e^{i\chi_0}\left\{-i\lambda K(1-i\chi_0-e^{-i\chi_0}) + \frac{1}{K^2}\left(1+B\frac{\partial}{\partial B}\right)\int_0^{\infty} dz\,V^2(\vec{R}-\vec{r})\right\} . \tag{3.7}$$

Combining $\mathfrak{F}_{fi}^{G}$ and $\mathfrak{F}_{fi}^{C1}$, and exponentiating the correction terms and noting that the λ-dependent terms in (3.7) are nearly canceling each other to order χ_0, we have

$$\mathfrak{F}_{fi}^{G}+\mathfrak{F}_{fi}^{C1}\approx iK\int d\vec{r}\,|\psi_0|^2\int d\vec{B}\,e^{i\vec{q}\cdot\vec{B}}(e^{i\chi_0+i\chi_1}-1), \tag{3.8}$$

where

$$\chi_1=-\frac{1}{K^3}\left(1+B\frac{\partial}{\partial B}\right)\int_0^\infty dz\,V^2(\vec{R},\vec{r}\,). \tag{3.9}$$

Equation (3.8) is, at this stage, only valid to first order in χ_1, and we should remember that the form for χ_1 above assumes (3.6). The results (3.7) and (3.8) are essentially the generalization[13] of the earlier study of the potential scattering by Wallace.[12]

(b) Similar correction can also be obtained with the choice (i) of (2.10) for A and C. Instead of (2.27) and (3.5), we now obtain

$$\mathfrak{F}_{fi}^{Ga}=iK_a\int d\vec{r}\,|\psi_0|^2\int d\vec{B}\,e^{i\vec{q}\cdot\vec{B}}(e^{i\chi_a}-1), \tag{3.10}$$

where

$$\chi_a=-\frac{1}{K_a}\int_{-\infty}^{\infty}V\,dz=\frac{K}{K_a}\chi_0$$

and

$$\mathfrak{F}_{fi}^{C1a}=-\int d\vec{r}\,|\psi_0|^2\int d\vec{B}\,e^{i\vec{q}\cdot\vec{B}}\tfrac{1}{2}e^{i\chi_a}\times\int_{-\infty}^{\infty}dz(\vec{\nabla}_{\vec{R}}\chi_{a-})\cdot(\vec{\nabla}_{\vec{R}}\chi_{a+})\,. \tag{3.11}$$

Notably, (3.10) and (3.11) are simpler, but it is not known whether in general

$$\mathfrak{F}_{fi}^{Ga}+\mathfrak{F}_{fi}^{C1a} \tag{3.8a}$$

is a better approximation than (3.8) and also whether the series expansion with the choice (2.10) converges faster than that with (2.12). In the potential scattering and for $q\lesssim K$, a simple study described below indicates that (3.8) is better than (3.8a). Table II shows the comparison between the two amplitudes for a Gaussian potential of the form

$$V=\tfrac{1}{2}g\,e^{-\alpha R^2}(1+\rho R^2)\,, \tag{3.12}$$

with $g=-0.4$, $\alpha=0.2$, and $\rho=0.3$ in the units $M=\hbar=c=1$ ($K=2.0$).

(c) It has been noted by Wallace[12] that the expansion of the form (2.15) in $(G_{\rm eik}\,C)$ in the potential scattering is a slowly converging series in the region $q\gtrsim K$. Since $\mathfrak{F}_{fi}^{Cn}$, with $n>1$, become very complicated to evaluate, the expansion (2.15) is less useful unless these higher-order terms are negligible. In the region of q values where higher order terms are needed, therefore, it is desirable to consider an alternative to the expansion of G, as was done in (2.15). For this purpose, we go back to G and write it in the form

$$\begin{aligned}G&=G_{\rm eik}+G_{\rm eik}(C+D)G\\&=G_{\rm eik}+G_{\rm eik}(C+D)G_t+G_{\rm eik}(C+D)G_t(G_t^{-1}-G^{-1})G,\end{aligned} \tag{3.13}$$

where we have used the trivial identity

$$G=G_t+G_t(G_t^{-1}-G^{-1})G\,, \tag{3.14}$$

with G_t an arbitrary Green's function yet to be

TABLE II. The differential cross sections calculated by the different approximations are given for the Gaussian potential (3.12). The first diffraction minimum occurs at $q\approx1.5$, and the second maximum at $q\approx1.5$, for $K=2.0$. The column GL is the Glauber cross section, while GLM denotes the values obtained with the leading correction χ_1 included. AI is the cross section with $\vec{K}\to\vec{K}_a$, while AIM includes the χ_{1a} correction as defined by (3.8a). SC denotes the semiclassical approximation of Ref. 22, and EX is the exact cross section calculated by the partial waves.

q	θ(deg)	GL	GLM	AI	AIM	SC	EX
0.0	0.0	40.31	41.63	40.31	41.63	41.29	41.49
0.4	11.5	22.24	22.75	22.23	22.75	22.58	22.70
0.8	23.1	3.260	3.159	3.257	3.152	3.135	3.163
1.2	35.2	0.0611	0.0430	0.0638	0.0442	0.0415	0.0434
1.4	41.0	0.0200	0.0273	0.0221	0.0304	0.0257	0.0262
1.6	47.2	0.0205	0.0275	0.0214	0.0297	0.0263	0.0266
1.8	53.4	0.0109	0.0129	0.0113	0.0138	0.0124	0.0125
2.0	60.0	0.00390	0.00381	0.00414	0.00406	0.00356	0.00360
2.2	66.7	0.00118	0.00089	0.00138	0.00103	0.00076	0.00078
2.4	73.6	0.00035	0.00025	0.00048	0.00039	0.00015	0.00017
2.6	81.1	0.00010	0.00009	0.00017	0.00020	0.00003	0.00004

specified. Certainly we can also expand G_{eik} in G_t as

$$G_{eik} = G_t + G_t(G_t^{-1} - G_{eik}^{-1})G_{eik}$$
$$= G_t + G_t(G_t^{-1} - G_{eik}^{-1})G_t + \cdots . \quad (3.15)$$

Therefore, to the lowest order we may try the form

$$G \approx G_{eik} + G_{eik}(C+D)G_t \quad (3.16)$$

or

$$G \approx G_{eik} + G_t(C+D)G_t \quad (3.17)$$

in the wave function Ψ_i of (2.16) and in T of (2.18).

The form for G_t is dictated by the need for a correction at large angles, where the linearization procedure and the evaluation of the phase change along a straight line trajectory make little sense. That is, we are eventually forced to a treatment in the spherical coordinate rather than the cylindrical variables adopted in G_{eik}. Thus, for example, we can try a WKB form[21]

$$G_t \approx -\frac{1}{2\pi}\frac{e^{iK|\vec{R}-\vec{R}'|}}{|\vec{R}-\vec{R}'|}e^{i\chi(\vec{R},\vec{R}';\vec{r})} \equiv G_{WKB}, \quad (3.18)$$

with

$$\chi = -\frac{1}{K}\int_{\vec{R}'}^{\vec{R}} V(\vec{R}, s; \vec{r})ds, \quad \vec{s} \equiv \vec{R} - \vec{R}',$$

where ds is along the line $\vec{R}-\vec{R}'$. A still simpler form for G_t may be used, in the form[22]

$$G_t \approx -\frac{1}{2\pi}\frac{e^{iQ(t)|\vec{R}-\vec{R}'|}}{|\vec{R}-\vec{R}'|} \equiv G_{sc} , \quad (3.19)$$

where

$$Q(t) = [K^2 - 2V(\vec{t}, \vec{r})]^{1/2}, \quad \vec{t} \equiv \tfrac{1}{2}(\vec{R}+\vec{R}') .$$

In fact, the form G_{sc} has been shown to be extremely effective when used in the place of G in the amplitude directly without G_{eik} at all. The essential feature of G_{WKB} and G_{sc} is their dependence on $s = |\vec{R}-\vec{R}'|$ rather than on $|z-z'|$. Therefore, even the free Green's function may improve the large angle behavior, with

$$G_t \approx -\frac{1}{2\pi}\frac{e^{iK|\vec{R}-\vec{R}'|}}{|\vec{R}-\vec{R}'|} \equiv G_0 . \quad (3.20)$$

One undesirable point with the above G_t, is, of course, that the use of such G_t requires the $d\vec{R}'$ integration in Ψ_i, as it appears in the amplitude

$$\mathcal{F}_{fi} \approx \mathcal{F}_{fi}^{G} + (\tilde{\Psi}_{f,t}|C|\tilde{\Psi}_i^{eik}) , \quad (3.21)$$

where

$$\tilde{\Psi}_i^{eik} = G_{eik}V\Psi_{i,0} = e^{i\vec{K}_i\cdot\vec{R}}\psi_0(\vec{r})(e^{i\chi_+} - 1) \quad (3.22a)$$

and

$$\tilde{\Psi}_{f,t} = G_t V\Psi_{f,0} \approx \int d\vec{R}'\, G_{WKB}(\vec{R}, \vec{R}'; \vec{r})V\Psi_{f,0}$$

or

$$\tilde{\Psi}_{f,t} \approx \int d\vec{R}'\, G_{sc}(\vec{R}, \vec{R}'; \vec{r})V\Psi_{f,0} . \quad (3.22b)$$

The integral in (3.22b) can be carried out readily in some cases when a simple V is involved. A direct evaluation of $\tilde{\Psi}_{f,t}$ from the scattering equation in the spherical coordinate is also possible which avoids the explicit use of G_t. An approximate way to carry out the integrations involved in (3.21) with G_{WKB} or G_{sc} has been given,[11] which is based on the angle-averaging procedure.

(d) Finally, we consider the question of the range of validity of the Glauber amplitude given in the form (2.32). It is often argued that, since $e^{i\chi_0}$ in the original derivation is replaced by the Γ_i's corresponding to the projectile-target nucleon interactions which are presumably valid for all q, the resulting $\mathcal{F}_{fi}^{G}$ should be better than the form with χ_0. It has never been clear, however, under what conditions this is valid. To study this question, we go back to (3.9) and consider the contribution from χ_1. [Although the form (3.5) may be more valid, (3.9) is sufficient to illustrate the main point.]. We write χ_1 in two terms as[13]

$$\chi_1 = -\frac{1}{K^3}\left(1 + B\frac{\partial}{\partial B}\right)\sum_{i=1}^{N}\int_0^\infty V_i^2 dz$$
$$-\frac{1}{K^3}\left(1 + B\frac{\partial}{\partial B}\right)\sum_{i\neq j}^{N}\int_0^\infty V_i V_j dz$$
$$\equiv \chi_1^{diag} + \chi_1^{off} . \quad (3.23)$$

For the first term χ_1^{diag}, and similarly for the diagonal parts in the higher-order terms χ_n^{diag}, we can still carry out the replacement (2.29) as

$$\Gamma \approx 1 - e^{i\chi^{diag}} = 1 - \prod_{i=1}^{N}(1 - \Gamma_i) , \quad (3.24)$$

where

$$\chi^{diag} = \sum_{n=0}^{\infty}\chi_n^{diag} = \sum_{n=0}^{\infty}\sum_{i=1}^{N}\chi_n^{diag}(i) ,$$

for the diagonal contribution; therefore, by adjusting the individual Γ_i correctly, we can improve the behavior of Γ and thus $\mathcal{F}_{fi}^{G}$ at large angles. This is thus related to the additivity of χ's in $\mathcal{F}_{fi}^{G}$. Obviously, when χ_1^{off} is not negligible in the case of overlapping potentials, the simple additivity assumption breaks down,[14,9,13] and we no longer expect that a mere substitution of (3.24) will improve the behavior of $\mathcal{F}_{fi}^{G}$. Perhaps only when χ_n^{off}, and other related corrections, con-

tribute to χ with sufficiently random phases and large cancellations, we expect that $\chi \approx \chi^{\text{diag}}$ is valid.

It was noted previously[14] that the additivity assumption on χ allows (3.24), and the assumption breaks down with the overlapping potentials; as we discussed above, the question of whether (3.24) would improve $\mathcal{F}_{fi}^{G}$ also depends very critically on the effect of the overlapping interactions and thus on the additivity assumption.

Now, an important point on the application of (3.24). If we rely entirely on the form (3.24) in explaining the scattering data involving nuclear target, then we are essentially missing the crux of the analysis; that is, we would like to learn eventually something about the particle correlations of the target nucleons and thus about χ_1^{off}. On the other hand, if, by inserting the "correct" projectile-nucleon profile function Γ_i, one finds a descrepancy between the cross sections from the theory and experiments, then it may most likely imply two things: (i) the effect of the operator D to be estimated in the next section may be important, and/or (ii) the effect of χ_1^{off} should be included. Of course, this argument presupposes that we have chosen the region of K and q where the amplitudes (2.20), (2.21), and (2.22) dominate. In the above sense, the Glauber theory could be of potential importance in extracting the structure information of the target system. The form (3.5) specifies exactly how such information could come into play in the elastic cross section. On the other hand, if (3.24) alone fits the data well, it only implies, at least in principle, that either the correlation effect is small or Γ_i is parametrized incorrectly, especially its off-shell behavior.

(e) We now consider briefly the convergence property of the series (2.15). The rate of convergence depends on the eigenvalues of the states generated by the kernel

$$\mathcal{K} \equiv G_{\text{eik}}(C+D) \equiv \mathcal{K}_C + \mathcal{K}_D \ . \tag{3.25}$$

Limiting our discussion to the effect of C, we define the strength eigenvalue problem by

$$\mathcal{K}_C \,|\, \zeta_n^C \rangle = \zeta_n^C \,|\, \zeta_n^C \rangle \tag{3.26}$$

with the ϵ factor in G_{eik} kept at $\epsilon > 0$. Then the corresponding series can be characterized by

$$\begin{aligned}(1+GC)_{D=0}\,|\,\zeta_n^C\rangle &= (1+\mathcal{K}_C+\mathcal{K}_C{}^2+\cdots)\,|\,\zeta_n^C\rangle \\ &= (1-\zeta_n^C)^{-1}\,|\,\zeta_n^C\rangle \ ,\end{aligned} \tag{3.27}$$

where the last step is obtained assuming that $|\zeta_n^C| < 1$. Therefore, for each state $|\zeta_n^C\rangle$ we expect the series to diverge whenever its eigenvalue leaves the unit circle, i.e., $|\zeta_n^C| \geq 1$. The magnitude of $\mathcal{K}_C$ may be estimated roughly from (2.14) and (2.12) as

$$\mathcal{K}_C \equiv G_{\text{eik}}\, C = \frac{C}{A + i\epsilon - V}$$

and

$$\|\mathcal{K}_C\| \sim |\chi_1/\chi_0| \ . \tag{3.28}$$

Noting the form (3.9) for χ_1, we then have approximately

$$|\chi_0| \sim |\bar{V}a/K| \tag{3.29a}$$

and

$$|\chi_1| \sim |\bar{V}a/K|\,[\,|\bar{V}/E| + |q/2K|^2\,] \ , \tag{3.29b}$$

thus

$$|\chi_1/\chi_0| \sim |\bar{V}/E| + [\,q/2K\,]^2 \ . \tag{3.30}$$

In (3.29a) and (3.29b), the parameter a corresponds to the range of the average interaction $\bar{V}$. The form (3.30) is very rough, but indicates that the higher-order corrections to χ_0 at small q may go like $|V/E|$, while at large q it may go like $(q/2K)$,[2] which dominates over $|\bar{V}/E|$ at high energies. It also suggests that the series may even be only asymptotically convergent for intermediate values of q. In this connection, it is of some formal interest to study the convergence property more rigorously in the potential scattering.

IV. CORRECTIONS TO THE FIXED-SCATTERER APPROXIMATION

In this section, we consider in more detail the second correction term (2.22) which arises because of the assumption in $\mathcal{F}_{fi}^{G}$ that the scatterer nucleons in the target are fixed during the scattering except for their total center of mass motion relative to the projectile particle. The meaning of this correction is made more transparent by examining the contents of the solution Ψ_i^{eik} and by showing explicitly to what extent the distortion effect is included in the solution. A study similar to this has been carried out by Foldy and Walecka[15] and by others from an opposite point of view in the course of justification of the Glauber procedure. One nice by-product of our study will be the explicit form of the inelastic component of the wave function which may be useful for many other purposes, as illustrated in Sec. V.

(a) The leading correction to the Glauber amplitude $\mathcal{F}_{fi}^{G}$ for the motion of the target particles is

$$\mathcal{F}_{fi}^{D1} \approx (\tilde{\Psi}_f^{\text{eik}}\,|\,D\,|\,\tilde{\Psi}_i^{\text{eik}}) \ , \tag{4.1}$$

where

$$D = H_T - E_0 = T_{\vec{r}} + U(\vec{r}) - E_0, \tag{4.2}$$

$$\tilde{\Psi}_i^{\text{eik}} \equiv \Psi_i^{\text{eik}} - \Psi_{i,0} = e^{i\vec{K}_i\cdot\vec{R}}\psi_0(\vec{r})(e^{i\chi_+} - 1), \tag{4.3}$$

$$\tilde{\Psi}_f^{\text{eik}} \equiv \Psi_f^{\text{eik}} - \Psi_{f,0} = e^{i\vec{K}_f\cdot\vec{R}}\psi_0(\vec{r})(e^{-i\chi_-^*} - 1). \tag{4.4}$$

First of all, we note that D satisfies the orthogonality property

$$PD = DP = 0, \quad \text{thus } D = QDQ, \tag{4.5}$$

where

$$Q = 1 - P, \quad P = |\psi_0(\vec{r})\rangle\langle\psi_0^*(\vec{r}')|. \tag{4.6}$$

That is, the property of D as given by (4.5) shows that the correction (4.1) is concerned only with the inelastic channel component (the Q-space part) of the scattering functions Ψ_i^{eik} and Ψ_f^{eik}. [This does *not* necessarily mean that the elastic component $P\Psi_i^{\text{eik}}$, for example, is correct. It only suggests that the $P\Psi_i^{\text{eik}}$ part has to be corrected through the factor $e^{i\chi_+}$ which is the only component in Ψ_i^{eik} that does not commute with the operator Q.] Explicitly,

$$\mathcal{T}_{fi}^{D1} = \int d\vec{r}\,\psi_0^*(\vec{r})\int d\vec{R}\,e^{-i\vec{K}_f\cdot\vec{R}}(e^{i\chi_-} - 1) \times [H_T - E_0]e^{i\vec{K}_i\cdot\vec{R}}(e^{i\chi_+} - 1)\psi_0, \tag{4.7}$$

where we have included the terms "-1" in the profile functions $(e^{i\chi_-} - 1)$ and $(e^{i\chi_+} - 1)$, although they can be dropped because of the property (4.5). These terms are essential in maintaining the convergence of the $d\vec{R}$ integration. Using the explicit form for χ_+ and H_T as given by (2.25) and (2.3), respectively, (4.7) can be reduced further to a form

$$\mathcal{T}_{fi}^{D1} = \int d\vec{R}\,e^{i\vec{q}\cdot\vec{B}}\int d\vec{r}\,(e^{i\chi_0} - e^{i\chi_+})\sum_{i=1}^{N}\left\{-\tfrac{1}{2}i(\nabla_i^2\chi_+)|\psi_0|^2 + \tfrac{1}{2}|\vec{\nabla}_i\chi_+|^2\cdot|\psi_0|^2 - i\psi_0^*(\vec{\nabla}_i\psi_0)\cdot(\vec{\nabla}_i\chi_+)\right\}, \tag{4.8}$$

where $\vec{\nabla}_i$ is the gradient with respect to the variable $\vec{r}_i$ for the ith target nucleon, and subsequently

$$\mathcal{T}_{fi}^{D1} = \int d\vec{B}\,e^{i\vec{q}\cdot\vec{B}}\int d\vec{r}\sum_{i=1}^{N}\int_{-\infty}^{\infty}dz(e^{i\chi_0} - e^{i\chi_+})\left\{\frac{i}{2K}\left|\psi_0\right|^2\int_{-\infty}^{z}(\nabla_i^2 V)dz' + \frac{1}{2K^2}|\psi_0|^2\left|\int_{-\infty}^{z}\vec{\nabla}_i V\,dz'\right|^2 + \frac{i}{K}\psi_0^*\int_{-\infty}^{z}(\vec{\nabla}_i\psi_0)\cdot(\vec{\nabla}_i V)dz'\right\}. \tag{4.9}$$

Note that the terms in the curly brackets vanish in the limit $z \to -\infty$, while the factor $(e^{i\chi_0} - e^{i\chi_+})$ vanishes at $z \to +\infty$. Also note that (4.9) is independent of the target binding potential U. Its effect is therefore entirely contained in the function ψ_0 insofar as the elastic amplitude is concerned, and manifests itself mainly through the last term in (4.9) involving $(\vec{\nabla}_i\psi_0)$.

The correction amplitude (4.9), together with (3.5), is the main result of the paper. For a given form of the interaction V and ψ_0, both these corrections can be estimated. It is important, however, to apply them only to the region where these corrections dominate. After all, we are eventually interested in learning about the correct V and ψ_0 themselves, *assuming* that $\mathcal{T}_{fi}^{G} + \mathcal{T}_{fi}^{C1} + \mathcal{T}_{fi}^{D1}$ is valid in a limited region of q values, i.e., $q/K \lesssim 1$, or $\Theta \lesssim \frac{1}{3}\pi$.

(b) To further clarify the effectiveness of Ψ_i^{eik} in representing the inelastic effect and the physical contents of the correction term $\mathcal{T}_{fi}^{D1}$, we consider Ψ_i^{eik} in the form

$$\Psi_i^{\text{eik}} = P\Psi_i^{\text{eik}} + Q\Psi_i^{\text{eik}}. \tag{4.10}$$

From (2.24) for Ψ_i^{eik}, we have explicitly

$$P\Psi_i^{\text{eik}} = e^{i\vec{K}_i\cdot\vec{R}}\psi_0(\vec{r})\langle\psi_0|e^{i\chi_+}|\psi_0\rangle \tag{4.11}$$

and

$$Q\Psi_i^{\text{eik}} = e^{i\vec{K}_i\cdot\vec{R}}\psi_0(\vec{r})(e^{i\chi_+} - \langle\psi_0|e^{i\chi_+}|\psi_0\rangle). \tag{4.12}$$

Expansion of the exponential $e^{i\chi_+}$ in (4.11) gives

$$\langle\psi_0|e^{i\chi_+}|\psi_0\rangle = 1 + i\langle\chi_+\rangle + \left(\frac{i^2}{2!}\right)\langle\chi_+\rangle^2 + \left(\frac{i^3}{3!}\right)\langle\chi_+\rangle^3 + \cdots + \frac{i^2}{2!}\langle\chi_+ Q\chi_+\rangle + \frac{i^4}{4!}\langle\chi_+ Q\chi_+\rangle^2 + \cdots + \frac{2i^3}{3!}\langle\chi_+\rangle\langle\chi_+ Q\chi_+\rangle + \frac{3i^4}{4!}\langle\chi_+\rangle^2\langle\chi_+ Q\chi_+\rangle + \cdots + \frac{i^3}{3!}\langle\chi_+ Q\chi_+ Q\chi_+\rangle + \cdots + \frac{2i^4}{4!}\langle\chi_+\rangle\langle\chi_+ Q\chi_+ Q\chi_+\rangle + \frac{i^4}{4!}\langle\chi_+ Q\chi_+ Q\chi_+ Q\chi_+\rangle + \cdots, \tag{4.13}$$

where we used the notation

$$\langle \mathcal{O} \rangle \equiv \langle \psi_0 | \mathcal{O} | \psi_0 \rangle .$$

Obviously, $P\Psi_i^{eik}$ thus contains a great deal of the inelastic components, all in the approximation of linearized Green's function G_{eik}, as each χ_+ involves a path-ordered integration over dz. How much of the Q component is left out in Ψ_i^{eik} is simply given to the leading order by $\mathfrak{F}_{fi}^{D_1}$ of (4.9).

The optical potential approach[16,19,20] to elastic scattering incorporates the effect of inelastic channels through a nonlocal potential. Although Ψ_i^{eik} is given in an extremely simple form, we have seen in (4.13) that the solution contains almost all the information carried usually by the optical potential. To show this further explicitly, we write (2.23) as a set of two coupled equations

$$[\vec{K}_0 \cdot (i\vec{\nabla}_{\vec{r}} + \vec{K}_a) - V_{PP}] P\Psi_i^{eik} = PVQ\Psi_i^{eik} , \tag{4.14a}$$

$$[\vec{K}_0 \cdot (i\vec{\nabla}_{\vec{r}} + \vec{K}_a) - V_{QQ}] Q\Psi_i^{eik} = QVP\Psi_i^{eik} , \tag{4.14b}$$

We can seek a solution of (4.14) in the form

$$P\Psi_i^{eik} \equiv e^{i\vec{K}_i \cdot \vec{R}} \psi_0(\vec{r}) e^{i\chi_+^P} \varphi_P(\vec{R}) , \tag{4.15a}$$

$$Q\Psi_i^{eik} \equiv e^{i\vec{K}_i \cdot \vec{R}} e^{i\chi_+^Q} \varphi_Q(\vec{R}, \vec{r}) \equiv G_{eik}^Q QVP\Psi_i^{eik} , \tag{4.15b}$$

where

$$\chi_+^P = -\frac{1}{K} \int_{-\infty}^{z} V_{PP}\, dz' \quad \text{and} \quad \chi_+^Q = -\frac{1}{K} \int_{-\infty}^{z} V_{QQ}\, dz' .$$

Substitution of (4.15) into (4.14) gives a set of coupled equations for φ_P and φ_Q, as

$$i\vec{K}_0 \cdot \vec{\nabla}_{\vec{R}} \varphi_P(\vec{R}) = e^{-i\vec{K}_i \cdot \vec{R}} e^{-i\chi_+^P} \langle \psi_0 | V | Q\Psi_i^{eik} \rangle$$

$$= e^{-i\chi_+^P} \langle \psi_0 | V e^{i\chi_+^Q} | \varphi_Q \rangle \tag{4.16a}$$

and

$$i\vec{K}_0 \cdot \vec{\nabla}_{\vec{R}} \varphi_Q(\vec{R}, \vec{r}) = e^{-i\vec{K}_i \cdot \vec{R}} e^{-i\chi_+^Q} QVP\Psi_i^{eik}$$

$$= e^{-i\chi_+^Q} QVP\psi_0 e^{i\chi_+^P} \varphi_P(\vec{R}) . \tag{4.16b}$$

Converting to the integral equations, (4.16) become

$$\varphi_P(\vec{R}) = 1 - \frac{i}{K} \int_{-\infty}^{z} dz' e^{-i\vec{K}_i \cdot \vec{R}'} e^{-i\chi_+^{P'}} \langle \psi_0 | V | Q\Psi_i^{eik} \rangle$$

$$= 1 - \frac{i}{K} \int_{-\infty}^{z} dz' e^{-i\chi_+^{P'}} \langle \psi_0 | V e^{i\chi_+^{Q'}} | \varphi_Q' \rangle \tag{4.17a}$$

and

$$\varphi_Q(\vec{R}, \vec{r}) = -\frac{i}{K} \int_{-\infty}^{z} dz' e^{-i\chi_+^{Q'}} QVP\psi_0 e^{i\chi_+^{P'}} \varphi_P(\vec{R}') . \tag{4.17b}$$

Uncoupling the equations, we have finally

$$\varphi_P = 1 - \frac{i}{K} \int_{-\infty}^{z} dz' e^{-i\vec{K}_i \cdot \vec{R}'} e^{-i\chi_+^{P'}} \langle \psi_0 | VG_{eik}^Q V | \psi_0 \rangle e^{i\vec{K}_i \cdot \vec{R}''} e^{i\chi_+^{P''}} \varphi_P(\vec{R}'')$$

$$= 1 - \frac{1}{K^2} \int_{-\infty}^{z} dz' e^{-i\chi_+^{P'}} \langle \psi_0 | PVQe^{i\chi_+^{Q'}} \int_{-\infty}^{z'} dz'' e^{-i\chi_+^{Q''}} QVP | \psi_0 \rangle e^{i\chi_+^{P''}} \varphi_P(\vec{R}'') \tag{4.18a}$$

and

$$\varphi_Q = -\frac{i}{K} \int_{-\infty}^{z} dz' e^{-i\chi_+^{Q'}} QVP\psi_0 e^{i\chi_+^{P'}} - \frac{1}{K^2} \int_{-\infty}^{z} dz' e^{-i\chi_+^{Q'}} QVP | \psi_0 \rangle e^{i\chi_+^{P'}} \int_{-\infty}^{z'} dz'' e^{-i\chi_+^{P''}} \langle \psi_0 | PVQe^{i\chi_+^{Q''}} \varphi_Q(\vec{R}'', \vec{r}') . \tag{4.18b}$$

This set of equations is obviously much more involved than (2.23), and in general requires an iterative procedure to obtain solutions. The reason why a simple solution of the form (2.24) can be obtained is of course that the kernels in (4.18) are of very special types.

First of all, we note in (4.18) that all the factors $\exp(\pm i\vec{K}_i \cdot \vec{R})$ cancelled out conveniently. This is a result of the approximation $D = H_T - E_0 \approx 0$, i.e., $\overline{QH_TQ} \equiv \bar{E}_Q \approx E_0$. In general, the wave number in the Q channel will be affected somewhat, but $\bar{E}_Q \approx E_0$ may still be a reasonable approximation if $E \gg E_0, \bar{E}_Q$. Secondly, $P\Psi_i^{eik}$ is affected by the Q component through φ_P being dependent on G_{eik}^Q in (4.18a). In fact, the factor $\langle \psi_0 | VG_{eik}^Q V | \psi_0 \rangle$ is the dispersion correction to V_{PP}, in the eikonal approximation. Thus, the original Ψ_i^{eik} fully incorporates the optical potential within the eikonal formalism. Thirdly, it is important to note, from a practical point of view, that (4.18a) *can* be readily solved for φ_P, because it is a function of one variable $\vec{R}$. This is *not* the case with φ_Q, which depends on both $\vec{R}$ and $\vec{r}$ variables. In any case φ_Q, which satisfies (4.18b), is such that

(4.15b) should be identical to (4.12). Thus, the solution of (4.18a) can be written simply as

$$\varphi_P = 1 - \frac{i}{K}\int_{-\infty}^{z} dz' e^{-i\chi_+^{P'}} \langle \psi_0 | V e^{i\chi_+'} | \psi_0 \rangle + \frac{i}{K}\int_{-\infty}^{z} dz' e^{-i\chi_+^{P'}} V_{PP} \langle \psi_0 | e^{i\chi_+'} | \psi_0 \rangle . \quad (4.19)$$

where, of course,

$$V_{PP} \equiv \langle \psi_0 | V | \psi_0 \rangle .$$

It is now a simple matter to check that (4.19) reduces to the original φ_P,

$$\varphi_P = \langle \psi_0 | e^{i\chi_+} | \psi_0 \rangle e^{-i\chi_+^P} \quad (4.20)$$

as expected from (4.11).

The equations (4.17) and (4.18) will be discussed more fully in the next section in comparing with the optical potential approach. One of the main difficulties in the optical potential approach is in accurately estimating the effect of the Q channel. It is therefore of potential importance to have here the explicit representation of $Q\Psi$ in the form $Q\Psi_i^{\text{eik}}$, (4.12).

(c) The convergence of the series (2.15) corresponding to the perturbation D depends, as with the operator C, on the kernel $\mathcal{K}_D$ defined by

$$\mathcal{K}_D \equiv G_{\text{eik}} D . \quad (4.21)$$

Thus, by defining the eigenvalue problem

$$\mathcal{K}_D | \zeta_n^D \rangle = \zeta_n^D | \zeta_n^D \rangle , \quad (4.22)$$

we have, for each state,

$$[1 + GD]_{C=0} | \zeta_n^D \rangle = [1 + \mathcal{K}_D + \mathcal{K}_D^2 + \cdots] | \zeta_n^D \rangle = [1 + \zeta_n^D + (\zeta_n^D)^2 + \cdots] | \zeta_n^D \rangle = (1 - \zeta_n^D)^{-1} | \zeta_n^D \rangle , \quad (4.23)$$

where the last expression in (4.23) is obtained only if $|\zeta_n^D| < 1$. Therefore, for each $|\zeta_n^D|$ which leaves the unit circle, we expect the series to diverge. Actual estimate of ζ_n^D is more involved due to the complexity of the expression (4.9). However, physically we expect that the effect of D is again expected to be large for the large momentum transfer region, where the high momentum target nucleons can collide with the projectile to cause a large q transfer. Thus, the average quantity

$$\bar{D} = \bar{E}_Q - E_0$$

is only expected to be large for large q. [We conjecture that $\bar{D}$ may have the similar q dependence as (3.30), but more detailed information is not yet available to make a definite statement, except the fact that the above picture is consistent with the behavior of $Q\Psi_i^{\text{eik}}$ which is rather effective in the small q region.]

The original expansion of G, Eq. (2.15), contains, besides (3.27) and (4.23), also the mixed terms involving both the operators C and D. It would be interesting to examine if, for certain cases, the C and D corrections may partially cancel.

V. MULTIPLE SCATTERING THEORY AND OPTICAL POTENTIALS: AN ALTERNATIVE APPROACH

The multiple diffraction theory of Glauber, as given by (2.32), has been applied to many scattering systems. As many of the corrections to the theory were not known, there have been attempts to understand the unexpected success of the theory in many cases and to estimate the corrections when deviations occurred. Paralleling these attempts, there has been a renewed interest in the multiple scattering theory of Watson,[19] Kerman, McManus, and Thaler,[20] and the optical potential approach. More recently, the optical potential has been converted by Feshbach *et al.*[16] into an additional set of coupled equations to be combined with the elastic equation. The various parameters in this set of equations are evaluated in terms of the target-particle correlation functions. A more systematic construction of the scattering equations has been given,[18] where many of the details have been further refined, and the theory compared with a still another approach[17] to the coupled-channel method. The distinct advantage of this recent approach with the optical potential is that the target structure part of the problem is more or less treated separately in the evaluation of parameters in the coupled equations. The scattering part of the problem is then treated as a second step in which the target variable is completely integrated out. This is in strong contrast to the Glauber theory in which the structure and scattering problems are both treated in a combined way, thus making it difficult to obtain corrections.

In the present section, we compare these two seemingly different approaches and show that they are in fact the same theory formulated in two different ways. We have seen a rough indication of this already in the analysis presented in Sec. IV.

For elastic scattering by a composite target, the scattering equation can be rewritten such that the elastic (P) and the inelastic (Q) components of Ψ are treated separately by a set of coupled equations. Following Feshbach,[23] we have

$$P[H - E]P\Psi = -PVQ\Psi , \quad (5.1)$$

$$Q[H - E]Q\Psi = -QVP\Psi \quad (5.2)$$

The above differential equations are converted into integral equations by introducing the appropriate Green's functions, as

$$P\Psi = P\Psi^P + PG^P PVQ\Psi\,, \tag{5.3a}$$

$$Q\Psi = G^Q QVP\Psi\,, \tag{5.3b}$$

where

$$P(H-E)P\Psi^P = 0\,,$$

$$P(H-E)PG^P P = -P \quad \text{or } G^P = [P(E+i\epsilon - H)P]^{-1} \tag{5.4}$$

and

$$G^Q = [Q(E+i\epsilon - H)Q]^{-1}\,. \tag{5.5}$$

The uncoupling of the equations is carried out using (5.3), as

$$P[H-E+VG^Q V]P\Psi = 0, \qquad \mathcal{U}_{\rm opt} = \langle\psi_0|\, V + VG^Q V\,|\psi_0\rangle \tag{5.6a}$$

$$Q[H-E+VG^P V]Q\Psi = -QVP\Psi^P \tag{5.6b}$$

or

$$P\Psi = P\Psi^P + PG^P PVG^Q VP\Psi\,, \tag{5.7a}$$

$$Q\Psi = G^Q QVP\Psi^P + G^Q QVPG^P PVQ\Psi\,. \tag{5.7b}$$

To complete the formal manipulations, we write

$$P\Psi^P = P\Psi_0 + G^P PVP\Psi_0\,, \tag{5.8}$$

where

$$P[T_{\vec{R}} - E_0']P\Psi_0 = 0, \quad P\Psi_0 \equiv \Psi_{i,0}\,.$$

All these results are well known.

We immediately find that the linearization of G^P and G^Q reduces, with

$$D \equiv H_T - E \to 0\,, \tag{5.9}$$

$$\begin{aligned} G^P &\to PG^P_{\rm eik}P, \\ G^Q &\to QG^Q_{\rm eik}Q\,, \end{aligned} \tag{5.10}$$

and the relevant equations in Sec. V approach the corresponding equations in Sec. IV. This establishes the explicit connection between the Glauber scattering function $\Psi_i^{\rm eik}$ given by (2.24) and (2.25), and the optical potential approach of (5.6). A similar discussion was also given by Foldy and Walecka,[15] in the reverse order, in the derivation of (2.24).

The multiple scattering theory further converts (5.1) and (5.2) into a set which involves the scattering operator τ in the place of V, where τ is defined by ($\mathcal{U} \equiv V_i$)

$$\tau = \mathcal{U} + \mathcal{U}\frac{\mathcal{Q}}{E+i\epsilon - H_0}\tau, \quad H_0 \equiv H - V = T_{\vec{R}} + H_T(\vec{r})\,.$$

The modified set of equations are[16]

$$\begin{aligned} P[T_{\vec{R}} + H_T + (N-1)\tau - E]P\Psi^\tau &= -(N-1)P\tau Q\Psi^\tau\,, \\ Q[T_{\vec{R}} + H_T + (N-1)\tau - E]Q\Psi^\tau &= -(N-1)Q\tau P\Psi^\tau \end{aligned} \tag{5.11}$$

with the amplitude now defined by

$$\mathcal{F}_{fi} = N\langle\Psi_{f,0}|\,\tau\,|\Psi^\tau\rangle\,. \tag{5.12}$$

The parametrization of τ is done by first converting it in terms of t_i, defined by

$$\tau \equiv \frac{1}{N}\sum_{i=1}^{N}\tau_i = \frac{1}{N}\sum_i t_i\,\mathcal{Q} + \frac{1}{N}\sum_i t_i\frac{\mathcal{Q}-1}{E+i\epsilon - H_0}\tau_i\,, \tag{5.13}$$

and also by

$$t_i = \mathcal{U}_i + \mathcal{U}_i\frac{1}{E+i\epsilon - H_0}t_i\,, \tag{5.14}$$

which is still a $(N+1)$-body operator, but without the antisymmetrization for the intermediate target states. Finally, t_i is replaced by its impulse approximation

$$t_i \sim t_i^F = V_i + V_i\frac{1}{E^{(+)} - T_{\vec{R}} - T_{F_i}}t_i^F\,, \tag{5.15}$$

where t_i^F is now a free two-body operator.

Obviously, the amplitude t_i^F is to be obtained approximately from the on-shell amplitude, and thus its off-shell extension involves the usual ambiguity. However, it seems that (5.15) is much more *direct* in parametrizing V_i for use in (5.1) and (5.2). This is the procedure followed for example in Ref. 17.

In any case, the equation for $Q\Psi^\tau$ in (5.11) cannot be solved so readily, because the Q space involves all the states of $H_T(\vec{r})$ except ψ_0. This problem is avoided in the coupled channel formulation[16-18] by introducing an averaging procedure such that all the inelastic channels projected by Q are replaced by a single "effective" channel. After the τ operator is manipulated away in terms of t_i, and expanding the optical potential in $(N-1)t_i$ and resumming it to *all* orders after making certain separable approximations, we obtain[16-18]

$$\begin{aligned} [T_{\vec{R}} + V_{00}^t - E_0']u_0(\vec{R}) &= -A_C(\vec{R})w(\vec{R})\,, \\ [T_{\vec{R}} + \bar{V}_Q - \bar{E}_Q']w(\vec{R}) &= -A_C(\vec{R})u_0(\vec{R})\,. \end{aligned} \tag{5.16}$$

The main part of the theory is thus in the determination of the parameters $A_C, \bar{V}_Q$, and $\bar{E}_Q'$. They are related to the two- and three-particle correlation functions for the target nucleons. However, once these parameters are determined, the solution of (5.16) can be obtained without difficulty by any of the several methods, including the partial

wave analysis.

The obvious drawback of the theory is, however, in the determination of these parameters in (5.16). A certain amount of ambiguity persists in the derivation of the coupled equations, and various alternatives and their corrections were considered in Ref. 18. The problem is obviously in handling the Q-space effect, and, in view of the simple form for $Q\Psi$ given by (4.12) in the eikonal approximation, we may modify the coupled-channel approach in the following way by combining the desirable feature of the both theories.

Apparently, the multiple diffraction theory obtains the $Q\Psi$ part very simply, while the optical potential approach explicitly includes the $P\Psi$ part at all angles. Combining these features of the two approaches, we may set (5.1) as

$$P(H-E)P\Psi \approx -PVQ\Psi_i^{\mathrm{eik}} \tag{5.17}$$

and

$$Q\Psi \approx Q\Psi_i^{\mathrm{eik}} = e^{i\vec{K}_i\cdot\vec{R}}\psi_0(\vec{r})[e^{i\chi_+} - \langle\psi_0|e^{i\chi_+}|\psi_0\rangle]\,. \tag{5.18}$$

Thus,

$$P\Psi \approx P\Psi^P + G^P PVQ\Psi_i^{\mathrm{eik}} \equiv P\Psi^P + P\Psi^{Q\,\mathrm{eik}}, \tag{5.19}$$

where both $P\Psi^P$ and G^P are for two-particle scattering, and thus can be evaluated very accurately, either using the eikonal perturbation theory,[12] the partial-wave expansion, or using the semiclassical approximation.[22] The system (5.18) and (5.19) should improve the large-angle behavior of $\mathcal{F}_{fi}$ as well as eliminate the complicated and often ambiguous procedure of evaluating the $Q\Psi$ part. Furthermore, a previous study[17] of the contribution of $Q\Psi$ indicated that the V_{PP} part seems to dominate the large-angle behavior while the $Q\Psi$ part mainly influences the cross section at small angles through the imaginary part of the elastic amplitude. If this trend persists for scattering systems in general, then the treatment of the $Q\Psi$ part by the linearized propagator is partially justified. (This feature depends on the coupling potentials QVP and PVQ, and also on the ImV.)

If we let

$$P\Psi = \psi_0(\vec{r})u_0(\vec{R}), \quad P\Psi^P = \psi_0(\vec{r})u_0^P(\vec{R}),$$
$$G^P = \psi_0(\vec{r})g^P(\vec{R},\vec{R}')\psi_0^*(\vec{r}'),$$

then

$$\begin{aligned} u_0(\vec{R}) &= u_0^P(\vec{R}) + \int d\vec{R}'\,g^P[\langle\psi_0|Ve^{i\chi_+}|\psi_0\rangle \\ &\qquad - V_{PP}\langle\psi_0|e^{i\chi_+}|\psi_0\rangle]e^{i\vec{K}_i\cdot\vec{R}'} \\ &= u_0^P(\vec{R}) + \int d\vec{R}'\,g^P(\vec{R},\vec{R}')W_{00}(\vec{R}')e^{i\vec{K}_i\cdot\vec{R}'}, \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} W_{00} &= \langle\psi_0|Ve^{i\chi_+}|\psi_0\rangle - V_{PP}\langle\psi_0|e^{i\chi_+}|\psi_0\rangle \\ &= iKe^{i\chi_+^P}\frac{d}{dz}[e^{-i\chi_+^P}\langle\psi_0|e^{i\chi_+}|\psi_0\rangle], \end{aligned} \tag{5.21}$$

and also

$$u_0^P(\vec{R}) = e^{i\vec{K}_i\cdot\vec{R}} + \int d\vec{R}'\,g^P(\vec{R},\vec{R}')V_{00}(\vec{R}')e^{i\vec{K}_i\cdot\vec{R}'}. \tag{5.22}$$

As noted earlier, both u_0 and u_0^P can be evaluated by first partial-wave analyzing the quantities in (5.20) and (5.22). On the other hand, we can always obtain a reasonable representation of g^P and u_0^P by the eikonalization, with the corrections of χ_1 type as discussed in Sec. III. This procedure is much simpler here because only V_{PP} is involved rather than the full $V(\vec{r},\vec{R})$. Still another possibility is to use the g^P in the semiclassical approximation[22] as

$$g^P \approx -\frac{1}{2\pi}e^{i\kappa^P(t)|\vec{R}-\vec{R}'|}/|\vec{R}-\vec{R}'|, \tag{5.23}$$

where

$$\kappa^P(t) = [K^2 - 2V_{PP}(t)]^{1/2}, \quad t = \tfrac{1}{2}|\vec{R}+\vec{R}'|.$$

This form has been shown to work extremely well at all angles,[22,11] especially when the scattering energy is high enough so that $\kappa^P(t)$ is almost always real and positive (so that no part of the configuration space is forbidden classically).

The elastic amplitude is now given in three pieces, as

$$\mathcal{F}_{fi} \approx \mathcal{F}_{fi}^P + \mathcal{F}_{fi}^{PQ} + \mathcal{F}_{fi}^Q, \tag{5.24}$$

where

$$\begin{aligned} \mathcal{F}_{fi}^P &= (\Psi_{f,0}^-|V|P\Psi^P) = \int d\vec{R}\,e^{-i\vec{K}_f\cdot\vec{R}}V_{00}u_0^P \\ &= \int d\vec{R}\,e^{-i\vec{K}_f\cdot\vec{R}}V_{00}\left[1+\int d\vec{R}'g^PV_{00}\right]e^{i\vec{K}_i\cdot\vec{R}'}, \end{aligned} \tag{5.25}$$

$$\begin{aligned} \mathcal{F}_{fi}^{PQ} &= (\Psi_{f,0}^-|V|P\Psi^{Q,\mathrm{eik}}) \\ &= \int d\vec{R}\,e^{-i\vec{K}_f\cdot\vec{R}}V_{00}\int d\vec{R}'g^P(\vec{R},\vec{R}')W_{00}(\vec{R}')e^{i\vec{K}_i\cdot\vec{R}'}, \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} \mathcal{F}_{fi}^Q &= (\Psi_{f,0}^-|V|Q\Psi_i^{\mathrm{eik}}) \\ &= \int d\vec{R}\,e^{-i\vec{K}_f\cdot\vec{R}}W_{00}e^{i\vec{K}_i\cdot\vec{R}}\,. \end{aligned} \tag{5.27}$$

[In (5.26) and (5.27), additional distortions could be included.[17]]

A crude estimate of the difference between (5.24) and the Glauber amplitude may be obtained by including the corrections of the type (2.21) only for the P part. That is, (5.24) may be approximated, for the region $q/K \lesssim 1$, by

$$\mathfrak{F}_{fi} \approx \mathfrak{F}_{fi}^{G} + \mathfrak{F}_{fi}^{C1P}, \tag{5.24a}$$

where the first term on the right hand side is the Glauber amplitude defined by (2.21), and

$$\mathfrak{F}_{fi}^{C1P} = \mathfrak{F}_{fi}^{C1}(V \to V_{PP})$$
$$= \frac{1}{K^2}\int d\vec{B}\, e^{i\vec{q}\cdot\vec{B}}\, e^{i\chi_0^P}\left(1 + B\frac{\partial}{\partial B}\right)$$
$$\times \int_0^\infty dz\, V_{PP}{}^2(\vec{B}, Z),$$

which follows from (3.7). Note that in this correction term, the dependence on the internal target variables is completely eliminated.

Some indications on the effectiveness of the system (5.18) and (5.19) may be seen from a recent comparison of the experiment with the various theoretical predictions made by Teubner, Lloyd, and Weigold.[24] For the electron-hydrogen elastic scattering at 50 eV energy, the Glauber amplitude calculated by Tai, Teubner, and Bassel[25] is seen to fit better at small angles, while the close-coupling calculation of Burke, Shey, and Smith[26] gave a much better fit at large angles ($\Theta \gtrsim 50°$). This strongly suggests that, at large angles, the $P\Psi^P$ part in (5.19) may not have been treated accurately enough in the Glauber amplitude, while the $Q\Psi$ part given by (5.18) may be adequate. The result seems to favor the separate treatment of the $P\Psi$ and $Q\Psi$ parts, as defined by (5.18) and (5.19). A recent calculation[17] using a set of coupled equations and effective channels also obtains the behavior similar to the result of Ref. 26 at large angles.

Note that, throughout the expressions (5.25)–(5.27), the $\vec{r}$ dependence is completely eliminated and only two potentials, $V_{PP} \equiv V_{00}$ and W_{00}, enter. There are no more equations to solve, and the amplitude $\mathfrak{F}_{fi}$ is completely reduced to quadratures. Admittedly, (5.24) is more involved than $\mathfrak{F}_{fi}^{G}$ of (2.27), but, presumably, we have in (5.24) already incorporated the major part of the corrections coming from the operator C to all orders (once a precise form of g^P is obtained). In view of the fact that $\mathfrak{F}_{fi}^{C1}$ and higher order contributions are rather involved to evaluate and also the fact that such a series may not converge very fast at larger angles, the slight complication in (5.24) may be more than justified.

If we were to compare (5.24) with (3.24), instead, which presumably contains the higher-order corrections from C insofar as the nonoverlapping part of the potential is concerned, the overlapping potential effect is correctly treated in (5.24) mainly through the g^P part of the wave function.

Thus, the main remaining correction to the amplitude (5.24) should come from $\mathfrak{F}_{fi}^{D1}$ as given in Sec. IV. As evident from the form for $Q\Psi_i^{eik}$, the adjustment in the energy was not made for the inelastic channel so that the oscillating part of $Q\Psi_i^{eik}$ is still given by $\exp(i\vec{K}_i\cdot\vec{R})$, which is definitely not correct. In this connection it is of interest to examine whether $Q\Psi_i^{eik}$ of (2.24) may be replaced by a slightly different form obtained with the choice (2.10). That is, following (3.10),

$$Q\Psi_{a,i}^{eik} = e^{i\vec{K}_a\cdot\vec{R}}\psi_0(\vec{r})[e^{i\chi_{+a}} - \langle\psi_0|e^{i\chi_{+a}}|\psi_0\rangle]. \tag{5.28}$$

We conjecture that perhaps the gross effect of the operator D may be incorporated by a further modification of $Q\Psi_a^{eik}$ as

$$Q\Psi_i^{eik} = e^{i\vec{K}_i\cdot\vec{R}}\psi_0(\vec{r})[e^{i\chi_{+i}} - \langle\psi_0|e^{i\chi_{+i}}|\psi_0\rangle], \tag{5.29}$$

where

$$K_i = K_Q\hat{K}_a, \quad K_Q = [2(E - \bar{E}_Q)]^{1/2},$$
$$\chi_{+i} = -\frac{1}{K_i}\int_{-\infty}^{z} V\, dz'. \tag{5.30}$$

In (5.30), $\bar{E}_Q$ is an average inelastic excitation energy. An estimate of $\bar{E}_Q$ may be obtained from the expression[17]

$$\bar{E}_Q \approx \frac{(P\Psi_i^P|P\hat{V}QH_TQ\hat{V}P|P\Psi_i^P)}{(P\Psi_i^P|P\hat{V}Q\hat{V}P|P\Psi_i^P)}, \tag{5.31}$$

where $\hat{V}$ is a smooth function similar to V and couples the P and Q components. The new approach to high-energy scattering by composite targets presented above, and possible further modifications suggested by (5.29), for example, is certainly an attractive possibility. But its effectiveness and potential usefulness are yet to be carefully examined, perhaps by applying the formalism to specific systems and to some realistic model problems. It is of crucial importance to investigate the range of validity of the theory in energy and momentum transfer variables.

Extensions of the system of equations (5.18) and (5.19) to include the exchange effect may be made by explicitly antisymmetrizing the $P\Psi$ part, but neglecting it in the $Q\Psi$ part. Such a procedure is justified only if the $Q\Psi$ contribution comes mainly from the long range part of the interaction affecting only the small momentum transfer region.

We can immediately extend the above treatment

of the elastic scattering to inelastic excitation of the target. We denote the initial and final target state projections by

$$P_i = |\psi_i\rangle\langle\psi_i^*| \quad \text{and} \quad P_f = |\psi_f\rangle\langle\psi_f^*| \tag{5.32}$$

and divide the full target space into two parts as

$$1 = P + Q\,, \tag{5.33}$$

where

$$P = P_i + P_f\,, \quad P_i P_f = 0\,, \quad P_i Q = P_f Q = 0\,.$$

Then (5.1) and (5.2) are now replaced by "three" coupled equations:

$$P_i(H-E)P_i\Psi = -P_i V P_f \Psi - P_i V Q \Psi\,, \tag{5.34a}$$

$$P_f(H-E)P_f\Psi = -P_f V P_i \Psi - P_f V Q \Psi\,, \tag{5.34b}$$

and

$$Q(H-E)Q\Psi = -QVP_i\Psi - QVP_f\Psi\,. \tag{5.34c}$$

Again the problem with the set (5.34) is the $Q\Psi$ which satisfies (5.34c), itself an infinite set of coupled equations.

Following the general philosophy of the approach advocated in (5.18) and (5.19), we again adopt the $Q\Psi$ part from the eikonal approximation and set

$$Q\Psi \approx Q\Psi_i$$

$$= e^{i\vec{K}_i\cdot\vec{R}}\psi_i(\vec{r})[e^{i\chi_+} - \langle\psi_i|e^{i\chi_+}|\psi_i\rangle - \langle\psi_f|e^{i\chi_+}|\psi_i\rangle]\,, \tag{5.35}$$

which is to be inserted into the right hand side of (5.34a) and (5.34b), and obtain a coupled set of integral equations

$$P_i\Psi \approx P_i\Psi^{P_i} + P_i G^{P_i} P_i V P_f \Psi + P_i G^{P_i} P_i V Q \Psi_i^{eik}\,,$$

$$P_f\Psi \approx P_f\Psi^{P_f} + P_f G^{P_f} P_f V P_i \Psi + P_f G^{P_f} P_f V Q \Psi_i^{eik}\,. \tag{5.36}$$

The set (5.36) *can* be solved for $P_i\Psi$ and $P_f\Psi$ rather accurately for all angles by one of several methods, just as with (5.19). Therefore, again, we have treated the direct channel components $P_i\Psi$ and $P_f\Psi$ very accurately, while the complicated $Q\Psi$ effect is simply obtained from the diffraction result. Further refinements of the above procedure are possible, as indicated already in connection with the elastic scattering.

We should remark on possible advantages of the above approach when the effects of isospin and spin effects are to be included. Since χ_+ and $Q\Psi_i^{eik}$ are extremely complicated quantities, we think that the $Q\Psi$ part may be effectively treated by a spin, isospin averaged procedure, while the P_i and P_f components incorporate these effects explicitly. Furthermore, the formalism such as (5.35) and (5.36) should be useful in cases when one or more resonance states dominate the scattering in a certain energy region, especially when such a resonance is caused in effect by the projectile-nucleon interactions contained in V (as in the pion-nucleus scattering.) We will elaborate on this in a later report.

Finally, we note that the system of equations (5.18) and (5.19) derived here is closely related to the formalism presented in the second paper of Ref. 17. To simulate the high-energy fast scattering, we postulated there an ansatz for the scattering function in the form

$$\Psi \approx \psi_0(\vec{r})u_0(\vec{R}) + y_t(\vec{R},\vec{r})\chi(\vec{r})\,, \tag{5.37}$$

where the impulse function y_t is to be constructed, for example, as

$$y_t \approx \int d\vec{R}'\, g_0(\vec{R},\vec{R}')[V - \langle V\rangle_{\vec{R}}]e^{i\vec{K}_t\cdot\vec{R}' - \gamma R'}\,. \tag{5.38}$$

In (5.38), g_0 is the free Green's function and $\langle V\rangle_{\vec{R}}$, defined by

$$\langle V\rangle_{\vec{R}} = \int d\vec{R}\, V(\vec{r},\vec{R})e^{i\vec{q}\cdot\vec{R}}\,,$$

is added to minimize the double counting with u_0. Evidently, with the approximation $g_0 \to g_0^{eik}$ and $\chi = \psi_0$, we recover the result (5.18) and (5.19). By requiring u_0 and χ to satisfy the coupled scattering equations further, however, we will again improve on the Glauber amplitude and (5.24).

VI. DISCUSSION

We have obtained in Sec. II a formally exact expansion of the Green's function and the elastic scattering amplitude, the leading term of which is designed to give the Glauber amplitude, while the next dominant terms provide a simple estimate of the corrections needed to extend the region of applicability of the theory to angles $\Theta \lesssim 60°$ (or $q \lesssim K$). Our main result is the expansions (2.15) and (2.18), and the explicit correction amplitudes (3.5) and (4.9), which are to be combined with (2.27).

In connection with the correction to the linearization, it has also been pointed out that the particular diffraction amplitude (3.24) may contain some parts of the C-type corrections, thus possibly extending the angular region of validity of $\mathfrak{F}_{fi}^{G}$. For this to be effective, the contribution from the overlapping interactions should have sufficiently random phases to cancel each other out. This seems to preclude the possibility of studying the correlation effect using (3.24). But, on the con-

trary, it suggests that any discrepancy between the prediction of (3.24) and the experiment may be parametrized in terms of the correlations, provided the individual profile functions Γ_i are "correctly" inserted.

The apparently slow convergence of the expansion of G and $\mathcal{T}$ in terms of $G_{\rm elk}$ and C at large angles may still pose some difficulty in applications as the higher-order terms are difficult to estimate. Several alternate procedures we have considered in Sec. III should be explored, especially along the lines suggested by (3.16) and (3.17).

The analysis carried out in Sec. IV of the Glauber scattering function should help clarify further some of the reasons for the effectiveness of (2.27) and (3.24). The discussion also helped to compare, in Sec. V, the diffraction theory of Secs. II–IV and the optical potential approach which has been vigorously pursued during the last several years as an alternative. As a result of the study in Sec. V we have suggested presumably an improved version of the theory, as given by (5.24).

Specific applications of the result presented here to pion-nucleus and proton-nucleus scatterings would be of great future interest, as more precise experimental data are becoming available at rapid paces. Specifically, estimates of the correction terms (3.5) and (4.9) will clarify some of the ambiguities and shortcomings of the previous analyses which employed the Glauber theory. Furthermore, additional studies of the theory outlined by (5.24) and its subsequent improvements would help further unify the two approaches.

We have neglected in our discussion the effect of spins and isospins, as well as some of the known relativistic kinematic corrections[21] and binding corrections.[20] These effects are of course important,[16] and should be the next urgent task to be investigated. We can then more seriously consider the off-shell extension of the hadron-nucleon amplitude which is required in the evaluation of the amplitude, but so far neglected in our discussion. Detailed extensions of the similar considerations to inelastic scatterings and to the rearrangement and breakup processes are yet to be worked out.

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